

NARROW DOMAINS AND THE HARNACK INEQUALITY FOR ELLIPTIC EQUATIONS

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Dedicated to Nina N. Ural'tseva

ABSTRACT. We present a direct proof of Moser's Harnack inequality that does not involve iterations. The method is based on a recursive estimate for solutions in domains of small measure. Such estimates can also be useful for other applications.

§1. INTRODUCTION

We consider second order elliptic operators L in the *divergence* form

$$(D) \quad Lu := (D, aDu) := D_i(a_{ij}D_ju).$$

Here and throughout the paper, the summation convention over repeated indices is enforced, D denotes a symbolic vector with components $D_i := \partial/\partial x_i$, and $a := [a_{ij}]$ is an $(n \times n)$ -matrix of real *measurable* coefficients satisfying the *uniform ellipticity* condition

$$(U) \quad (a\xi, \xi) := a_{ij}\xi_i\xi_j \geq \nu |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \quad \text{and} \quad \sum_{i,j} |a_{ij}|^2 \leq \nu^{-2}n,$$

with an *ellipticity constant* $\nu \in (0, 1]$. It is easily seen that $\nu = 1$ corresponds to $L = \Delta$.

In the classical theory (see [LU, 1.2] or [GT, 8.1]), the equations $Lu = 0$ or inequalities $Lu \geq 0$, $Lu \leq 0$ in an open set $\Omega \subset \mathbb{R}^n$ are understood in an integral sense, and the operators L (which can also include lower order terms) are applied to functions u in the Sobolev space $W^{1,2}(\Omega)$, i.e., $u, D_iu \in L^2(\Omega)$. The pointwise estimates for such functions, such as the estimates for Hölder constants and the Harnack inequality, hold true under an implicit assumption that the values of u are appropriately modified on a set of zero measure. Fortunately, the most difficult part is to establish such kind of estimates under the additional assumption that all the functions a_{ij} and u are smooth. It is important only that the key estimates depend only on the prescribed constants, such as n and ν , and do not depend on the “additional” smoothness of a_{ij} and u . We shall make this point clearer below, in §6. For this reason, in order to make the exposition of our method more transparent, we always assume $u \in C(\bar{\Omega})$, and, in the main part of the text, $u \in C^1(\bar{\Omega})$ (this is really important in the proof of Theorem 3.4).

In the pioneering work by De Giorgi [DG57], as an intermediate step in the proof of the Hölder continuity of solutions of $Lu = 0$, the following pointwise estimate was derived (in an equivalent formulation).

Theorem 1.1. *Let $B_R := B_R(x_0)$ be a ball of radius $R > 0$ centered at some point $x_0 \in \mathbb{R}^n$. Suppose that u is a function in $W^{1,2}(B_R) \cap C(\bar{B}_R)$ satisfying $u \geq 0$ and*

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$Lu := (D, aDu) \geq 0$ in B_R . Then

$$(1.1) \quad M := \sup_{B_{R/2}} u \leq N \cdot \left(\frac{1}{|B_R|} \int_{B_R} u^2 dx \right)^{1/2}, \quad \text{where } N = N(n, \nu).$$

The approach in [DG57] takes into consideration the sets

$$A_{k,r} := \{x \in B_r : u(x) > k\}, \quad \text{where } k > 0 \text{ and } r \in (R/2, R].$$

It was shown there that for some sequences $k_j \nearrow M$ and $r_j \searrow R/2$, the measures

$$(1.2) \quad |A_{k_j, r_j}| = |B_{r_j} \cap \{u > k_j\}| \quad \text{tend to } 0 \quad \text{as } j \rightarrow \infty,$$

which obviously implies $u \leq M$ on $B_{R/2}$.

Loosely speaking, (1.2) means that the sets A_{k_j, r_j} become *narrow* for large j .

In the general theory of linear and quasilinear elliptic and parabolic equation, which was created by Ladyzhenskaya and Ural'tseva (see [LU, LSU] and the references therein), the key estimates for solutions are given in terms of the measures $|A_{k,r}|$. It was pointed out in [LU, II.6] that the authors considered their general technique as a further development of De Giorgi's method.

Estimate (1.1) also appears as a byproduct of the original proof of Moser's Harnack inequality [M60, M61, M64]. Namely, in Moser's approach, there exist sequences $p_j \nearrow \infty$ and $r_j \searrow R/2$, such that

$$(1.3) \quad \|u\|_{L^{p_j}(B_{r_j})} \leq M \quad \text{for all } j,$$

so that the norm of u in $L^\infty(B_{R/2})$ does not exceed M .

The reader can also find a direct exposition of the methods by De Giorgi and Moser in application to (1.1), the Hölder estimates, and the Harnack inequality in the book by Han and Lin [HL, Chapter 4].

A different kind of iterations appears in the proof of similar results for operators L in the *nondivergence* form

$$(ND) \quad Lu := (aD, Du) := a_{ij} D_i D_j u$$

with measurable a_{ij} satisfying (U). As in (D), for simplicity we do not include lower order terms, and also we do not consider more general parabolic operators. In our joint work with Krylov [KS80, S80] (see also [K85, Chapter 4]), we proved the Hölder estimates and the Harnack inequality in the (ND) case on the grounds of the following facts.

Theorem 1.2. *Let an open set $\Omega \subset \mathbb{R}^n$, $x_0 \in \Omega$, and $R > 0$ be such that*

$$(1.4) \quad |\Omega_R| := |\Omega \cap B_R(x_0)| \leq \mu \cdot |B_R|, \quad \text{where } \mu = \text{const} \in (0, 1).$$

(i) *Let $u \in W^{2,n}(\Omega) \cap C(\bar{\Omega})$ satisfy*

$$(1.5) \quad u \geq 0, \quad Lu := (aDu, Du) \geq 0 \quad \text{in } \Omega,$$

and $u = 0$ on $(\partial\Omega) \cap B_R(x_0)$. Then

$$(1.6) \quad u(x_0) \leq \beta \cdot \sup_{\Omega_R} u, \quad \text{where } \beta = \beta(n, \nu, \mu) \in (0, 1).$$

(ii) *(Estimate for narrow domains). Moreover,*

$$(1.7) \quad \beta(n, \nu, \mu) \rightarrow 0^+ \quad \text{as } \mu \rightarrow 0^+.$$

Such estimates (also called *growth lemmas*) have been introduced by Landis [La71] for an alternative proof of the results by De Giorgi and Moser mentioned above. In this theorem, part (ii) is proved first: it follows easily from Aleksandrov's estimate in [A63] (see also [GT, Theorem 9.1]). Then the more difficult part (i) follows by iteration, by

using a special covering lemma in order to increase the “critical” value of μ for which (i) holds true.

It was shown in [FS01] that this approach also works in the case of (D), only instead of Aleksandrov’s estimate one should use the energy estimate. Note that (1.7) also follows immediately from (1.1). Indeed, by extending $u \equiv 0$ to $B_R \setminus \Omega$, we get a function u satisfying all the assumptions of Theorem 1.1. Hence,

$$(1.8) \quad u(x_0) \leq M \leq N \cdot \left(\frac{1}{|B_R|} \int_{B_R} u^2 dx \right)^{1/2} \leq N \cdot \left(\frac{|\Omega_R|}{|B_R|} \right)^{1/2} \cdot \sup_{\Omega_R} u,$$

and (1.4) implies (1.7) with $\beta := N\mu^{1/2}$. This argument is especially simple in the case where $L = \Delta$, the Laplace operator. Then the extended function is subharmonic in B_R , by the mean value theorem we have

$$u(x_0) \leq \frac{1}{|B_R|} \int_{\Omega_R} u dx \leq \frac{|\Omega_R|}{|B_R|} \cdot \sup_{\Omega_R} u,$$

and (1.7) follows with $\beta = \mu$.

Note that though $\beta = \mu$ looks “better” than $\beta = N\mu^{1/2}$ for small μ , both these expressions are far from being optimal. In §2, we use an elementary “scaling” argument to show that from the strict inequality $\beta(n, \nu, \mu_0) < 1$ for some $\mu_0 \in (0, 1)$ it follows that in fact convergence in (1.7) is much faster:

$$(1.9) \quad \beta \leq \exp(-c_0 \mu^{-1/n}) \text{ with } c_0 = \text{const} > 0.$$

In its turn, (1.9) follows from a simple recurrent relation $\beta(n, \nu, 2^{-n}\mu) \leq \beta^2(n, \nu, \mu)$. This relation is the core of our approach, it allows us to prove the Harnack inequality and other related facts without using iterations. It only requires a mild technical assumption $u \in C^1$. In §3, we prove a version of estimate (1.7) (Theorem 3.4) and show that it is equivalent to (1.1) (Theorem 3.5). The proof of this equivalence follows the lines of [FS01, Theorem 3.4], we only put aside the crucial part in Lemma 3.6, which is needed again in the proof of the Harnack inequality. Theorem 4.2 in §4 can be treated as a version of Theorem 1.2 after replacing u by $(1 - u)^+$ and taking $\beta := 1 - \beta_1$. In §5, we prove the Harnack inequality along with its standard applications. The proof is short, because all the essential elements are scattered among the previous sections. Finally, in §6 we show how to get rid of an *a priori* assumption $u \in C^1$ and extend the main results to $u \in W^{1,2}$.

Notation. The notation $A := B$, or $B =: A$, means “ $A = B$ by definition”.

$x = (x_1, \dots, x_n)$ are vectors or points in \mathbb{R}^n , $(x, y) := x_i y_i$ is the scalar product of $x, y \in \mathbb{R}^n$ (the summation convention is implied), and $|x| := (x, x)^{1/2}$ is the length of $x \in \mathbb{R}^n$.

$B_R(x) := \{y \in \mathbb{R}^n : |y - x| < R\}$ is a ball of radius $R > 0$ centered at $x \in \mathbb{R}^n$.

Ω is an open set in \mathbb{R}^n with boundary $\partial\Omega$ and Lebesgue measure $|\Omega|$. For fixed $x_0 \in \mathbb{R}^n$ and $R > 0$, $\Omega_R := \Omega_R(x_0) := \Omega \cap B_R(x_0)$.

$\Omega^\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$.

$a^+ := \max\{a, 0\}$. N (with or without indices) is a generic notation for various constants depending only on the given quantities, such as n, ν , etc. This dependence is indicated in the parentheses, e.g., $N = N(n, \nu, \dots)$.

§2. PRELIMINARY RESULTS

We exploit the fact that the constant β in (1.6) depends only on n, ν, μ , and does not depend on the other characteristics of Ω , the a_{ij} , and u , so that we can “optimize” the choice in a certain way.

Lemma 2.1. *For fixed n and ν , the constants $\beta(\mu) = \beta(n, \nu, \mu)$ in Theorem 1.2 satisfy*

$$(2.1) \quad \beta(2^{-n}\mu) \leq \beta^2(\mu) \quad \text{for } \mu \in (0, 1).$$

Proof. Let u satisfy (1.6), and suppose that (1.4) holds true with $2^{-n}\mu$ in place of μ , i.e.,

$$(2.2) \quad x_0 \in \Omega, \quad \text{and} \quad |\Omega_R| \leq 2^{-n}\mu \cdot |B_R|.$$

Since $2^{-n} \cdot |B_R| = |B_{R/2}|$, we can use (1.6) with $R/2$ in place of R :

$$(2.3) \quad u(x_0) \leq \beta(\mu) \cdot \sup_{\Omega_{R/2}(x_0)} u.$$

By the maximum principle, the supremum on the right-hand side is attained at some point $y_0 \in \Omega \cap \partial B_{R/2}(x_0)$, which also satisfies $B_{R/2}(y_0) \subset B_R(x_0)$, whence by (2.2)

$$|\Omega_{R/2}(y_0)| \leq |\Omega_R(x_0)| \leq 2^{-n}\mu \cdot |B_R| = \mu \cdot |B_{R/2}|.$$

Using (2.3) and once again (1.6) with $y_0, R/2$ in place of x_0, R (respectively), we get

$$u(x_0) \leq \beta(\mu) \cdot u(y_0) \leq \beta^2(\mu) \cdot \sup_{\Omega_{R/2}(y_0)} u \leq \beta^2(\mu) \cdot \sup_{\Omega_R(x_0)} u.$$

The lemma is proved. □

By itself, estimate (2.1) does not guarantee the convergence $\beta(\mu) \rightarrow 0^+$ as $\mu \rightarrow 0^+$, because it holds true for $\beta \equiv 1$. We really need $\beta(\mu_0) < 1$ for some $\mu_0 \in (0, 1)$.

Corollary 2.2. *Let $\beta(\mu)$ be a positive monotone nondecreasing function on $(0, 1)$ such that $\beta(\mu_0) \in (0, 1)$ for some $\mu_0 \in (0, 1)$. Then from (2.1) it follows that*

$$(2.4) \quad \beta(\mu) \leq \beta_0(\mu) := \exp(-c_0 \cdot \mu^{-1/n}) \quad \text{on } (0, \mu_0],$$

where

$$(2.5) \quad c_0 := -\frac{1}{2} \cdot \mu_0^{1/n} \ln \beta(\mu_0) > 0.$$

Proof. Obviously,

$$(0, \mu_0] = \bigcup_{j=0}^{\infty} \Delta_j, \quad \text{where } \Delta_j := [2^{-n(j+1)}\mu_0, 2^{-nj}\mu_0].$$

We have

$$\beta_0(2^{-n}\mu) = \exp(-2c_0\mu^{-1/n}) = \beta_0^2(\mu) \quad \text{on } (0, \mu_0],$$

and by the choice of c_0 in (2.5),

$$\beta_0(2^{-n}\mu_0) = \exp(-2c_0\mu_0^{-1/n}) = \exp(\ln(\beta(\mu_0))) = \beta(\mu_0).$$

Since both functions $\beta(\mu)$ and $\beta_0(\mu)$ are monotone nondecreasing, it follows that

$$\beta(\mu) \leq \beta(\mu_0) = \beta_0(2^{-n}\mu_0) \leq \beta_0(\mu) \quad \text{on } \Delta_0 := [2^{-n}\mu_0, \mu_0].$$

Suppose that $\beta(\mu) \leq \beta_0(\mu)$ on Δ_j for some integer $j \geq 0$. For such μ , we have

$$\beta(2^{-n}\mu) \leq \beta^2(\mu) \leq \beta_0^2(\mu) = \beta_0(2^{-n}\mu).$$

Since each point in Δ_{j+1} can be represented as $2^{-n}\mu$ with $\mu \in \Delta_j$, we have $\beta \leq \beta_0$ on Δ_{j+1} . By induction, $\beta \leq \beta_0$ on Δ_j for each integer $j \geq 0$, which completes the proof of (2.4). □

§3. ESTIMATES FOR SUBSOLUTIONS

Throughout this section, we fix an open set $\Omega \subset \mathbb{R}^n$, a point $x_0 \in \Omega$, a constant $R > 0$, and denote $B_R := B_R(x_0)$, $\Omega_R := \Omega \cap B_R$. Here we derive local estimates for functions u satisfying the inequality $Lu := (D, aDu) \geq 0$ in $\Omega \subset \mathbb{R}^n$ in a weak sense, i.e., by formal integration by parts,

$$(3.1) \quad \int_{\Omega} (aDu, D\eta) dx \leq 0 \quad \forall \eta \in C_0^\infty(\Omega), \quad \eta \geq 0.$$

By approximation, this property is easily extended to functions $\eta \in C^1(\bar{\Omega})$ satisfying $u = 0$ on $\partial\Omega$. In the classical theory (see [LU, III.4], [GT, Chapter 8]), these relations are considered for $u \in W^{1,2}(\Omega)$, so that $Du \in L^2(\Omega)$. For the simplicity of presentation, especially for the proof of Theorem 3.4, we impose the technical restriction $u \in C^1(\bar{\Omega})$. The extension of main results to $u \in W^{1,2}(\Omega)$ is pretty standard. For completeness and the reader's convenience, we provide more details at the end, in §6.

We start with an *energy estimate*, which is well known even for more general parabolic equations (see estimate (III.2.18) in [LSU]). It served as an initial step in Moser's iterations (see [M61, Lemma 4], and also estimate (IX.5.7) in [LU]).

Lemma 3.1 (Energy estimate). *Let u be a functions in $C^1(\bar{\Omega})$ satisfying*

$$(3.2) \quad u \geq 0, \quad Lu := (D, aDu) \geq 0 \quad \text{in } \Omega; \quad \text{and } u = 0 \quad \text{on } (\partial\Omega) \cap B_R.$$

Take a function $\zeta \in C^1(\bar{B}_R)$ satisfying

$$(3.3) \quad 0 \leq \zeta \leq 1, \quad |D\zeta| \leq \frac{4}{R} \quad \text{in } B_R; \quad \zeta \equiv 1 \quad \text{on } B_{R/2}, \quad \zeta = 0 \quad \text{on } \partial B_R.$$

Then

$$(3.4) \quad \int_{\Omega_R} |D(\zeta u)|^2 dx \leq \frac{N}{R^2} \int_{\Omega_R} u^2 dx, \quad \text{where } N = N(n, \nu).$$

Proof. The function $\eta \in C^1(\bar{\Omega})$ satisfies

$$D\eta = \zeta^2 Du + 2\zeta u D\zeta \quad \text{in } \Omega_R, \quad \text{and } \eta = 0 \quad \text{on } \partial\Omega_R.$$

From (3.1) it follows that

$$\nu \int_{\Omega_R} \zeta^2 |Du|^2 dx \leq \int_{\Omega_R} \zeta^2 (aDu, Du) dx \leq -2 \int_{\Omega_R} \zeta u (aDu, D\zeta) dx.$$

By the Cauchy–Schwarz inequality and the estimate for $|D\zeta|$ in (3.3), the last expression does not exceed

$$\frac{\nu}{2} \int_{\Omega_R} \zeta^2 |Du|^2 dx + \frac{N}{R^2} \int_{\Omega_R} u^2 dx, \quad \text{where } N = N(n, \nu) > 0.$$

This yields the estimate

$$\int_{\Omega_R} \zeta^2 |Du|^2 dx \leq \frac{N}{R^2} \int_{\Omega_R} u^2 dx.$$

Now the desired estimate (3.4) follows from

$$|D(\zeta u)|^2 = |\zeta Du + u D\zeta|^2 \leq 2\zeta^2 |Du|^2 + 2u^2 |D\zeta|^2.$$

The lemma is proved. □

The following lemma is a version of estimate (II.2.12) in [LU].

Lemma 3.2. *Let u be a function in $C^1(\bar{\Omega})$ satisfying $u = 0$ on $\partial\Omega$. Then*

$$(3.5) \quad \|u\|_{1,\Omega} := \int_{\Omega} |u| dx \leq |\Omega|^{\frac{1}{2} + \frac{1}{n}} \|Du\|_{2,\Omega}.$$

Proof. The Gagliardo–Nirenberg–Sobolev inequality (see the proof of Theorem 1 in [E, 5.6] for $p = 1$) can be written as

$$(3.6) \quad \|u\|_{q,\Omega} := \left(\int_{\Omega} |u|^q dx \right)^{1/q} \leq \|Du\|_{1,\Omega}, \quad \text{where } q := \frac{n}{n-1}.$$

In the case where $|\Omega| = 1$, by the Hölder inequality, the norm $\|u\|_{p,\Omega}$ is a monotone nondecreasing function of p , and (3.5) follows immediately from (3.6).

In the general case, take $\lambda := |\Omega|^{1/n}$ and substitute $x = \lambda y$, $dx = \lambda^n dy$. Then the function

$$v(y) := u(\lambda y) \quad \text{in } \Omega_0 := \{y \in \mathbb{R}^n : \lambda y \in \Omega\}$$

satisfies the above assumptions with $|\Omega_0| = \lambda^{-n} \cdot |\Omega| = 1$. Therefore,

$$\|u\|_{1,\Omega} = \lambda^n \cdot \|v\|_{1,\Omega_0} \leq \lambda^n \cdot \|Dv\|_{2,\Omega_0} = \lambda^{n+1} \cdot \|Du(\lambda y)\|_{2,\Omega_0} = \lambda^{\frac{n}{2}+1} \cdot \|Du\|_{2,\Omega}.$$

By the choice of λ , the last expression coincides with the right-hand side in (3.5). \square

Combining Lemmas 3.1 and 3.2, we derive the following useful estimate.

Lemma 3.3. *Let u be a function in $C^1(\bar{\Omega})$ satisfying the assumptions (3.2) in Lemma 3.1. Suppose that for some positive constants μ, β , and c , we have inequality (1.4) and*

$$(3.7) \quad |\Omega_{R/2} \cap \{u > \beta M\}| \geq c\mu \cdot |B_R|, \quad \text{where } M := \sup_{\Omega_R} u.$$

Then

$$(3.8) \quad c\beta \leq N_0 \mu^{\frac{1}{n}}, \quad \text{where } N_0 = N_0(n, \nu) > 0.$$

Proof. From properties (1.4) and (3.7) it follows that

$$(3.9) \quad c\beta M \cdot |\Omega_R| \leq c\mu\beta M \cdot |B_R| \leq \int_{\Omega_{R/2}} u dx \leq \int_{\Omega_R} \zeta u dx,$$

where ζ is a function in (3.3). Applying Lemma 3.2 to the function $\zeta u \in C^1(\bar{\Omega}_R)$, and then using the energy estimate (3.4), we obtain

$$\begin{aligned} \int_{\Omega_R} \zeta u dx &= \|\zeta u\|_{1,\Omega_R} \leq |\Omega_R|^{\frac{1}{2}+\frac{1}{n}} \cdot \|D(\zeta u)\|_{2,\Omega_R} \\ &\leq \frac{N}{R} \cdot |\Omega_R|^{\frac{1}{2}+\frac{1}{n}} \cdot \|u\|_{2,\Omega_R} \leq \frac{NM}{R} \cdot |\Omega_R|^{1+\frac{1}{n}}. \end{aligned}$$

Comparing with (3.9) and again using (1.4), we see that

$$c\beta \leq \frac{N}{R} \cdot |\Omega_R|^{\frac{1}{n}} \leq \frac{N}{R} \cdot \mu^{\frac{1}{n}} \cdot |B_R|^{\frac{1}{n}} =: N_0 \mu^{\frac{1}{n}}.$$

Thus, the lemma is proved. \square

The next theorem includes property (1.7) in Theorem 1.2 (ii) for small $\mu > 0$. Note that the rate of convergence $\beta \rightarrow 0$ as $\mu \rightarrow 0^+$ can be improved, in accordance with Lemma 2.1 and Corollary 2.2.

Theorem 3.4. *Suppose that the measure of Ω_R satisfies (1.4) with a small constant $\mu \in (0, 1)$, namely,*

$$(3.10) \quad \beta := 4^{n+1} N_0 \mu^{\frac{1}{n}} \leq \frac{1}{2},$$

where $N_0 = N_0(n, \nu)$ is a constant in (3.8). Then for any function $u \in C^1(\bar{\Omega})$ satisfying (3.2), we have the estimate

$$(3.11) \quad u(x_0) \leq \beta \cdot \sup_{\Omega_R} u.$$

Proof. Fix $\varepsilon > 0$. We show that for each ball $B_r(y) \subset B_R := B_R(x_0)$ and $k \geq 0$, from

$$(3.12) \quad |\Omega_r(y) \cap \{u > k\}| \leq \mu \cdot |B_r|$$

it follows that

$$(3.13) \quad (u - k)^+(y) \leq \varepsilon\sqrt{r} + \beta \cdot M, \text{ where } M := \sup_{\Omega_r(y)} (u - k)^+.$$

Since the constant β does not depend on ε , the estimate remains valid for $\varepsilon = 0$, and the desired estimate (3.11) follows by taking $k = 0$, $y = x_0$, and $r = R$.

Our assumption $u \in C^1(\bar{\Omega})$ together with (3.12) guarantees that

$$(u - k)^+(y) \leq r \cdot \sup |Du| \leq \varepsilon\sqrt{r}$$

for small $r > 0$, so that (3.13) is trivial. Therefore, it suffices to prove that (3.12) implies (3.13) assuming that we already have a similar property with $r/4$ in place of r , for arbitrary k, y , and r under consideration.

Based on (3.12), we can apply Lemma 3.3 to the function $u - k$ on the open set $\Omega_r(y) \cap \{u > k\}$, which satisfies (1.1) with $x_0 = y$, $R = r$. By our choice of β in (3.10), we have

$$4^{-n} \cdot \frac{\beta}{2} > N_0 \mu^{\frac{1}{n}},$$

i.e., inequality (3.8) fails with $c := 4^{-n}$ and $\beta/2$ in place of β . Therefore, (3.7) must fail as well:

$$(3.14) \quad \left| \Omega_{r/2}(y) \cap \left\{ u - k > \frac{\beta M}{2} \right\} \right| < 4^{-n} \mu \cdot |B_r| = \mu \cdot |B_{r/4}|.$$

Set $k_0 := k + \beta M/2$. Inequality (3.14) implies (3.12) with $k_0, r/4$ in place of k, r , respectively. Since we assume that (3.13) is valid for $r/4$, we get

$$(u - k_0)^+(y) \leq \frac{\varepsilon\sqrt{r}}{2} + \beta \cdot \sup_{\Omega_{r/4}(y)} (u - k_0)^+.$$

Further, note that for each point $z \in \Omega_{r/4}(y)$, we have $\Omega_{r/4}(z) \subset \Omega_{r/2}(y)$, so by the same argument, we get a rough estimate

$$(u - k_0)^+(z) \leq \frac{\varepsilon\sqrt{r}}{2} + \beta \cdot \sup_{\Omega_{r/4}(z)} (u - k_0)^+ < \frac{\varepsilon\sqrt{r}}{2} + \beta \cdot M.$$

Combining these estimates, we arrive at

$$(u - k_0)^+(y) < \varepsilon\sqrt{r} + \beta^2 M.$$

Finally, since $k_0 - k = \beta M/2$ and $\beta \leq 1/2$, we obtain estimate (3.13):

$$(u - k)^+(y) \leq (u - k_0)^+(y) + \frac{\beta M}{2} \leq \varepsilon\sqrt{r} + \beta \left(\beta + \frac{1}{2} \right) M \leq \varepsilon\sqrt{r} + \beta M.$$

As has already been pointed out, the desired estimate (3.11) follows. \square

Theorem 3.4 automatically yields a pointwise estimate in terms of L^p norms. This is an “elliptic” version of Theorem 3.4 in [FS01] for the parabolic case. Note that one can apply this theorem with $p = 2$ and an arbitrary $y \in \Omega_{R/2} := \Omega_{R/2}(x_0)$ in place of x_0 :

$$u^2(y) \leq \frac{N}{|B_R|} \int_{\Omega_{R/2}(y)} u^2 dx.$$

Since $\Omega_{R/2}(y) \subset \Omega_R$, estimate (1.1) in Theorem 1.1 follows.

Theorem 3.5. *Let $u \in C^1(\bar{\Omega})$ satisfy (3.2). Then for arbitrary $p > 0$ we have*

$$(3.15) \quad u^p(x_0) \leq \frac{N}{|B_R|} \int_{\Omega_R} u^p dx, \text{ where } N = N(n, \nu, p) > 0.$$

If $B_R \subset \Omega$, then $\Omega_R = B_R$ and there is no boundary condition in (3.2).

The proof of this theorem follows easily from the next claim.

Lemma 3.6. *Under the assumptions of the previous theorem, for arbitrary $\gamma > 0$, there is a constant $\mu_0 = \mu_0(n, \nu, \gamma) \in (0, 1)$ and a ball $B_\rho(y) \subset B_{2\rho}(y) \subset B_R$ such that*

$$(3.16) \quad |\Omega_\rho(y) \cap \{u > k\}| > \mu_0 \cdot |B_\rho|, \text{ where } k := 2^{-\gamma-1} \left(\frac{R}{\rho}\right)^\gamma u(x_0).$$

Proof of Theorem 3.5. Take $\gamma := n/p$. Then by (3.16) we have

$$\int_{\Omega_\rho(y)} u^p dx \geq \mu_0 \cdot |B_\rho| \cdot k^p = 2^{-n-p} \mu_0 \cdot |B_R| \cdot u^p(x_0),$$

and (3.15) follows with $N := 2^{n+p}/\mu_0$. \square

Proof of Lemma 3.6. Denote $d(x) := R - |x - x_0| = \text{dist}(x, \partial B_R)$ on Ω_R . Then $d(x_0) = R$, and

$$(3.17) \quad R^\gamma u(x_0) \leq M := \sup_{\Omega_R} d^\gamma u.$$

Since $d^\gamma u = 0$ on $\partial\Omega_R$, the supremum here is attained at some point $y \in \Omega_R$. Set $\rho := d(y)/2$, so that $M = (2\rho)^\gamma u(y)$. Obviously, $d \geq \rho$ on $\Omega_\rho(y)$, whence

$$\sup_{\Omega_\rho(y)} u \leq \rho^{-\gamma} \sup_{\Omega_\rho(y)} d^\gamma u = \rho^{-\gamma} (d^\gamma u)(y) = 2^\gamma u(y),$$

and the function $v(x) := u(x) - u(y)/2$ satisfies

$$(3.18) \quad v(y) = 2^{-1}u(y) \geq 2^{-\gamma-1} \sup_{\Omega_\rho(y)} u > 2^{-\gamma-1} \sup_{\Omega_\rho(y)} v.$$

Note that by (3.17),

$$(3.19) \quad k := 2^{-\gamma-1} \left(\frac{R}{\rho}\right)^\gamma u(x_0) \leq 2^{-\gamma-1} \rho^{-\gamma} M = \frac{u(y)}{2}.$$

Now we take $\mu_0 = \mu_0(n, \nu, \gamma)$ from the identity

$$2^{-\gamma-1} = 4^{n+1} N_0 \mu_0^{\frac{1}{n}},$$

where $N_0 = N_0(n, \nu)$ is a constant in (3.10) in Theorem 3.4, so that (3.10) holds true with $\beta := 2^{-\gamma-1}$, $\mu := \mu_0$. On the other hand, inequality (3.18) means that (3.11) fails for the function $v(x)$ in $\Omega_\rho(y) \cap \{v > 0\}$. Therefore, (1.1) fails as well, i.e.,

$$|\Omega_\rho(y) \cap \{v > 0\}| > \mu_0 \cdot |B_\rho|.$$

By (3.19), we have $\{v > 0\} \subset \{u > k\}$, and estimate (3.16) follows. \square

§4. ESTIMATES FOR SUPERSOLUTIONS

We need the following estimate.

Lemma 4.1. *Suppose that an open set Ω and a ball B_1 of radius 1 in \mathbb{R}^n satisfy*

$$|\Omega_1| := |\Omega \cap B_1| \leq \mu \cdot |B_1|, \quad \text{where } \mu \in (0, 1).$$

Then for any function $v \in C^1(\bar{\Omega})$ such that $v = 0$ on $(\partial\Omega) \cap B_1$, we have

$$(4.1) \quad \int_{\Omega_1} v^2 dx \leq N \cdot \int_{\Omega_1} |Dv|^2 dx, \quad \text{where } N = N(n, \mu) > 0.$$

This lemma follows immediately from [M60, Lemma 2] with $N := B_1 \setminus \Omega$, (see also [LU, Lemma II.3.8]).

One can also prove (4.1) by a contradiction argument, which does not provide an explicit expression for the constant $N(n, \mu)$ (see [HL, Lemma 4.8]). The following lemma is also a version of [HL, Theorem 4.9], see also the proof of (IX.5.15) in [LU].

Theorem 4.2. *Let u be a function in $C^1(\bar{B}_{2R})$ satisfying*

$$u \geq 0 \quad \text{and} \quad Lu := (D, aDu) \leq 0 \quad \text{in } B_{2R}$$

in a weak sense, i.e.,

$$(4.2) \quad \int_{B_{2R}} (aDu, D\eta_0) dx \geq 0 \quad \forall \eta_0 \in C_0^\infty(B_{2R}), \quad \eta_0 \geq 0.$$

Suppose that

$$(4.3) \quad |B_R \cap \{u < 1\}| \leq \mu \cdot |B_R|, \quad \text{where } \mu \in (0, 1).$$

Then

$$(4.4) \quad u \geq \beta_1 = \beta_1(n, \nu, \mu) > 0 \quad \text{on } B_{R/2}.$$

Proof. Step 1. By rescaling $x \rightarrow R^{-1}x$, the proof reduces to the case where $R = 1$. In addition, without loss of generality, we may assume that $u \geq \delta = \text{const} > 0$ on B_2 , so that

$$v := G(u) := -\ln u \in C^1(\bar{B}_2).$$

Note that since $G'(u) = -u^{-1} < 0$ and $G''(u) = u^{-2} > 0$, we have $Lv := (D, aDv) \geq 0$ in B_2 . Indeed, for an arbitrary $0 \leq \eta \in C^1(\bar{B}_2)$, $\eta = 0$ on ∂B_2 we also have $0 \leq \eta_0 := -G'(u)\eta \in C^1(\bar{B}_2)$, $\eta_0 = 0$ on ∂B_2 , and

$$D\eta_0 = -G'(u)D\eta - G''(u)\eta Du, \quad Dv = G'(u)Du.$$

Therefore,

$$\begin{aligned} \int_{B_2} (aDv, D\eta) dx &= \int_{B_2} (aDu, G'(u)D\eta) dx \\ &= - \int_{B_2} (aDu, D\eta_0) dx - \int_{B_2} G''(u)\eta \cdot (aDu, Du) dx \leq 0, \end{aligned}$$

because of (4.2) and the ellipticity condition (U). By definition, this means that $Lv \geq 0$ in B_2 in a weak sense. Moreover, since

$$G''(u)(aDu, Du) = (aDv, Dv) \geq \nu |Dv|^2,$$

we can write a stronger inequality

$$(4.5) \quad \int_{B_2} (aDv, D\eta) dx \leq -\nu \int_{B_2} \eta |Dv|^2 dx \quad \forall \eta \in C_0^\infty(B_2), \quad \eta \geq 0.$$

Take $\eta := \zeta^2$, where ζ is a standard cut-off function in (3.3) for the ball B_2 , i.e.,

$$0 \leq \zeta \leq 1, \quad |D\zeta| \leq 2 \text{ in } B_2; \quad \zeta \equiv 1 \text{ on } B_1.$$

Then from (4.5) it follows that

$$\nu \int_{B_2} \zeta^2 |Dv|^2 dx \leq -2 \int_{B_2} \zeta \cdot (aDv, D\zeta) dx \leq N \int_{B_2} \zeta |Dv| dx$$

with $N = N(n, \nu) > 0$. By the Cauchy–Schwarz inequality, the last expression does not exceed

$$\frac{\nu}{2} \int_{B_2} \zeta^2 |Dv|^2 dx + N.$$

Therefore,

$$(4.6) \quad \int_{B_1} |Dv|^2 dx \leq \int_{B_2} \zeta^2 |Dv|^2 dx \leq N = N(n, \nu).$$

Step 2. Setting $\Omega := B_2 \cap \{u < 1\} = B_2 \cap \{v > 0\}$, we observe that

$$v \geq 0, \quad Lv \geq 0 \text{ in } \Omega_1 := \Omega \cap B_1; \text{ and } v = 0 \text{ on } (\partial\Omega) \cap B_1.$$

Moreover, by (4.3), we also have $|\Omega_1| \leq \mu \cdot |B_1|$. Then we can use Lemma 4.1, which in combination with (4.6) provides the estimate

$$\int_{\Omega_1} v^2 dx \leq N \cdot \int_{\Omega_1} |Dv|^2 dx \leq N = N(n, \nu, \mu).$$

Step 3. Finally, for each point $y \in \Omega_{1/2}$, we can apply Theorem 3.5 to the function v on $\Omega_{1/2}(y) \subset \Omega_1$ with $p = 2$. Then

$$v^2(y) \leq N \int_{\Omega_{1/2}(y)} v^2 dx \leq N \int_{\Omega_1} v^2 dx \leq N = N(n, \nu, \mu).$$

Since this estimate is uniform for $y \in \Omega_{1/2} := B_{1/2} \cap \{u < 1\}$, we have

$$u = e^{-v} \geq \beta_1 = \beta_1(n, \nu, \mu) > 0 \text{ on } \Omega_{1/2}.$$

On the complementary set $B_{1/2} \setminus \Omega_{1/2}$, we have $u \geq 1$, so that the inequality $u \geq \beta_1$ actually holds true on the entire ball $B_{1/2}$, i.e., we have (4.4). \square

Corollary 4.3. *Suppose that $u \in C^1(\overline{B_{4r}})$ satisfies $u \geq 0$ and $Lu \leq 0$ in B_{4r} . Then*

$$(4.7) \quad m(\rho) := \inf_{B_\rho} u \leq \left(\frac{2r}{\rho}\right)^\gamma m(r) \text{ for } 0 < \rho \leq r, \text{ where } \gamma = \gamma(n, \nu) > 0.$$

Proof. Fix $\rho \in (0, r/2]$ and apply Theorem 4.2 to the function $u_0(x) := u(x)/m(\rho)$ with $R = 4\rho \leq 2r$. We have $u_0 \geq 1$ on $B_{R/4} = B_\rho$; therefore,

$$|B_R \cap \{u_0 < 1\}| \leq |B_R \setminus B_{R/4}| = \mu \cdot |B_R|, \text{ where } \mu := 1 - 4^{-n}.$$

By (4.4) we have $u_0 \geq \beta_1 > 0$ on $B_{R/2} = B_{2\rho}$, where $\beta_1 \in (0, 1)$ depends only on n and ν . Then $u \geq \beta_1 m(\rho)$ on $B_{2\rho}$, so that $m(2\rho) \geq \beta_1 m(\rho)$, and

$$(4.8) \quad m(\rho) \leq 2^\gamma m(2\rho), \text{ where } \gamma = \gamma(n, \nu) := -\log_2 \beta_1 > 0.$$

For each $\rho \in (0, r]$, one can write

$$2^{-j-1}r < \rho \leq 2^{-j}r, \quad 2^j \leq \frac{r}{\rho} \text{ for some integer } j \geq 0.$$

Iterating (4.8) gives the desired estimate (4.7):

$$m(\rho) \leq m(2^{-j-1}r) \leq 2^{(j+1)\gamma} m(r) \leq \left(\frac{2r}{\rho}\right)^\gamma m(r).$$

The corollary is proved. \square

§5. THE HARNACK INEQUALITY

Now we show how Lemma 3.6, Theorem 4.2, and Corollary 4.3 work for the proof of Moser's Harnack inequality [M61]. The results in this section are true without *a priori* assumption $u \in C^1$. We discuss this separately in §6.

Theorem 5.1 (Harnack Inequality). *Let $u \in C^1(\overline{B_{14R}})$ satisfy*

$$u \geq 0 \text{ and } Lu := (D, aDu) = 0 \text{ in } B_{14R}.$$

Then for a concentric ball B_R ,

$$(5.1) \quad \sup_{B_R} u \leq N_1 \cdot \inf_{B_R} u, \text{ where } N_1 = N_1(n, \nu) > 1.$$

Proof. Without loss of generality, we assume that $R = 1$. Fix an arbitrary point $x_0 \in B_1$. Let $\gamma = \gamma(n, \nu) > 0$ be a constant in Corollary 4.3. By Lemma 3.6, there is a constant $\mu_0 = \mu_0(n, \nu) \in (0, 1)$ and a ball $B_\rho(y) \subset B_{2\rho}(y) \subset B_1(x_0)$ such that

$$(5.2) \quad |B_\rho \cap \{u > k\}| > \mu_0 \cdot |B_\rho|, \text{ where } k := 2^{-\gamma-1} \rho^{-\gamma} u(x_0).$$

We can rewrite this inequality as

$$|B_\rho \cap \{k^{-1}u < 1\}| < \mu \cdot |B_\rho|, \text{ where } \mu := 1 - \mu_0 \in (0, 1),$$

and apply Theorem 4.2 with $R = \rho$ to the function $k^{-1}u$ on the ball $B_\rho(y) \subset B_{2\rho}(y) \subset B_1(x_0)$. This implies

$$u \geq \beta_1 k \text{ on } B_{\rho/2}(y), \text{ where } \beta_1 = \beta_1(n, \nu) > 0.$$

Note that $B_1 \subset B_3(y) \subset B_{12}(y) \subset B_{14}$, so that we can use Corollary 4.3 with $r = 3$:

$$\beta_1 k \leq \inf_{B_{\rho/2}(y)} u \leq \left(\frac{12}{\rho}\right)^\gamma \inf_{B_3(y)} u \leq \left(\frac{12}{\rho}\right)^\gamma \inf_{B_1} u.$$

From the expression for k in (5.2), we conclude that

$$\beta_1 u(x_0) \leq 2 \cdot 24^\gamma \cdot \inf_{B_1} u.$$

Since x_0 is an arbitrary point in B_1 , inequality (5.1) follows with $N_1 = N_1(n, \nu) := 2 \cdot 24^\gamma / \beta_1$.

The theorem is proved. \square

One can reformulate the Harnack inequality in the following equivalent way, which is more convenient for some applications.

Theorem 5.2. *Let Ω be a bounded open set in \mathbb{R}^n such that $\text{diam } \Omega \leq K\delta$, where K and δ are positive constants, and let $\Omega^\delta := \Omega \cap \{\text{dist}(x, \partial\Omega) > \delta\}$ be connected. Suppose that $u \in C^1(\Omega)$ satisfies $u \geq 0$ and $Lu = 0$ in Ω . Then*

$$(5.3) \quad \sup_{\Omega^\delta} u \leq N_2 \cdot \inf_{\Omega^\delta} u, \text{ where } N_2 = N_2(n, \nu, K) > 1.$$

Proof. Any two points $x, y \in \Omega^\delta$ can be connected in Ω^δ by a curve γ of length $|\gamma| \leq N\delta$, where $N = N(n, K)$. One can choose a finite sequence $x_0 = x, x_1, x_2, \dots, x_M = y$ in γ such that $|x_j - x_{j-1}| < \delta/14$ for all $j = 1, 2, \dots, M$, where M does not exceed a constant $N_0 = N_0(n, K)$. Iterating (3.2) with $R := \delta/14$, we get

$$u(x) = u(x_0) \leq N_1 u(x_1) \leq N_1^2 u(x_2) \leq \dots N_1^M u(x_M) = N_1^M u(y),$$

and (5.3) follows with $N_2 = N_2(n, K) := N_1^{N_0}$. \square

Corollary 5.3. *In Theorem 5.1, the ball B_{14R} can be replaced by B_{cR} with $c = \text{const} > 1$. Then (5.1) holds true with $N_1 = N_1(n, \nu, c) > 1$.*

Proof. In Theorem 5.2, take $\Omega := B_{cR}$ and $\delta := (1 - c)R$. Then

$$\Omega^\delta = B_R, \quad \text{diam } \Omega = 2cR + K\delta \text{ with } K := 2c \cdot (1 - c)^{-1},$$

and the statement follows from (5.3). \square

Another standard consequence of the Harnack inequality is the Hölder continuity of solutions (see [M61, Section 5]).

Theorem 5.4 (Hölder continuity). *Let $u \in C^1(\Omega)$ satisfy $Lu = 0$ in Ω . Then*

$$(5.4) \quad \omega(\rho) := \sup_{B_\rho} u - \inf_{B_\rho} u \leq \left(\frac{2\rho}{r}\right) \omega(r) \text{ for } 0 < \rho < r < R, \quad B_R \subset \Omega.$$

§6. SMOOTH APPROXIMATION

Now we show how the results of the preceding sections, which were proved for solutions $u \in C^1$, can be extended to the functions $u \in W^{1,2}(\Omega)$ satisfying $Lu := (D, aDu) = 0$ in Ω , i.e.,

$$(6.1) \quad \int_{\Omega} (aDu, D\eta) = 0 \quad \forall \eta \in C_0^\infty(\Omega).$$

Consider approximation of u and a_{ij} by standard mollifiers $u^{(\varepsilon)} := \varphi_\varepsilon * u$ and $a_{ij}^{(\varepsilon)} := \varphi_\varepsilon * a_{ij}$, where

$$0 \leq \varphi \in C_0^\infty(B_\varepsilon), \quad \int_{B_\varepsilon} \varphi_\varepsilon dx = 1.$$

For $0 < \varepsilon \leq \delta$, we have $u^{(\varepsilon)}, a_{ij}^{(\varepsilon)} \in C^\infty(\overline{\Omega^\delta})$,

$$(6.2) \quad u^{(\varepsilon)} \rightarrow u \text{ in } W^{1,2}(\Omega^\delta), \quad a_{ij}^{(\varepsilon)} \rightarrow a_{ij} \text{ a.e. in } \Omega^\delta, \text{ as } \varepsilon \rightarrow 0^+.$$

Moreover, the uniform ellipticity condition U for $a = [a_{ij}]$ is preserved for $a^{(\varepsilon)} = [a_{ij}^{(\varepsilon)}]$. See, e.g., Subsection 5.3.1 and Appendix C.5 in [E] for these and other properties of mollifiers.

We will rely on the existence of smooth solutions of the Dirichlet problem with smooth data. Namely, if $B_{R+\delta} \subset \Omega$, then there exists a unique solution $u_\varepsilon \in C^\infty(\overline{B_R})$ of the problem

$$(6.3) \quad L_\varepsilon u_\varepsilon := (D, a^{(\varepsilon)} Du_\varepsilon) = 0 \text{ in } B_R, \quad u_\varepsilon = u^{(\varepsilon)} \text{ on } \partial B_R.$$

For our purposes, it suffices to have $u_\varepsilon \in C^1(\overline{B_R})$.

Lemma 6.1. *We have $u_\varepsilon \rightarrow u$ in $W^{1,2}(\Omega)$, i.e.,*

$$(6.4) \quad \int_{B_R} |u_\varepsilon - u|^2 dx + \int_{B_R} |Du_\varepsilon - Du|^2 dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0^+.$$

Proof. We use the integral identities corresponding to $Lu = 0$ and $L_\varepsilon u_\varepsilon = 0$ in B_R , with a smooth function $\eta_\varepsilon := u^{(\varepsilon)} - u_\varepsilon$ that vanishes on ∂B_R :

$$\int_{B_R} (aDu, D\eta_\varepsilon) dx = 0 \quad \text{and} \quad \int_{B_R} (a^{(\varepsilon)} Du_\varepsilon, D\eta_\varepsilon) dx = 0.$$

Then

$$(6.5) \quad \nu \int_{B_R} |D\eta_\varepsilon|^2 \leq \int_{B_R} (a^{(\varepsilon)} \eta_\varepsilon, D\eta_\varepsilon) dx = \int_{B_R} (v_\varepsilon, D\eta_\varepsilon) dx,$$

where $v_\varepsilon := a^{(\varepsilon)}(Du^{(\varepsilon)} - aDu)$. We write

$$|v_\varepsilon| \leq |a^{(\varepsilon)}(Du^{(\varepsilon)} - Du)| + |(a^{(\varepsilon)} - a)Du| =: F_{1,\varepsilon} + F_{2,\varepsilon}.$$

By (6.2),

$$F_{1,\varepsilon} \leq N \cdot |Du^{(\varepsilon)} - Du| \rightarrow 0 \text{ in } L^2,$$

and also by the dominated convergence theorem we have $F_{2,\varepsilon} \rightarrow 0$ in L^2 . By the Cauchy–Schwarz inequality, from (6.5) it follows that $|D\eta_\varepsilon| \rightarrow 0$ in L^2 . Using also the Poincaré inequality, we get $\eta_\varepsilon \rightarrow 0$ in $W^{1,2}$. Together with the convergence $u^{(\varepsilon)} \rightarrow u$ in $W^{1,2}$, we finally get (6.4). \square

Lemma 6.2. *The functions $\{u_\varepsilon, 0 < \varepsilon \leq \delta\}$ are uniformly bounded on $B_{R-\delta}$.*

Proof. It is easy to check that $L_\varepsilon(u_\varepsilon^2) \geq 0$ in B_R . Indeed, for an arbitrary $0 \leq \eta \in C_0^\infty(B_R)$ we have

$$\int_{B_R} (a^{(\varepsilon)} D(u_\varepsilon^2), D\eta) = \int_{B_R} (a^{(\varepsilon)} Du_\varepsilon, 2u_\varepsilon D\eta).$$

Since

$$2u_\varepsilon D\eta = D\eta_0 - 2\eta Du_\varepsilon, \text{ where } \eta_0 := 2\eta u_\varepsilon \in C_0^\infty(B_R),$$

the previous expression is equal to

$$-2 \int_{B_R} \eta (a^{(\varepsilon)} Du_\varepsilon, Du_\varepsilon) \leq 0.$$

This means that the u_ε^2 are a nonnegative subsolution uniformly bounded in $L^1(B_R)$. Now we can apply Theorem 3.5 with $u = u_\varepsilon^2, p = 1$, and δ in place of R :

$$u_\varepsilon^2(x_0) \leq \frac{N}{|B_\delta|} \int_{B_\delta(x_0)} u_\varepsilon^2 \leq N \int_{B_R} u_\varepsilon^2 dx \leq N \quad \forall x_0 \in B_{R-\delta}.$$

Here N denotes various different constants independent on ε . The lemma is proved. \square

Lemma 6.3. *The functions u_ε converge uniformly on $B_{R-2\delta}$ as $\varepsilon \rightarrow 0^+$ to a Hölder continuous function u_0 that coincides with u a.e.*

Proof. Indeed, estimate (5.4) for u_ε in balls B_ρ with different centers guarantees the compactness of $\{u_\varepsilon\}$ in $C(\overline{B_{R-2\delta}})$ and convergence to a function u_0 . Since also $u_\varepsilon \rightarrow u$ in L^2 , the limit function u_0 coincides with u a.e., and u_0 satisfies (5.4). \square

On the grounds of Lemmas 6.1–6.3, the extension of results in §5 from $u_\varepsilon \in C^1$ to $u \in W^{1,2}$ is straightforward. Note that we only used the convergence $u_\varepsilon \rightarrow u$ in L^2 , though Lemma 6.1 provides convergence together with $Du_\varepsilon \rightarrow Du$ in L^2 .

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