L^p ESTIMATES FOR DEGENERATE ELLIPTIC SYSTEMS WITH VMO COEFFICIENTS

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ABSTRACT. Interior L^p gradient weighted estimates are proved for degenerate elliptic systems in divergence form with VMO coefficients.

§1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 3$) and ω a Muckenhoupt A_2 weight. Let us consider the system

$$(1) \quad L_{ij}u^j = -(a_{ij}^{\alpha\beta}u_{x_\alpha}^j)_{x_\beta} = (f_i^\beta)_{x_\beta}, \quad i, j = 1, 2, \dots, N, \quad \alpha, \beta = 1, 2, \dots, n,$$

where $\frac{f_i^\beta}{\omega}$ belongs to the weighted class $L_\omega^p(\Omega)$ for any $i = 1, \dots, N$, $\beta = 1, 2, \dots, n$, for $1 < p < \infty$. Our aim in this paper is to obtain weighted interior gradient estimates and smoothness for the weak solutions u of system (1), assuming some mild restrictions on the discontinuous coefficients $a_{ij}^{\alpha\beta}$. Namely, we assume the coefficient $a_{ij}^{\alpha\beta}$ to be in the class VMO. Moreover, we assume the system to be degenerate elliptic in the following sense:

$$(2) \quad \begin{aligned} \exists \lambda > 0 : \lambda^{-1}\omega(x)|\xi|^2 &\leq a_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j \leq \lambda\omega(x)|\xi|^2 \\ &\text{for all } \xi \in \mathbb{R}^{nN} \text{ and a.e. } x \in \Omega. \end{aligned}$$

In the uniformly elliptic case ($\omega \equiv 1$), condition (2) is not sufficient for the validity of gradient estimates even for smooth data. On the other hand, it is also known that the result is true under some additional assumptions on the coefficients. The problem has been investigated by many authors in the case of uniformly elliptic equations and systems. In [1] Agmon, Douglis, and Nirenberg showed estimates and regularity assuming the coefficients to be uniformly continuous functions. In [30] Miranda showed the same result assuming a weak differentiability hypothesis on the coefficients, namely, that they belong to the class $W^{1,n}$.

The work of Campanato should also be mentioned, where the problem has been investigated under various assumptions. In [6], the coefficients of the leading part satisfy a Cordes type assumption. In [7], the coefficients of the leading part are Hölder continuous. Later, in [8], Chiarenza and Franciosi showed regularity assuming a Morrey type condition on the first derivatives of the coefficients of the equation.

A significant change of direction was given by Chiarenza, Frasca, and Longo in [10] and [11], where a nonvariational linear elliptic equation was considered under the assumption that the coefficients a_{ij} belong to the Sarason class of vanishing mean oscillation functions (VMO), introduced in [33].

In [10] and [11] it was shown that the integrability of the second order derivatives of the solution is the same as the integrability of the data. The result of [10] and [11]

2010 *Mathematics Subject Classification*. Primary 35J47.

Key words and phrases. Muckenhoupt weights, degenerate elliptic systems, VMO.

was extended to several different settings. The parallel result for the variational case was given in [12]. In [21] and [20], it was proved that the second order derivatives of the solutions belong to the same Morrey class of the data (see also [32]). After that, many extensions have been performed in various directions. We recall an extension to a quasilinear nonvariational elliptic equation by Palagachev in [31], the extension to the Hörmander vector fields by Bramanti and Brandolini in [2]. The parabolic case was investigated by Bramanti and Cerutti in [3]. More recently, related questions were considered by Byun and Wang in [5] and by Krylov and Kim in [28] (see also [29]). In [9] Chiarenza, Franciosi, and Frasca generalized the result of [10] to the case of elliptic systems. Then, in [32] Ragusa extended the results of [9] to the case of elliptic systems with lower order terms. In [13] an elliptic system with respect to Hörmander vector fields has been studied. For more references see e.g. [4, 13, 14, 15, 17, 18, 21, 22, 23, 24, 27].

We establish interior weighted estimates for the gradient of weak solutions of the degenerate elliptic system (1) with A_2 type degeneracy. We refer to [26] for the first regularity theory in the case of A_2 degeneracy.

We follow the technique introduced in [10] (see also [12] and [17] for the variational case). The main point is a clever use of a representation formula for the first derivatives of the solution. Starting with a freezing argument, we obtain an implicit formula in which first derivatives appear on both sides. Then we use the weighted L^p boundedness of some singular operator arising in the formula, and we establish the result for smooth and compactly supported solutions. Finally, a standard covering argument allows us to get the result.

§2. PRELIMINARIES

In this section we collect some definitions and known results. In the sequel we shall denote the ball centered at x_0 with radius $r > 0$ by $B_r(x_0)$ or simply by B_r or B when the center or the radius are clear.

Definition 2.1. Let ω be a nonnegative function locally integrable in $\mathbb{R}^n$, and let $1 < p < +\infty$. We say that ω is an A_p weight and write $\omega \in A_p$ if

$$\sup_B \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B (\omega(x))^{\frac{-1}{p-1}} dx \right)^{p-1} \equiv C_0 < +\infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. The number C_0 is called the A_p constant of ω .

As it is well known, the A_p weights satisfy the doubling property.

Theorem 2.1. Let ω be an A_p weight in $\mathbb{R}^n$, $1 < p < \infty$. Then there exists $C_d > 1$ such that

$$\omega(B_{2r}(x_0)) \leq C_d \omega(B_r(x_0))$$

for every $x_0 \in \mathbb{R}^n$ and $r > 0$, where $\omega(B_r(x_0)) = \int_{B_r(x_0)} \omega dx$.

Any $\omega \in A_p$ gives rise to a measure. We define Lebesgue and Sobolev classes with respect to the measure $\omega(x) dx$.

Definition 2.2. Let Ω be a bounded domain in $\mathbb{R}^n$, let ω be an A_p weight with $1 < p < \infty$, and let k be a positive integer. Suppose $u: \Omega \rightarrow \mathbb{R}^N$ is a locally integrable function. We say that u belongs to the weighted Lebesgue space $L_\omega^p(\Omega, \mathbb{R}^N)$ if

$$(3) \quad \|u\|_{L_\omega^p(\Omega, \mathbb{R}^N)} = \left(\int_\Omega |u(x)|^p \omega(x) dx \right)^{\frac{1}{p}} < +\infty,$$

where $|u(x)|$ is the Euclidean norm in $\mathbb{R}^N$ of $u(x)$.

We say that u belongs to the weighted Sobolev space $W_\omega^{k,p}(\Omega, \mathbb{R}^N)$ if u and its distributional partial derivatives $u_{x_{j_1} x_{j_2} \dots x_{j_k}}$, $1 \leq j_i \leq n$, for all $i = 1, 2, \dots, k$, belong to the weighted Lebesgue class $L_\omega^p(\Omega, \mathbb{R}^N)$.

If $N = 1$, we denote $L_\omega^p(\Omega, \mathbb{R})$ by $L_\omega^p(\Omega)$ and $W_\omega^{k,p}(\Omega, \mathbb{R})$ by $W_\omega^{k,p}(\Omega)$.

The weighted Lebesgue classes $L_\omega^p(\Omega, \mathbb{R}^N)$ are Banach spaces with respect to the norm (3) and Hilbert spaces if $p = 2$. The weighted Sobolev classes $W_\omega^{k,p}(\Omega, \mathbb{R}^N)$ are Banach spaces with respect to the norm

$$\|u\|_{W_\omega^{k,p}(\Omega, \mathbb{R}^N)} = \|u\|_{L_\omega^p(\Omega, \mathbb{R}^N)} + \sum_{i=1}^k \sum_{\{j_1, j_2, \dots, j_i\}} \|u_{x_{j_1} x_{j_2} \dots x_{j_i}}\|_{L_\omega^p(\Omega, \mathbb{R}^N)}.$$

Now we state embedding theorems for weighted Sobolev spaces.

Theorem 2.2. *Let ω be an A_p weight. There exist constants c and $\delta > 0$ such that, for all $u \in C_0^\infty(\Omega)$ and τ satisfying $1 \leq \tau \leq \frac{n}{n-1} + \delta$, we have*

$$\|u\|_{L_\omega^\tau(\Omega)} \leq c \|\nabla u\|_{L_\omega^p(\Omega, \mathbb{R}^n)},$$

where c depends only on n , C_0 , and the diameter of Ω .

Proof. See Theorem 1.3 in [26]. □

Theorem 2.3. *Let $\omega \in A_p$. There exist constants c and $\delta > 0$ such that, for all $u \in C_0^\infty(B_r)$ and τ satisfying $1 \leq \tau \leq \frac{n}{n-1} + \delta$, we have*

$$\left\| \int_{B_r} \frac{u(y)\omega(y)}{|x-y|^{n-1}} dy \right\|_{L_\omega^\tau(\mathbb{R}^n)} \leq cr[\omega(B_r)]^{\frac{1}{p\tau} - \frac{1}{p}} \|u\|_{L_\omega^p(B_r)},$$

where c depends only on n and the A_p constant of ω .

Proof. It is a slight modification of Theorem 1.2 in [26]. □

We shall use the John–Nirenberg space BMO of functions of bounded mean oscillation and its subspace VMO introduced in [27] and [33] respectively.

Definition 2.3. A locally integrable function f in Ω belongs to the space BMO(Ω) if

$$[f] \equiv \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| dx < +\infty,$$

where $B \subset \Omega$ and $f_B \equiv \frac{1}{|B|} \int_B f(x) dx$.

Let $f \in \text{BMO}(\Omega)$. We set

$$\eta(f; r) \equiv \sup_{\rho \leq r} \frac{1}{|B_\rho|} \int_{B_\rho} |f(x) - f_{B_\rho}| dx.$$

We say that f belongs to VMO(Ω) if, moreover, $\lim_{r \rightarrow 0} \eta(f; r) = 0$, and we call η the VMO modulus of the function f .

Remark 2.1. It is well known that, given $f \in \text{VMO}$ with modulus $\eta(f; r)$, there exists a sequence of C^∞ functions $\{f_i\}$ with moduli $\eta(f_i; r)$ such that $\{f_i\}$ converges to f in VMO and $\eta(f_i; r) \leq \eta(f; r)$ for all $i \in \mathbb{N}$.

We recall the definition of a Calderón–Zygmund kernel.

Definition 2.4. Let $k: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ be a measurable function. We say that k is a Calderón–Zygmund kernel if

- i) $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$;
- ii) $k(x)$ is homogeneous of degree $-n$;
- iii) $\int_\Sigma k(x) d\sigma_x = 0$, where $\Sigma \equiv \{x : |x| = 1\}$.

We shall need boundedness of singular integral operators (i.e., integral operators whose kernel is a Calderón–Zygmund one) and related commutators.

Theorem 2.4. *Let $k: \Omega \times (\mathbb{R}^n \setminus \{0\}) \rightarrow \mathbb{R}$ be such that*

$$\begin{cases} \text{i)} & k(x, \cdot) \text{ is a Calderón–Zygmund kernel for a.e. } x \in \mathbb{R}^n; \\ \text{ii)} & \max_{|j| \leq 2n} \left\| \frac{\partial^j}{\partial y_j} k(x, y) \right\|_{L^\infty(\mathbb{R}^n \times \Sigma)} \equiv M < +\infty. \end{cases}$$

For $f \in L_\omega^p(\mathbb{R}^n)$, $1 < p < +\infty$, and $\varepsilon > 0$, set

$$K_\varepsilon f(x) = \int_{|x-y|>\varepsilon} k(x, x-y) f(y) \omega(y) dy, \quad x \in \mathbb{R}^n.$$

Then for any $f \in L_\omega^p(\mathbb{R}^n)$ there exists $Kf \in L_\omega^p(\mathbb{R}^n)$ such that

$$\lim_{\varepsilon \rightarrow 0} \|K_\varepsilon f - Kf\|_{L_\omega^p(\mathbb{R}^n)} = 0,$$

and moreover, there exists c depending on M , n , the A_p constant of ω , and the diameter of Ω , such that

$$\|Kf\|_{L_\omega^p(\mathbb{R}^n)} \leq c \|f\|_{L_\omega^p(\mathbb{R}^n)}.$$

Proof. The proof is similar to that of Theorem 2.10 in [10]. □

Theorem 2.5 (see [21, Theorem 2.1]). *Let $k: \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \rightarrow \mathbb{R}$ be such that*

$$\begin{cases} \text{i)} & k(x, \cdot) \text{ is a Calderón–Zygmund kernel for a.e. } x \in \mathbb{R}^n; \\ \text{ii)} & \max_{|j| \leq 2n} \left\| \frac{\partial^j}{\partial y_j} k(x, y) \right\|_{L^\infty(\mathbb{R}^n \times \Sigma)} \equiv M < +\infty. \end{cases}$$

For $f \in L_\omega^p(\mathbb{R}^n)$, $1 < p < +\infty$, $\phi \in \text{BMO}$, and $\varepsilon > 0$, set

$$C_\varepsilon[\phi, f](x) = \int_{|x-y|>\varepsilon} k(x, x-y) [\phi(x) - \phi(y)] f(y) dy, \quad x \in \mathbb{R}^n.$$

Then for any $f \in L_\omega^p(\mathbb{R}^n)$ there exists $C[\phi, f] \in L_\omega^p(\mathbb{R}^n)$ such that

$$\lim_{\varepsilon \rightarrow 0} \|C_\varepsilon[\phi, f] - C[\phi, f]\|_{L_\omega^p(\mathbb{R}^n)} = 0,$$

and moreover, there exists $c \equiv c(n, p, M)$ such that

$$\|C[\phi, f]\|_{L_\omega^p(\mathbb{R}^n)} \leq c[\phi] \|f\|_{L_\omega^p(\mathbb{R}^n)}.$$

As a consequence of the preceding result, we have the following theorem [21, Theorem 2.2].

Theorem 2.6. *Suppose k is as in Theorem 2.4, $1 < p < +\infty$, and $\phi \in \text{VMO}$. Then for any $\epsilon > 0$ there exists a positive number ρ_0 such that for any $r < \rho_0$ we have*

$$(4) \quad \|C[\phi, f]\|_{L_\omega^p(B_r)} \leq c\epsilon \|f\|_{L_\omega^p(B_r)} \quad \text{for all } f \in L_\omega^p(B_r), \quad 1 < p < +\infty.$$

§3. INTERIOR ESTIMATES

This section is devoted to interior estimates and a regularity result. We consider only the case where $p > 2$. The case of $1 < p < 2$ can be recovered by duality.

Suppose Ω is a bounded domain in $\mathbb{R}^n$ and $\omega \in A_2$.

We consider the system

$$(5) \quad L_{ij} u^j = -(a_{ij}^{\alpha\beta} u_{x_\alpha}^j)_{x_\beta} = (f_i^\beta)_{x_\beta}, \quad i = 1, 2, \dots, N, \quad \alpha, \beta = 1, 2, \dots, n.$$

Here and in the sequel α and β run from 1 to n , while i and j run from 1 to N .

We assume that

$$(6) \quad a_{ij}^{\alpha\beta} \in \text{VMO}(\Omega).$$

Regarding the known term, for $\mathbf{f} = \{\mathbf{f}_i\} = \{f_i^\alpha\}$ we assume that

$$(7) \quad \frac{\mathbf{f}}{\omega} \in L_\omega^p(\Omega, \mathbb{R}^{nN}), \quad 1 < p < +\infty.$$

The coefficients $a_{ij}^{\alpha\beta}$ satisfy the following ellipticity condition: there exists $\lambda > 0$ such that

$$(8) \quad \lambda^{-1}\omega(x)|\xi|^2 \leq a_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j \leq \lambda\omega(x)|\xi|^2 \text{ for all } \xi \in \mathbb{R}^{nN} \text{ and a.e. } x \in \Omega.$$

Definition 3.1. A solution of (5) is a function $u \in W_{\omega}^{1,p}(\Omega, \mathbb{R}^N)$ such that

$$\int_{\Omega} a_{ij}^{\alpha\beta} u_{x_\alpha}^j \varphi_{x_\beta}^i dx = - \int_{\Omega} f_i^\beta \varphi_{x_\beta}^i dx \text{ for all } \varphi \in C_0^\infty(\Omega, \mathbb{R}^N).$$

We recall some classical results.

Consider the system

$$(9) \quad \bar{L}_{ij} u^j \equiv -(\bar{a}_{ij}^{\alpha\beta} u_{x_\alpha}^j)_{x_\beta} = g_i,$$

where $g = \{g_i\}$ is a smooth function and the $\bar{a}_{ij}^{\alpha\beta}$ are real constants satisfying (8) with $\omega = 1$.

A fundamental matrix solution of (9) is a matrix $\Gamma^{jk}(x-y)$ such that, for fixed y , we have

$$\bar{L}_{ij} \Gamma^{jk} = \delta_{ik} \delta_0, \quad k = 1, \dots, N,$$

where δ_{ik} is Kronecker's delta and δ_0 is the Dirac mass at 0.

Theorem 3.1. *If u is a smooth vector-valued function with compact support in a ball $B \subset \subset \Omega$, then*

$$u^k(x) = \int_B \Gamma^{ki}(x-\xi) \bar{L}_{ij} u^j(\xi) d\xi, \quad k = 1, 2, \dots, N,$$

where

- i) $\bar{L}_{ij}$ is an elliptic operator with constant coefficients;
- ii) Γ^{ki} is a fundamental matrix solution for the operator $\bar{L}_{ij}$.

Our main result, Theorem 3.2, will be proved as a consequence of Lemma 3.2. In order to prove this lemma, we need a representation formula and an estimate for the gradient of the solutions of the system

$$(10) \quad L_{ij} u^j = g_i + \operatorname{div} \mathbf{f}_i,$$

where the g_i and $\mathbf{f}_i$ are compactly supported functions.

For this, we introduce some more notation. Let x_0 be a point of Ω at which (8) is true, and let $\Gamma^{ki}(x_0, t)$ be a fundamental matrix solution for the constant coefficients elliptic operator

$$\bar{L}_{ij} u^j = -(a_{ij}^{\alpha\beta}(x_0) u_{x_\alpha}^j)_{x_\beta}.$$

We set

$$\Gamma_\alpha^{ki}(x_0, t) = \frac{\partial}{\partial t_\alpha} \Gamma^{ki}(x_0, t), \quad \Gamma_{\alpha\beta}^{ki}(x_0, t) = \frac{\partial^2}{\partial t_\alpha \partial t_\beta} \Gamma^{ki}(x_0, t).$$

Lemma 3.1. *Let $B \subset \Omega$. Let $u \in C_0^\infty(B, \mathbb{R}^N)$ be a solution of system (10) in B . Suppose (6) and (8) are true, $a_{ij}^{\alpha\beta} \in C^\infty(\Omega)$, $g \in C_0^\infty(B, \mathbb{R}^N)$, and $\mathbf{f} \in C_0^\infty(B, \mathbb{R}^{nN})$. Then for all $x \in B$, all $k = 1, \dots, N$, and all $\alpha = 1, \dots, n$, we have*

$$(11) \quad \begin{aligned} u_{x_\alpha}^k(x) = & - \text{p.v.} \int_B \Gamma_{\alpha\beta}^{ki}(x, x-y) \left\{ [a_{ij}^{\alpha\beta}(x) - a_{ij}^{\alpha\beta}(y)] u_{x_\alpha}^j(y) - f_j^\beta(y) \right\} dy \\ & + \int_B g_i(y) \Gamma_\alpha^{ki}(x, x-y) dy + c_{\alpha\beta}^{ki}(x) f_i^\beta(x), \end{aligned}$$

where

$$c_{\alpha\beta}^{ki}(x) = \int_{|t|=1} \Gamma_{\alpha}^{ki}(x, t) t_{\beta} d\sigma_t.$$

Moreover, there exists $\sigma > 0$ and c independent of u , $\mathbf{f}$, and g such that

$$(12) \quad \|\nabla u\|_{L_{\omega}^p(B_{\sigma}, \mathbb{R}^{nN})} \leq c \left(\left\| \frac{\mathbf{f}}{\omega} \right\|_{L_{\omega}^p(B_{\sigma}, \mathbb{R}^{nN})} + \sigma [\omega(B_{\sigma})]^{\frac{1}{p} - \frac{1}{p_*}} \left\| \frac{g}{\omega} \right\|_{L_{\omega}^{p_*}(B_{\sigma}, \mathbb{R}^{nN})} \right),$$

where $p_* = \frac{n-1}{n}p$.

Proof. Let $x_0 \in B$. For $i = 1, \dots, N$, we have

$$\begin{aligned} -(a_{ij}^{\alpha\beta}(x_0)u_{x_{\alpha}}^j(x))_{x_{\beta}} &= -((a_{ij}^{\alpha\beta}(x_0) - a_{ij}^{\alpha\beta}(x))u_{x_{\alpha}}^j(x) - f_i^{\beta}(x))_{x_{\beta}} + g_i(x) \\ &\equiv -(\lambda_i^{\beta, x_0}(x))_{x_{\beta}} + g_i(x). \end{aligned}$$

Since u , g , and $\mathbf{f}$ are compactly supported in B , if $x \in B$ and $k = 1, \dots, N$, then

$$(13) \quad u^k(x) = - \int_B \Gamma_{\beta}^{ki}(x_0, x - y) \lambda_i^{\beta, x_0}(y) dy + \int_B g_i(y) \Gamma^{ki}(x_0, x - y) dy.$$

Differentiating (13) we get

$$\begin{aligned} (14) \quad u_{x_{\alpha}}^k(x) &= - \text{p.v.} \int_B \Gamma_{\alpha\beta}^{ki}(x_0, x - y) [a_{ij}^{h\beta}(x_0) - a_{ij}^{h\beta}(y)] u_{x_h}^j(y) \\ &\quad + \text{p.v.} \int_B \Gamma_{\alpha\beta}^{ki}(x_0, x - y) f_i^{\beta}(y) dy \\ &\quad + \int_B g_i(y) \Gamma_{\alpha}^{ki}(x_0, x - y) dy + c_{\alpha\beta}^{ki}(x_0) \lambda_i^{\beta, x_0}(x). \end{aligned}$$

Formula (14) is true for all $x \in B$. We can take $x = x_0$ to obtain

$$\begin{aligned} (15) \quad u_{x_{\alpha}}^k(x) &= - \text{p.v.} \int_B \Gamma_{\alpha\beta}^{ki}(x, x - y) [a_{ij}^{h\beta}(x) - a_{ij}^{h\beta}(y)] u_{x_h}^j(y) \\ &\quad + \text{p.v.} \int_B \Gamma_{\alpha\beta}^{ki}(x, x - y) f_i^{\beta}(y) dy \\ &\quad + \int_B g_i(y) \Gamma_{\alpha}^{ki}(x, x - y) dy + c_{\alpha\beta}^{ki}(x) f_i^{\beta}(x) \end{aligned}$$

which yields (11).

Now, since $\omega c_{\alpha\beta}^{ki} \in L^{\infty}$, we estimate the $L_{\omega}^p(B_{\sigma})$ norm of the function in (15). Then, using Theorem 2.6, Theorem 2.4, and Theorem 2.3, we arrive at (12). $\square$

Lemma 3.2. *Let $u \in C^{\infty}(\Omega, \mathbb{R}^N)$ be a solution of system (5) with $a_{ij}^{\alpha\beta} \in C^{\infty}(\Omega)$ and $\mathbf{f} \in C^{\infty}(\Omega, \mathbb{R}^{nN})$. Suppose (6) and (8) are true. Then there exist $\sigma > 0$ and c independent of u such that*

$$(16) \quad \|\nabla u\|_{L_{\omega}^p(B_{\sigma/2}, \mathbb{R}^{nN})} \leq c \left(\sigma [\omega(B_{\sigma})]^{\frac{1}{p} - \frac{1}{2}} \|\nabla u\|_{L_{\omega}^2(B_{\sigma}, \mathbb{R}^{nN})} + \left\| \frac{\mathbf{f}}{\omega} \right\|_{L_{\omega}^p(B_{\sigma}, \mathbb{R}^{nN})} + \|u\|_{L_{\omega}^p(B_{\sigma}, \mathbb{R}^N)} \right)$$

for every $B_{\sigma} \subset \subset \Omega$.

Proof. Let $0 < \gamma < 1$, and let σ be as in Lemma 3.1. Let $\theta \in C_0^{\infty}(B_{\sigma})$ be a cut-off function such that $\theta \equiv 1$ in $B_{\gamma\sigma}$, $0 \leq \theta \leq 1$, and $|\nabla \theta| \leq \frac{c}{(1-\gamma)\sigma}$. The function θu is supported in B_{σ} and solves the equation

$$(17) \quad -(a_{ij}^{\alpha\beta}(\theta u^j)_{x_{\alpha}})_{x_{\beta}} = -(a_{ij}^{\alpha\beta} \theta_{x_{\alpha}} u^j - \theta f_i^{\beta})_{x_{\beta}} - (a_{ij}^{\alpha\beta} \theta_{x_{\alpha}} u_{x_{\beta}}^j + \theta_{x_{\beta}} f_i^{\beta}).$$

We set $q^* = \frac{nq}{n-1}$ for $q > 1$. First, we prove (16) in the case where $2 < p \leq 2^*$. We get the result applying Lemma 3.1 to equation (17), along with the Hölder inequality ($p_* \leq 2$). We have

$$\begin{aligned}
 \|\nabla u\|_{L_\omega^p(B_{\gamma\sigma}, \mathbb{R}^{nN})} &\leq c \sum_{i=1}^N \left(\sum_{\beta=1}^n \left\| \frac{a_{ij}^{\alpha\beta} \theta_{x_\alpha} u^j - \theta f_i^\beta}{\omega} \right\|_{L_\omega^p(B_\sigma)} \right. \\
 &\quad \left. + \sigma[\omega(B_\sigma)]^{\frac{1}{p} - \frac{1}{p_*}} \left\| \frac{a_{ij}^{\alpha\beta} \theta_{x_\alpha} u_{x_\beta}^j + \theta_{x_\beta} f_i^\beta}{\omega} \right\|_{L_\omega^{p_*}(B_\sigma)} \right) \\
 (18) \quad &\leq c \left[\|u\|_{L_\omega^p(B_\sigma, \mathbb{R}^N)} + \left\| \frac{\mathbf{f}}{\omega} \right\|_{L_\omega^p(B_\sigma, \mathbb{R}^{nN})} \right. \\
 &\quad \left. + \sigma[\omega(B_\sigma)]^{\frac{1}{p} - \frac{1}{p_*}} \left(\|\nabla u\|_{L_\omega^{p_*}(B_\sigma, \mathbb{R}^{nN})} + \left\| \frac{\mathbf{f}}{\omega} \right\|_{L_\omega^{p_*}(B_\sigma, \mathbb{R}^{nN})} \right) \right] \\
 &\leq c \left(\|u\|_{L_\omega^p(B_\sigma, \mathbb{R}^N)} + \left\| \frac{\mathbf{f}}{\omega} \right\|_{L_\omega^p(B_\sigma, \mathbb{R}^{nN})} + \sigma[\omega(B_\sigma)]^{\frac{1}{p} - \frac{1}{2}} \|\nabla u\|_{L_\omega^2(B_\sigma, \mathbb{R}^{nN})} \right),
 \end{aligned}$$

and to get (16) it remains to choose $\gamma = \frac{1}{2}$.

Now we prove (16) for $2^* < p \leq 2^{**}$. Let θ be a cut-off function, $\theta \equiv 1$ in $B_{\gamma^2\sigma}$, supported in $B_{\gamma\sigma}$ and such that $0 \leq \theta \leq 1$ and $|\nabla \theta| \leq \frac{c}{\gamma\sigma(1-\gamma)}$. Applying Lemma 3.1 to equation (17), we obtain

$$\begin{aligned}
 \|\nabla u\|_{L_\omega^p(B_{\gamma^2\sigma}, \mathbb{R}^{nN})} &\leq c \left[\|u\|_{L_\omega^p(B_{\gamma\sigma}, \mathbb{R}^N)} + \left\| \frac{\mathbf{f}}{\omega} \right\|_{L_\omega^p(B_{\gamma\sigma}, \mathbb{R}^{nN})} \right. \\
 &\quad \left. + \gamma\sigma[\omega(B_{\gamma\sigma})]^{\frac{1}{p} - \frac{1}{p_*}} \left(\|\nabla u\|_{L_\omega^{p_*}(B_{\gamma\sigma}, \mathbb{R}^{nN})} + \left\| \frac{\mathbf{f}}{\omega} \right\|_{L_\omega^{p_*}(B_{\gamma\sigma}, \mathbb{R}^{nN})} \right) \right] \\
 &\leq c \left(\|u\|_{L_\omega^p(B_\sigma, \mathbb{R}^N)} + \left\| \frac{\mathbf{f}}{\omega} \right\|_{L_\omega^p(B_\sigma, \mathbb{R}^{nN})} + \gamma\sigma[\omega(B_{\gamma\sigma})]^{\frac{1}{p} - \frac{1}{2^*}} \|\nabla u\|_{L_\omega^{2^*}(B_{\gamma\sigma}, \mathbb{R}^{nN})} \right).
 \end{aligned}$$

Now we apply (18) with $p = 2^*$ to majorize the term $\|\nabla u\|_{2^*}$. The Hölder inequality yields

$$\|\nabla u\|_{L_\omega^p(B_{\gamma^2\sigma}, \mathbb{R}^{nN})} \leq c \left(\|u\|_{L_\omega^p(B_\sigma, \mathbb{R}^N)} + \left\| \frac{\mathbf{f}}{\omega} \right\|_{L_\omega^p(B_\sigma, \mathbb{R}^{nN})} + \sigma[\omega(B_\sigma)]^{\frac{1}{p} - \frac{1}{2}} \|\nabla u\|_{L_\omega^2(B_\sigma, \mathbb{R}^{nN})} \right)$$

and for $\gamma^2 = \frac{1}{2}$ we get (16). In order to show the result for any $p > 2$, it suffices to iterate

the procedure choosing $h \in \mathbb{N}$ such that $2^{*\overbrace{** \cdots *}^{h-1}} < p \leq 2^{*\overbrace{** \cdots *}^h}$ and $\gamma^h = \frac{1}{2}$. □

Theorem 3.2. *Let u be a solution of (5). If (6), (7), (8) are fulfilled, then for any $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ there exists a constant C depending on n, N, p , the A_2 constant of ω , and $\text{dist}(\Omega', \Omega'')$ such that*

$$\|\nabla u\|_{L_\omega^p(\Omega', \mathbb{R}^{nN})} \leq C \left(\|\nabla u\|_{L_\omega^2(\Omega'', \mathbb{R}^{nN})} + \left\| \frac{\mathbf{f}}{\omega} \right\|_{L_\omega^p(\Omega'', \mathbb{R}^{nN})} + \|u\|_{L_\omega^p(\Omega'', \mathbb{R}^N)} \right).$$

Proof. The result can be obtained from Lemma 3.2 by a density argument (see Remark 2.1). □

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Received 5/JAN/2013

Originally published in English