ALGEBRAIC PROPERTIES OF BI-POLYMATROIDAL IDEALS

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ABSTRACT. Classes of monomial ideals are considered in the polynomial ring in two sets of variables $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$. Some algebraic properties of bi-polymatroidal ideals of R are studied. More precisely, the behavior of the monomial localization of such ideals is investigated.

INTRODUCTION

Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be the polynomial ring in two sets of variables over a field K . Classes of monomial ideals of R have been studied intensively (see [10]). In this paper, we are interested in a special class of monomial ideals of R , namely, in the bi-polymatroidal ideals. In [7], the author introduced the generalized notion of a discrete bi-polymatroid and the corresponding notion of a bi-polymatroidal ideal in R . Let L be a monomial ideal of R generated in a single degree, and let $G(L)$ be its unique set of minimal generators. L is called a *bi-polymatroidal ideal* if for any two monomials $u, v \in G(L)$ such that $\deg_{X_i}(u) > \deg_{X_i}(v)$ or $\deg_{Y_k}(u) > \deg_{Y_k}(v)$, there exists $j \in [n]$ with $\deg_{X_j}(u) < \deg_{X_j}(v)$ or $l \in [m]$ with $\deg_{Y_l}(u) < \deg_{Y_l}(v)$ such that $X_j(u/X_i) \in G(L)$ or $Y_l(u/Y_k) \in G(L)$. An interesting class of discrete bi-polymatroids is the discrete bi-polymatroid of Veronese type, whose set of bases is

$$B_{q,s} = \{(a; b) \in \mathbb{Z}_+^{n+m} : |a| = k, |b| = r, 0 \leq a_i, b_j \leq s\}, \quad s \leq q.$$

The corresponding bi-polymatroidal ideals of R are the ideals of Veronese bi-type $L_{q,s}$. They are an extension of the ideals of Veronese type in a polynomial ring in two sets of variables. More precisely, the Veronese bi-type ideals are monomial ideals of R generated in the same degree: $L_{q,s} = \sum_{k+r=q} I_{k,s} J_{r,s}$, with $k, r \geq 1$, $s \leq q$, where $I_{k,s}$ is the Veronese-type ideal generated in degree k by the set

$$\left\{ X_1^{a_{i_1}} \dots X_n^{a_{i_n}} \mid \sum_{j=1}^n a_{i_j} = k, 0 \leq a_{i_j} \leq s, s \in \{1, \dots, k\} \right\}$$

and $J_{r,s}$ is the Veronese-type ideal generated in degree r by the set

$$\left\{ Y_1^{b_{i_1}} \dots Y_m^{b_{i_m}} \mid \sum_{j=1}^m b_{i_j} = r, 0 \leq b_{i_j} \leq s, s \in \{1, \dots, r\} \right\},$$

see [5, 6, 7, 8].

Now we are interested in a problem introduced in [1] and [4], where the authors proved that a monomial localization of a polymatroidal ideal is a polymatroidal ideal.

Let $\wp$ be a prime ideal of R . The monomial localization of a monomial ideal L with respect to $\wp$ is the monomial ideal $L(\wp)$ obtained from L by substituting 1 for the variables that do not belong to $\wp$. More precisely, setting $\wp = (X_{i_1}, \dots, X_{i_r}, Y_{j_1}, \dots, Y_{j_t})$, we

2010 *Mathematics Subject Classification*. Primary 13F20.

Key words and phrases. Bi-polymatroidal ideals, monomial localization.

denote by $L(\wp)$ the monomial ideal in the polynomial ring $K[X_{i_1}, \dots, X_{i_r}, Y_{j_1}, \dots, Y_{j_t}]$, where $L(\wp) = L_C$ with $C = [n+m] \setminus \{i_1, \dots, i_r, j_1, \dots, j_t\}$. If L is a square-free monomial ideal, then $L(\wp) = L : X_C$, where $X_C = \prod_{i \in C} X_i$.

In this paper, we generalize the properties studied in [1] about the monomial localization to all bi-polymatroidal ideals of R ; in particular, we verify that a monomial localization $L(\wp)$ of a bi-polymatroidal ideal L is again a bi-polymatroidal ideal.

The paper is organized as follows. In §1, some preliminary notions on the bi-polymatroidal ideals are given. In §2 we study a certain permanence property of the bi-polymatroidal ideals of R . We show that the property of L to be a bi-polymatroidal ideal is equivalent to the fact that $L : u$ is a bi-polymatroidal ideal for all monomials u of R . Moreover, we prove that a monomial localization of a bi-polymatroidal ideal is again a bi-polymatroidal ideal.

§1. PRELIMINARY NOTIONS

Let $n, m > 0$ be integers, and let $[n+m] = \{1, 2, \dots, n+m\}$. Let $\mathbb{Z}_+$ be the set of nonnegative integers and $\mathbb{Z}_+^{n+m}$ the set of the vectors $(a; b)$ with $a \in \mathbb{Z}_+^n$ and $b \in \mathbb{Z}_+^m$, i.e., $(a; b) = (a_1, \dots, a_n; b_1, \dots, b_m) \in \mathbb{Z}_+^{n+m}$ with $a_i \geq 0, b_j \geq 0$ for all i, j .

The modulus of the vector $(a; b)$ is the number $|(a; b)| = |a| + |b| = \sum_{i=1}^n a_i + \sum_{j=1}^m b_j$. Let $(a; b)$ and $(c; d)$ be two vectors of $\mathbb{Z}_+^{n+m}$, we have $(a; b) \geq (c; d)$ if all components $(a_i - c_i; b_j - d_j)$ of the vector $(a - c; b - d)$ are nonnegative. If $(a; b) \geq (c; d)$ and $(a; b) \neq (c; d)$, we write $(a; b) > (c; d)$. We say that $(c; d)$ is a subvector of $(a; b)$ if $(a; b) \geq (c; d)$. Moreover, we set $(a; b) \vee (c; d) = (\max\{a_1, c_1\}, \dots, \max\{a_n, c_n\}; \max\{b_1, d_1\}, \dots, \max\{b_m, d_m\})$.

Hence $(a; b) \leq (a; b) \vee (c; d)$ and $(c; d) \leq (a; b) \vee (c; d)$.

In [3], the following combinatorial concept of a discrete polymatroid was introduced.

Definition 1.1. A *discrete polymatroid* on the set $[n] = \{1, 2, \dots, n\}$ is a nonempty finite subset $P \subset \mathbb{Z}_+^n$ satisfying the following conditions:

- 1) together with each $u \in P$, the set P contains all its integral subvectors, that is, if $u \in P$ and $v \in \mathbb{Z}_+^n$ with $v \leq u$, then $v \in P$;
- 2) for all $u, v \in P$ with $|u| < |v|$, there is a vector $w \in P$ such that $u < w < u \vee v$.

In [7], the following notion of a discrete bi-polymatroid was introduced as a generalization of the discrete polymatroid.

Definition 1.2. A *discrete bi-polymatroid* on the set $[n+m]$ is a nonempty finite subset $P \subset \mathbb{Z}_+^{n+m}$ satisfying the following conditions:

- 1) together with each $(a; b) \in P$, the set P contains all its integral subvectors, that is, if $(a; b) \in P$ and $(c; d) \in \mathbb{Z}_+^{n+m}$ with $(c; d) \leq (a; b)$, then $(c; d) \in P$;
- 2) for all $(a; b), (c; d) \in P$ with $|(a; b)| < |(c; d)|$, there is a vector $(u; v) \in P$ such that $(a; b) < (u; v) < (a; b) \vee (c; d)$.

A *base* of a discrete bi-polymatroid P is a vector $(a; b) \in P$ such that $(a; b) < (c; d)$ for no $(c; d) \in P$. The set of bases of P is denoted by $B(P)$.

Remark 1.3. Each base of a discrete bi-polymatroid P has the same modulus, called the *rank* of P . Indeed, if $(a; b)$ and $(c; d)$ are bases of P with $|(a; b)| < |(c; d)|$, then by Definition 1.2 there exists $(u; v) \in P$ with $(a; b) < (u; v) < (a; b) \vee (c; d)$. This contradicts the maximality of $(a; b) \in B(P)$.

A characterization of discrete bi-polymatroids in terms of their set of bases looks like this.

Theorem 1.4 (Bi-exchange property, see [7]). *P is a discrete bi-polymatroid if and only if the following is true: if $(a; b), (c; d) \in B(P)$ with $a_i > c_i$ or $b_k > d_k$, then there exist $j \in \{1, \dots, n\}$ and $l \in \{1, \dots, m\}$ with $a_j < c_j$ or $b_l < d_l$ such that $(a; b) - (e_i; 0) + (e_j; 0) \in B(P)$ or $(a; b) - (0; e'_k) + (0; e'_l) \in B(P)$, where the e_i, e'_j denote the standard basis vectors of $\mathbb{Z}_+^n, \mathbb{Z}_+^m$, respectively.*

Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be the polynomial ring in two sets of variables with $\deg(X_i) = \deg(Y_j) = 1$ for all $i = 1, \dots, n, j = 1, \dots, m$. For a monomial ideal $L \subset R$, we denote by $G(L)$ its unique set of minimal generators.

A monomial ideal generated by all the monomials corresponding to the set $B(P)$ of bases of a discrete bi-polymatroid is called a *bi-polymatroidal ideal* and is generated by all the monomials $\underline{X}^a \underline{Y}^b$ with $(a, b) \in B(P)$, where $\underline{X}^a \underline{Y}^b$ stands for $X_1^{a_1} \dots X_n^{a_n} Y_1^{b_1} \dots Y_m^{b_m}$.

In [7], the following definition of a bi-polymatroidal ideal was given as a consequence of Theorem 1.4.

Definition 1.5. A monomial ideal L of R generated in a single degree is said to be *bi-polymatroidal* if the following condition is satisfied: for all monomials $u = X_1^{a_1} \dots X_n^{a_n} Y_1^{b_1} \dots Y_m^{b_m}$ and $v = X_1^{c_1} \dots X_n^{c_n} Y_1^{d_1} \dots Y_m^{d_m}$ in $G(L)$ and for each i with $a_i > c_i$ or k with $b_k > d_k$, we have $j \in [n]$ with $a_j < c_j$ or $l \in [m]$ with $b_l < d_l$ such that $X_j(u/X_i) \in G(L)$ or $Y_l(u/Y_k) \in G(L)$.

It follows that for any two monomials $u, v \in G(L)$ such that $\deg_{X_i}(u) > \deg_{X_i}(v)$ or $\deg_{Y_k}(u) > \deg_{Y_k}(v)$ there exist $j \in [n]$ with $\deg_{X_j}(u) < \deg_{X_j}(v)$ or $l \in [m]$ with $\deg_{Y_l}(u) < \deg_{Y_l}(v)$ such that $X_j(u/X_i) \in G(L)$ or $Y_l(u/Y_k) \in G(L)$.

Theorem 1.6 (see [7]). *The bi-polymatroidal ideals of $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ have linear quotients.*

We shall describe a special class of bi-polymatroidal ideals. Let q, s, r, k be nonnegative integers such that $k + r = q$ and $s \leq q$; then

$$B_{q,s} = \{(a; b) \in \mathbb{Z}_+^{n+m} : |a| = k, |b| = r, 0 \leq a_i, b_j \leq s\}$$

is the set of bases of a discrete bi-polymatroid of rank q .

In fact, let $(a; b), (c; d) \in B_{q,s}$ with $a_i > c_i$ or $b_k > d_k$. Then for some j, l such that $a_j < c_j$ or $b_l < d_l$, we have $(a; b) - (e_i; 0) + (e_j; 0) = (a_1, \dots, a_i - 1, \dots, a_j + 1, \dots, a_n; b_1, \dots, b_m)$ and $(a; b) - (0; e'_k) + (0; e'_l) = (a_1, \dots, a_n; b_1, \dots, b_k - 1, \dots, b_l + 1, \dots, b_m)$. Hence, $(a; b) - (e_i; 0) + (e_j; 0) \in B_{q,s}$ and $(a; b) - (0; e'_k) + (0; e'_l) \in B_{q,s}$ by definition. This discrete bi-polymatroid is called the *discrete bi-polymatroid of Veronese bi-type*; we denote it by $P_{q,s}$.

Example 1.7. Let $n = m = 2$ and $s = 2$. The set of bases of a discrete bi-polymatroid of rank 3 is

$$B_{3,2} \{(2, 0; 1, 0), (2, 0; 0, 1), (1, 1; 1, 0), (1, 1; 0, 1), (0, 2; 1, 0), \\ (0, 2; 0, 1), (1, 0; 2, 0), (1, 0; 0, 2), (1, 0; 1, 1), (0, 1; 2, 0), (0, 1; 0, 2), (0, 1; 1, 1)\}.$$

The bi-polymatroidal ideal corresponding to the set $B_{q,s}$ is a Veronese bi-type ideal:

$$L_{q,s} = \sum_{k+r=q} I_{k,s} J_{r,s}, \quad k, r \geq 1, \quad s \leq q,$$

where $I_{k,s}$ is the Veronese type ideal of degree k in the variables $X_1, \dots, X_n$ generated by the set $\{X_1^{a_1} \dots X_n^{a_n} \mid \sum_{i=1}^n a_i = k, 0 \leq a_i \leq s\}$, and $J_{r,s}$ is the Veronese type ideal of degree r in the variables $Y_1, \dots, Y_m$ generated by $\{Y_1^{b_1} \dots Y_m^{b_m} \mid \sum_{j=1}^m b_j = r, 0 \leq b_j \leq s\}$,

$$L_{q,s} = \left(\left\{ X_1^{a_1} \dots X_n^{a_n} Y_1^{b_1} \dots Y_m^{b_m} \mid \sum_{i=1}^n a_i + \sum_{j=1}^m b_j = q, 0 \leq a_i, b_j \leq s \right\} \right)$$

(see [5, 6, 7]).

Remark 1.8. By definition, the ideal $L_{q,s}$ is not trivial for

$$2 \leq q \leq s(n+m) - 1, \quad s \leq q.$$

Example 1.9. Let $R = K[X_1, X_2; Y_1, Y_2]$.

$$\begin{aligned} L_{4,2} &= I_{3,2}J_{1,2} + I_{1,2}J_{3,2} + I_{2,2}J_{2,2} = I_{3,2}J_1 + I_1J_{3,2} + I_2J_2 \\ &= (X_1^2X_2Y_1, X_1^2X_2Y_2, X_1X_2^2Y_1, X_1X_2^2Y_2, X_1Y_1^2Y_2, X_2Y_1^2Y_2, X_1Y_1Y_2^2, X_2Y_1Y_2^2, \\ &\quad X_1^2Y_1^2, X_1^2Y_1Y_2, X_1^2Y_2^2, X_2^2Y_1^2, X_2^2Y_2^2, X_2^2Y_1Y_2, X_1X_2Y_1^2, X_1X_2Y_2^2, X_1X_2Y_1Y_2). \end{aligned}$$

Remark 1.10. In general, $I_{k,s} \subseteq I_k$, where I_k is the *Veronese ideal* of degree k generated by all monomials in the variables $X_1, \dots, X_n$ of degree k (see [10]).

We have $I_{k,s} = I_k$ for any $k \leq s$. If $s = 1$, $I_{k,1}$ is the square-free Veronese ideal of degree k generated by all the square-free monomials in the variables $X_1, \dots, X_n$ of degree k . Similar considerations are valid for $J_{r,s} \subset K[Y_1, \dots, Y_m]$.

§2. MONOMIAL LOCALIZATION

In this section we study some permanence properties of the bi-polymatroidal ideals of R .

Theorem 2.1. *Let $L \subset R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be a monomial ideal and u a monomial of R . The following conditions are equivalent.*

1. L is a bi-polymatroidal ideal.
2. $L : u$ is a bi-polymatroidal ideal for all monomials u .
3. $L : u$ is generated in a single degree and has linear quotients for all monomials u .
4. $L : u$ has a linear resolution for all monomials u .

Proof. $1 \Rightarrow 2$. Let L be a bi-polymatroidal ideal. It suffices to prove that for all variables X_i (respectively, Y_k), $L : X_i$ (respectively, $L : Y_k$) is a bi-polymatroidal ideal of R . Set $J = L : X_i$. Let $u, v \in G(J)$, then $X_i u, X_i v \in X_i J \subseteq L$. If $\deg_{X_i}(u) = \deg_{X_i}(v)$, then $X_i u$ and $X_i v$ have the property described in Definition 1.5. Hence, this property is enjoyed by u and v . If $\deg_{X_i}(u) > \deg_{X_i}(v)$, then X_i divides u . For a variable X_t with $\deg_{X_t}(u) > \deg_{X_t}(v)$, we prove that there exists a variable X_j with $\deg_{X_j}(v) > \deg_{X_j}(u)$ and such that $X_j(u/X_t) \in G(J)$. Since $\deg_{X_t}(X_i u) > \deg_{X_t}(X_i v)$ and L is a bi-polymatroidal ideal, it follows that there exists a variable X_j with $\deg_{X_j}(X_i u) < \deg_{X_j}(X_i v)$ such that $X_j(X_i u/X_t) \in G(L)$. Then $X_j(u/X_t) \in J$. The same result holds true for $\deg_{X_i}(u) < \deg_{X_i}(v)$.

In similar way, if we set $J = L : Y_k$, the claim follows with the same argument. We conclude that $L : u$ is bi-polymatroidal.

$2 \Rightarrow 3$. Since any bi-polymatroidal ideal is generated in a single degree by definition and has linear quotients (Theorem 1.6), the same is true for $L : u$.

$3 \Rightarrow 4$. This follows from the general fact that the ideals generated in a single degree with linear quotients have a linear resolution (see [2, Lemma 4.1]).

$4 \Rightarrow 1$. If $L : u$ has a linear resolution, then it is generated in a single degree for all monomials u of R . Let $v, w \in G(L)$ with $\deg_{X_i}(v) > \deg_{X_i}(w)$ or $\deg_{Y_k}(v) > \deg_{Y_k}(w)$. We prove that there exists a variable X_j such that $\deg_{X_j}(w) > \deg_{X_j}(v)$ and $X_j(v/X_i) \in G(L)$, or there exists Y_l such that $\deg_{Y_l}(w) > \deg_{Y_l}(v)$ and $Y_l(u/Y_k) \in G(L)$.

First, suppose that $\deg_{X_i}(v) > \deg_{X_i}(w)$, then, by the hypothesis, $L : \frac{v}{X_i}$ is generated in the same degree. Since $X_i \in G(L : \frac{v}{X_i})$, it follows that $L : \frac{v}{X_i}$ is generated in degree 1. Since $w/GCD(w, \frac{v}{X_i}) \in L : \frac{v}{X_i}$, there exists $z \in G(L)$ such that $X_j = z/GCD(z, \frac{v}{X_i})$ for some j and X_j divides $w/GCD(w, \frac{v}{X_i})$. Then $\deg_{X_j}(w) > \deg_{X_j}(\frac{v}{X_i})$. Since $\deg_{X_i}(v) > \deg_{X_i}(w)$, we have $X_i \neq X_j$. Hence, $\deg_{X_j}(w) > \deg_{X_j}(\frac{v}{X_i}) = \deg_{X_j}(v)$. Our hypothesis implies that L is generated in a single degree. Hence, $\deg(z) = \deg(v)$. Moreover, since $X_j = z/GCD(z, \frac{v}{X_i})$, it follows that $\deg_{X_t}(z) \leq \deg_{X_t}(\frac{v}{X_i}) = \deg_{X_t}(v)$ for all $t \neq i, j$, $\deg_{X_j}(z) = \deg_{X_j}(\frac{v}{X_i}) + 1 = \deg_{X_j}(v) + 1$, and $\deg_{X_i}(z) \leq \deg_{X_i}(\frac{v}{X_i}) = \deg_{X_i}(v) - 1$. Consequently, $z = X_j(v/X_i)$, whence $X_j(v/X_i) \in G(L)$.

Now we suppose that $\deg_{Y_k}(v) > \deg_{Y_k}(w)$, then, by the hypothesis, $L : \frac{v}{Y_k}$ is generated in the same degree. Using the same argument, we can prove that $Y_l(v/Y_k) \in G(L)$ for some $l \in [m]$. The claim follows. $\square$

Example 2.2. $R = K[X_1, X_2; Y_1, Y_2]$. Let

$$L = L_{3,2} = (X_1Y_1Y_2, X_2Y_1Y_2, X_1Y_1^2, X_2Y_1^2, X_1Y_2^2, X_2Y_2^2, \\ X_1X_2Y_1, X_1X_2Y_2, X_1^2Y_1, X_1^2Y_2, X_2^2Y_1, X_2^2Y_2)$$

be a bi-polymatroidal ideal of Veronese bi-type.

Set $u = X_1$. Then

$$L : u = (Y_1Y_2, X_2Y_1Y_2, Y_1^2, X_2Y_1^2, Y_2^2, X_2Y_2^2, X_2Y_1, X_2Y_2, X_1Y_1, X_1Y_2, X_2^2Y_1, X_2^2Y_2).$$

Hence,

$$L : u = (Y_1Y_2, Y_1^2, Y_2^2, X_2Y_1, X_2Y_2, X_1Y_1, X_1Y_2),$$

which is again a bi-polymatroidal ideal.

We denote the set of monomial prime ideals of R by $\mathcal{P}(R)$. Let $\wp \in \mathcal{P}(R)$ be a monomial prime ideal. The *monomial localization* of a monomial ideal L with respect to $\wp$ is the monomial ideal $L(\wp)$ obtained from L by substituting 1 for the variables that do not belong to $\wp$. More precisely, set $\wp = (X_{i_1}, \dots, X_{i_r}, Y_{j_1}, \dots, Y_{j_t})$; we denote by $L(\wp)$ the monomial ideal in the polynomial ring $K[X_{i_1}, \dots, X_{i_r}, Y_{j_1}, \dots, Y_{j_t}]$, where $L(\wp) = L_C$ with $C = [n+m] \setminus \{i_1, \dots, i_r, j_1, \dots, j_t\}$. If L is a square-free monomial ideal, then $L(\wp) = L : X_C$, where $X_C = \prod_{i \in C} X_i$.

Example 2.3. Let $R = K[X_1, X_2; Y_1, Y_2]$. Let

$$L = L_{3,2} = (X_1Y_1Y_2, X_2Y_1Y_2, X_1Y_1^2, X_2Y_1^2, X_1Y_2^2, X_2Y_2^2, \\ X_1X_2Y_1, X_1X_2Y_2, X_1^2Y_1, X_1^2Y_2, X_2^2Y_1, X_2^2Y_2)$$

be a bi-polymatroidal ideal of Veronese bi-type.

Set $\wp = (X_2, Y_1, Y_2)$. We compute the monomial localization of L with respect to $\wp$, i.e., $L(\wp) = L_C$ with $C = \{1\}$:

$$L(\wp) = (Y_1Y_2, X_2Y_1Y_2, Y_1^2, X_2Y_1^2, Y_2^2, X_2Y_2^2, X_2Y_1, X_2Y_2, Y_1, Y_2, X_2^2Y_1, X_2^2Y_2) = (Y_1, Y_2),$$

obtained from L by substituting 1 for the variable X_1 that does not belong to $\wp$.

Remark 2.4. For square-free bi-polymatroidal ideals, the monomial localization is a bi-polymatroidal ideal, by Theorem 2.1 (because $L(\wp) = L : u$, where $u = \prod_{i \in C} X_i$).

For the non-square-free bi-polymatroidal ideals L , we give the following result.

Theorem 2.5. *If $L \subset R$ is a bi-polymatroidal ideal, then:*

- 1) *for all $\wp \in \mathcal{P}(R)$, the ideal $L(\wp)$ is a bi-polymatroidal ideal;*
- 2) *for all $\wp \in \mathcal{P}(R)$, $L(\wp)$ has linear quotients;*
- 3) *for all $\wp \in \mathcal{P}(R)$, $L(\wp)$ has a linear resolution.*

Proof. 1. Let $L \subset R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be a bi-polymatroidal ideal generated in degree q . For any $i \in [n+m]$, let $L_{\{i\}}$ be the monomial ideal obtained from L by substituting 1 for the variable with index i . Let $\underline{X}^a \underline{Y}^b \in L$, where $\underline{X}^a \underline{Y}^b = X_1^{a_1} \dots X_n^{a_n} Y_1^{b_1} \dots Y_m^{b_m}$ and $(a; b) = (a_1, \dots, a_n; b_1, \dots, b_m) \in \mathbb{Z}_+^{n+m}$. Then $L = (\{\underline{X}^a \underline{Y}^b \mid (a; b) \in B\})$. Suppose $i \in [n]$; this means that $(L)_{\{i\}}$ is the monomial ideal obtained from L by substituting 1 for the variable X_i . It follows that $(L)_{\{i\}} = (\{\underline{X}^{a'} \underline{Y}^b \mid (a; b) \in B\})$, where for all $(a; b) \in B$ we put $\underline{X}^{a'} \underline{Y}^b = \underline{X}^a \underline{Y}^b / X_i^{a_i}$. We prove that $(L)_{\{i\}}$ is a bi-polymatroidal ideal.

The first step is to prove that $(L)_{\{i\}}$ is generated in a single degree. If $h_i = \max\{a_i \mid (a; b) \in B\}$, then we prove that $G((L)_{\{i\}}) = \{\underline{X}^a \underline{Y}^b / X_i^{h_i} \mid (a; b) \in B, a_i = h_i\}$. Let $(c; d) \in B$, then $c_i < h_i$. Now we show that there exists $(u, v) \in B$ with $u_i = h_i$ and such that $\underline{X}^{u'} \underline{Y}^v$ divides $\underline{X}^{c'} \underline{Y}^d$. We proceed by induction on $h_i - c_i$.

If $h_i - c_i = 0$, all is clear. Suppose now that $h_i - c_i > 0$, i.e., $c_i < h_i$. Let $(a, b) \in B$ with $a_i = h_i$. By Theorem 1.4, there exists an integer $j \in [n]$ with $a_j < c_j$ such that $(a; b) - (e_j; 0) + (e_j; 0) \in B$, and, by symmetry, $(c, d) - (e_j; 0) + (e_i; 0) \in B$. Set $(z, t) = (c, d) - (e_j; 0) + (e_i; 0)$. Consequently, $\underline{X}^{z'} \underline{Y}^t$ divides $\underline{X}^{c'} \underline{Y}^d$. Since $h_i - z_i < h_i - c_i$, the inductive hypothesis shows that there exists $(u, v) \in B$ with $u_i = h_i$ such that $\underline{X}^{u'} \underline{Y}^v$ divides $\underline{X}^{z'} \underline{Y}^t$. It follows that $\underline{X}^{u'} \underline{Y}^v$ divides $\underline{X}^{c'} \underline{Y}^d$, yielding the desired conclusion.

The second step is to show that the set $B' = \{(a', b) \mid \underline{X}^{a'} \underline{Y}^b \in G((L)_{\{i\}})\}$ is the set of bases of a discrete bi-polymatroid P' of rank $q - h_i$ on the set $[n+m] \setminus \{i\}$.

For all $(a'; b) \in B'$ we have $|(a', b)| = q - h_i$. Now we verify the bi-exchange property: if $(a', b), (c', d) \in B'$ with $a'_k > c'_k$, then $k \neq i$. By assumption for $(a, b), (c, d) \in B$, if $a_k = a'_k > c'_k = c_k$, then there exists $l \in [n]$ such that $a_l < c_l$ and $(a; b) - (e_k; 0) + (e_l; 0) \in B$. Set $(x, y) = (a; b) - (e_k; 0) + (e_l; 0)$; since $a_i = c_i = h_i$, we have $l \neq i$ and $x_i = a_i$. Hence, $(x', y) \in B'$, i.e., $(a'; b) - (e_k; 0) + (e_l; 0) \in B'$ as required. Then B' is a discrete bi-polymatroid on the set $[n+m] \setminus \{i\}$. It follows that $(L)_{\{i\}}$ is a bi-polymatroidal ideal.

If $i \in [m]$, the claim follows by the same argument. In all cases $L_{\{i\}}$ is a bi-polymatroidal ideal. As a consequence, if $\wp = (X_{i_1}, \dots, X_{i_r}, Y_{j_1}, \dots, Y_{j_r})$ is a prime ideal of R , then the monomial localization $L(\wp) = L_C$ with $C = [n+m] \setminus \{i_1, \dots, i_r, j_1, \dots, j_r\}$ is a bi-polymatroidal ideal.

2. This follows by Theorem 1.6.

3. It is a general fact that the ideals generated in the same degree with linear quotients have a linear resolution (see [2, Lemma 4.1]). $\square$

Example 2.6. Let $R = K[X_1, X_2; Y_1, Y_2]$. Consider the bi-polymatroidal ideal of Example 2.3:

$$L = (X_1 Y_1 Y_2, X_2 Y_1 Y_2, X_1 Y_1^2, X_2 Y_1^2, X_1 Y_2^2, X_2 Y_2^2, \\ X_1 X_2 Y_1, X_1 X_2 Y_2, X_1^2 Y_1, X_1^2 Y_2, X_2^2 Y_1, X_2^2 Y_2)$$

and put $\wp = (X_2, Y_1, Y_2)$, $C = \{1\}$. Then

$$L(\wp) = L_C = (Y_1, Y_2),$$

which is a bi-polymatroidal ideal of R .

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Received 5/SEP/2013

Originally published in English