ON SOLUTIONS OF CONVOLUTION EQUATIONS IN SPACES OF ULTRADIFFERENTIABLE FUNCTIONS

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ABSTRACT. A representation for a particular and the general solution is established for convolution equations in nonquasianalytic Beurling spaces ultradifferentiable functions of mean type on a finite interval. As a particular case, differential equations of infinite order with constant coefficients are studied.

§1. INTRODUCTION

Let ω be a weight function, and let φ_ω^* be the Young conjugate to the function $\varphi_\omega(x) = \omega(e^x)$. Finally, let $I = (-a, a)$ be a finite interval in $\mathbb{R}$.

The paper is devoted to a description of solutions of the convolution equation

$$(1.1) \quad T_\mu f = g$$

in the nonquasianalytic Beurling space

$$\mathcal{E}_{(\omega)}^1(I) = \left\{ f \in C^\infty(I) : \forall q \in (0, 1), \forall l \in (0, a) : \|f\|_{\omega, q, l} = \sup_{j \in \mathbb{N}_0} \sup_{|x| \leq l} \frac{|f^{(j)}(x)|}{e^{q\varphi_\omega^*(j/q)}} < \infty \right\}$$

of ultradifferentiable functions (UDF) of mean type determined by the weight ω on I . Here T_μ is the surjective convolution operator that acts linearly and boundedly on $\mathcal{E}_{(\omega)}^1(I)$ by the rule

$$(T_\mu f)(x) = \langle \psi_\mu, f(x + \cdot) \rangle, \quad f \in \mathcal{E}_{(\omega)}^1(I), \quad x \in I;$$

ψ_μ is a continuous linear functional in $(\mathcal{E}_{(\omega)}^1(I))'$ called usually the symbol of T_μ . For simplicity, by the symbol of T_μ we shall mean the Fourier–Laplace transform μ of ψ_μ rather than this functional itself. So, the symbol is an entire function that is a multiplier for the weighted space of entire functions

$$H_{(\omega), I}^1 = \left\{ f \in H(\mathbb{C}) : \exists q \in (0, 1), \exists l \in (0, a) : \|f\|_{\omega, q, l} = \sup_{z \in \mathbb{C}} \frac{|f(z)|}{e^{q\omega(z) + l|\operatorname{Im} z|}} < \infty \right\},$$

which is isomorphic to $(\mathcal{E}_{(\omega)}^1(I))'$.

It is known (see [1, Theorem 2]) that, as a partial case, equations (1.1) include differential equations of infinite order with constant coefficients

$$(1.2) \quad \sum_{k=0}^{\infty} a_k f^{(k)} = g, \quad f, g \in \mathcal{E}_{(\omega)}^1(I).$$

Earlier, in [1], all symbols μ of the surjective operators T_μ were described, i.e., of all convolution equations (1.1) solvable in $\mathcal{E}_{(\omega)}^1(I)$ for every right-hand side $g \in \mathcal{E}_{(\omega)}^1(I)$.

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Next, in [2] an exponential polynomial basis was constructed in $\ker T_\mu$, i.e., in the space of solutions of the homogeneous differential equation $T_\mu f = 0$.

In the present paper, we establish the form of a particular solution of (1.1). Together with the results of [2], this enables us to write out the general solution of equations (1.1) and (1.2) in $\mathcal{E}_{(\omega)}^1(I)$.

Presently, convolution equations have been studied fairly well also in Beurling spaces of UDF of maximal type (see, e.g., [3, 4, 5]). However, the form of a particular and the general solution in such spaces has not been established.

A particular solution for equation (1.1) will be sought in the form $f = \sum_{j=1}^{\infty} f_j e^{-i\nu_j x}$, where the exponents ν_j , $j \in \mathbb{N}$, are chosen in such a way that the system $\{e^{-i\nu_j x} : j \in \mathbb{N}\}$ be an absolutely representative system in $\mathcal{E}_{(\omega)}^1(I)$ (an ARS, for short) and the $|\mu(\nu_j)|$, $j \in \mathbb{N}$, obey appropriate lower estimates. Finally, we shall prove that if $\sum_{j=1}^{\infty} g_j e^{-i\nu_j x}$ is an expansion of the right-hand side g in (1.1) in an absolutely convergent series, then the function

$$(1.3) \quad f = \sum_{j=1}^{\infty} \frac{g_j}{\mu(\nu_j)} e^{-i\nu_j x}$$

is a solution for (1.1) in $\mathcal{E}_{(\omega)}^1(I)$. This is the principal result of the paper.

In order to construct the system $\{\nu_j : j \in \mathbb{N}\}$, we use the theory of weakly sufficient sets, the theory of ARS, and the relationship between them. The idea of finding a particular solution in a form similar to that indicated above was proposed by Leont'ev in [6]. Later, the method was applied by Korobeĭnik in [7]. This method was also used in [8] for the space $\mathcal{E}_{(\omega)}^1(\mathbb{R})$ of Beurling UDF of mean type on the real line.

It should be noted that, in general, the system $\{\nu_j : j \in \mathbb{N}\}$ is found in a nonconstructive way. In this connection, we shall also study the solutions of convolution equations in the spaces $\mathcal{E}_{(\omega)}^1(I)$ determined by the weights $\omega(t) = t^{\rho(t)}$, where $\rho(t) \rightarrow \rho \in (0, 1)$ is a proximate order. It is known (see [1]) that all convolution equations (1.1) in these spaces are differential equations (1.2) with constant coefficients. If $\rho(t) \equiv \rho \in (0, 1)$, then the corresponding spaces $\mathcal{E}_{(\omega)}^1(I)$ are projective analogs of the well-known Gevrey classes.

For the class of weights indicated above, the system $\{\nu_j : j \in \mathbb{N}\}$ is constructed explicitly in terms of the symbol μ of equation (1.2); moreover, the ν_j can be chosen nonnegative. In this construction, we agree to determine μ by its simple zeros $(\lambda_s)_s$ that form an R -set (this ensures the surjectivity of T_μ). Finally, we shall prove that the function (1.3) is a particular solution of equation (1.2), and its general solution has the form

$$(1.4) \quad f = \sum_{s=1}^{\infty} a_s e^{-i\lambda_s x} + \sum_{j=1}^{\infty} \frac{g_j}{\mu(\nu_j)} e^{-i\nu_j x}, \quad a_s \in \mathbb{C}, \quad s = 1, 2, \dots$$

The paper is organized as follows. In §2, we introduce weight functions, the corresponding Beurling spaces of UDF of mean type, and the dual spaces $H_{(\omega),I}^1$, $H_{(\omega),I}^1$ and the corresponding ARS of exponentials in $\mathcal{E}_{(\omega)}^1(I)$. In §4, we prove the main result, i.e., the theorem about a partial solution for equation (1.1) in $\mathcal{E}_{(\omega)}^1(I)$. §§5, 6 are devoted to solutions of convolution equations in generalized projective Gevrey classes.

§2. WEIGHT FUNCTIONS, SPACES OF UDF, AND CONVOLUTION OPERATORS

Let ω be a nonquasianalytic *weight function*, i.e., a monotone nondecreasing continuous function $\omega: [0, \infty) \rightarrow [0, \infty)$ satisfying the following conditions:

$$(\alpha) \quad \forall p > 1 \exists C > 0 : \omega(x+y) \leq p(\omega(x) + \omega(y)) + C, \quad \forall x, y \geq 0;$$

- (β) $\int_1^\infty \frac{\omega(t)}{t^2} dt < \infty$;
 (γ) $\ln t = o(\omega(t))$, $t \rightarrow \infty$;
 (δ) the function $\varphi_\omega(x) := \omega(e^x)$ is convex on $[0, \infty)$.

There is no loss of generality in assuming that $\omega(1) = 0$. We put $\omega(z) := \omega(|z|)$, $z \in \mathbb{C}$.

We describe the properties of weight functions required in what follows. It is known that $\omega(t) = o(t)$ as $t \rightarrow \infty$ and

$$(2.1) \quad \lim_{r \downarrow 1} \limsup_{t \rightarrow \infty} \frac{\omega(rt)}{\omega(t)} = 1.$$

Next, for any nonquasianalytic weight ω , an important role is played by its harmonic extension to the upper and lower half-planes:

$$P_\omega(x + iy) := \begin{cases} \frac{|y|}{\pi} \int_{-\infty}^\infty \frac{\omega(t)}{(t-x)^2 + y^2} dt & \text{if } y \neq 0, \\ \omega(x) & \text{if } y = 0. \end{cases}$$

The function P_ω is continuous and subharmonic on the entire plane, moreover, $P_\omega(z) \geq \omega(z)$ for all $z \in \mathbb{C}$. We shall need the upper estimates for $P_\omega(z)$ established in the next lemma.

Lemma 1. *Let P_ω be the harmonic extension of a nonquasianalytic weight ω . Then for every $\varepsilon > 0$ there exists $C > 0$ such that*

$$(2.2) \quad P_\omega(x + iy) \leq (1 + \varepsilon)\omega(x) + \varepsilon|y| + C, \quad x + iy \in \mathbb{C}.$$

Proof. Fix $\varepsilon \in (0, \pi/2 - 1)$. By (α), there exists $C_1 > 0$ with

$$\omega(x + y) \leq (1 + \varepsilon)(\omega(x) + \omega(y)) + C_1, \quad x, y \geq 0.$$

Clearly, it suffices to prove (2.2) for $y > 0$. For such y , we have

$$(2.3) \quad P_\omega(x + iy) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{\omega(x + yt)}{t^2 + 1} dt \leq (1 + \varepsilon)\omega(x) + (1 + \varepsilon)\frac{2}{\pi} \int_0^\infty \frac{\omega(yt)}{t^2 + 1} dt + C_1.$$

Next,

$$\frac{2}{\pi} \int_0^\infty \frac{\omega(yt)}{t^2 + 1} dt = \frac{2}{\pi} \int_0^1 \frac{\omega(yt)}{t^2 + 1} dt + \frac{2}{\pi} \int_1^\infty \frac{\omega(yt)}{t^2 + 1} dt \leq \frac{\omega(y)}{2} + \frac{2y}{\pi} \int_y^\infty \frac{\omega(s)}{s^2} ds.$$

Since $\omega(y) = o(y)$ as $y \rightarrow \infty$ and $\int_y^\infty \frac{\omega(s)}{s^2} ds \rightarrow 0$ as $y \rightarrow \infty$ by (β), we see that there exists $C_2 > 0$ such that

$$\frac{\omega(y)}{2} + \frac{2y}{\pi} \int_y^\infty \frac{\omega(s)}{s^2} ds \leq \frac{\varepsilon}{1 + \varepsilon} y + C_2$$

for all $y > 0$. Returning to (2.3), finally we obtain

$$P_\omega(x + iy) \leq (1 + \varepsilon)\omega(x) + \varepsilon y + C_1 + (1 + \varepsilon)C_2, \quad x \in \mathbb{R}, \quad y > 0,$$

proving the lemma. $\square$

We pass to the construction of spaces of UDF. Let $\varphi_\omega^*(y) := \sup\{xy - \varphi_\omega(x) : x \geq 0\}$, $y \geq 0$, be the Young conjugate function to φ_ω . The *Beurling space of UDF of mean type* on a finite interval $I = (-a, a)$ in $\mathbb{R}$ is defined to be the following weighted space of infinitely differentiable functions on I :

$$\mathcal{E}_{(\omega)}^1(I) = \left\{ f \in C^\infty(I) : \forall q \in (0, 1), \forall l \in (0, a) : |f|_{\omega, q, l} = \sup_{j \in \mathbb{N}_0} \sup_{|x| \leq l} \frac{|f^{(j)}(x)|}{e^{q\varphi_\omega^*(j/q)}} < \infty \right\}.$$

This space possesses a natural topology determined by the collection of seminorms $\{|\cdot|_{\omega,q,l} : q \in (0,1), l \in (0,a)\}$, and it is an (FS) -space with this topology.

As is well known (see [9, Theorem 1]), the Fourier–Laplace transformation of the functionals

$$F : \varphi \in (\mathcal{E}_{(\omega)}^1(I))' \mapsto \widehat{\varphi}(z) := \varphi_x(e^{-izx}), \quad z \in \mathbb{C},$$

establishes a topological isomorphism between the strong dual $(\mathcal{E}_{(\omega)}^1(I))'_\beta$ to $\mathcal{E}_{(\omega)}^1(I)$ and the following space of entire functions:

$$H_{(\omega),I}^1 = \left\{ f \in H(\mathbb{C}) \mid \exists q \in (0,1), \exists l \in (0,a) : \|f\|_{\omega,q,l} = \sup_{z \in \mathbb{C}} \frac{|f(z)|}{e^{q\omega(z)+l|\operatorname{Im} z|}} < \infty \right\}.$$

The space $H_{(\omega),I}^1$ is endowed with the natural topology of the inductive limit

$$\operatorname{ind}_{q \in (0,1)} \operatorname{ind}_{l \in (0,a)} H_{\omega,q,l}$$

of the Banach spaces $H_{\omega,q,l} = \{f \in H(\mathbb{C}) : \|f\|_{\omega,q,l} < \infty\}$ and becomes a (DFS) -space.

It is easily seen that $H_{(\omega),I}^1$ is not closed under multiplication of functions. In accordance with [1, Proposition 1], the set of all multipliers of $H_{(\omega),I}^1$, i.e., of all entire functions μ with $\mu H_{(\omega),I}^1 \subset H_{(\omega),I}^1$ coincides with

$$M_{(\omega)}^1 = \left\{ \mu \in H(\mathbb{C}) : \forall \varepsilon > 0 \quad \sup_{z \in \mathbb{C}} \frac{|\mu(z)|}{e^{\varepsilon\omega(z)+\varepsilon|\operatorname{Im} z|}} < \infty \right\}.$$

Moreover, every multiplier $\mu \in M_{(\omega)}^1$ is continuous, i.e., the multiplication operator $\Lambda_\mu : f \mapsto \mu f$ acts continuously on $H_{(\omega),I}^1$.

Lemma 1 in [1] allows us to replace $\omega(z)$ by $\omega(\operatorname{Re} z)$ in the definitions of the spaces $H_{(\omega),I}^1$ and $M_{(\omega)}^1$.

A convolution operator in $\mathcal{E}_{(\omega)}^1(I)$ is defined as the adjoint to the multiplication operator Λ_μ , which acts linearly and continuously in $H_{(\omega),I}^1 \simeq (\mathcal{E}_{(\omega)}^1(I))'_\beta$. Specifically, for a nontrivial multiplier $\mu \in M_{(\omega)}^1$, we find the continuous linear functional $\psi_\mu := F^{-1}(\mu)$ on $\mathcal{E}_{(\omega)}^1(I)$ and for $f \in \mathcal{E}_{(\omega)}^1(I)$ put

$$(T_\mu f)(x) := \langle \psi_\mu, f(x + \cdot) \rangle, \quad x \in I.$$

This operator T_μ is called the *convolution operator with symbol μ* . We have $T_\mu = (F^{-1} \circ \Lambda_\mu \circ F)'$, so T_μ acts linearly and continuously in $\mathcal{E}_{(\omega)}^1(I)$.

Note that T_μ acts in the following way on the elements of the system $\{e^{-izx} : z \in \mathbb{C}\}$ (which is complete in $\mathcal{E}_{(\omega)}^1(I)$):

$$T_\mu(e^{-izx}) = \mu(z) e^{-izx}, \quad x \in I, \quad z \in \mathbb{C}.$$

It is known (see [1, Theorem 2]) that if an entire function $\mu(z) = \sum_{k=0}^\infty a_k(-i)^k z^k$ is a *strong multiplier* of $H_{(\omega),I}^1$, i.e., satisfies the condition

$$\sup_{z \in \mathbb{C}} \frac{|\mu(z)|}{e^{\varepsilon\omega(z)}} < \infty \quad \text{for every } \varepsilon > 0,$$

then $T_\mu f = \sum_{k=0}^\infty a_k f^{(k)}$ for all $f \in \mathcal{E}_{(\omega)}^1(I)$, i.e., the convolution operator is a differential operator of infinite order with constant coefficients.

In [1, Theorem 3] it was shown that in the case of a *strict* nonquasianalytic weight (i.e., a weight satisfying $\limsup_{t \rightarrow \infty} \frac{\omega(Kt)}{\omega(t)} < K$ with some $K > 1$) all multipliers in $M_{(\omega)}^1$ are strong, and all convolution operators are differential operators with constant coefficients.

In [1, Theorem 1] several equivalent conditions were found that are necessary and sufficient for T_μ to be a surjection of $\mathcal{E}_{(\omega)}^1(I)$ onto $\mathcal{E}_{(\omega)}^1(I)$. We reproduce one of them, which will be required in what follows:

(SC) *for every $\varepsilon > 0$ and every $\delta > 0$, there exists $r_0 > 0$ such that for every $z = x + iy \in \mathbb{C}$ with $|x| \geq r_0$ and $|y| \leq \delta|x|$ there is a circle C_z of radius $R_z \leq \delta\omega(x) + \delta|y|$ enclosing z and such that all points ζ on this circle satisfy*

$$|\mu(\zeta)| \geq e^{-\varepsilon\omega(\operatorname{Re} \zeta) - \varepsilon|\operatorname{Im} \zeta|}.$$

Condition (SC) is equivalent to the statement that μ is a *divisor* for $H_{(\omega),I}^1$. This means that if $f \in H_{(\omega),I}^1$ and $\frac{f}{\mu} \in H(\mathbb{C})$, then $\frac{f}{\mu} \in H_{(\omega),I}^1$ (see the paper quoted above).

So, let μ be a divisor for $H_{(\omega),I}^1$. Then the convolution equation (1.1) has a solution in $\mathcal{E}_{(\omega)}^1(I)$ for every right-hand side $g \in \mathcal{E}_{(\omega)}^1(I)$.

For every $g \in \mathcal{E}_{(\omega)}^1(I)$, the general solution of equation (1.1) is the sum of some particular solution and the general solution of the corresponding homogeneous equation

$$(2.4) \quad T_\mu f = 0.$$

The general solution of (2.4) was constructed in [2]. Namely, it is easily seen that if λ_s is a zero of the symbol μ of the equation in question and k_s is the multiplicity of this zero, then the exponential monomials

$$(2.5) \quad x^l e^{-i\lambda_s x}, \quad l = 0, 1, \dots, k_s - 1,$$

are solutions of the homogeneous equation (2.4). In [2], a special open covering $(U_j)_j$ of the zero set $(\lambda_s)_s$ for μ was constructed. Then it was proved that the space of solutions for the homogeneous equation (2.4) has an absolute basis formed by linear combinations of solutions (2.5): the linear combinations are composed of solutions that correspond to the zeros λ_s belonging to one and the same U_j .

Thus, the construction of the general solution for (1.1) reduces to the construction of its particular solution.

§3. WEAKLY SUFFICIENT SETS FOR $H_{(\omega),I}^1$

The present section is devoted to the construction of a weakly sufficient set for $H_{(\omega),I}^1$ on which $|\mu|$ admits an appropriate lower estimate. Here μ is a divisor $H_{(\omega),I}^1$, i.e., the symbol of a resolvable convolution equation (1.1).

Recall that, by [10], a set $S \subset \mathbb{C}$ is said to be *weakly sufficient* for $H_{(\omega),I}^1$ if the topology $\bigcap_{q \in (0,1)} \bigcap_{l \in (0,a)} H_{\omega,q,l;S}$ coincides with the initial topology $\bigcap_{q \in (0,1)} \bigcap_{l \in (0,a)} H_{\omega,q,l}$ of $H_{(\omega),I}^1$; here

$$H_{\omega,q,l;S} = \left\{ f \in H_{(\omega),I}^1 : \|f\|_{\omega,q,l;S} = \sup_{z \in S} \frac{|f(z)|}{e^{q\omega(\operatorname{Re} z) + l|\operatorname{Im} z|}} < \infty \right\}.$$

The required weakly sufficient set for $H_{(\omega),I}^1$ will be constructed in two stages.

Lemma 2. *The set $S_0 := \mathbb{R} \cup (i\mathbb{R})$ (i.e., the union of the real and imaginary axes) is weakly sufficient for $H_{(\omega),I}^1$.*

Proof. The condition that S_0 is weakly sufficient for $H_{(\omega),I}^1$ is equivalent to the requirement that for every $q \in (0,1)$ and $l \in (0,a)$ there exist $\tilde{q} \in (0,1)$, $\tilde{l} \in (0,a)$, and $C > 0$ such that

$$\|f\|_{\omega,\tilde{q},\tilde{l}} \leq C \cdot \|f\|_{\omega,q,l;S_0} \quad \text{for all } f \in H_{\omega,q,l;S_0}.$$

Fix $q \in (0, 1)$ and $l \in (0, a)$. For $f \in H_{\omega, q, l; S_0}$ we have

$$(3.1) \quad |f(x)| \leq \|f\|_{\omega, q, l; S_0} \cdot e^{q\omega(x)}, \quad x \in \mathbb{R};$$

$$(3.2) \quad |f(iy)| \leq \|f\|_{\omega, q, l; S_0} \cdot e^{l|y|}, \quad y \in \mathbb{R}.$$

By the Phragmen–Lindelöf principle (see [11, Theorem 6.5.4]), for all $z = x + iy$ with $y \neq 0$ we have

$$(3.3) \quad \ln |f(x + iy)| \leq \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\ln |f(t)|}{(x - t)^2 + y^2} dt + D|y|,$$

where $D = \limsup_{r \rightarrow \infty} \frac{2}{\pi r} \int_0^\pi \ln |f(re^{i\theta})| \sin \theta d\theta$.

By (3.1), we obtain

$$(3.4) \quad \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\ln |f(t)|}{(x - t)^2 + y^2} dt \leq \ln \|f\|_{\omega, q, l; S_0} + qP_\omega(x + iy).$$

To estimate D , we consider the indicator $h_f(\theta)$ (at the order 1) of the entire function f . Since $\frac{\omega(r)}{r} \rightarrow 0$ as $r \rightarrow \infty$, from (3.1) and (3.2) it follows that $h_f(0) = h_f(\pi) = 0$, and $h_f(\frac{\pi}{2}) \leq l$. Recalling that the indicator is trigonometrically convex, we conclude that $h_f(\theta) \leq l \sin \theta$ for all $\theta \in [0, \pi]$. Consequently, for every $\varepsilon > 0$ there exists $r_\varepsilon > 0$ with

$$\ln |f(re^{i\theta})| \leq (l \sin \theta + \varepsilon)r, \quad r \geq r_\varepsilon, \quad \theta \in [0, \pi].$$

Therefore, for $r \geq r_\varepsilon$ we have

$$\frac{2}{\pi r} \int_0^\pi \ln |f(re^{i\theta})| \sin \theta d\theta \leq \frac{2}{\pi} \int_0^\pi (l \sin \theta + \varepsilon) \sin \theta d\theta = l + \frac{4\varepsilon}{\pi}.$$

Thus, $D \leq l$.

Returning to (3.3) and taking (3.4) into account, we see that

$$\ln |f(x + iy)| \leq \ln \|f\|_{\omega, q, l; S_0} + qP_\omega(x + iy) + l|y|$$

for all $x + iy \in \mathbb{C}$ with $y \neq 0$. Take $\varepsilon > 0$ in such a way that $(1 + \varepsilon)q < 1$ and $l + q\varepsilon < a$. Using Lemma 1, we fix $C > 0$ such that (2.2) is fulfilled. As a result, we arrive at

$$\ln |f(x + iy)| \leq (1 + \varepsilon)q\omega(x) + (l + q\varepsilon)|y| + C + \ln \|f\|_{\omega, q, l; S_0}, \quad y \neq 0.$$

Putting $\tilde{q} := (1 + \varepsilon)q$ and $\tilde{l} := l + q\varepsilon$, finally we obtain

$$|f(x + iy)| \leq e^C \cdot \|f\|_{\omega, q, l; S_0} \cdot e^{\tilde{q}\omega(x) + \tilde{l}|y|}, \quad y \neq 0.$$

By continuity, this estimate is true also for $y = 0$.

Thus,

$$\|f\|_{\omega, \tilde{q}, \tilde{l}} \leq e^C \cdot \|f\|_{\omega, q, l; S_0},$$

and the lemma follows. $\square$

Now, taking two sequences $\varepsilon_k \downarrow 0$ and $\delta_k \downarrow 0$, we use condition (SC) to find the corresponding $r_k \uparrow \infty$. We assume that all r_k are so large that $\omega(t) \leq t$ for $t \geq r_1$ and $2\delta_k r_k \geq r_1$.

By (SC), every point $z = \operatorname{Re} z$ of the real axis can be put inside a circle C_z^{Re} of radius $R_z \leq \delta_k \omega(\operatorname{Re} z)$ all points of which satisfy

$$(3.5) \quad |\mu(\zeta)| \geq e^{-\varepsilon_k \omega(\operatorname{Re} \zeta) - \varepsilon_k |\operatorname{Im} \zeta|}.$$

Moreover, for $\zeta \in C_z^{\operatorname{Re}}$ we also have

$$(3.6) \quad |\operatorname{Re} \zeta| \leq |\operatorname{Re} z| + 2R_z \leq |\operatorname{Re} z| + 2\delta_k \omega(\operatorname{Re} z) \leq (1 + 2\delta_k) |\operatorname{Re} z|;$$

$$(3.7) \quad |\operatorname{Im} \zeta| \leq 2R_z \leq 2\delta_k \omega(\operatorname{Re} z).$$

Next, the entire function μ has zero type at the order 1, so it is a function of completely regular growth. Thus, outside a certain exceptional set $\bigcup_j E_j$ of zero linear density, E_j is the disk $\{z : |z - \xi_j| < \rho_j\}$, we have $\frac{\ln |\mu(z)|}{|z|} \rightarrow 0$ as $|z| \rightarrow \infty$. It is well known that the disks E_j may be taken mutually disjoint. So, we may assume that the r_k , $k \in \mathbb{N}$, are chosen so as to obey the conditions

$$|\mu(z)| \geq e^{-\varepsilon_k |z|}, \quad |z| \geq r_k, \quad z \notin \bigcup_j E_j.$$

In particular, for all points $z = i \operatorname{Im} z$ on the imaginary axis with $r_k \leq |\operatorname{Im} z| < r_{k+1}$, $k \in \mathbb{N}$, $z \notin \bigcup_j E_j$, we have

$$(3.8) \quad |\mu(i \operatorname{Im} z)| \geq e^{-\varepsilon_k |\operatorname{Im} z|}.$$

Now let $z \in \bigcup_j E_j$. We denote by $j(z)$ the index of the disk containing z . Since the set E_j of the disks has zero linear density, we have

$$\lim_{\substack{z \rightarrow \infty \\ z \in \bigcup_j E_j}} \frac{\rho_{j(z)}(z)}{|z|} = 0.$$

Consequently, we may assume that

$$(3.9) \quad \rho_{j(z)}(z) \leq \delta_k |z|, \quad |z| \geq r_k, \quad z \in E_{j(z)}.$$

Lemma 3. *The set*

$$S = [-r_1, r_1] \cup [-ir_1, ir_1] \cup \left(\bigcup_{k \in \mathbb{N}} \bigcup_{\substack{z = \operatorname{Re} z \\ r_k \leq |\operatorname{Re} z| < r_{k+1}}} C_z^{\operatorname{Re}} \right) \cup \left(i\mathbb{R} \setminus \bigcup_j E_j \right)$$

is weakly sufficient for $H_{(\omega), I}^1$.

Proof. By Lemma 2, it suffices to prove that for every $q \in (0, 1)$ and $l \in (0, a)$ there exist $\tilde{q} \in (0, 1)$, $\tilde{l} \in (0, a)$, and $C > 0$ such that

$$\|f\|_{\omega, \tilde{q}, \tilde{l}; S_0} \leq C \cdot \|f\|_{\omega, q, l; S}, \quad f \in H_{\omega, q, l; S}.$$

Fix $q \in (0, 1)$ and $l \in (0, a)$. Let f be an arbitrary function in $H_{\omega, q, l; S}$. Then

$$(3.10) \quad |f(\zeta)| \leq \|f\|_{\omega, q, l; S} \cdot e^{q\omega(\operatorname{Re} \zeta) + l|\operatorname{Im} \zeta|}, \quad \zeta \in S.$$

Take $\varepsilon > 0$ with $q + 2\varepsilon < 1$ and $l + 2\varepsilon < a$. Next, find $k_0 = k_0(q, l)$ such that

$$(3.11) \quad 2l\delta_k \leq \varepsilon,$$

$$(3.12) \quad 2q\delta_k + l(1 + 2\delta_k) \leq l + \varepsilon,$$

$$(3.13) \quad q\omega((1 + 2\delta_k)t) \leq (q + \varepsilon)\omega(t) + C_1, \quad t \geq 0,$$

for all $k \geq k_0$. The last condition can be ensured in view of (2.1).

Let z be an arbitrary point of S_0 , i.e., an arbitrary point of the real or the imaginary axis. If $z \in [-r_1, r_1] \cup [-ir_1, ir_1] \cup (i\mathbb{R} \setminus \bigcup_j E_j)$, then $z \in S$, so that $|f(z)|$ satisfies estimate (3.10) with $\zeta = z$.

Consider the remaining cases.

1) Let $z = \operatorname{Re} z$, $|z| > r_1$. We find $k \in \mathbb{N}$ such that $r_k \leq |\operatorname{Re} z| < r_{k+1}$. On the corresponding circle C_z^{Re} , we choose a point ζ such that $|f(\zeta)| = \sup \{|f(w)| : w \text{ is inside } C_z^{\operatorname{Re}}\}$. Then (3.10), (3.6), and (3.7) show that

$$|f(z)| \leq |f(\zeta)| \leq \|f\|_{\omega, q, l; S} \cdot \exp(q\omega((1 + 2\delta_k)\operatorname{Re} z) + 2l\delta_k\omega(\operatorname{Re} z)).$$

For $k \geq k_0$, by (3.11) and (3.13) this estimate can be continued as follows:

$$|f(z)| \leq \|f\|_{\omega, q, l; S} \cdot e^{C_1} \cdot e^{(q+2\varepsilon)\omega(\operatorname{Re} z)}.$$

Putting $C_2 := \exp(C_1 + q\omega((1 + 2\delta_1)r_{k_0}) + 2l\delta_1\omega(r_{k_0}))$, we obtain

$$(3.14) \quad |f(z)| \leq C_2 \cdot \|f\|_{\omega, q, l; S} \cdot e^{(q+2\varepsilon)\omega(\operatorname{Re} z)}, \quad z = \operatorname{Re} z, \quad |z| > r_1.$$

2) If $z = i \operatorname{Im} z$, $|z| > r_1$, and $z \in \bigcup_j E_j$, then, as above, we find $j(z) \in \mathbb{N}$ and $k \in \mathbb{N}$ with $z \in E_{j(z)}$ and $r_k \leq |z| < r_{k+1}$. On the boundary of the disk $E_{j(z)}$, we take a point ζ with $|f(\zeta)| = \sup\{|f(w)| : w \in E_{j(z)}\}$. Then

$$|\operatorname{Re} \zeta| \leq 2\rho_{j(z)}, \quad |\operatorname{Im} \zeta| \leq |\operatorname{Im} z| + 2\rho_{j(z)}.$$

By (3.10) and (3.9), we have

$$\begin{aligned} |f(z)| &\leq |f(\zeta)| \leq \|f\|_{\omega, q, l; S} \cdot \exp\{q\omega(2\rho_{j(z)}) + l(|\operatorname{Im} z| + 2\rho_{j(z)})\} \\ &\leq \|f\|_{\omega, q, l; S} \cdot \exp\{q\omega(2\delta_k \operatorname{Im} z) + l(1 + 2\delta_k)|\operatorname{Im} z|\} \\ &\leq \|f\|_{\omega, q, l; S} \cdot \exp\{(2q\delta_k + l(1 + 2\delta_k))|\operatorname{Im} z|\}. \end{aligned}$$

Since $k \geq k_0$, by (3.12) we obtain

$$|f(z)| \leq \|f\|_{\omega, q, l; S} \cdot e^{(l+\varepsilon)|\operatorname{Im} z|}.$$

Putting $\rho_0 := \sup\{\rho_{j(z)} : |z| < r_{k_0}\}$ and $C_3 := \exp\{q\omega(2\rho_0) + l(r_{k_0} + 2\rho_0)\}$, we finally conclude that

$$(3.15) \quad |f(z)| \leq C_3 \cdot \|f\|_{\omega, q, l; S} \cdot e^{(l+\varepsilon)|\operatorname{Im} z|}, \quad z = i \operatorname{Im} z, \quad |z| \geq r_1, \quad z \in \bigcup_j E_j.$$

Combining (3.14) and (3.15), we see that

$$|f(z)| \leq C \cdot \|f\|_{\omega, q, l; S} \cdot \exp\{(q + 2\varepsilon)\omega(\operatorname{Re} z) + (l + \varepsilon)|\operatorname{Im} z|\}$$

for $z \in S_0 \setminus S$, where $C = \max\{C_2, C_3\}$ is independent of f .

Put $\tilde{q} = q + 2\varepsilon$ and $\tilde{l} = l + \varepsilon$. Then $\|f\|_{\omega, \tilde{q}, \tilde{l}; S_0} \leq C \cdot \|f\|_{\omega, q, l; S}$. This proves the lemma. $\square$

By Epifanov's result (see [12, Theorem 1]) concerning discretization of weakly sufficient sets, the set S contains a subset $(\nu_j)_{j=1}^\infty$, $|\nu_j| \uparrow \infty$, weakly sufficient for $H_{(\omega), I}^1$. Moreover, from (3.5) and (3.8) it follows that

$$(3.16) \quad |\mu(\nu_j)| \geq e^{-\varepsilon_k \omega(\operatorname{Re} \nu_j) - \varepsilon_k |\operatorname{Im} \nu_j|}, \quad j \geq j(k),$$

where $j(k)$ is chosen in such a way that $|\nu_{j(k)}| \geq r_k$.

In conclusion of this section, we apply the result of Korobeĭnik about the relationship between weakly sufficient sets and ARS. We recall (see [7]) that a sequence $(x_j)_{j=1}^\infty$ of elements of a locally convex space H is called an ARS in H if every element $x \in H$ expands in an absolutely convergent series $x = \sum_{j=1}^\infty c_j x_j$. By [13, Theorem K], since the set $(\nu_j)_{j=1}^\infty$ is weakly sufficient for $H_{(\omega), I}^1$, the system of exponentials $\{e^{-i\nu_j x}\}_{j=1}^\infty$ is an ARS in $\mathcal{E}_{(\omega)}^1(I)$.

§4. THEOREM ON A PARTICULAR SOLUTION OF A CONVOLUTION EQUATION

Theorem 1. *Let μ be a divisor for $H_{(\omega), I}^1$, and let $\{e^{-i\nu_j x}\}_{j=1}^\infty$ be the corresponding ARS in $\mathcal{E}_{(\omega)}^1(I)$. If the right-hand side g of the convolution equation (1.1) is expanded in a series $g = \sum_{j=1}^\infty g_j e^{-i\nu_j x}$ convergent absolutely in $\mathcal{E}_{(\omega)}^1(I)$, then the function*

$$(4.1) \quad f = \sum_{j=1}^\infty \frac{g_j}{\mu(\nu_j)} e^{-i\nu_j x}$$

is a particular solution of the convolution equation (1.1) in $\mathcal{E}_{(\omega)}^1(I)$ (the last series converges absolutely in $\mathcal{E}_{(\omega)}^1(I)$).

Proof. We show that the series (4.1) converges absolutely in $\mathcal{E}_{(\omega)}^1(I)$. It is known (see [9, Lemma 3]), that

$$(4.2) \quad \frac{1}{1 + |\nu_j|} \cdot e^{q\omega(\nu_j) + l|\operatorname{Im} \nu_j|} \leq |e^{-i\nu_j x}|_{\omega, q, l} \leq e^{q\omega(\nu_j) + l|\operatorname{Im} \nu_j|}, \quad j \in \mathbb{N},$$

for all $q \in (0, 1)$ and $l \in (0, a)$.

We fix $q \in (0, 1)$ and $l \in (0, a)$, and denote $a_j := \frac{|g_j|}{|\mu(\nu_j)|} \cdot |e^{-i\nu_j x}|_{\omega, q, l}$. Take $\varepsilon > 0$ with $q + \varepsilon < 1$, $l + \varepsilon < a$. Then find an index k_0 such that $\varepsilon_k < \varepsilon$ for $k \geq k_0$ and choose $j_0 = j(k_0)$ such that (3.16) is fulfilled for $j \geq j_0$.

We estimate a_j for $j \geq j_0$. Let $k \geq k_0$ be such that $r_k \leq |\nu_j| < r_{k+1}$. By (3.16) and (4.2), we have

$$a_j \leq |g_j| e^{(q+\varepsilon)\omega(\nu_j) + (l+\varepsilon)|\operatorname{Im} \nu_j|} \leq |g_j| \cdot |e^{-i\nu_j x}|_{\omega, q+2\varepsilon, l+2\varepsilon} \cdot e^{\ln(1+|\nu_j|) - \varepsilon\omega(\nu_j) - \varepsilon|\operatorname{Im} \nu_j|}.$$

By the restriction (γ) on the weight ω , there exists $C > 1$ with

$$\ln(1+t) \leq \varepsilon\omega(t) + \ln C, \quad t \geq 0.$$

Therefore, $a_j \leq C \cdot |g_j| \cdot |e^{-i\nu_j x}|_{\omega, q+2\varepsilon, l+2\varepsilon}$, $j \geq j_0$. Since

$$\sum_{j=1}^{\infty} |g_j| \cdot |e^{-i\nu_j x}|_{\omega, q+2\varepsilon, l+2\varepsilon} < \infty,$$

we see that $\sum_{j=1}^{\infty} a_j < \infty$, that is, the series (4.1) converges absolutely in $\mathcal{E}_{(\omega)}^1(I)$. Since the space $\mathcal{E}_{(\omega)}^1(I)$ is complete, the sum f of this series is a function in $\mathcal{E}_{(\omega)}^1(I)$; clearly, it is a solution of the convolution equation (1.1). $\square$

§5. A PARTICULAR AND THE GENERAL SOLUTION OF THE CONVOLUTION EQUATION IN THE CASE WHERE $\omega(t) = t^{\rho(t)}$

Consider the convolution equation (1.1) in the space $\mathcal{E}_{(\omega)}^1(I)$ with $\omega(t) = t^{\rho(t)}$, where $\rho(t) \rightarrow \rho \in (0, 1)$ is some proximate order. We put $\rho(z) := \rho(|z|)$, $z \in \mathbb{C}$.

Every such weight is strict (see §2), so that all multipliers of the corresponding space $H_{(\omega), I}^1$ are strong and all convolution equations (1.1) are differential equations (1.2) with constant coefficients.

Suppose that the symbol μ of equation (1.2) is determined by its simple zeros:

$$\mu(z) = \prod_{s=1}^{\infty} \left(1 - \frac{z}{\lambda_s}\right), \quad |\lambda_s| \uparrow \infty,$$

and, moreover, the sequence of zeros satisfies the condition

$$(5.1) \quad \lim_{s \rightarrow \infty} \frac{s}{|\lambda_s|^{\rho(\lambda_s)}} = 0$$

and forms an R-set (see [14]). The last claim means that the points λ_s lie inside certain angles with common vertex at zero that have no other points in common, and moreover, the points inside one particular angle, when enumerated in the order of increase of their moduli, satisfy

$$(5.2) \quad |\lambda_{s+1}| - |\lambda_s| > d|\lambda_s|^{1-\rho(\lambda_s)}, \quad s \in \mathbb{N}, \quad \text{for some } d > 0.$$

Formula (5.1) is equivalent to the property that μ has zero type at the order $\rho(r)$, i.e., is a multiplier for $H_{(\omega), I}^1$. In its turn, condition (5.2) ensures that μ is a divisor of $H_{(\omega), I}^1$, i.e., it satisfies (SC). Indeed, if (5.2) is fulfilled, then μ is of completely regular growth at

the order $\rho(r)$, that is, outside some exceptional disks $C_s = \{z : |z - \lambda_s| < r_s\}$, $s \in \mathbb{N}$, we have

$$(5.3) \quad \lim_{z \rightarrow \infty} \frac{\ln |\mu(z)|}{|z|^{\rho(z)}} = 0.$$

It is well known (see [15, the remark after Theorem 1.2.6]), in the case in question any fixed $\gamma > 0$ can be taken for the role of all radii r_s . If γ is sufficiently small, the disks C_s are mutually disjoint.

Similar symbols μ that determine resolvable convolution equations in $\mathcal{E}_{(\omega)}^1(I)$ were treated in [16].

Clearly, in the case under study, the general solution of the homogeneous equation $\sum_{k=0}^{\infty} a_k f^{(k)} = 0$ corresponding to (1.2) in $\mathcal{E}_{(\omega)}^1(I)$ has the form

$$(5.4) \quad f = \sum_{s=1}^{\infty} \alpha_s e^{-i\lambda_s x}, \quad \alpha_s \in \mathbb{C}, \quad s \in \mathbb{N},$$

and the series in (5.4) converges absolutely in $\mathcal{E}_{(\omega)}^1(I)$. Indeed, since the zeros λ_s of the symbol μ lie sufficiently far away from one another, there is no need in gathering the corresponding elementary solutions $e^{-i\lambda_s x}$ into groups (see §2 and [2]).

Our aim in this and the next section is to explicitly construct the exponents ν_j of the system $\{e^{-i\nu_j x}\}_{j=1}^{\infty}$ of Theorem 1 in terms of the zeros λ_s , $s \in \mathbb{N}$ of μ . It should be noted that the ν_j will be chosen on the positive real semiaxis.

In the present section, we shall describe a way of constructing the system $(\nu_j)_{j=1}^{\infty}$ and formulate the results about a particular solution and the general solution for the convolution equation. All justifications will be presented in the next (final) section because they are bulky.

Choose a sufficiently small $\gamma < \frac{1}{4a}$. Let $C_s = \{z : |z - \lambda_s| < \gamma\}$ be the corresponding exceptional disks such that (5.3) holds true outside them.

The positive zeros $\frac{\pi n}{a}$, $n \in \mathbb{N}$, of the function $\sin az$ will be split in two monotone increasing sequences $\xi_1, \xi_2, \dots$ and $\eta_1, \eta_2, \dots$ in accordance with the following rule: the points $\pm \eta_j$ do not belong to any of C_s , whereas either ξ_j or $-\xi_j$ belongs to some C_s . We shall analyze the most difficult case where there are infinitely many points ξ_j . Put $K_j^{\pm} = \{z : |z \mp \xi_j| < \gamma\}$, $j \in \mathbb{N}$.

Next, we apply [15, Theorem 1.2.3] to extract a subsequence $(\zeta_k)_{k=1}^{\infty}$ from the sequence $(\frac{\pi}{2a} + \frac{\pi n}{a})_{n=1}^{\infty}$ so that all points of this subsequence lie outside $\bigcup_s C_s$ and satisfy

$$(5.5) \quad \lim_{k \rightarrow \infty} \frac{k}{\zeta_k^{\rho(\zeta_k)}} = \frac{1}{\pi} \tan \frac{\pi \rho}{2};$$

$$(5.6) \quad \zeta_{k+1} - \zeta_k > d_1 \zeta_k^{1-\rho(\zeta_k)}, \quad k \in \mathbb{N}, \text{ for some } d_1 > 0.$$

A fortiori, $\zeta_k \notin \bigcup_j K_j^{\pm}$, $k \in \mathbb{N}$.

We define the required sequence $(\nu_j)_{j=1}^{\infty}$ to be the union of $(\eta_k)_{k=1}^{\infty}$ and $(\zeta_k)_{k=1}^{\infty}$.

In §6 it will be proved that, with this choice of the ν_j , $j \in \mathbb{N}$, the system $\{e^{-i\nu_j x}\}_{j=1}^{\infty}$ will be an ARC in $\mathcal{E}_{(\omega)}^1(I)$ and, moreover, the quantities $|\mu(\nu_j)|$ satisfy (3.16). This will establish the following statement.

Theorem 2. *Let $\omega(t) = t^{\rho(t)}$, where $\rho(t) \rightarrow \rho \in (0, 1)$ is some proximate order. Let $\mu(z) = \prod_{s=1}^{\infty} (1 - \frac{z}{\lambda_s}) = \sum_{k=0}^{\infty} a_k (-i)^k z^k$, where the sequence $(\lambda_s)_{s=1}^{\infty}$ satisfies (5.1) and forms an R-set. Finally, let $(\nu_j)_{j=1}^{\infty}$ be the sequence constructed above by the points λ_s .*

Suppose that the right-hand side g of equation (1.2) is represented by the series $g = \sum_{j=1}^{\infty} g_j e^{-i\nu_j x}$ absolutely convergent in $\mathcal{E}_{(\omega)}^1(I)$. Then the function $f = \sum_{j=1}^{\infty} \frac{g_j}{\mu(\nu_j)} e^{-i\nu_j x}$

is a particular solution of this equation in the space $\mathcal{E}_{(\omega)}^1(I)$, and its general solution has the form

$$f = \sum_{s=1}^{\infty} \alpha_s e^{-i\lambda_s x} + \sum_{j=1}^{\infty} \frac{g_j}{\mu(\nu_j)} e^{-i\nu_j x}, \quad \alpha_s \in \mathbb{C}, \quad s \in \mathbb{N}.$$

Remark 1. If the zeros λ_s of the symbol μ satisfy the condition

$$(5.7) \quad |\operatorname{Im} \lambda_s| \geq \delta_0, \quad s \geq s_0,$$

we can choose the η_k as follows: $\eta_k = \frac{\pi k}{a}$, $k \in \mathbb{N}$.

We show an example of the construction of the sequence $(\nu_j)_{j=1}^{\infty}$.

Example 1. Let $\omega(t) = t^{1/2}$, and let $\mu(z) = \prod_{s=1}^{\infty} (1 - \frac{z}{\lambda_s})$, where $(\lambda_s)_{s=1}^{\infty}$ satisfies (5.1), (5.2), and (5.7). Then the sequence $(\nu_j)_{j=1}^{\infty}$ can be chosen as the union of the sequences

$$\eta_k = \frac{\pi k}{a}, \quad k = 1, \dots, \quad \text{and} \quad \zeta_k = \frac{\pi}{2a} + \frac{\pi[a\pi k^2]}{a}, \quad k = 1, \dots$$

Here, as usual, $[a\pi k^2]$ is the integral part of $a\pi k^2$, $k \in \mathbb{N}$. In fact, conditions (5.5) and (5.6) are easily verified for $(\zeta_k)_{k=1}^{\infty}$ in this case.

§6. PROOF OF THE RESULTS OF §5

Our main purpose in the present section is to prove that if the ν_j , $j \in \mathbb{N}$, are chosen as described in §5, then the system $\{e^{-i\nu_j x}\}_{j=1}^{\infty}$ is an ARS in $\mathcal{E}_{(\omega)}^1(I)$, and, moreover, the quantities $|\mu(\nu_j)|$, $j \in \mathbb{N}$, satisfy (3.16).

By [17, Theorem 3], the system $\{e^{-i\nu_j x}\}_{j=1}^{\infty}$ is an ARS in $\mathcal{E}_{(\omega)}^1(I)$ if and only if there exists an entire function $L(z)$ with the following properties:

(A) for every $\varepsilon > 0$ there exists $C > 0$ such that

$$|L(z)| \leq C \cdot \exp \{ (1 + \varepsilon) |z|^{\rho(z)} + (a + \varepsilon) |\operatorname{Im} z| \}, \quad z \in \mathbb{C};$$

(B) there is a sequence $R_n \uparrow \infty$ such that for every $\varepsilon > 0$ there exists an index $n_0(\varepsilon)$ with

$$|L(z)| \geq \exp \{ (1 - \varepsilon) |z|^{\rho(z)} + (a - \varepsilon) |\operatorname{Im} z| \}, \quad |z| = R_n, \quad n \geq n_0(\varepsilon);$$

(Γ) for all $q \in (0, 1)$ we have

$$\sum_{j=1}^{\infty} \frac{1}{|L'(\nu_j)|} \exp q \nu_j^{\rho(\nu_j)} < \infty.$$

An entire function $L(z)$ with the above properties will be constructed in the form $L(z) = L_1(z) \cdot L_2(z)$, where

$$L_1(z) = az \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\eta_k^2} \right), \quad L_2(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\zeta_k^2} \right)$$

(the points η_k and ζ_k , $k \in \mathbb{N}$, were chosen in §5).

We show that $L(z)$ satisfies (A), (B), and (Γ).

First, we study the function $L_1(z)$. By construction, $L_1(z) = \frac{\sin az}{\varphi(z)}$, where $\varphi(z) = \prod_{j=1}^{\infty} (1 - \frac{z^2}{\xi_j^2})$; here the ξ_j , $j \in \mathbb{N}$, are the positive zeros of $\sin az$ with the property that either ξ_j or $-\xi_j$ belongs to some exceptional disk C_s for the symbol μ . Since the diameter of C_s is $2\gamma < \frac{1}{a}$, every C_s may contain at most one ξ_j . Clearly, this implies that, like $(\lambda_s)_{s=1}^{\infty}$, the sequence $(\xi_j)_{j=1}^{\infty}$ forms an R-set and $\lim_{j \rightarrow \infty} \frac{j}{\xi_j^{\rho(\xi_j)}} = 0$. Therefore, φ has

zero type at the order $\rho(r)$ and is of completely regular growth at this order. Accordingly, outside the disks $K_j^\pm = \{z : |z \mp \xi_j| < \gamma\}$ we have

$$\lim_{z \rightarrow \infty} \frac{\ln |\varphi(z)|}{|z|^{\rho(z)}} = 0.$$

Thus, for every $\varepsilon > 0$ there exists $r_0(\varepsilon) > 0$ such that

$$(6.1) \quad |\varphi(z)| \leq e^{\varepsilon|z|^{\rho(z)}}, \quad |z| \geq r_0(\varepsilon);$$

$$(6.2) \quad |\varphi(z)| \geq e^{-\varepsilon|z|^{\rho(z)}}, \quad |z| \geq r_0(\varepsilon), \quad z \notin \bigcup_j K_j^\pm.$$

Let an index $k_0(\varepsilon)$ be chosen in such a way that $\eta_k \geq r_0(\varepsilon)$ and $\zeta_k \geq r_0(\varepsilon)$, for $k \geq k_0(\varepsilon)$. By (6.1), we obtain

$$(6.3) \quad |L'_1(\eta_k)| = \frac{a}{|\varphi(\eta_k)|} \geq a e^{-\varepsilon \eta_k^{\rho(\eta_k)}}, \quad k \geq k_0(\varepsilon).$$

Next, by (6.1) and the choice of the ζ_k , we see that

$$(6.4) \quad |L_1(\zeta_k)| = \frac{1}{|\varphi(\zeta_k)|} \geq e^{-\varepsilon \zeta_k^{\rho(\zeta_k)}}, \quad k \geq k_0(\varepsilon).$$

Other properties of the function $L_1(z)$ are collected in the following lemmas.

Lemma 4. *The function $L_1(z)$ obeys the upper estimate*

$$(6.5) \quad |L_1(z)| \leq e^{a\gamma} \cdot \exp \{2\varepsilon|z|^{\rho(z)} + a|\operatorname{Im} z|\}, \quad |z| \geq r_0(\varepsilon).$$

Proof. Since $|\sin az| \leq e^{a|\operatorname{Im} z|}$ for all $z \in \mathbb{C}$, by (6.2) we see that

$$|L_1(z)| \leq \exp \{\varepsilon|z|^{\rho(z)} + a|\operatorname{Im} z|\}, \quad |z| \geq r_0(\varepsilon), \quad z \notin K_j^\pm.$$

Now, let z belong to some of the exceptional disks $K_j^\pm$, for instance, to $K_j^+ = \{t : |t - \xi_j| < \gamma\}$, and let $|z| \geq r_0(\varepsilon)$. On the circle $|t - \xi_j| = \gamma$, we find a point w such that $|L_1(w)| = \max\{|L_1(t)| : |t - \xi_j| \leq \gamma\}$. At the same time, we have $|w| \leq |z| + 2\gamma$, $|\operatorname{Im} w| \leq |\operatorname{Im} z| + \gamma$. Increasing $r_0(\varepsilon)$ if necessary, we may and do assume that

$$(t + 2\gamma)^{\rho(t+2\gamma)} \leq 2t^{\rho(t)}, \quad t \geq r_0(\varepsilon).$$

Consequently,

$$|L_1(z)| \leq |L_1(w)| \leq \exp \{\varepsilon|w|^{\rho(w)} + a|\operatorname{Im} w|\} \leq e^{a\gamma} \cdot \exp \{2\varepsilon|z|^{\rho(z)} + a|\operatorname{Im} z|\}.$$

This proves the lemma. $\square$

Lemma 5. *On the circles $|z| = R_n$, where $R_n = \frac{\pi}{4a} + \frac{\pi n}{a}$, $n \in \mathbb{N}$, we have*

$$(6.6) \quad |L_1(z)| \geq M \cdot \exp \{-\varepsilon|z|^{\rho(z)} + a|\operatorname{Im} z|\}, \quad n \geq n_0(\varepsilon),$$

where $M := \frac{1}{2}(1 - e^{-\frac{\pi}{4}})$.

Proof. Let $n_0(\varepsilon)$ be such that $R_n \geq r_0(\varepsilon)$ for $n \geq n_0(\varepsilon)$. Then $|\varphi(z)|$ obeys (6.1) on the circles $|z| = R_n$, $n \geq n_0(\varepsilon)$.

We estimate $|\sin are^{i\theta}|$ for $r = R_n$, $n \in \mathbb{N}$. Clearly, it suffices to consider $\theta \in [0, \frac{\pi}{2}]$. We have

$$|\sin are^{i\theta}| = \frac{1}{2} A(r, \theta) \cdot e^{ar \sin \theta}, \quad r \geq 0, \quad \theta \in [0, \frac{\pi}{2}],$$

where $A(r, \theta) = |1 - e^{-2ar(\sin \theta - i \cos \theta)}|$. The quantity $A(r, \theta)$ satisfies the following inequalities:

$$A(r, \theta) \geq 1 - e^{-2ar \sin \theta}; \quad A(r, \theta) \geq \sqrt{1 - \cos^2(2ar \cos \theta)}.$$

Fix $n \in \mathbb{N}$. If $\theta \in [0, \frac{\pi}{2}]$ is such that $2aR_n \sin \theta \leq \frac{\pi}{4}$, then

$$\frac{\pi}{4} + 2\pi n = 2aR_n - \frac{\pi}{4} \leq 2aR_n \cos \theta \leq 2aR_n = \frac{\pi}{2} + 2\pi n,$$

so that $A(R_n, \theta) \geq \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$ for such θ . But if $\theta \in [0, \frac{\pi}{2}]$ is such that $2aR_n \sin \theta > \frac{\pi}{4}$, then $A(R_n, \theta) \geq 1 - e^{-\frac{\pi}{4}}$. In any case, $A(R_n, \theta) \geq 2M$, $\theta \in [0, \frac{\pi}{2}]$. Thus,

$$|\sin az| \geq M e^{a|\operatorname{Im} z|}, \quad |z| = R_n, \quad n \in \mathbb{N}.$$

Combining this with (6.1), we arrive at (6.6). $\square$

We pass to the study of $L_2(z)$. By (5.6), $L_2(z)$ is a function of completely regular growth at the order $\rho(r)$. Next, taking (5.5) and [15, Theorem 1.2.7] into account, we conclude that the indicator of $L_2(z)$ at this order is equal to

$$H_2(\theta) = \frac{\cos \rho(\theta - \frac{\pi}{2})}{\cos \frac{\rho\pi}{2}}, \quad \theta \in (0, \pi).$$

So, increasing $r_0(\varepsilon)$ if necessary, we may assume that

$$(6.7) \quad |L_2(re^{i\theta})| \leq \exp \left\{ \left(\frac{\cos \rho(\theta - \frac{\pi}{2})}{\cos \frac{\rho\pi}{2}} + \varepsilon \right) r^{\rho(r)} \right\}, \quad r \geq r_0(\varepsilon), \quad \theta \in [0, \pi];$$

$$(6.8) \quad |L_2(re^{i\theta})| \geq \exp \left\{ \left(\frac{\cos \rho(\theta - \frac{\pi}{2})}{\cos \frac{\rho\pi}{2}} - \varepsilon \right) r^{\rho(r)} \right\},$$

$$r \geq r_0(\varepsilon), \quad \theta \in [0, \pi], \quad re^{i\theta} \notin \bigcup_l B_l^\pm,$$

where the $B_l^\pm = \{z : |z \mp \zeta_l| < \gamma\}$, $l \in \mathbb{N}$, are the exceptional disks for $L_2(z)$. In particular, since no η_k lies in $B_l^\pm$, $k, l \in \mathbb{N}$, we obtain

$$(6.9) \quad |L_2(\eta_k)| \geq e^{(1-\varepsilon)\eta_k^{\rho(\eta_k)}}, \quad k \geq k_0(\varepsilon).$$

Next, by [15, Theorem 1.2.8] we have

$$\lim_{k \rightarrow \infty} \frac{\ln \frac{1}{|L'_2(\zeta_k)|}}{\zeta_k^{\rho(\zeta_k)}} = -1.$$

Consequently, $k_0(\varepsilon)$ can be chosen so large that

$$(6.10) \quad |L'_2(\zeta_k)| > e^{(1-\varepsilon)\zeta_k^{\rho(\zeta_k)}}, \quad k \geq k_0(\varepsilon).$$

Finally, we prove that the function $L(z) = L_1(z) \cdot L_2(z)$ possesses the required properties.

Lemma 6. *The function $L(z) = L_1(z) \cdot L_2(z)$ has properties (A), (B), and (Γ).*

Proof. (A) Combining (6.5) and (6.7), we see that

$$|L(re^{i\theta})| \leq e^{a\gamma} \cdot \exp \left\{ \left(\frac{\cos \rho(\theta - \frac{\pi}{2})}{\cos \frac{\rho\pi}{2}} + 3\varepsilon \right) r^{\rho(r)} + ar \sin \theta \right\}$$

for all $r \geq r_0(\varepsilon)$ and $\theta \in [0, \pi]$. We find $\theta_0(\varepsilon) \in (0, \frac{\pi}{2})$ with

$$(6.11) \quad 1 - \varepsilon < \frac{\cos \rho(\theta - \frac{\pi}{2})}{\cos \frac{\rho\pi}{2}} < 1 + \varepsilon, \quad \theta \in [0, \theta_0(\varepsilon)] \cup [\pi - \theta_0(\varepsilon), \pi].$$

Then for θ as in this formula, we have

$$|L(re^{i\theta})| \leq e^{a\gamma} \cdot \exp \{ (1 + 4\varepsilon)r^{\rho(r)} + ar \sin \theta \}, \quad r \geq r_0(\varepsilon).$$

But if $\theta \in [\theta_0(\varepsilon), \pi - \theta_0(\varepsilon)]$, then, again increasing $r_0(\varepsilon)$ if necessary, we obtain

$$\left(\frac{\cos \rho(\theta - \frac{\pi}{2})}{\cos \frac{\rho\pi}{2}} + 3\varepsilon \right) r^{\rho(r)} \leq \left(\frac{1}{\cos \frac{\rho\pi}{2}} + 3\varepsilon \right) r^{\rho(r)} \leq \varepsilon r \sin \theta_0(\varepsilon) \leq \varepsilon r \sin \theta$$

whenever $r \geq r_0(\varepsilon)$. Thus, in this case

$$|L(re^{i\theta})| \leq e^{a\gamma} \cdot e^{(a+\varepsilon)r \sin \theta}, \quad r \geq r_0(\varepsilon).$$

This proves (A).

(B) Consider the circles $|z| = R_n$, $R_n = \frac{\pi}{4a} + \frac{\pi n}{a}$, $n \in \mathbb{N}$. Clearly, they do not intersect the exceptional disks $B_l^\pm$, $l \in \mathbb{N}$. Consequently, by (6.6) and (6.8), for $n \geq n_0(\varepsilon)$ on this circles we have the inequalities

$$|L(R_n e^{i\theta})| \geq M \exp \left\{ \left(\frac{\cos \rho(\theta - \frac{\pi}{2})}{\cos \frac{\rho\pi}{2}} - 2\varepsilon \right) R_n^{\rho(R_n)} + a R_n \sin \theta \right\}, \quad \theta \in [0, \pi].$$

Using the left inequality in (6.11), we see that

$$|L(R_n e^{i\theta})| \geq M \exp \{ (1 - 3\varepsilon) R_n^{\rho(R_n)} + a R_n \sin \theta \}$$

for

$$\theta \in [0, \theta_0(\varepsilon)] \cup [\pi - \theta_0(\varepsilon), \pi]$$

and $n \geq n_0(\varepsilon)$.

But if $\theta \in [\theta_0(\varepsilon), \pi - \theta_0(\varepsilon)]$, then, taking $n_0(\varepsilon)$ so large that

$$(1 + 2\varepsilon) R_n^{\rho(R_n)} < \varepsilon R_n \sin \theta_0(\varepsilon), \quad n \geq n_0(\varepsilon),$$

we get

$$\begin{aligned} & \left(\frac{\cos \rho(\theta - \frac{\pi}{2})}{\cos \frac{\rho\pi}{2}} - 2\varepsilon \right) R_n^{\rho(R_n)} + a R_n \sin \theta \geq -2\varepsilon R_n^{\rho(R_n)} + a R_n \sin \theta \\ & \geq R_n^{\rho(R_n)} + (a - \varepsilon) R_n \sin \theta - (1 + 2\varepsilon) R_n^{\rho(R_n)} + \varepsilon R_n \sin \theta_0(\varepsilon) \\ & \geq R_n^{\rho(R_n)} + (a - \varepsilon) R_n \sin \theta. \end{aligned}$$

Combining the two cases considered above and using again the identity $L(-z) = L(z)$, $z \in \mathbb{C}$, finally we obtain the estimate

$$|L(z)| \geq M \exp \{ (1 - 3\varepsilon) |z|^{\rho(z)} + (a - \varepsilon) |\operatorname{Im} z| \}$$

on the circles $|z| = R_n$, $n \geq n_0(\varepsilon)$. Thus, $L(z)$ obeys condition (B).

(Γ) Finally, we estimate $|L'(\eta_k)|$ and $|L'(\zeta_k)|$, $k \in \mathbb{N}$. We have

$$L'(\eta_k) = L'_1(\eta_k) \cdot L_2(\eta_k), \quad L'(\zeta_k) = L_1(\zeta_k) \cdot L'_2(\zeta_k), \quad k \in \mathbb{N}.$$

By (6.3) and (6.9), we obtain

$$|L'(\eta_k)| \geq a \cdot e^{(1-2\varepsilon)\eta_k^{\rho(\eta_k)}}, \quad k \geq k_0(\varepsilon).$$

Similarly, (6.4) and (6.10) yield

$$|L'(\zeta_k)| \geq e^{(1-2\varepsilon)\zeta_k^{\rho(\zeta_k)}}, \quad k \geq k_0(\varepsilon).$$

Clearly, these inequalities ensure (Γ), and the lemma follows. $\square$

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