ON THE SOLVABILITY OF INITIAL-BOUNDARY VALUE PROBLEMS FOR A VISCOUS COMPRESSIBLE FLUID IN AN INFINITE TIME INTERVAL

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*Dedicated to Nina Nicolaevna Ural'tseva
with great admiration*

ABSTRACT. The solution of the first boundary-value problem for the Navier–Stokes equations is estimated in the case of a compressible fluid in an infinite time interval; the solvability of the problem is proved, together with the exponential decay of the solution as $t \rightarrow \infty$. The proof is based on the “free work” method due to Prof. M. Padula. It is shown that the method is applicable to the analysis of free boundary problems.

§1. INTRODUCTION

This paper is devoted to the problem

$$(1.1) \quad \begin{cases} \rho(x, t)(\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}) - \nabla \cdot T(\mathbf{v}) + \nabla p(\rho) = 0, \\ \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, & x \in \Omega, \quad t > 0, \\ \mathbf{v}(x, t) = 0, & x \in S = \partial\Omega, \quad t > 0, \\ \mathbf{v}(x, 0) = \mathbf{v}_0(x), \quad \rho(x, 0) = \rho_0(x), & x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^3$, $\mathbf{v}$ and ρ are the velocity and the density of the fluid, $p(\rho)$ is a positive and strictly monotone increasing pressure function, $T(\mathbf{v}) = \mu S(\mathbf{v}) + \mu_1 I \nabla \cdot \mathbf{v}$ is a viscous part of the stress tensor, $S(\mathbf{v}) = (\nabla \mathbf{v}) + (\nabla \mathbf{v})^T$ is the doubled rate-of-strain tensor, and μ and μ_1 are positive constant viscosity coefficients.

Problem (1.1) is very well studied (see [13, 14, 15, 16, 17, 18, 19]). In [1, 2], for arbitrarily large initial data M. Padula proved the inequality

$$(1.2) \quad e^{2\beta t} (\|\mathbf{v}(\cdot, t)\|_{L_2(\Omega)}^2 + \|\rho(\cdot, t) - \bar{\rho}\|_{L_2(\Omega)}^2) \leq c (\|\mathbf{v}_0\|_{L_2(\Omega)}^2 + \|\rho_0 - \bar{\rho}_0\|_{L_2(\Omega)}^2), \quad t > 0,$$

where $\bar{\rho} = |\Omega|^{-1} \int_{\Omega} \rho(x, t) dx$ is the mean value of the density, $\beta, c = \text{const} > 0$. This inequality was obtained as a result of estimation of a certain auxiliary Lyapunov type function called “modified energy”. It was assumed in [1, 2] that $p(\rho) = k\rho$. A similar estimate, in the case of a general $p(\rho)$, was obtained by Denisova in [3] for the problem of motion of two liquids of different types (compressible and incompressible) separated by a free interface and contained in a bounded vessel. Our goal is to obtain similar estimates for higher order derivatives of the solution. It has turned out that the same method is applicable for doing this. We also establish the solvability of the problem in an infinite

2010 *Mathematics Subject Classification.* Primary 35Q30.

Key words and phrases. Navier–Stokes equations, viscosity, anisotropic Sobolev–Slobodetski spaces.

The author is thankful to E. V. Frolova, W. M. Zajaczkowski, and V. Kalantarov for useful suggestions and discussions. The work was partially supported by the EU project FLUX 319012.

time interval. But, in contrast to the result of M. Padula, we are constrained to require the smallness of the initial data.

We work in anisotropic Sobolev–Slobodetski spaces

$$W_2^{r,r/2}(\mathcal{D}_T) = W_2^{r,0}(\mathcal{D}_T) \cap W_2^{0,r/2}(\mathcal{D}_T),$$

where $\mathcal{D}_T = \mathcal{D} \times (0, T)$, $T \leq \infty$, $\mathcal{D}$ is a domain in $\mathbb{R}^n$,

$$W_2^{r,0}(\mathcal{D}_T) = L_2(0, T; W_2^r(\mathcal{D})), \quad W_2^{0,s}(\mathcal{D}_T) = W_2^s(0, T; L_2(\mathcal{D})).$$

By $W_2^l(\mathcal{D})$ we mean an isotropic Sobolev space with the norm given by

$$\|u\|_{W_2^l(\mathcal{D})}^2 = \sum_{|j| \leq l} \|D^j u\|_{L_2(\mathcal{D})}^2$$

if l is an integer, or

$$\|u\|_{W_2^l(\mathcal{D})}^2 = \|u\|_{W_2^{[l]}(\mathcal{D})}^2 + \sum_{|j|=l} \int_{\mathcal{D}} \int_{\mathcal{D}} |D^j u(x) - D^j u(y)|^2 \frac{dx dy}{|x - y|^{n+2\lambda}},$$

if $l = [l] + \lambda$, $0 < \lambda < 1$.

We write problem (1.1) in the Lagrangian coordinates $\xi \in \Omega$ related to the Eulerian coordinates $x \in \Omega$ by

$$(1.3) \quad x = \xi + \int_0^t \mathbf{u}(\xi, t) d\xi \equiv X(\xi, t) = \xi + \mathbf{U}(\xi, t),$$

where $\mathbf{u}(\xi, t) = \mathbf{v}(X(\xi, t), t)$. The transformation (1.3) converts the problem (1.1) into

$$(1.4) \quad \begin{cases} r\mathbf{u}_t - \nabla_u \cdot T_u(\mathbf{u}) + \nabla_u p(r) = 0, \\ r_t + r\nabla_u \cdot \mathbf{u} = 0, \quad \xi \in \Omega, \quad \mathbf{u}(\xi, t) = 0, \quad \xi \in S, \\ \mathbf{u}(\xi, 0) = \mathbf{v}_0(\xi), \quad r(\xi, 0) = \rho_0(\xi), \quad \xi \in \Omega, \end{cases}$$

where $r(\xi, t) = \rho(X(\xi, t), t)$, $\nabla_u = \mathcal{L}^{-T} \nabla = J_u^{-1} A(\xi, t) \nabla$ is the transformed gradient ∇_x , $\mathcal{L} = (\frac{\partial X}{\partial \xi})$. $J_u = \det \mathcal{L}$, A is the co-factors matrix of $\mathcal{L}$, and $T_u(\mathbf{u}) = \mu S_u(\mathbf{u}) + \mu_1 I \nabla_u \cdot \mathbf{u}$, $S_u(\mathbf{u}) = (\nabla_u \mathbf{u}) + (\nabla_u \mathbf{u})^T$. The reason why the Lagrangian coordinates are used is purely technical: in these coordinates the space derivatives of r disappear from the continuity equation.

We set $r = \bar{\rho} + \theta$, where $\bar{\rho} = M/|\Omega|$ is the mean density and $M = \int_{\Omega} \rho(x, t) dx = \int_{\Omega} r(\xi, t) J_u(\xi, t) d\xi = \text{const}$ is the total mass of the fluid. Clearly, $\int_{\Omega} \theta(\xi, t) J_u(\xi, t) d\xi = 0$. System (1.4) can be written in the form

$$(1.5) \quad \begin{cases} \bar{\rho} \mathbf{u}_t - \nabla \cdot T(\mathbf{u}) + p'(\bar{\rho}) \nabla \theta = \mathbf{l}_1(\mathbf{u}, \theta), \\ \theta_t + \bar{\rho} \nabla \cdot \mathbf{u} = l_2(\mathbf{u}, \theta), \quad \xi \in \Omega, \\ \mathbf{u}(\xi, t) = 0, \quad \xi \in S, \\ \mathbf{u}(\xi, 0) = \mathbf{v}_0(\xi), \quad \theta(\xi, 0) = \theta_0(\xi) = \rho_0(\xi) - \bar{\rho}, \end{cases}$$

where

$$(1.6) \quad \begin{aligned} \mathbf{l}_1 &= -\theta \mathbf{u}_t + (\nabla_u \cdot T_u(\mathbf{u}) - \nabla \cdot T(\mathbf{u})) + p'(\bar{\rho}) \nabla \theta - p'(\bar{\rho} + \theta) \nabla_u \theta, \\ l_2 &= -r \nabla_u \cdot \mathbf{u} + \bar{\rho} \nabla \cdot \mathbf{u} = \bar{\rho} (\nabla - \nabla_u) \cdot \mathbf{u} - \theta \nabla_u \cdot \mathbf{u}. \end{aligned}$$

Replacing $\mathbf{l}_1$ and l_2 by some given functions $\mathbf{f}$ and h , we obtain a linear problem

$$(1.7) \quad \begin{cases} \bar{\rho} \mathbf{v}_t - \nabla \cdot T(\mathbf{v}) + p'(\bar{\rho}) \nabla \theta = \mathbf{f}(x, t), \\ \theta_t + \bar{\rho} \nabla \cdot \mathbf{v} = h(x, t), \quad x \in \Omega, \quad t > 0, \\ \mathbf{v}(x, t) = 0, \quad x \in S, \quad t > 0, \\ \mathbf{v}(x, 0) = \mathbf{v}_0(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in \Omega. \end{cases}$$

The main result of the paper is as follows.

Theorem 1. Suppose $S \in W_2^{3/2+l}$, $1/2 < l < 1$, $p \in C^2(\bar{\rho}/2, 3\bar{\rho}/2)$, $\bar{\rho}, p'(\bar{\rho}) > 0$, and $|\rho_0(x)| \geq c_0 > 0$. For arbitrary $\mathbf{v}_0 \in W_2^{l+1}(\Omega)$ and $\theta_0 \in W_2^{l+1}(\Omega)$ satisfying the conditions

$$(1.8) \quad \mathbf{v}_0(x, 0) = 0, \quad x \in S, \quad \int_{\Omega} \theta_0(x) dx = 0,$$

$$(1.9) \quad \|\mathbf{v}_0\|_{W_2^{l+1}(\Omega)} + \|\theta_0\|_{W_2^{l+1}(\Omega)} \leq \epsilon \ll 1,$$

problem (1.4) has a unique solution defined for $t \geq 0$, and

$$(1.10) \quad \|e^{\beta t} \mathbf{u}\|_{W_2^{l+2, l/2+1}(Q_T)} + \|e^{\beta t} \theta\|_{W_2^{l+1, 0}(Q_T)} + \|e^{\beta t} \theta_t\|_{W_2^{l+1, 0}(Q_T)} \leq c(\|\mathbf{v}_0\|_{W_2^{l+1}(\Omega)} + \|\theta_0\|_{W_2^{l+1}(\Omega)}),$$

where $Q_T = \Omega \times (0, T)$, $T \leq \infty$, $\beta > 0$.

The proof of this theorem is given in §3; §2 contains necessary estimates of the solution of a linearized problem. In §4, a somewhat more technical method of the analysis of the free boundary problem for two fluids is outlined. The results of §2 (in particular, estimate (2.2)) enable us to take account of the mass forces $\mathbf{f}(x, t)$ decaying exponentially as $t \rightarrow \infty$. We have not done it for the sake of simplicity.

§2. LINEAR PROBLEM

In this section we obtain the following result.

Theorem 2. Suppose that $\bar{\rho}, p'(\bar{\rho}) > 0$, $\mathbf{v}_0 \in W_2^{l+1}(\Omega)$, $\theta_0 \in W_2^{l+1}(\Omega)$, $\mathbf{f} \in W_2^{l, l/2}(Q_T)$, $h \in W_2^{l+1, 0}(Q_T)$, $T \leq \infty$, $1/2 < l < 1$, and

$$(2.1) \quad \int_{\Omega} h(x, t) dx = 0, \quad \int_{\Omega} \theta_0(x) dx = 0.$$

Then problem (1.7) has a unique solution

$$\mathbf{v} \in W_2^{2+l, 1+l/2}(Q_T), \quad \theta, \theta_t \in W_2^{l+1, 0}(Q_T),$$

and

$$(2.2) \quad \|e^{\beta t} \mathbf{v}\|_{W_2^{l+2, l/2+1}(Q_T)} + \|e^{\beta t} \theta\|_{W_2^{l+1, 0}(Q_T)} + \|e^{\beta t} \theta_t\|_{W_2^{l+1, 0}(Q_T)} \leq c(\|\mathbf{v}_0\|_{W_2^{l+1}(\Omega)} + \|\theta_0\|_{W_2^{l+1}(\Omega)} + \|e^{\beta t} \mathbf{f}\|_{W_2^{l, l/2}(Q_T)} + \|e^{\beta t} h\|_{W_2^{l+1, 0}(Q_T)})$$

with $\beta > 0$ and c independent of $T \leq \infty$.

The proof of this theorem occupies the rest of the section; our main attention is given to the proof of (2.2). We start with reproducing an analog of M. Padula's result for the linear problem (1.7).

Proposition 1. If conditions (2.1) are fulfilled, then the solution of problem (1.7) satisfies the inequality

$$(2.3) \quad \sup_{t < T} e^{2\beta t} (\|\mathbf{v}(\cdot, t)\|_{L_2(\Omega)}^2 + \|\theta(\cdot, t)\|_{L_2(\Omega)}^2) + \int_0^T e^{2\beta t} (\|\mathbf{v}\|_{W_2^1(\Omega)}^2 + \|\theta\|_{L_2(\Omega)}^2) dt \leq c \left(\|\mathbf{v}_0\|_{L_2(\Omega)}^2 + \|\theta_0\|_{L_2(\Omega)}^2 + \int_0^T e^{2\beta t} (\|\mathbf{f}\|_{L_2(\Omega)}^2 + \|h\|_{L_2(\Omega)}^2) dt \right), \quad T \leq \infty.$$

Proof. We note that conditions (2.1) imply $\int_{\Omega} \theta(x, t) dx = 0$, $t > 0$. By standard calculation, the energy relation

$$\frac{1}{2} \frac{d}{dt} \bar{\rho} \|\mathbf{v}\|_{L_2(\Omega)}^2 + \int_{\Omega} T(\mathbf{v}) : \nabla \mathbf{v} dx - p'(\bar{\rho}) \int_{\Omega} \theta \nabla \cdot \mathbf{v} dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx$$

can be obtained, which implies

$$(2.4) \quad \frac{1}{2} \frac{d}{dt} \left(\bar{\rho} \|\mathbf{v}\|_{L_2(\Omega)}^2 + \frac{p'(\bar{\rho})}{\bar{\rho}} \|\theta\|_{L_2(\Omega)}^2 \right) + \frac{\mu}{2} \|S(\mathbf{v})\|_{L_2(\Omega)}^2 + \mu_1 \|\nabla \cdot \mathbf{v}\|_{L_2(\Omega)}^2 = \int_{\Omega} \left(\mathbf{f} \cdot \mathbf{v} + \frac{p'(\bar{\rho})}{\bar{\rho}} h \theta \right) dx.$$

Next, we introduce the vector field $\mathbf{V}(x, t)$ such that

$$(2.5) \quad \begin{aligned} \nabla \cdot \mathbf{V}(x, t) &= -\theta(x, t), \quad x \in \Omega, \quad \mathbf{V}(x, t) = 0, \quad x \in S, \\ \|\mathbf{V}\|_{W_2^1(\Omega)} &\leq c \|\theta\|_{L_2(\Omega)}, \\ \|\mathbf{V}_t\|_{W_2^1(\Omega)} &\leq c \|\theta_t\|_{L_2(\Omega)} \leq c(\|\mathbf{v}\|_{W_2^1(\Omega)} + \|h\|_{L_2(\Omega)}) \end{aligned}$$

(problem (2.5) has been analyzed by many authors, see, e.g., [4, 5, 6]). We multiply the first equation in (1.7) by $\mathbf{V}$ and integrate over Ω , obtaining

$$(2.6) \quad \bar{\rho} \left(\frac{d}{dt} \int_{\Omega} \mathbf{v} \cdot \mathbf{V} dx - \int_{\Omega} \mathbf{v} \cdot \mathbf{V}_t dx \right) + \int_{\Omega} T(\mathbf{v}) : \nabla \mathbf{V} dx + p'(\bar{\rho}) \|\theta\|_{L_2(\Omega)}^2 = \int_{\Omega} \mathbf{f} \cdot \mathbf{V} dx.$$

Finally, we multiply (2.6) by a small positive γ and add to (2.4), which yields

$$(2.7) \quad \frac{1}{2} \frac{dE_0(t)}{dt} + E_1(t) = \int_{\Omega} \left(\mathbf{f} \cdot (\mathbf{v} + \gamma \mathbf{V}) + \frac{p'(\bar{\rho})}{\bar{\rho}} h \theta \right) dx + \bar{\rho} \gamma \int_{\Omega} \mathbf{v} \cdot \mathbf{V}_t dx,$$

where

$$(2.8) \quad \begin{aligned} E_0(t) &= \bar{\rho} \|\mathbf{v}\|_{L_2(\Omega)}^2 + \frac{p'(\bar{\rho})}{\bar{\rho}} \|\theta\|_{L_2(\Omega)}^2 + 2\bar{\rho} \gamma \int_{\Omega} \mathbf{v} \cdot \mathbf{V} dx, \\ E_1(t) &= \frac{\mu}{2} \|S(\mathbf{v})\|_{L_2(\Omega)}^2 + \mu_1 \|\nabla \cdot \mathbf{v}\|_{L_2(\Omega)}^2 + \gamma p'(\bar{\rho}) \|\theta\|_{L_2(\Omega)}^2 + \gamma \int_{\Omega} T(\mathbf{v}) : \nabla \mathbf{V} dx. \end{aligned}$$

Due to (2.5), (2.8) and the Korn inequality, we have

$$(2.9) \quad \begin{aligned} c_1 (\|\mathbf{v}\|_{L_2(\Omega)}^2 + \|\theta\|_{L_2(\Omega)}^2) &\leq E_0(t) \leq c_2 (\|\mathbf{v}\|_{L_2(\Omega)}^2 + \|\theta\|_{L_2(\Omega)}^2), \\ E_1(t) &\geq 2\beta E_0(t), \quad \beta > 0, \\ E_1(t) &\geq c_3 (\|\mathbf{v}\|_{W_2^1(\Omega)}^2 + \|\theta\|_{L_2(\Omega)}^2), \end{aligned}$$

provided γ is sufficiently small. Hence,

$$\frac{dE_0(t)}{dt} + 2\beta E_0(t) + E_1(t) \leq I(t),$$

where $\beta > 0$, and

$$I(t) = 2 \int_{\Omega} \left(\mathbf{f} \cdot (\mathbf{v} + \gamma \mathbf{V}) + \frac{p'(\bar{\rho})}{\bar{\rho}} h \theta \right) dx + 2\gamma \bar{\rho} \int_{\Omega} \mathbf{v} \cdot \mathbf{V}_t dx.$$

It follows that

$$(2.10) \quad e^{2\beta t} E_0(t) + \int_0^t e^{2\beta \tau} E_1(\tau) d\tau \leq E_0(0) + \int_0^t e^{2\beta \tau} |I(\tau)| d\tau,$$

which leads to (2.3) (in view of (2.9)) after easy calculations.

In order to estimate higher order norms of $\mathbf{v}$ and θ , we use a classical localization method. Let $\{\varphi_k(x)\}$, $k = 1, 2, \dots, m$, be a partition of unity in Ω subordinate to the covering of Ω with the balls $\omega_k = \{|x - x_k| \leq d, x_k \in S\}$, $k = 1, \dots, m-1$, of a small radius d plus a strictly interior subdomain ω_m of Ω . We assume that every point of Ω can belong to not more than a fixed number m_1 of the ω_k , moreover, the intersections $\omega_k \cap \omega_m$, $k < m$, contain balls B_k of radius αd , $\alpha < 1$, and we are given smooth functions $\zeta_k(x)$

with $\text{supp } \zeta_k \subset B_k$, $\int_{B_k} \zeta_k(x) dx = 1$. Let $\theta_k(x, t) = \theta(x, t)\varphi_k(x)$, $k \leq m$, $\omega'_k = \omega_k \cap \Omega$, and let

$$(2.11) \quad \begin{aligned} \vartheta_k(x, t) &= \theta_k(x, t) - \zeta_k(x) \int_{\omega'_k} \theta_k(y, t) dy, \quad k < m, \\ \vartheta_m(x, t) &= \theta_m(x, t) + \sum_{k=1}^{m-1} \zeta_k(x) \int_{\omega'_k} \theta_k(y, t) dy. \end{aligned}$$

It is clear that $\text{supp } \vartheta_k \subset \omega_k$, $\sum_{k=1}^m \vartheta_k(x, t) = \sum_{k=1}^m \theta_k = \theta(x, t)$, and

$$(2.12) \quad \begin{aligned} \int_{\omega'_k} \vartheta_k(x, t) dx &= 0, \quad k < m, \\ \int_{\omega_m} \vartheta_m(x, t) dx &= \int_{\omega_m} \theta_m(x, t) dx + \sum_{k=1}^{m-1} \int_{\omega'_k} \theta_k(x, t) dx = 0. \end{aligned}$$

We also set $\mathbf{v}_k = \mathbf{v}\varphi_k$,

$$(2.13) \quad \begin{aligned} \mathbf{w}_k(x, t) &= \mathbf{v}_k(x, t) - \zeta_k(x) \int_{\omega'_k} \mathbf{v}_k(x, t) dx, \quad k < m, \\ \mathbf{w}_m(x, t) &= \mathbf{v}_m(x, t) + \sum_{k=1}^{m-1} \zeta_k(x) \int_{\omega'_k} \mathbf{v}_k(x, t) dx. \end{aligned}$$

and compute the expressions

$$(2.14) \quad \begin{aligned} \bar{\rho} \mathbf{w}_{k,t} - \nabla \cdot T(\mathbf{w}_k) + p'(\bar{\rho}) \nabla \vartheta_k &= \mathbf{f}_k + \mathbf{f}_k, \\ \vartheta_{k,t} + \bar{\rho} \nabla \cdot \mathbf{w}_k &= h_k + \mathbf{h}_k, \end{aligned}$$

where

$$(2.15) \quad \begin{aligned} \mathbf{f}_k &= \mathbf{f}\varphi_k - \zeta_k \int_{\omega'_k} \mathbf{f}\varphi_k dx, \quad h_k = h\varphi_k - \zeta_k \int_{\omega'_k} h\varphi_k dx, \quad k < m, \\ \mathbf{f}_m &= \mathbf{f}\varphi_m + \sum_{k=1}^{m-1} \zeta_k \int_{\omega'_k} \mathbf{f}\varphi_k dx, \quad h_m = h\varphi_m + \sum_{k=1}^{m-1} \zeta_k \int_{\omega'_k} h\varphi_k dx, \end{aligned}$$

and the $\mathbf{f}_k$, $\mathbf{h}_k$ are functions containing the derivatives of the φ_k , ζ_k and, consequently, only lower order derivatives of $\mathbf{v}$ and θ .

To obtain interior estimates for the derivatives of the solution, we consider system (2.14) with $k = m$. Since $\text{supp } \mathbf{w}_m, \text{supp } \vartheta_m \subset \omega_m$, we extend $\mathbf{w}_m$ and ϑ_m by zero to a certain cube $\Omega = \{|x_j - x_{jm}| \leq D\}$, $j = 1, 2, 3$, containing ω_m (and outside Ω). Inequality (2.10) for system (2.14) in Ω implies

$$(2.16) \quad \begin{aligned} &\int_0^T e^{2\beta t} (\|\mathbf{w}_m(\cdot, t)\|_{W_2^1(\omega_m)}^2 + \|\vartheta_m(\cdot, t)\|_{L_2(\omega_m)}^2) dt \\ &\leq c \left(\|\mathbf{w}_m(\cdot, 0)\|_{L_2(\omega_m)}^2 + \|\vartheta_m(\cdot, 0)\|_{L_2(\omega_m)}^2 + \left| \int_0^T e^{2\beta t} I_m(t) dt \right| \right) \end{aligned}$$

and, as a consequence,

$$(2.17) \quad \begin{aligned} &\int_0^T e^{2\beta t} (\|\mathbf{w}_m(\cdot, t)\|_{W_2^1(\omega_m)}^2 + \|\vartheta_m(\cdot, t)\|_{L_2(\omega_m)}^2) dt \\ &\leq c \left(\|\mathbf{w}_m(\cdot, 0)\|_{L_2(\Omega)}^2 + \|\vartheta_m(\cdot, 0)\|_{L_2(\Omega)}^2 \right. \\ &\quad \left. + \int_0^T e^{2\beta t} (\|\mathbf{f}_m + \mathbf{f}_m\|_{L_2(\omega_m)}^2 + \|h_m + \mathbf{h}_m\|_{L_2(\omega_m)}^2) dt \right). \end{aligned}$$

Let

$$\Delta_j^q(z)f(x) = \sum_{k=0}^q (-1)^{q-k} \binom{q}{k} f(x + k\mathbf{e}_j z), \quad \mathbf{e}_j = (\delta_{ij})_{i=1,2,3},$$

be the q th finite difference of $f(x)$ with respect to the variable x_j , $q > l + 1$. Taking the finite differences of equations (2.14) with $k = m$, we obtain

$$(2.18) \quad \begin{aligned} \bar{\rho} \Delta_j^q(z) \mathbf{w}_{m,t} - \nabla \cdot T(\Delta_j^q(z) \mathbf{w}_m) + p'(\bar{\rho}) \nabla \Delta_j^q(z) \vartheta_m &= \Delta_j^q(z) (\mathbf{f}_m + \mathbf{f}_m), \\ \Delta_j^q(z) \vartheta_{m,t} + \bar{\rho} \nabla \cdot \Delta_j^q(z) \mathbf{w}_m &= \Delta_j^q(z) (h_m + \mathbf{h}_m). \end{aligned}$$

We assume that $0 < z < z_0$ with z_0 so small (or D so large) that $\text{supp } \Delta_j^q(z) \mathbf{w}_m$, $\text{supp } \Delta_j^q(z) \vartheta_m$ are contained in Ω . Since $\int_{\Omega} \Delta_j^q(z) \vartheta_m dx = 0$, we can introduce a vector field $\mathbf{W}_m(x, t)$ such that

$$(2.19) \quad \begin{aligned} \nabla \cdot \mathbf{W}_m(x, t) &= -\Delta_j^q(z) \vartheta_m(x, t), \quad x \in \Omega, \\ \|\mathbf{W}_m\|_{W_2^1(\Omega)} &\leq c \|\Delta_j^q(z) \vartheta_m\|_{L_2(\Omega)}, \\ \|\mathbf{W}_{m,t}\|_{L_2(\Omega)} &\leq c (\|\Delta_j^q(z) \nabla \cdot \mathbf{w}_m\|_{L_2(\Omega)} + \|\Delta_j^q(z) (h_m + \mathbf{h}_m)\|_{L_2(\Omega)}). \end{aligned}$$

By Proposition 1 applied to system (2.18), we have

$$(2.20) \quad \begin{aligned} &\int_0^T e^{2\beta t} (\|\Delta_j^q(z) \mathbf{w}_m(\cdot, t)\|_{W_2^1(\Omega)}^2 + \|\Delta_j^q(z) \vartheta_m(\cdot, t)\|_{L_2(\Omega)}^2) dt \\ &\leq c \left(\|\Delta_j^q(z) \mathbf{w}_m(\cdot, 0)\|_{L_2(\Omega)}^2 + \|\Delta_j^q(z) \vartheta_m(\cdot, 0)\|_{L_2(\Omega)}^2 \right. \\ &\quad \left. + 2 \left| \int_0^T e^{2\beta t} dt \int_{\Omega} (\Delta_j^q(z) (\mathbf{f}_m + \mathbf{f}_m) \cdot (\Delta_j^q(z) \mathbf{w}_m + \gamma \mathbf{W}_m(x, t)) \right. \right. \\ &\quad \left. \left. + \Delta_j^q(z) (h_m + \mathbf{h}_m) \Delta_j^q(z) \vartheta_m dx \right| \right. \\ &\quad \left. + 2\bar{\rho}\gamma \left| \int_0^T e^{2\beta t} dt \int_{\Omega} \Delta_j^q(z) \mathbf{w}_m \cdot \mathbf{W}_{m,t}(x, t) dx \right| \right). \end{aligned}$$

We make use of the identity

$$(2.21) \quad \begin{aligned} &\int_0^T e^{2\beta t} dt \int_{\Omega} (\Delta_j^q(z) (\mathbf{f}_m + \mathbf{f}_m) \cdot (\Delta_j^q(z) \mathbf{w}_m + \gamma \mathbf{W}_m) dx \\ &= - \int_0^T e^{2\beta t} dt \int_{\Omega} (\Delta_j^{q-1}(z) (\mathbf{f}_m + \mathbf{f}_m) \cdot (\Delta_j^{q+1} \mathbf{w}_m + \Delta_j(z) \mathbf{W}_m) dx \end{aligned}$$

and estimate the penultimate integral in (2.20) (denoted by J_m) as follows:

$$(2.22) \quad \begin{aligned} |J_m| &\leq c \left(\int_0^T e^{2\beta t} (\|\Delta_j^{q-1}(z) (\mathbf{f}_m + \mathbf{f}_m)\|_{L_2(\Omega)} (\|\Delta_j^{q+1} \mathbf{w}_m\|_{L_2(\Omega)} + |z| \|\nabla \mathbf{W}_m\|_{L_2(\Omega)}) dt \right. \\ &\quad \left. + \int_0^T e^{2\beta t} (\|\Delta_j^q(z) (h_m + \mathbf{h}_m)\|_{L_2(\Omega)} \|\Delta_j^q(z) \vartheta_m\|_{L_2(\Omega)} dt) \right) \\ &\leq c \left(\int_0^T e^{2\beta t} \|\Delta_j^{q-1}(z) (\mathbf{f}_m + \mathbf{f}_m)\|_{L_2(\Omega)} (\|\Delta_j^{q+1} \mathbf{w}_m\|_{L_2(\Omega)} + |z| \|\Delta_j^q(z) \vartheta_m\|_{L_2(\Omega)}) dt \right. \\ &\quad \left. + \int_0^T e^{2\beta t} \|\Delta_j^q(z) (h_m + \mathbf{h}_m)\|_{L_2(\Omega)} \|\Delta_j^q(z) \vartheta_m\|_{L_2(\Omega)} dt \right). \end{aligned}$$

Next, we divide (2.20) by $z^{1+2(l+1)}$ and integrate over $z \in (0, z_0)$. Since the norms

$$\|u\|_{L_2(\Omega)} + \sum_{j=1}^3 \left(\int_0^{z_0} \|\Delta_j^s(z) u\|_{L_2(\Omega)}^2 \frac{dz}{z^{1+2r}} \right)^{1/2}$$

and $\|u\|_{W_2^r(\Omega)}$ are equivalent for all $s > r > 0$ (see [7]), our calculations lead to

$$\begin{aligned}
 & \|e^{\beta t} \mathbf{w}_m\|_{W_2^{l+2,0}(\Omega_T)}^2 + \|e^{\beta t} \vartheta_m\|_{W_2^{l+1,0}(\Omega_T)}^2 \\
 (2.23) \quad & \leq c(\|e^{\beta t}(\mathbf{f}_m + \mathbf{f}_m)\|_{W_2^{l,l/2}(\Omega_T)}^2 + \|e^{\beta t}(h_m + \mathbf{h}_m)\|_{W_2^{l+1,0}(\Omega_T)}^2 \\
 & \quad + \|\mathbf{w}_m(\cdot, 0)\|_{W_2^{l+1}(\Omega)}^2 + \|\vartheta_m(\cdot, 0)\|_{W_2^{l+1}(\Omega)}^2) \\
 & \equiv A_m,
 \end{aligned}$$

where $\Omega_T = \Omega \times (0, T)$. Moreover, we have

$$\begin{aligned}
 & \|e^{\beta t} \vartheta_{m,t}\|_{W_2^{l+1,0}(\Omega_T)}^2 \\
 (2.24) \quad & \leq c(\|e^{\beta t} \nabla \cdot \mathbf{w}_m\|_{W_2^{l+1,0}(\Omega_T)}^2 + \|e^{\beta t}(h_m + \mathbf{h}_m)\|_{W_2^{l+1,0}(\Omega_T)}^2) \leq cA_m, \\
 & \|e^{\beta t} \mathbf{w}_{m,t}\|_{W_2^{0,l/2}(\Omega_T)}^2 \leq c(\|e^{\beta t} D_x^2 \mathbf{w}_m\|_{W_2^{0,l/2}(\Omega_T)}^2 + \|e^{\beta t} \nabla \vartheta_m\|_{W_2^{0,l/2}(\Omega_T)}^2 \\
 & \quad + \|e^{\beta t}(\mathbf{f}_m + \mathbf{f}_m)\|_{W_2^{0,l/2}(\Omega_T)}^2).
 \end{aligned}$$

By the interpolation inequalities, the right-hand side in the last estimate is not less than

$$\begin{aligned}
 & \delta(\|e^{\beta t} \mathbf{w}_{m,t}\|_{W_2^{0,l/2}(\Omega_T)}^2 + \|e^{\beta t} \vartheta_{m,t}\|_{W_2^{l+1,0}(\Omega_T)}^2) \\
 & \quad + c(\delta)(\|e^{\beta t} \mathbf{w}_m\|_{W_2^{2+l,0}(\Omega_T)}^2 + \|e^{\beta t} \vartheta_m\|_{W_2^{l+1,0}(\Omega_T)}^2) \\
 & \quad + c(\|e^{\beta t} \vartheta_{m,t}\|_{W_2^{l+1,0}(\Omega_T)}^2 + \|e^{\beta t}(\mathbf{f}_m + \mathbf{f}_m)\|_{W_2^{0,l/2}(\Omega_T)}^2).
 \end{aligned}$$

Collecting the estimates and choosing δ sufficiently small, we obtain the desired inequality

$$(2.25) \quad \|e^{\beta t} \mathbf{w}_m\|_{W_2^{l+2,l/2+1}(\Omega_T)}^2 + \|e^{\beta t} \vartheta_m\|_{W_2^{l+1,0}(\Omega_T)}^2 + \|e^{\beta t} \vartheta_{m,t}\|_{W_2^{l+1,0}(\Omega_T)}^2 \leq cA_m.$$

We pass to estimates for $\mathbf{v}$ and θ near the boundary; for this we consider equations (2.14) with $k < m$. Without loss of generality we may assume that the point x_k coincides with the origin and the x_3 -axis is directed along the interior normal $-\mathbf{n}(x_k)$. Let $x_3 = \phi(x')$, $x' = (x_1, x_2) \in K$, be the equation of the boundary S near the origin; by K we mean the disk $|x| \leq d_1$, $d_1 > d$.

We study system (2.14) assuming for simplicity that $S \in W_2^{2+l}$. We make the change of variables

$$(2.26) \quad x_1 = y_1, \quad x_2 = y_2, \quad x_3 = y_3 + \phi(y'),$$

where $\phi \in W_2^{2+l}(K)$. The Jacobi matrix of this transformation is given by

$$\mathfrak{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \phi_{y_1} & \phi_{y_2} & 1 \end{pmatrix},$$

whence $\det \mathfrak{L} = 1$, and $\mathfrak{L}^{-1}$ coincides with the co-factors matrix $\hat{\mathfrak{L}}$. The transformation (2.26) converts $\omega_k \cap \Omega$ (where the $\mathbf{w}_k$ and ϑ_k are given) into a certain $\tilde{\omega}_k \subset \mathbb{R}_+^3 = \{x_3 > 0\}$. We extend $\mathbf{w}_k$ and ϑ_k by zero to a larger domain

$$\Omega^+ = \{0 < y_3 < 2d_0\}, \quad y' \in \Omega' = \{|y_\alpha| < d_0\}, \quad \alpha = 1, 2,$$

where $d_0 > d$. In the new coordinates y , system (2.14) takes the form

$$\begin{aligned}
 (2.27) \quad & \bar{\rho} \mathbf{w}_{k,t} - \tilde{\nabla} \cdot \tilde{T}(\mathbf{w}_k) + p'(\bar{\rho}) \tilde{\nabla} \vartheta_k = \mathbf{f}_k + \mathbf{f}_k, \\
 & \vartheta_{k,t} + \bar{\rho} \tilde{\nabla} \cdot \mathbf{w}_k = h_k + \mathbf{h}_k, \quad y \in \Omega^+,
 \end{aligned}$$

where $\tilde{\nabla} = \mathfrak{L}^{-T} \nabla$ is the transformed gradient ∇_x , $\tilde{T}(\mathbf{u}) = \mu \tilde{S}(\mathbf{u}) + \mu_1 I \tilde{\nabla} \cdot \mathbf{u}$ is the transformed stress tensor, and $\tilde{S}(\mathbf{u}) = \tilde{\nabla} \mathbf{u} + (\tilde{\nabla} \mathbf{u})^T$ is the transformed rate-of-strain tensor. We write system (2.27), completed with a boundary condition on Ω' , in the form

$$(2.28) \quad \begin{aligned} \bar{\rho} \mathbf{w}_{k,t} - \nabla \cdot T(\mathbf{w}_k) + p'(\bar{\rho}) \nabla \vartheta_k &= \mathbf{f}_k + \tilde{\mathbf{f}}_k, \\ \vartheta_{k,t} + \bar{\rho} \nabla \cdot \mathbf{w}_k &= h_k + \tilde{h}_k, \quad y \in \Omega^+, \quad \mathbf{w}_k = 0, \quad y' \in \Omega', \quad y_3 = 0, \end{aligned}$$

where the functions

$$(2.29) \quad \begin{aligned} \tilde{\mathbf{f}}_k &= \mathbf{f}_k + \tilde{\nabla} \cdot \tilde{T}(\mathbf{w}_k) - \nabla \cdot T(\mathbf{w}_k) - p'(\bar{\rho}) \tilde{\nabla} \vartheta_k + p'(\bar{\rho}) \nabla \vartheta_k, \\ \tilde{h}_k &= h_k - \bar{\rho} \tilde{\nabla} \cdot \mathbf{w}_k + \bar{\rho} \nabla \cdot \mathbf{w}_k \end{aligned}$$

involve also higher order derivatives of $\mathbf{w}_k$ and ϑ_k with small coefficients proportional to ϕ or to the derivatives of ϕ . We note that

$$(2.30) \quad \int_{\Omega^+} \vartheta_k(y, t) dy = 0. \quad \square$$

Remark. If $\phi \in W_2^{3/2+l}(K)$, then, instead of (2.26), the transformation

$$(2.31) \quad x' = y', \quad x_3 = y_3 + \phi^*(y)$$

should be used, where ϕ^* is the extension of ϕ to $K \times (0, 2d_0)$ such that

$$(2.32) \quad \begin{aligned} \phi^*(y) &= \phi(y'), \quad \frac{\partial \phi^*}{\partial y_3} = 0, \quad y_3 = 0, \\ \|\phi^*\|_{W_2^{l+2}(K \times (0, 2d_0))} &\leq c \|\phi\|_{W_2^{l+3/2}(K)}. \end{aligned}$$

Since $\phi^*(0) = 0$ and $\nabla \phi^*(0) = 0$, we have

$$(2.33) \quad |\phi^*(y)| \leq c|y|^{1+\alpha}, \quad |\nabla \phi^*(y)| \leq c|y|^\alpha, \quad \alpha = l - 1/2.$$

The Jacobi matrix of the transformation (2.31) equals

$$\mathfrak{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \phi_{y_1}^* & \phi_{y_2}^* & 1 + \phi_{y_3}^* \end{pmatrix},$$

and $\det \mathfrak{L} = 1 + \phi_{y_3}^*$. Consequently,

$$\int_{\tilde{\omega}_k} \vartheta_k(y, t) dy = \int_{\tilde{\omega}_k} \vartheta_k(y, t) (1 - \det \mathfrak{L}) dy = - \int_{\tilde{\omega}_k} \vartheta_k(y, t) \phi_{y_3}^* dy.$$

In order to conserve condition (2.30), we can introduce a new function of the form

$$\vartheta'_k(y, t) = \vartheta_k(y, t) - \Sigma_k(y) \int_{\omega_k} \vartheta_k(x, t) (1 - \det \mathfrak{L}) dx,$$

where $\Sigma_k(y)$ is smooth, $\text{supp } \Sigma_k \subset \Omega^+$, and $\int_{\text{supp } \Sigma_k} \Sigma_k(x) dx = 1$. Clearly, the $(\mathbf{w}_k, \vartheta')$ satisfy a system of the type (2.28) with other $\tilde{\mathbf{f}}_k$ and $\tilde{h}_k$, but possessing the same properties. Since the new system is treated in the same way as (2.28), we omit the details.

We proceed by studying equations (2.28). We follow the arguments in the proof of (2.23), keeping in mind that finite differences can be taken this time only with respect to tangential variables y_1, y_2 . Therefore, instead of (2.23) we obtain

$$(2.34) \quad \begin{aligned} &\|e^{\beta t} \nabla \mathbf{w}_k\|_{W_{2, \tan}^{l+1, 0}(\Omega_T^+)}^2 + \|e^{\beta t} \mathbf{w}_k\|_{W_{2, \tan}^{l+1, 0}(\Omega_T^+)}^2 + \|e^{\beta t} \vartheta_k\|_{W_{2, \tan}^{l+1}(\Omega_T^+)}^2 \\ &\leq c(\|\mathbf{w}_k(\cdot, 0)\|_{W_{2, \tan}^{l+1}(\Omega^+)}^2 + \|\vartheta_k(\cdot, 0)\|_{W_{2, \tan}^{l+1}(\Omega^+)}^2 \\ &\quad + \|e^{\beta t}(\mathbf{f}_k + \tilde{\mathbf{f}}_k)\|_{W_{2, \tan}^{l, 0}(\Omega_T^+)}^2 + \|e^{\beta t}(h_k + \tilde{h}_k)\|_{W_{2, \tan}^{l+1, 0}(\Omega_T^+)}^2), \end{aligned}$$

where $\mathfrak{Q}_T^+ = \mathfrak{Q}^+ \times (0, T)$ and

$$W_{2,\tan}^r(\mathfrak{Q}^+) = L_2(0, 2d_0; W_2^r(\mathfrak{Q}')).$$

To estimate the derivatives of w_k, ϑ_k with respect to y_3 and t , we rewrite (2.28) leaving on the left-hand side only these derivatives and carrying all the remaining terms over to the right. In this way we arrive at a problem that can be viewed as one-dimensional in the interval $(0, 2d_0)$, namely,

$$(2.35) \quad \begin{cases} \bar{\rho} w_{\alpha,t} - \mu \frac{\partial^2 w_{\alpha}}{\partial y_3^2} = f_{\alpha} + \tilde{f}_{\alpha} + f'_{\alpha} \equiv g_{\alpha}, & \alpha = 1, 2, \\ w_{\alpha}|_{t=0} = w_{\alpha,0}, & w_{\alpha}|_{y_3=0, 2d_0} = 0, \end{cases}$$

$$(2.36) \quad \begin{cases} \bar{\rho} w_{3,t} - (2\mu + \mu_1) \frac{\partial^2 w_3}{\partial y_3^2} + p'(\bar{\rho}) \frac{\partial \vartheta}{\partial y_3} = f_3 + \tilde{f}_3 + f'_3 \equiv g_3, \\ \vartheta_t + \bar{\rho} \frac{\partial w_3}{\partial y_3} = h + \tilde{h} + h' \equiv e, \\ w_3|_{y_3=0} = w_3|_{y_3=2d_0} = 0, & w_3|_{t=0} = w_{3,0}, \quad \vartheta|_{t=0} = \vartheta_0, \end{cases}$$

(we have omitted the index k for a moment). The functions $f'_i, h', i = 1, 2, 3$, stem from the transposed terms estimated in (2.34).

We consider problem (2.36) for w_3 and ϑ . We start by reducing (2.36) to a similar problem with homogeneous initial data. We set $w_3^{(1)}(x, t) = u(x, t)\chi(t)$, where $\chi(t)$ is a smooth cut-off function equal to 1 for $t \in [0, 1]$ and to zero for $t \geq 2$, and we require that

$$(2.37) \quad \begin{aligned} u(x, 0) &= w_{3,0}(x), \quad u(x, t) = 0, \quad x_3 = 0, 2d_0, \\ \|u\|_{W_2^{2+l, 1+l/2}((0, 2d_0) \times (0, 2))} &\leq c \|w_{3,0}\|_{W_2^{1+l}(0, 2d_0)} \end{aligned}$$

for arbitrary fixed $x' \in \mathfrak{Q}'$; the function $w_3^{(1)}$ satisfies a similar inequality. The function $\vartheta^{(1)}$ is defined by $\vartheta^{(1)}(x, t) = \vartheta_0(x)\chi(t)$; it is clear that

$$(2.38) \quad \|\vartheta^{(1)}\|_{W_2^{1+l, 0}((0, 2d_0) \times (0, 2))} + \|\vartheta_t^{(1)}\|_{W_2^{1+l, 0}((0, 2d_0) \times (0, 2))} \leq c \|\vartheta_0\|_{W_2^{1+l}(0, 2d_0)}.$$

For $w_3^{(2)} = w_3 - w_3^{(1)}$ and $\vartheta^{(2)} = \vartheta - \vartheta^{(1)}$ we obtain the problem

$$(2.39) \quad \begin{cases} \bar{\rho} w_{3,t}^{(2)} - (2\mu + \mu_1) \frac{\partial^2 w_3^{(2)}}{\partial y_3^2} + p'(\bar{\rho}) \frac{\partial \vartheta^{(2)}}{\partial y_3} = g_3^{(1)}, \\ \vartheta_t^{(2)} + \bar{\rho} \frac{\partial w_3^{(2)}}{\partial y_3} = e^{(1)}, & y_3 \in (0, 2d_0), \\ w_3 = 0, & y_3 = 0, 2d_0, \quad w_3|_{t=0} = 0, \quad \vartheta|_{t=0} = 0, \end{cases}$$

where

$$(2.40) \quad \begin{aligned} g_3^{(1)} &= g_3 - \bar{\rho} w_{3,t}^{(1)} + (2\mu + \mu_1) \frac{\partial^2 w_3^{(1)}}{\partial y_3^2} - p'(\bar{\rho}) \frac{\partial \vartheta^{(1)}}{\partial y_3}, \\ e^{(1)} &= e - \vartheta_t^{(1)} - \bar{\rho} \frac{\partial w_3^{(1)}}{\partial y_3}. \end{aligned}$$

We extend all the functions in (2.39) by zero to the domain $t < 0$ and apply the Laplace transformation defined by a standard formula

$$\tilde{u}(s) = \int_0^\infty e^{-st} u(t) dt.$$

For the transformed functions $\tilde{w}_3^{(2)}$ and $\tilde{\vartheta}^{(2)}$ we obtain the equations

$$\begin{cases} \bar{\rho} s \tilde{w}_3^{(2)} - (2\mu + \mu_1) \frac{d^2 \tilde{w}_3^{(2)}}{dy_3^2} + p'(\bar{\rho}) \frac{d\tilde{\vartheta}^{(2)}}{dy_3} = \tilde{g}_3^{(1)}, \\ s \tilde{\vartheta}^{(2)} + \bar{\rho} \frac{d\tilde{w}_3^{(2)}}{dy_3} = \tilde{e}^{(1)}, \\ \tilde{w}_3^{(2)} = 0, & y_3 = 0, 2d_0. \end{cases}$$

Upon eliminating $\tilde{v}^{(2)}$, we arrive at the Sturm–Liouville problem

$$\begin{cases} \bar{\rho}s\tilde{w}_3^{(2)} - \left(2\mu + \mu_1 + \frac{\bar{\rho}p'(\bar{\rho})}{s}\right) \frac{d^2\tilde{w}_3^{(2)}}{dy_3^2} = \tilde{g}_3^{(1)} - \frac{p'(\bar{\rho})}{s} \frac{d\tilde{e}^{(1)}}{dy_3}, \\ \tilde{w}_3^{(2)} = 0, \quad y_3 = 0, 2d_0, \end{cases}$$

or

$$(2.41) \quad \begin{cases} R(s)\tilde{w}_3^{(2)} - \frac{d^2\tilde{w}_3^{(2)}}{dy_3^2} = \tilde{G}, \\ \tilde{w}_3^{(2)} = 0, \quad y_3 = 0, 2d_0, \end{cases}$$

where

$$(2.42) \quad \begin{aligned} R(s) &= \frac{\bar{\rho}s^2}{as+b}, \quad a = 2\mu + \mu_1 > 0, \quad b = \bar{\rho}p'(\bar{\rho}) > 0, \\ \tilde{G} &= \frac{s}{as+b}\tilde{g}_3^{(1)} - \frac{p'(\bar{\rho})}{as+b} \frac{d\tilde{e}^{(1)}}{dy_3}. \end{aligned}$$

The solution of (2.41) is given by the well-known formula (see [8])

$$(2.43) \quad \tilde{w}_3^{(2)}(y_3) = C^{-1} \left(\int_{y_3}^{2d_0} z_2(\eta) z_1(y_3) \tilde{G}(\eta) d\eta + \int_0^{y_3} z_2(y_3) z_1(\eta) \tilde{G}(\eta) d\eta \right),$$

where

$$z_1(\eta) = e^{r\eta} - e^{-r\eta}, \quad z_2 = e^{(2d_0-\eta)r} - e^{-(2d_0-\eta)r}, \quad r = \sqrt{R(s)} = \sqrt{\bar{\rho}} \frac{s}{\sqrt{as+b}},$$

are solutions of the homogeneous equation $R(s)z - z'' = 0$ vanishing for $\eta = 0$ and $\eta = 2d_0$, respectively, while C is the Wronskian of z_1 and z_2 :

$$C = z_2'(\eta)z_1(\eta) - z_1'(\eta)z_2(\eta) = 2r(e^{-2d_0r} - e^{2d_0r}).$$

Since $r(s)$ is an analytic function for $\operatorname{Re} s > -b/a$, we can fix $\operatorname{Re} s = s_1 \in (-b/a, 0)$. It is not hard to verify that under this condition the function (2.43) satisfies the inequality

$$(2.44) \quad |s|^2 \|\tilde{w}_3^{(2)}\|_{L_2(0,2d_0)}^2 \leq c(s_1) \|\tilde{G}\|_{L_2(0,2d_0)}^2.$$

We multiply (2.44) by $|s|^l$ and integrate over the line $\operatorname{Re} s = s_1$. This leads to an inequality equivalent to

$$(2.45) \quad \|e^{\beta t} w_3^{(2)}\|_{W_2^{0,1+l/2}((0,2d_0) \times (0,\infty))}^2 \leq c \|e^{\beta t} G\|_{W_2^{0,l/2}((0,2d_0) \times (0,\infty))}^2$$

with $\beta = -s_1 > 0$. Integrating (2.45) over $\mathfrak{Q}'$, we obtain

$$(2.46) \quad \|e^{\beta t} w_3^{(2)}\|_{W_2^{0,1+l/2}(\mathfrak{Q}_\infty^+)}^2 \leq c \|e^{\beta t} G\|_{W_2^{0,l/2}(\mathfrak{Q}_\infty^+)}^2.$$

Next, we estimate the $W_2^{2+l}(0, 2d_0)$ -norm of $\tilde{w}_3^{(2)}$ by using equation (2.41). We have

$$\|D_{y_3}^2 \tilde{w}_3^{(2)}\|_{W_2^l(0,2d_0)} \leq c(|R(s)| \|\tilde{w}_3^{(2)}\|_{W_2^l(0,2d_0)} + \|\tilde{G}\|_{W_2^l(0,2d_0)}),$$

which implies, by the interpolation inequality,

$$(2.47) \quad \begin{aligned} & \|e^{\beta t} w_3^{(2)}\|_{W_2^{2+l,0}((0,2d_0) \times (0,\infty))} \\ & \leq c \left(\|e^{\beta t} D_t w_3^{(2)}\|_{W_2^{l,0}((0,2d_0) \times (0,\infty))} + \|e^{\beta t} G\|_{W_2^{l,0}((0,2d_0) \times (0,\infty))} \right) \\ & \leq \delta \|e^{\beta t} w_3^{(2)}\|_{W_2^{2+l,0}((0,2d_0) \times (0,\infty))} + c(\delta) \|e^{\beta t} w_3^{(2)}\|_{W_2^{0,1+l/2}((0,2d_0) \times (0,\infty))} \\ & \quad + c \|e^{\beta t} G\|_{W_2^{l,l/2}((0,2d_0) \times (0,\infty))}. \end{aligned}$$

Choosing δ sufficiently small and taking (2.45) into account, we obtain

$$(2.48) \quad \|e^{\beta t} w_3^{(2)}\|_{W_2^{2+l,1+l/2}((0,2d_0) \times (0,\infty))}^2 \leq c \|e^{\beta t} G\|_{W_2^{l,l/2}((0,2d_0) \times (0,\infty))}^2.$$

Finally, we use (2.37), (2.38), and (2.45), to get

$$\begin{aligned}
 & \|e^{\beta t} w_3\|_{L_2(\Omega', W_2^{2+l, 1+l/2}((0, 2d_0) \times (0, \infty)))}^2 + \|e^{\beta t} \vartheta\|_{L_2(\Omega', W_2^{1+l, 0}((0, 2d_0) \times (0, \infty)))}^2 \\
 & \quad + \|e^{\beta t} \vartheta_t\|_{L_2(\Omega', W_2^{1+l, 0}((0, 2d_0) \times (0, \infty)))}^2 \\
 (2.49) \quad & \leq c \left(\|e^{\beta t} g_3\|_{L_2(\Omega', W_2^{l, l/2}((0, 2d_0) \times (0, \infty)))}^2 + \|e^{\beta t} \frac{\partial e}{\partial y_3}\|_{L_2(\Omega', W_2^{l, 0}((0, 2d_0) \times (0, \infty)))}^2 \right. \\
 & \quad \left. + \|w_{3,0}\|_{L_2(\Omega', W_2^{l+1}(0, 2d_0))}^2 + \|\vartheta_0\|_{L_2(\Omega', W_2^{l+1}(0, 2d_0))}^2 \right).
 \end{aligned}$$

The functions w_α can be estimated by similar (in fact, more elementary) calculations, so that we have

$$\begin{aligned}
 & \|e^{\beta t} \mathbf{w}\|_{L_2(\Omega', W_2^{2+l, 1+l/2}((0, 2d_0) \times (0, \infty)))}^2 + \|e^{\beta t} \vartheta\|_{L_2(\Omega', W_2^{1+l, 0}((0, 2d_0) \times (0, \infty)))}^2 \\
 & \quad + \|e^{\beta t} \vartheta_t\|_{L_2(\Omega', W_2^{1+l, 0}((0, 2d_0) \times (0, \infty)))}^2 \\
 (2.50) \quad & \leq c \left(\|e^{\beta t} \mathbf{g}\|_{L_2(\Omega', W_2^{l, l/2}((0, 2d_0) \times (0, \infty)))}^2 + \|e^{\beta t} \nabla e\|_{L_2(\Omega', W_2^{l, 0}((0, 2d_0) \times (0, \infty)))}^2 \right. \\
 & \quad \left. + \|\mathbf{w}_0\|_{L_2(\Omega', W_2^{l+1}(0, 2d_0))}^2 + \|\vartheta_0\|_{L_2(\Omega', W_2^{l+1}(0, 2d_0))}^2 \right).
 \end{aligned}$$

Since the interval $(0, \infty)$ in (2.45)–(2.50) can be replaced by $(0, T)$, as in parabolic problems [20], inequalities (2.50) and (2.34) imply

$$\begin{aligned}
 & \|e^{2\beta t} \mathbf{w}_k\|_{W_2^{l+2, 1+l/2}(\Omega_T^+)}^2 + \|e^{2\beta t} \vartheta_k\|_{W_2^{l+1, 0}(\Omega_T^+)}^2 + \|e^{2\beta t} \vartheta_{k,t}\|_{W_2^{l+1, 0}(\Omega_T^+)}^2 \\
 (2.51) \quad & \leq c \left(\|\mathbf{g}_k\|_{W_2^{l, l/2}(\Omega_T^+)}^2 + \|e_k\|_{W_2^{l+1, 0}(\Omega_T^+)}^2 + \|\mathbf{w}_k(\cdot, 0)\|_{W_2^{l+1}(\Omega^+)}^2 + \|\vartheta_k(\cdot, 0)\|_{W_2^{l+1}(\Omega^+)}^2 \right),
 \end{aligned}$$

where

$$\mathbf{g}_k = \mathbf{f}_k + \tilde{\mathbf{f}}_k + \mathbf{f}'_k, \quad e_k = h_k + \tilde{h}_k + h'_k.$$

We recall that the functions $\mathbf{f}'_k$ and h'_k are linear combinations of the derivatives of $\mathbf{w}_k$ and ϑ_k , except for $\mathbf{w}_{k,t}$, $\vartheta_{k,t}$, $D_{y_3}^2 \mathbf{w}_k$, $D_{y_3} \vartheta_k$. They can be estimated by interpolation inequalities, which yields

$$\begin{aligned}
 & \|e^{\beta t} \mathbf{f}'_k\|_{L_2(\Omega', W_2^{l, l/2}((0, 2d_0) \times (0, T)))}^2 + \|e^{\beta t} h'_k\|_{L_2(\Omega', W_2^{l+1, 0}((0, 2d_0) \times (0, T)))}^2 \\
 (2.52) \quad & \leq \delta \left(\|e^{\beta t} \mathbf{w}_k\|_{W_2^{l+2, 1+l/2}(\Omega_T^+)}^2 + \|e^{\beta t} \vartheta_k\|_{W_2^{l+1, 0}(\Omega_T^+)}^2 + \|e^{\beta t} \vartheta_{k,t}\|_{W_2^{l+1, 0}(\Omega_T^+)}^2 \right) \\
 & \quad + c(\delta) \left(\|e^{\beta t} \mathbf{w}_k\|_{W_2^{l+2, 0}(\Omega_T^+)}^2 + \|e^{\beta t} \vartheta_k\|_{W_2^{l+1, 0}(\Omega_T^+)}^2 \right).
 \end{aligned}$$

Hence, for small δ we have

$$\begin{aligned}
 & \|e^{\beta t} \mathbf{w}_k\|_{W_2^{l+2, 1+l/2}(\Omega_T^+)}^2 + \|e^{\beta t} \vartheta_k\|_{W_2^{l+1, 0}(\Omega_T^+)}^2 + \|e^{\beta t} \vartheta_{k,t}\|_{W_2^{l+1, 0}(\Omega_T^+)}^2 \\
 (2.53) \quad & \leq c \left(\|e^{\beta t} (\mathbf{f}_k + \tilde{\mathbf{f}}_k)\|_{W_2^{l, l/2}(\Omega_T^+)}^2 + \|e^{\beta t} (h_k + \tilde{h}_k)\|_{W_2^{l+1, 0}(\Omega_T^+)}^2 \right. \\
 & \quad \left. + \|\mathbf{w}_k(\cdot, 0)\|_{W_2^{l+1}(\Omega^+)}^2 + \|\vartheta_k(\cdot, 0)\|_{W_2^{l+1}(\Omega^+)}^2 \right),
 \end{aligned}$$

Since $\mathbf{v} = \sum_{k=1}^m \mathbf{v} \varphi_k$, the norms $\|e^{\beta t} \mathbf{v}\|_{W_2^{2+l, 1+l/2}(Q_T)}$ and

$$\sum_{k=1}^m \|e^{\beta t} \mathbf{v}_k\|_{W_2^{2+l, 1+l/2}(\omega_k \times (0, T))}^2$$

are equivalent. The differences $\mathbf{w}_k - \mathbf{v}_k = -\zeta_k \int_{\omega_k} \mathbf{v}_k dx$, $k < m$, involve no higher order terms and can be estimated by interpolation inequalities, like $\mathbf{f}_k$ and h_k . The same is

true for $\vartheta_k - \theta_k$. Hence,

$$\begin{aligned}
 & \sum_{k=1}^m \|e^{\beta t} \mathbf{v}_k\|_{W_2^{2+l,1+l/2}(\omega_k \times (0,T))}^2 + \sum_{k=1}^m \|e^{\beta t} \theta_k\|_{W_2^{1+l,0}(\omega_k \times (0,T))}^2 \\
 & + \sum_{k=1}^m \|e^{\beta t} \theta_{k,t}\|_{W_2^{1+l,0}(\omega_k \times (0,T))}^2 \\
 (2.54) \quad & \leq c \left(\|\mathbf{v}_0\|_{W_2^{l+1}(\Omega)}^2 + \|\theta_0\|_{W_2^{l+1}(\Omega)}^2 + \|e^{\beta t} \mathbf{f}\|_{W_2^{l,l/2}(Q_T)}^2 + \|e^{\beta t} h\|_{W_2^{l+1,0}(Q_\infty)}^2 \right) \\
 & + \delta \left(\|e^{\beta t} \mathbf{v}\|_{W_2^{2+l,1+l/2}(Q_\infty)}^2 + \|e^{\beta t} \theta_t\|_{W_2^{1+l,0}(Q_T)}^2 + \|e^{\beta t} \theta\|_{W_2^{1+l,0}(Q_T)}^2 \right) \\
 & + c(\delta) \left(\|e^{\beta t} \mathbf{v}\|_{L_2(Q_T)}^2 + \|e^{\beta t} \theta\|_{L_2(Q_T)}^2 \right).
 \end{aligned}$$

Since the weighted L_2 -norms of $\mathbf{v}$ and θ were already estimated in (2.3), inequalities (2.3) and (2.54) imply (2.2) if δ is chosen small. This completes the proof of inequality (2.2).

As for the solvability of problem (1.7) in an arbitrary finite time interval $(0, T)$, it suffices to prove this in the case of zero initial data. Eliminating θ , we can write (1.7) in the form

$$\begin{cases} \bar{\rho} \mathbf{v}_t - \nabla \cdot T(\mathbf{v}) = \mathbf{f} + p'(\bar{\rho}) \int_0^t \nabla(\nabla \cdot \mathbf{v}) d\tau + p'(\bar{\rho}) \bar{\rho} \int_0^t \nabla h d\tau, & x \in \Omega, \\ \mathbf{v}(x, t) = 0, & x \in S, \quad t > 0, \\ \mathbf{v}(x, 0) = 0, & x \in \Omega, \end{cases}$$

which shows that the solvability of (1.7) follows from that of a similar problem for the Lamé system

$$\begin{cases} \bar{\rho} \mathbf{v}_t - \nabla \cdot T(\mathbf{v}) = \mathbf{f}, \\ \mathbf{v}(x, t) = 0, & x \in S, \quad t > 0, \\ \mathbf{v}(x, 0) = 0, & x \in \Omega. \end{cases}$$

The latter solvability can be established in the same way (and even much easier) as in [12] for the case of the more complicated boundary conditions

$$(T(\mathbf{v}) \mathbf{n}_0)_\tau = 0, \quad \mathbf{n}_0 \cdot T(\mathbf{v}) \mathbf{n}_0 - \sigma \mathbf{n}_0 \int_0^t \Delta_0 \mathbf{v} d\tau = 0,$$

where Δ_0 is the Laplace–Beltrami operator on Γ_0 .

In conclusion, we consider problem (1.7) with initial data that do not obey (2.1).

Theorem 3. For arbitrary $\mathbf{v}_0 \in W_2^{l+1}(\Omega)$, $\rho_0 \in W_2^{l+1}(\Omega)$, $\mathbf{f} \in W_2^{l,l/2}(Q_T)$, and $h \in W_2^{l+1,0}(Q_T)$ satisfying $\mathbf{v}_0(x) = 0$, $x \in S$, problem (1.7) has a unique solution $\mathbf{v} \in W_2^{2+l,1+l/2}(Q_T)$, $\theta, \theta_t \in W_2^{l+1,0}(Q_T)$, $T < \infty$, and

$$\begin{aligned}
 (2.55) \quad & \|\mathbf{v}\|_{W_2^{l+2,l/2+1}(Q_T)} + \|\theta\|_{W_2^{l+1,0}(Q_T)} + \|\theta_t\|_{W_2^{l+1,0}(Q_T)} \\
 & \leq c(T) (\|\mathbf{v}_0\|_{W_2^{l+1}(\Omega)} + \|\theta_0\|_{W_2^{l+1}(\Omega)} + \|\mathbf{f}\|_{W_2^{l,l/2}(Q_T)} + \|h\|_{W_2^{l+1,0}(Q_T)}).
 \end{aligned}$$

This theorem follows from Theorem 2, because the functions $\mathbf{v}$ and

$$\theta' = \theta - \frac{1}{|\Omega|} \left(\int_\Omega \theta_0 dx + \int_0^t \int_\Omega h dx d\tau \right) = \theta - \frac{1}{|\Omega|} \int_\Omega \theta dx$$

solve problem (1.7) with the initial data

$$\left(\mathbf{v}_0, \theta'_0 = \theta_0 - \frac{1}{|\Omega|} \int_\Omega \theta_0 dx \right)$$

and with $h' = h - |\Omega|^{-1} \int_{\Omega} h \, dx$ in place of h on the right-hand side. In this problem conditions (2.1) are satisfied.

§3. NONLINEAR PROBLEM

In this section, we outline the proof of Theorem 1. It is based on Theorem 2 and on the following estimates of nonlinear terms.

Proposition 2. *Let $p \in C^2(\bar{\rho}/2, 3\bar{\rho}/2)$, $\mathbf{U}(\xi, t) = \int_0^t \mathbf{u}(\xi, \tau) \, d\tau$. If*

$$(3.1) \quad \begin{aligned} \sup_{t < T} \|\mathbf{U}(\cdot, t)\|_{W_2^{2+l}(\Omega)} &\leq \delta_1, \quad \delta_1 > 0, \\ \sup_{Q_T} |\theta(x, t)| &\leq \bar{\rho}/2, \end{aligned}$$

then

$$(3.2) \quad Z(T) \leq c(\delta_1 Y(T) + Y^2(T)),$$

where $T > 1$ and

$$(3.3) \quad \begin{aligned} Z(T) &\equiv Z(\mathbf{u}, q) = \|l_1(\mathbf{u}, \theta)\|_{W_2^{l, l/2}(Q_T)} + \|l_2(\mathbf{u}, \theta)\|_{W_2^{l+1, 0}(Q_T)}, \\ Y(T) &\equiv Y(\mathbf{u}, q) = \|\mathbf{u}\|_{W_2^{2+l, 1+l/2}(Q_T)} + \|\theta\|_{W_2^{l+1, 0}(Q_T)} + \|\theta_t\|_{W_2^{l+1, 0}(Q_T)}, \\ l &\in (1/2, 1). \end{aligned}$$

Proof. We use the auxiliary inequalities

$$(3.4) \quad \|fg\|_{W_2^l(\Omega)} \leq c\|f\|_{W_2^l(\Omega)}\|g\|_{W_2^{l+1}(\Omega)},$$

$$(3.5) \quad \|fg\|_{L_2(\Omega)} \leq c\|f\|_{L_p(\Omega)}\|g\|_{L_q(\Omega)} \leq c\|f\|_{W_2^l(\Omega)}\|g\|_{W_2^1(\Omega)},$$

where $1/p + 1/q = 1/2$, $l - 3/2 + 3/p = 0$. The condition $l > 1/2$ implies that $1 - 3/2 + 3/q > 0$. Setting

$$\|u\|_{W_2^{\lambda}(0, T)}^2 = \|u\|_{L_2(0, T)}^2 + \int_0^1 \frac{dh}{h^{1+2\lambda}} \int_h^T |\Delta_t(-h)u(t)|^2 \, dt, \quad 0 < \lambda < 1,$$

where $\Delta_t(-h)u = u(t-h) - u(t)$, we obtain

$$(3.6) \quad \begin{aligned} \|\theta \mathbf{u}_t\|_{W_2^{l, 0}(Q_T)} &\leq c \sup_{t < T} \|\theta(\cdot, t)\|_{W_2^{l+1}(\Omega)} \|\mathbf{u}_t\|_{W_2^{l, 0}(Q_T)}, \\ \|\Delta_t(-h)(\theta \mathbf{u}_t)\|_{L_2(\Omega)} &\leq c \left(\sup_{\Omega} |\theta(x, t)| \|\Delta_t(-h)\mathbf{u}_t\|_{L_2(\Omega)} + \|\mathbf{u}_t\|_{W_2^1(\Omega)} \int_0^h \|\theta_t(\cdot, t-\tau)\|_{W_2^1(\Omega)} \, d\tau \right), \end{aligned}$$

which implies

$$(3.7) \quad \|\theta \mathbf{u}_t\|_{W_2^{l, l/2}(Q_T)} \leq c \|\mathbf{u}_t\|_{W_2^{l, l/2}(Q_T)} (\sup_{t < T} \|\theta\|_{W_2^{l+1}(\Omega)} + \|\theta_t\|_{W_2^{l+1, 0}(Q_T)}).$$

Next, we consider the term $\nabla_u \cdot T_u - \nabla \cdot T(\mathbf{u}) = (\nabla_u - \nabla) \cdot T(\mathbf{u}) + \nabla_u \cdot (T_u(\mathbf{u}) - T(\mathbf{u}))$. Since $\nabla_u - \nabla = (J_u^{-1}A - I)\nabla$ and

$$\|J_u^{-1} - I\|_{W_2^{l+1}(\Omega)} + \|A - 1\|_{W_2^{l+1}(\Omega)} \leq c\delta_1,$$

we have

$$(3.8) \quad \|(\nabla_u - \nabla) \cdot T(\mathbf{u})\|_{W_2^{l, 0}(Q_T)} \leq c\delta_1 \|D^2 \mathbf{u}\|_{W_2^{l, 0}(Q_T)}.$$

Moreover, from the estimate

$$\|\Delta_t(-h)(J_u^{-1}A - I)\|_{W_2^1(\Omega)} \leq c \int_0^h \|\mathbf{u}\|_{W_2^2(\Omega)} \, dt \leq c\sqrt{h} \|\mathbf{u}\|_{W_2^{2, 0}(Q_T)},$$

it follows that

$$(3.9) \quad \|(\nabla_u - \nabla) \cdot T(\mathbf{u})\|_{W_2^{0,l/2}(Q_T)} \leq c(\delta_1 + \|\mathbf{u}\|_{W_2^{2,0}(Q_T)}) \|D_x^2 \mathbf{u}\|_{W_2^{l,l/2}(Q_T)}.$$

The expression $\nabla \cdot (T_u(\mathbf{u}) - T(\mathbf{u}))$ is estimated in a similar way.

Now we estimate the last term $-P(\theta)\nabla\theta + p(\bar{\rho} + \theta)(\nabla - \nabla_u)\theta$ in $\mathbf{l}_1$, where $P(\theta) = p'(\bar{\rho} + \theta) - p'(\bar{\rho})$. We have

$$(3.10) \quad \|P(\theta)\nabla\theta\|_{L_2(Q_T)} \leq c \sup_{Q_T} |\theta(x, t)| \|\nabla\theta\|_{L_2(Q_T)}.$$

Assuming that θ is extended to $\mathbb{R}^3$ with preservation of class, we estimate the difference

$$(3.11) \quad \begin{aligned} & P(\theta(x+z, t))\nabla\theta(x+z, t) - P(\theta(x, t))\nabla\theta(x, t) \\ &= (P(\theta(x+z, t)) - P(\theta(x, t)))\nabla\theta(x+z, t) + P(\theta(x, t))(\nabla\theta(x+z, t) - \nabla\theta(x, t)). \end{aligned}$$

Since

$$\begin{aligned} |P(\theta(x+z, t)) - P(\theta(x, t))| &\leq c|\theta(x+z, t) - \theta(x, t)|, \\ |P(\theta(x, t))| &\leq c|\theta(x, t)|, \end{aligned}$$

the L_2 -norm of the difference (3.11) is controlled by

$$(3.12) \quad \begin{aligned} & c(\|\theta(x+z, t) - \theta(x, t)\|_{L_q(\Omega)} \|\nabla\theta\|_{L_p(\Omega)} \\ &+ \sup_{\Omega} |\theta(x, t)| \|\nabla\theta(x+z, t) - \nabla\theta(x, t)\|_{L_2(\Omega)}) \\ &\leq c\|\theta(x+z, t) - \theta(x, t)\|_{W_2^1(\Omega)} \|\theta(\cdot, t)\|_{W_2^{l+1}(\Omega)}. \end{aligned}$$

Multiplying (3.11), (3.12) by $|z|^{-3-2l}$ and integrating with respect to z , we obtain

$$(3.13) \quad \|P(\theta)\nabla\theta\|_{W_2^{l,0}(Q_T)} \leq c \sup_{Q_T} \|\theta\|_{W_2^{l+1}(\Omega)} \|\theta\|_{W_2^{l+1,0}(Q_T)}.$$

The finite difference of $P(\theta)\nabla\theta$ with respect to t is estimated in the same way:

$$(3.14) \quad \begin{aligned} & \|P(\theta)\nabla\theta\|_{W_2^{0,l/2}(Q_T)} \\ &\leq c(\sup_{Q_T} |\theta(x, t)| \|\nabla\theta\|_{W_2^{0,l/2}(Q_T)} + \|\theta\|_{W_2^{l/2}(0,T), W_2^1(\Omega)} \sup_{t < T} \|\theta\|_{W_2^{l+1}(\Omega)}) \\ &\leq c \sup_{t < T} \|\theta\|_{W_2^{l+1}(\Omega)} (\|\theta_t\|_{W_2^{l+1,0}(Q_T)} + \|\theta\|_{W_2^{l+1,0}(Q_T)}), \end{aligned}$$

whence

$$(3.15) \quad \|P(\theta)\nabla\theta\|_{W_2^{l,l/2}(Q_T)} \leq c \sup_{t < T} \|\theta\|_{W_2^{l+1}(\Omega)} (\|\theta\|_{W_2^{l+1,0}(Q_T)} + \|\theta_t\|_{W_2^{l+1,0}(Q_T)}).$$

It can be shown by the same arguments that the expression $p(\bar{\rho} + \theta)(\nabla_u - \nabla)\theta$ satisfies the same inequality with the additional term

$$c\delta_1 (\|\theta\|_{W_2^{l+1,0}(Q_T)} + \|\theta_t\|_{W_2^{l+1,0}(Q_T)}) + c \sup_{t < T} \|\theta\|_{W_2^{l+1}(\Omega)} \|\mathbf{u}\|_{W_2^{2,0}(Q_T)}$$

on the right hand side.

As for $l_2 = \theta\nabla_u \cdot \mathbf{u} + \bar{\rho}(\nabla_u - \nabla)\mathbf{u}$, we have

$$(3.16) \quad \|l_2(\mathbf{u}, \theta)\|_{W_2^{l+1,0}(Q_T)} \leq c(\delta_1 + \sup_{t < T} \|\theta\|_{W_2^{l+1}(\Omega)}) \|\mathbf{u}\|_{W_2^{l+2,0}(Q_T)}.$$

Estimate (3.2) is a consequence of (3.6)–(3.16). The proposition is proved. $\square$

We describe the procedure of constructing the solution of a nonlinear problem (1.5) in the time interval $t > 0$. Observe that the solution of this problem satisfies the condition

$$(3.17) \quad \int_{\Omega} \theta(x, t) J_u(x, t) dx = 0$$

(in view of the mass conservation), while in the case of a linear problem we have

$$(3.18) \quad \int_{\Omega} \theta(x, t) dx = 0.$$

Therefore, we shall construct the solution step by step, from the time interval $[(k-1)T, kT]$ to $[kT, (k+1)T]$, with a certain sufficiently large but finite T , eliminating at every step the discrepancy between conditions (3.17) and (3.18). This procedure was proposed in [9].

We set $\mathbf{u} = \mathbf{u}' + \mathbf{u}''$, $\theta = \theta' + \theta''$, represent $(\mathbf{u}_0, \theta_0)$ in the form $\mathbf{u}_0 = \mathbf{u}'_0 + \mathbf{u}''_0$, $\theta_0 = \theta'_0 + \theta''_0$, where

$$(3.19) \quad \mathbf{u}'_0 = \mathbf{u}_0, \quad \mathbf{u}''_0 = 0, \quad \theta'_0 = \theta_0, \quad \theta''_0 = 0,$$

and define $(\mathbf{u}', \theta')$ and $(\mathbf{u}'', \theta'')$ as solutions of the problems

$$(3.20) \quad \begin{cases} \bar{\rho} \mathbf{u}'_t - \nabla \cdot T(\mathbf{u}') + p'(\bar{\rho}) \nabla \theta' = 0, \\ \theta'_t + \bar{\rho} \nabla \cdot \mathbf{u}' = 0, \quad \xi \in \Omega, \\ \mathbf{u}'(\xi, t) = 0, \quad x \in S, \\ \mathbf{u}'(\xi, 0) = \mathbf{u}_0(\xi), \quad \theta'(\xi, 0) = \theta_0(\xi), \quad \xi \in \Omega, \end{cases}$$

$$(3.21) \quad \begin{cases} \bar{\rho} \mathbf{u}''_t - \nabla \cdot T(\mathbf{u}'') + p'(\bar{\rho}) \nabla \theta'' = \mathbf{l}_1(\mathbf{u}' + \mathbf{u}'', \theta' + \theta''), \\ \theta''_t + \bar{\rho} \nabla \cdot \mathbf{u}'' = \mathbf{l}_2(\mathbf{u}' + \mathbf{u}'', \theta' + \theta''), \quad \xi \in \Omega, \\ \mathbf{u}''(\xi, t) = 0, \quad x \in S, \\ \mathbf{u}''(\xi, 0) = 0, \quad \theta''(\xi, 0) = 0, \quad \xi \in \Omega, \end{cases}$$

Since $\int_{\Omega} \theta_0(x) dx = 0$ (because $X(\xi, 0) = I$ and $J_u(\xi, 0) = 1$), problem (3.20) has a global solution satisfying (2.2) (with $\mathbf{f} = 0$, $h = 0$) for arbitrary $T > 0$, whereas (3.21) is solvable in the time interval $(0, T)$, when $\epsilon = \epsilon(T)$ in (1.9) is sufficiently small. By (2.2) and (2.55),

$$(3.22) \quad \begin{aligned} e^{\beta T} N(\mathbf{u}'(\cdot, T), \theta'(\cdot, T)) &\leq cN(\mathbf{u}_0, \theta_0), \\ Y(\mathbf{u}'', \theta'') &\leq c(\delta_1 Y(\mathbf{u}'' + \mathbf{u}', \theta'' + \theta')) + Y^2(\mathbf{u}'' + \mathbf{u}', \theta'' + \theta') \\ &\leq c(\delta_1 Y(\mathbf{u}'', \theta'') + Y^2(\mathbf{u}'', \theta'') + \varepsilon N(\mathbf{u}_0, \theta_0)) \end{aligned}$$

(under the assumptions (3.1)), where

$$N(\mathbf{u}_0, \theta_0) = \|\mathbf{u}_0\|_{W_2^{l+1}(\Omega)} + \|\theta_0\|_{W_2^{l+1}(\Omega)}.$$

If $\varepsilon N(\mathbf{u}_0, \theta_0)$ is sufficiently small, then (3.1) is true, problem (3.21) is solvable for $t \in (0, T)$, and

$$N(\mathbf{u}''(\cdot, T), \theta''(\cdot, T)) \leq cY(\mathbf{u}'', \theta'') \leq c\varepsilon N(\mathbf{u}_0, \theta_0).$$

Hence,

$$(3.23) \quad \begin{aligned} N(\mathbf{u}(\cdot, T), \theta(\cdot, T)) &\leq N(\mathbf{u}'(\cdot, T), \theta'(\cdot, T)) + N(\mathbf{u}''(\cdot, T), \theta''(\cdot, T)) \\ &\leq c(\varepsilon + e^{-\beta T})N(\mathbf{u}_0, \theta_0) \leq \frac{1}{8}N(\mathbf{u}_0, \theta_0), \end{aligned}$$

provided

$$(3.24) \quad c(\varepsilon + e^{-\beta T}) \leq \frac{1}{8}.$$

We fix T satisfying this condition.

Assume that the solution of (1.5) is found for $t < kT$, and that

$$(3.25) \quad N_k \leq \frac{1}{8}N_{k-1} \leq \cdots \leq \frac{1}{8^k}N_0,$$

where $N_k = N(\mathbf{u}(\cdot, kT), \theta(\cdot, kT))$. The function $\theta(\xi, kT)$ satisfies

$$\int_{\Omega} \theta(\xi, kT) J_u(\xi, kT) d\xi = 0,$$

and we set $\theta(\xi, kT) = \theta'_k(\xi) + \theta''_k(\xi)$, where

$$(3.26) \quad \begin{aligned} \theta'_k(\xi) &= \theta(\xi, kT) - |\Omega|^{-1} \int_{\Omega} \theta(\xi, kT) d\xi, \\ \theta''_k(\xi) &= |\Omega|^{-1} \int_{\Omega} \theta(\xi, kT) (1 - J_u(\xi, kT)) d\xi. \end{aligned}$$

We consider problems (3.20) and (3.21) in the interval $t \in (kT, (k+1)T)$, taking $(\mathbf{u}'_k = \mathbf{u}(\xi, kT), \theta'_k(\xi))$ and $(\mathbf{u}''_k = 0, \theta''_k(\xi))$, respectively, as initial data. It is clear that

$$(3.27) \quad \begin{aligned} N(\mathbf{u}'(\cdot, (k+1)T), \theta'(\cdot, (k+1)T)) &\leq e^{-\beta T} N(\mathbf{u}(\cdot, kT), \theta'(\cdot, kT)) \\ &\leq ce^{-\beta T} N(\mathbf{u}(\cdot, kT), \theta(\cdot, kT)), \end{aligned}$$

moreover, if conditions (3.1) are satisfied for $t \leq kT$, then we can estimate the nonlinear terms in accordance with (3.2), proving that

$$(3.28) \quad N(\mathbf{u}''(\cdot, (k+1)T), \theta''(\cdot, (k+1)T)) \leq cY_k(\mathbf{u}'', \theta'') \leq c\delta_1 N(\mathbf{u}(\cdot, kT), \theta(\cdot, kT)),$$

where Y_j is the sum of norms defined in (3.3) and computed in the interval $t \in (jT, (j+1)T)$. By (3.27) and (3.28),

$$N(\mathbf{u}(\cdot, (k+1)T), \theta(\cdot, (k+1)T)) \leq \frac{1}{8} N(\mathbf{u}(\cdot, kT), \theta(\cdot, kT)).$$

For every interval $t \in (jT, (j+1)T)$, $j = 1, \dots, k$, we have

$$Y_j \leq (Y'_j + Y''_j) \leq cN_j \leq c8^{-j}N_0.$$

If β is so small that $e^{\beta T} \leq 4$, then

$$\sum_{j=0}^k Y_j^2 e^{2\beta jT} \leq c \sum_{j=0}^k \frac{e^{2\beta jT}}{8^j} N_0^2 \leq cN_0^2,$$

which is equivalent to (1.10).

The verification of condition (3.1) for $\mathbf{U}$ in the interval $(0, kT)$ reduces to

$$\sup_{t < kT} \|\mathbf{U}(\cdot, t)\|_{W_2^{l+1}(\Omega)} \leq c \left(\int_0^{kT} e^{2\beta t} \|\mathbf{u}\|_{W_2^{l+2}(\Omega)}^2 dt \right)^{1/2} \leq cN_0 \leq c\epsilon,$$

hence we can set $\delta_1 = c\epsilon$. The inequality $\|\theta\|_{W_2^{l+1}(\Omega)} \leq \bar{\rho}/2$ is verified in a similar way.

More detailed arguments can be found in [10]. By the same method, the solution of the nonhomogeneous Navier–Stokes equation with mass forces $\mathbf{f}(x, t)$ can be constructed, provided $\mathbf{f}$ decays exponentially as $t \rightarrow \infty$ (see [10, 11]).

§4. ON THE FREE BOUNDARY PROBLEM

In this section, we consider the free boundary problem

$$(4.1) \quad \begin{cases} \rho(x, t)(\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}) - \nabla \cdot T(\mathbf{v}) + \nabla p(\rho) = 0, \\ \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \quad x \in \Omega_t^+ \cup \Omega_t^-, \quad t > 0, \\ [\mathbf{v}] = 0, \quad [-p(\rho)\mathbf{n} + T(\mathbf{v})\mathbf{n}] = 0, \quad V_n = \mathbf{v} \cdot \mathbf{n}, \quad x \in \Gamma_t, \\ \mathbf{v}^-(x, t) = 0, \quad x \in S, \\ \mathbf{v}(x, 0) = \mathbf{v}_0(x), \quad \rho(x, 0) = \rho_0(x), \quad x \in \Omega_0^+ \cup \Omega_0^-, \end{cases}$$

where $\mathbf{v}(x, t) = \mathbf{v}^\pm(x, t)$, $\rho(x, t) = \rho^\pm(x, t)$ for $x \in \Omega_t^\pm$, Ω_t^+ and Ω_t^- are bounded domains separated by a free interface $\Gamma_t = \partial\Omega_t^+$ that is given for $t = 0$ and should be found for $t > 0$. The domain $\Omega = \Omega_t^+ \cup \Gamma_t \cup \Omega_t^-$ is fixed; the surface $S = \partial\Omega$ is bounded away from Γ_t . By $T(\mathbf{v}) \equiv T^\pm(\mathbf{v})$ we mean the viscous part of the stress tensor:

$$T^\pm(\mathbf{v}) = \mu^\pm S(\mathbf{v}^\pm) + \mu_1^\pm I \nabla \cdot \mathbf{v}^\pm, \quad x \in \Omega^\pm,$$

$\mathbf{n}$ is the normal to Γ_t exterior with respect to Ω^+ , and V_n is the velocity of evolution of Γ_t in the direction $\mathbf{n}$, $\mu^\pm, \mu_1^\pm = \text{const} > 0$, $[\mathbf{u}]$ is the jump of $\mathbf{u}$ on Γ_t . We assume that $p^+(\rho) = p^-(\rho) = p(\rho)$.

In the Lagrangian coordinates, problem (4.1) takes the form

$$(4.2) \quad \begin{cases} r(\xi, t) \mathbf{u}_t - \nabla_u \cdot T_u(\mathbf{u}) + \nabla_u p(r) = 0, \\ r_t + r \nabla_u \cdot \mathbf{u} = 0, \quad \xi \in \Omega_0^+ \cup \Omega_0^-, \quad t > 0, \\ [\mathbf{u}] = 0, \quad [-p(r) \mathbf{n} + T_u(\mathbf{u}) \mathbf{n}] = 0, \quad \xi \in \Gamma_t, \\ \mathbf{u}^- = 0, \quad \xi \in S, \\ \mathbf{u}(\xi, 0) = \mathbf{v}_0(\xi), \quad r(\xi, 0) = \rho_0(\xi), \quad \xi \in \Omega_0^+ \cup \Omega_0^-, \end{cases}$$

where $r = \rho(X(\xi, t), t)$, $\mathbf{u}(\xi, t) = \mathbf{v}(X, t)$,

$$X(\xi, t) = \xi + \int_0^t \mathbf{u}(\xi, \tau) d\tau \equiv \xi + \mathbf{U}(\xi, t), \quad \xi \in \Omega_0^- \cup \Omega_0^+.$$

As in §2, we set $\nabla_u = J_u^{-1} A \nabla$, $S_u(\mathbf{u}) = (\nabla_u \mathbf{u}) + (\nabla_u \mathbf{u})^T$, $T_u(\mathbf{u}) = \mu S_u(\mathbf{u}) + \mu_1 I \nabla_u \cdot \mathbf{u}$, $J_u = \det \mathcal{L}$, $\mathcal{L} = \left(\frac{\partial X}{\partial \xi} \right)$, $\mathbf{n}(X) = \frac{J_u^{-1} A \mathbf{n}_0}{|J_u^{-1} A \mathbf{n}_0|}$. The elements of the matrix A are co-factors of the elements $a_{ij} = \delta_{ij} + \int_0^t \frac{\partial u_i}{\partial \xi_j} d\tau$ of the matrix $\mathcal{L}$.

Moreover, we set $\rho = \bar{\rho} + \theta$, $\bar{\rho} = M/|\Omega|$,

$$M = \int_\Omega \rho(x, t) dx = \int_\Omega r(\xi, t) J_u(\xi, t) d\xi = \text{const} > 0,$$

which implies

$$\int_\Omega \theta(\xi, t) J_u(\xi, t) d\xi = 0,$$

and we write (4.2) in the form

$$(4.3) \quad \begin{cases} \bar{\rho} \mathbf{u}_t - \nabla \cdot T(\mathbf{u}) + p'(\bar{\rho}) \nabla \theta = \mathbf{l}_1(\mathbf{u}, \theta), \\ \theta_t + \bar{\rho} \nabla \cdot \mathbf{u} = l_2(\mathbf{u}, \theta), \quad \xi \in \Omega_0^+ \cup \Omega_0^-, \\ \mathbf{u}(\xi, t) = 0, \quad \xi \in S, \\ [\mathbf{u}] = 0, \quad [\mu^\pm \Pi_0 S(\mathbf{u}) \mathbf{n}_0] = \mathbf{l}_3(\mathbf{u}), \\ [-p'(\bar{\rho}) \theta + \mathbf{n}_0 \cdot T(\mathbf{u}) \mathbf{n}_0] = l_4(\mathbf{u}, \theta), \quad \xi \in \Gamma_0, \\ \mathbf{u}(\xi, 0) = \mathbf{v}_0(\xi), \quad \theta(\xi, 0) = \theta_0(\xi) = \rho_0(\xi) - \bar{\rho}, \quad \xi \in \Omega_0, \end{cases}$$

where

$$(4.4) \quad \begin{aligned} \mathbf{l}_1 &= -\vartheta \mathbf{u}_t + (\nabla_u \cdot T_u(\mathbf{u}) - \nabla \cdot T(\mathbf{u})) + p'(\bar{\rho}) \nabla \theta - p'(\bar{\rho} + \theta) \nabla_u \theta, \\ l_2 &= -r \nabla_u \cdot \mathbf{u} + \bar{\rho} \nabla \cdot \mathbf{u}, \\ \mathbf{l}_3 &= [\mu(\Pi_0^2 S(\mathbf{u}) \mathbf{n}_0 - \Pi_0 \Pi S_u(\mathbf{u}) \mathbf{n})], \\ l_4 &= -[\mathbf{n} \cdot T_u(\mathbf{u}) \mathbf{n} - \mathbf{n}_0 \cdot T(\mathbf{u}) \mathbf{n}_0] + [p(\bar{\rho} + \theta) - p(\bar{\rho}) - p'(\bar{\rho}) \theta], \end{aligned}$$

$$\Pi \mathbf{f} = \mathbf{f} - \mathbf{n}(\mathbf{f} \cdot \mathbf{n}), \quad \Pi_0 \mathbf{f} = \mathbf{f} - \mathbf{n}_0(\mathbf{f} \cdot \mathbf{n}_0).$$

Along with (4.3), we consider the linear problem

$$(4.5) \quad \begin{cases} \bar{\rho} \mathbf{v}_t - \nabla \cdot T(\mathbf{v}) + p'(\bar{\rho}) \nabla \theta = \mathbf{f}(x, t), \\ \theta_t + \bar{\rho} \nabla \cdot \mathbf{v} = h(x, t), \quad \xi \in \Omega_0^+ \cup \Omega_0^-, \quad t > 0, \\ \mathbf{v}(x, t) = 0, \quad x \in S, \\ [\mathbf{v}] = 0, \quad [-p'(\bar{\rho}) \theta \mathbf{n}_0 + T(\mathbf{u}) \mathbf{n}_0] = \mathbf{b}(x, t), \quad x \in \Gamma_0, \\ \mathbf{v}(x, 0) = \mathbf{v}_0(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in \Omega. \end{cases}$$

The main attention is paid to the analysis of problem (4.5). We state analogs of Theorems 2, 3 and Proposition 1.

Theorem 4. 1. For arbitrary $\mathbf{v}_0 \in W_2^{l+1}(\Omega^\pm)$, $\theta_0 \in W_2^{l+1}(\Omega^\pm)$, $\mathbf{f} \in W_2^{l,l/2}(Q_T^\pm)$, $h \in W_2^{l+1,0}(Q_T^\pm)$, and $\mathbf{b} \in W_2^{l+1/2,l/2+1/4}(G_T)$ satisfying the conditions

$$(4.6) \quad \begin{aligned} \mathbf{v}_0(x) &= 0, \quad x \in S, \\ [\mathbf{v}_0] &= 0, \quad [-p'(\bar{\rho}) \theta_0 \mathbf{n}_0 + T(\mathbf{u}_0) \mathbf{n}_0] = \mathbf{b}(x, 0), \quad x \in \Gamma_0, \end{aligned}$$

problem (4.5) has a unique solution $\mathbf{v} \in W_2^{2+l,1+l/2}(Q_T^\pm)$, $\theta, \theta_t \in W_2^{l+1,0}(Q_T^\pm)$, $T < \infty$, and

$$(4.7) \quad \begin{aligned} & \sum_{\pm} (\|\mathbf{v}\|_{W_2^{l+2,l/2+1}(Q_T^\pm)} + \|\theta\|_{W_2^{l+1,0}(Q_T^\pm)} + \|\theta_t\|_{W_2^{l+1,0}(Q_T^\pm)}) \\ & \leq c(T) \left(\sum_{\pm} (\|\mathbf{v}_0\|_{W_2^{l+1}(\Omega^\pm)} + \|\theta_0\|_{W_2^{l+1}(\Omega^\pm)} + \|\mathbf{f}\|_{W_2^{l,l/2}(Q_T^\pm)} \right. \\ & \quad \left. + \|h\|_{W_2^{l+1,0}(Q_T^\pm)}) + \|\mathbf{b}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \right), \end{aligned}$$

where $G_T = \Gamma_0 \times (0, T)$.

2. Moreover, if

$$(4.8) \quad \int_{\omega} \theta_0(x) dx = 0, \quad \int_{\Omega} h(x, t) dx = 0, \quad t > 0,$$

then the solution is defined for all $t > 0$, the condition $\int_{\Omega} \theta(x, t) dx = 0$ is fulfilled, and

$$(4.9) \quad \begin{aligned} & \sum_{\pm} (\|e^{\beta t} \mathbf{v}\|_{W_2^{l+2,l/2+1}(Q_T^\pm)} + \|e^{\beta t} \theta\|_{W_2^{l+1,0}(Q_T^\pm)} + \|e^{\beta t} \theta_t\|_{W_2^{l+1,0}(Q_T^\pm)}) \\ & \leq c \left(\sum_{\pm} (\|\mathbf{v}_0\|_{W_2^{l+1}(\Omega^\pm)} + \|\theta_0\|_{W_2^{l+1}(\Omega^\pm)} + \|e^{\beta t} \mathbf{f}\|_{W_2^{l,l/2}(Q_T^\pm)} \right. \\ & \quad \left. + \|e^{\beta t} h\|_{W_2^{l+1,0}(Q_T^\pm)}) + \|e^{\beta t} \mathbf{b}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \right) \end{aligned}$$

with $\beta > 0$ and c independent of $T \leq \infty$.

Proposition 3. If (4.8) is true, then the solution of problem (4.5) satisfies the condition $\int_{\Omega} \theta(x, t) dx = 0$, $t \geq 0$, and

$$(4.10) \quad \begin{aligned} & \sup_{t < T} e^{2\beta t} (\|\mathbf{v}(\cdot, t)\|_{L_2(\Omega)}^2 + \|\theta(\cdot, t)\|_{L_2(\Omega)}^2) + \int_0^T e^{2\beta t} (\|\mathbf{v}\|_{W_2^1(\Omega)}^2 + \|\theta\|_{L_2(\Omega)}^2) dt \\ & \leq c \left(\|\mathbf{v}_0\|_{L_2(\Omega)}^2 + \|\theta_0\|_{L_2(\Omega)}^2 + \int_0^T e^{2\beta t} (\|\mathbf{f}\|_{L_2(\Omega)}^2 + \|h\|_{L_2(\Omega)}^2 + \|\mathbf{b}\|_{L_2(\Gamma_0)}^2) dt \right). \end{aligned}$$

Estimate (4.10) is proved in the same way as (2.2). Now we briefly describe the proof of (4.9) by the method of localization.

Let $\{\omega_k\}$ be a covering of Ω of the following form: the ω_k , $k = 1, \dots, m^- - 1$ are the balls $|x - x_k| \leq d$ with $x_k \in S$ (as in §2), the ω_k , $k = 1 + m^-, \dots, m^- + m^+ - 1$, are the balls $|x - x_k| \leq d$ with $x_k \in \Gamma_0$; the ω_{m^-} and $\omega_{m^-+m^+}$ are strictly interior

subdomains of Ω^- and Ω^+ , respectively. Let $\{\varphi_k\}$ be a partition of unity subordinate to this covering, and let ζ_k , $k = 1, \dots, m^- - 1$, $k = m^- + 1, \dots, m^- + m^+ - 1$, be functions with $\text{supp } \zeta_k \subset B_k \subset \omega_k \cap \omega_{m^-}$ or $B_k \subset \omega_k \cap \omega_{m^-+m^+}$ such that $\int_{B_k} \zeta_k(x) dx = 1$, where the B_k are balls with radius αd , $\alpha < 1$. We set $\mathbf{v}_k = \mathbf{v}_{\zeta_k}$, $\theta_k = \theta_{\zeta_k}$, $\omega'_k = \omega_k \cap \Omega$, and

$$\mathbf{w}_k(x, t) = \mathbf{v}_k(x, t) - \zeta_k(x) \int_{\omega'_k} \mathbf{v}_k(y, t) dy, \quad \vartheta_k = \theta_k - \zeta_k \int_{\omega'_k} \theta_k(y, t) dy$$

for $k \neq m^-, m^- + m^+$, and

$$\begin{aligned} \tilde{\mathbf{w}}_{m^-}(x, t) &= \mathbf{v}_{m^-}(x, t) + \sum_{k=1}^{m^- - 1} \zeta_k(x) \int_{\omega'_k} \mathbf{v}_k(y, t) dy, \\ \tilde{\mathbf{w}}_{m^-+m^+} &= \mathbf{v}_{m^-+m^+} + \sum_{k=m^-+1}^{m^-+m^+ - 1} \zeta_k \int_{\omega'_k} \mathbf{v}_k dy, \\ \tilde{\vartheta}_{m^-}(x, t) &= \theta_{m^-}(x, t) + \sum_{k=1}^{m^- - 1} \zeta_k(x) \int_{\omega'_k} \theta_k(y, t) dy, \\ \tilde{\vartheta}_{m^-+m^+} &= \theta_{m^-+m^+} + \sum_{k=m^-+1}^{m^-+m^+ - 1} \zeta_k \int_{\omega'_k} \theta_k(y) dy. \end{aligned}$$

Moreover, let $\tilde{\omega} = \Omega \setminus (\bar{\omega}_{m^-} \cup \bar{\omega}_{m^-+m^+})$, $\tilde{\omega}_{m^-} = \omega_{m^-} \cup \tilde{\omega}$, $\tilde{\omega}_{m^-+m^+} = \omega_{m^-+m^+} \cup \tilde{\omega}$, and let

$$\begin{aligned} \mathbf{w}_{m^-}(x, t) &= \tilde{\mathbf{w}}_{m^-} - \tilde{\zeta}(x) \int_{\tilde{\omega}_{m^-}} \tilde{\mathbf{w}}_{m^-}(y, t) dy, \\ \mathbf{w}_{m^-+m^+}(x, t) &= \tilde{\mathbf{w}}_{m^-+m^+} + \tilde{\zeta}(x) \int_{\tilde{\omega}_{m^-}} \tilde{\mathbf{w}}_{m^-}(y, t) dy, \\ \vartheta_{m^-}(x, t) &= \tilde{\vartheta}_{m^-} - \tilde{\zeta}(x) \int_{\tilde{\omega}_{m^-}} \tilde{\vartheta}_{m^-}(y, t) dy, \\ \vartheta_{m^-+m^+}(x, t) &= \tilde{\vartheta}_{m^-+m^+} + \tilde{\zeta}(x) \int_{\tilde{\omega}_{m^-}} \tilde{\vartheta}_{m^-}(y, t) dy, \end{aligned}$$

where $\tilde{\zeta}(x)$ is a smooth function with $\text{supp } \tilde{\zeta} \subset \tilde{\omega}$ and $\int_{\tilde{\omega}} \tilde{\zeta}(x) dx = 1$. It is clear that

$$\begin{aligned} \int_{\omega'_k} \vartheta_k(x, t) dx &= 0, \quad k \neq m^-, m^- + m^+, \\ \int_{\omega_{m^-}} \tilde{\vartheta}_{m^-} dx + \int_{\omega_{m^-+m^+}} \tilde{\vartheta}_{m^-+m^+} dx &= 0, \\ \int_{\tilde{\omega}_{m^-}} \vartheta_{m^-} dx &= 0, \quad \int_{\tilde{\omega}_{m^-+m^+}} \vartheta_{m^-+m^+} dx = 0. \end{aligned}$$

As in §2, we have

$$\begin{aligned} (4.11) \quad \bar{\rho} \mathbf{w}_{k,t} - \nabla \cdot T(\mathbf{w}_k) + p'(\bar{\rho}) \nabla \vartheta_k &= \mathbf{f}_k + \mathbf{f}_k, \\ \vartheta_{k,t} + \bar{\rho} \nabla \cdot \mathbf{w}_k &= h_k + \mathbf{h}_k, \quad k \neq m^-, m^- + m^+, \end{aligned}$$

$$\begin{aligned} (4.12) \quad \bar{\rho} \mathbf{w}_{m^-,t} - \nabla \cdot T(\mathbf{w}_{m^-}) + p'(\bar{\rho}) \nabla \vartheta_{m^-} &= \mathbf{f}_{m^-} + \mathbf{f}_{m^-}, \\ \vartheta_{m^-,t} + \bar{\rho} \nabla \cdot \mathbf{w}_{m^-} &= h_{m^-} + \mathbf{h}_{m^-}, \end{aligned}$$

$$\begin{aligned} (4.13) \quad \bar{\rho} \mathbf{w}_{m^-+m^+,t} - \nabla \cdot T(\mathbf{w}_{m^-+m^+}) + p'(\bar{\rho}) \nabla \vartheta_{m^-+m^+} &= \mathbf{f}_{m^-+m^+} + \mathbf{f}_{m^-+m^+}, \\ \vartheta_{m^-+m^+,t} + \bar{\rho} \nabla \cdot \mathbf{w}_{m^-+m^+} &= h_{m^-+m^+} + \mathbf{h}_{m^-+m^+}, \end{aligned}$$

where

$$\begin{aligned}
 \mathbf{f}_k &= \mathbf{f}\varphi_k - \zeta_k \int_{\omega'_k} \mathbf{f}\varphi_k dx, \quad h_k = h\varphi_k - \zeta_k \int_{\omega'_k} h\varphi_k dx, \quad k \neq m^-, m^- + m^+, \\
 \tilde{\mathbf{f}}_{m^-} &= \mathbf{f}\varphi_{m^-} + \sum_{k=1}^{m^- - 1} \zeta_k \int_{\omega'_k} \mathbf{f}\varphi_k dx, \quad \tilde{h}_{m^-} = h\varphi_{m^-} + \sum_{k=1}^{m^- - 1} \zeta_k \int_{\omega'_k} h\varphi_k dx, \\
 \tilde{\mathbf{f}}_{m^- + m^+} &= \mathbf{f}\varphi_{m^- + m^+} + \sum_{k=1+m^-}^{m^- + m^+ - 1} \zeta_k \int_{\omega'_k} \mathbf{f}\varphi_k dx, \\
 \tilde{h}_{m^- + m^+} &= h\varphi_{m^- + m^+} + \sum_{k=1+m^-}^{m^- + m^+ - 1} \zeta_k \int_{\omega'_k} h\varphi_k dx, \\
 \mathbf{f}_{m^-}(x, t) &= \tilde{\mathbf{f}}_{m^-}(x, t) - \tilde{\zeta}(x) \int_{\tilde{\omega}_{m^-}} \tilde{\mathbf{f}}_{m^-} dy, \\
 \mathbf{f}_{m^- + m^+}(x, t) &= \tilde{\mathbf{f}}_{m^- + m^+}(x, t) + \tilde{\zeta}(x) \int_{\tilde{\omega}_{m^-}} \tilde{\mathbf{f}}_{m^-} dy, \\
 h_{m^-} &= \tilde{h}_{m^-} - \tilde{\zeta}(x) \int_{\tilde{\omega}_{m^-}} \tilde{h}_{m^-} dy, \quad h_{m^- + m^+} = \tilde{h}_{m^- + m^+} + \tilde{\zeta}(x) \int_{\tilde{\omega}_{m^-}} \tilde{h}_{m^-} dy.
 \end{aligned}
 \tag{4.14}$$

The functions $\mathbf{f}_k, h_k$ involve derivatives of the cut-off functions φ_k, ζ_k and, consequently, only lower order derivatives of $\mathbf{v}$ and θ .

In order to estimate $\mathbf{v}, \theta$ in the interior of $\Omega^\pm$ or in the neighborhood of S , we need to analyze equations (4.12) and (4.13) completed by the initial conditions for $\mathbf{w}_k, \vartheta_k$, or equations (4.11) completed by the initial and boundary conditions $\mathbf{w}_k = 0, x \in S$. This is done as in §2.

To estimate $\mathbf{v}, \theta$ near Γ_0 , we turn to equations (4.11) with $k = m^- + 1, \dots, m^- + m^+ - 1$, completed by the conditions

$$\begin{aligned}
 [\mathbf{w}_k] &= 0, \quad [-p'(\bar{\rho})\vartheta_k \mathbf{n}_0 + T(\mathbf{w}_k)\mathbf{n}_0] = \mathbf{b}_k + \mathbf{b}_k, \quad x \in \Gamma_0, \\
 \mathbf{w}_k(x, 0) &= \mathbf{v}_k(x, 0) - \zeta_k(x) \int_{\omega'_k} \mathbf{v}_k(\eta, 0) d\eta \equiv \mathbf{w}_0^{(k)}(x), \\
 \vartheta_k(x, 0) &= \theta_k(x, 0) - \zeta_k(x) \int_{\omega'_k} \theta_k(\eta, 0) d\eta \equiv \vartheta_0^{(k)}(x)
 \end{aligned}
 \tag{4.15}$$

with $\mathbf{b}_k(x, t) = \mathbf{b}(x, t)\varphi_k(x)$ and $\mathbf{b}_k$ containing lower order derivatives of $\mathbf{v}$ and θ . After “rectifying” Γ_0 near x_k and “freezing” the coefficients at this point, we arrive at the following model problem (an analog of (2.28)):

$$\begin{cases}
 \bar{\rho}\mathbf{w}_t - \nabla \cdot T(\mathbf{w}) + p'(\bar{\rho})\nabla\vartheta = \mathbf{f} + \tilde{\mathbf{f}}, \\
 \vartheta_t + \bar{\rho}\nabla \cdot \mathbf{w} = h + \tilde{h}, \quad y \in \hat{\Omega}, \\
 [\mathbf{w}] = 0, \quad [\mu S_{\alpha 3}(\mathbf{w})] = b_\alpha + \tilde{b}_\alpha, \quad \alpha = 1, 2, \\
 [-p'(\bar{\rho})\vartheta + T_{33}(\mathbf{w})] = b_3 + \tilde{b}_3, \quad y_3 = 0, \quad y' \in \Omega', \\
 \mathbf{w}(y, t) = 0, \quad y_3 = \pm 2d_0, \quad y' \in \Omega', \\
 \mathbf{w}(y, 0) = \mathbf{w}_0(y), \quad \vartheta(y, 0) = \vartheta_0(y), \quad y \in \hat{\Omega},
 \end{cases}
 \tag{4.16}$$

where

$$\hat{\Omega} = \Omega' \times (-2d_0, 2d_0), \quad \Omega' = \{|y_\alpha| \leq d_0, \quad y_3 = 0\}, \quad \alpha = 1, 2.$$

We take the finite differences of (4.16) with respect to y_1, y_2 and, like in §2, obtain the following estimate similar to (2.34):

$$(4.17) \quad \begin{aligned} & \|e^{\beta t} \nabla \mathbf{w}\|_{W_{2,\tan}^{l+1,0}(\hat{\Omega}_T)}^2 + \|e^{\beta t} \mathbf{w}\|_{W_{2,\tan}^{l+1,0}(\hat{\Omega}_T)}^2 + \|e^{\beta t} \vartheta\|_{W_{2,\tan}^{l+1}(\hat{\Omega}_T)}^2 \\ & \leq c(\|e^{\beta t}(\mathbf{f} + \tilde{\mathbf{f}})\|_{W_{2,\tan}^{l,0}(\hat{\Omega}_T)}^2 + \|e^{\beta t}(h + \tilde{h})\|_{W_{2,\tan}^{l+1,0}(\hat{\Omega}_T)}^2 \\ & \quad + \|e^{\beta t}(\mathbf{b} + \tilde{\mathbf{b}})\|_{W_{2,\tan}^{l+1/2,0}(\hat{\Omega}_T)}^2 + \|\mathbf{w}(\cdot, 0)\|_{W_{2,\tan}^{l+1}(\hat{\Omega})}^2 + \|\vartheta(\cdot, 0)\|_{W_{2,\tan}^{l+1}(\hat{\Omega})}^2). \end{aligned}$$

To estimate the derivatives of $\mathbf{w}$ and ϑ with respect to t and y_3 , we treat (4.16) as a union of one-dimensional problems (cf. (2.35), (2.36)):

$$(4.18) \quad \begin{cases} \bar{\rho} w_{\alpha,t} - \mu \frac{\partial^2 w_{\alpha}}{\partial y_3^2} = g_{\alpha}, & |y_3| < 2d_0, \quad \alpha = 1, 2, \\ [w_{\alpha}] = 0, \quad \left[\mu \frac{\partial w_{\alpha}}{\partial y_3} \right] = a_{\alpha}(y', t), & y_3 = 0, \\ w_{\alpha}(y, t) = 0, & y_3 = \pm 2d_0, \\ w_{\alpha}(y, 0) = w_{\alpha 0}(y), & |y_3| \leq 2d_0, \end{cases}$$

$$(4.19) \quad \begin{cases} \bar{\rho} w_{3,t} - (2\mu + \mu_1) \frac{\partial^2 w_3}{\partial y_3^2} + p'(\bar{\rho}) \frac{\partial \vartheta}{\partial y_3} = g_3, \\ \vartheta_t + \bar{\rho} \frac{\partial w_3}{\partial y_3} = e, & |y_3| < 2d_0, \\ [w_3] = 0, \quad \left[-p'(\bar{\rho}) \vartheta + (2\mu + \mu_1) \frac{\partial w_3}{\partial y_3} \right] = a_3, & y_3 = 0, \\ w_3 = 0, & y_3 = \pm 2d_0, \\ w_3(y, 0) = w_{30}(y), \quad \vartheta(y, 0) = \vartheta_0(y), & |y_3| \leq 2d_0. \end{cases}$$

Both w_{α} and (w_3, ϑ) can be constructed and estimated as in §2. Consider problem (4.19). We introduce the function $u_1(x, t) = u_1^{\pm}(x, t)$, $x \in I^{\pm} = \{\pm x_3 \in (0, 2d_0)\}$ satisfying the conditions

$$u_1^{\pm}(x, 0) = w_{3,0}^{\pm}(x), \quad x_3 \in I^{\pm}, \quad u_1^{\pm}(x, t) = 0, \quad x_3 = \pm 2d_0$$

$u_1^{\pm}(x, t) = 0$ for $t > 2$, and the inequality

$$\|u_1^{\pm}\|_{W_2^{l+2, l/2+1}(I^{\pm} \times (0, 2))} \leq c \|w_{3,0}\|_{W_2^{l+1}(I^{\pm})}.$$

It follows that

$$\sum_{\pm} \|u_1^{\pm}\|_{L_2(\Omega', W_2^{2+l, 1+l/2}(I^{\pm} \times (0, 2)))} \leq \|w_3\|_{L_2(\Omega', W_2^{1+l}(I^{\pm}))}.$$

The function $\vartheta^{(1)}(x, t) = \vartheta_0(x) \chi(t)$ satisfies

$$\sum_{\pm} (\|\vartheta^{(1)}\|_{W_2^{l+1,0}(I^{\pm} \times (0, 2))} + \|\vartheta_t^{(1)}\|_{W_2^{l+1,0}(I^{\pm} \times (0, 2))}) \leq c \sum_{\pm} \|\vartheta_0\|_{W_2^{l+1}(I^{\pm})}.$$

For the differences $u_2 = w_3 - u_1$, $\vartheta^{(2)} = \vartheta - \vartheta^{(1)}$ we obtain the problem

$$(4.20) \quad \begin{cases} \bar{\rho} u_{2,t} - (2\mu + \mu_1) \frac{\partial^2 u_2}{\partial y_3^2} + p'(\bar{\rho}) \frac{\partial \vartheta^{(2)}}{\partial y_3} = g_3^{(1)}, \\ \vartheta_t^{(2)} + \bar{\rho} \frac{\partial u_2}{\partial y_3} = e^{(1)}, & |y_3| < 2d_0, \\ [w_3] = 0, \quad \left[-p'(\bar{\rho}) \vartheta^{(2)} + (2\mu + \mu_1) \frac{\partial u_2}{\partial y_3} \right] = a_3^{(1)}, & y_3 = 0, \\ w_3 = 0, & y_3 = \pm 2d_0, \\ u_2(y, 0) = 0, \quad \vartheta^{(2)}(y, 0) = 0, & |y_3| \leq 2d_0, \end{cases}$$

with $g_3^{(1)}$, $e^{(1)}$, $a^{(1)}$ satisfying

$$\begin{aligned} & \sum_{\pm} (\|e^{\beta t} g^{(1)}\|_{W_2^{l,l/2}(I^{\pm} \times (0,T))} + \|e^{\beta t} e^{(1)}\|_{W_2^{l+1,0}(I^{\pm} \times (0,T))}) + \|e^{\beta t} a^{(1)}\|_{W_2^{l/2+1/4}(0,T)} \\ & \leq c \left(\sum_{\pm} (\|e^{\beta t} g\|_{W_2^{l,l/2}(I^{\pm} \times (0,T))} + \|e^{\beta t} e\|_{W_2^{l+1,0}(I^{\pm} \times (0,T))}) + \|e^{\beta t} a\|_{W_2^{l/2+1/4}(0,T)} \right. \\ & \quad \left. + \sum_{\pm} (\|e^{\beta t} u_1\|_{W_2^{l+1,1+l/2}(I^{\pm} \times (0,T))} + \|e^{\beta t} \vartheta^{(1)}\|_{W_2^{l+1,0}(I^{\pm} \times (0,T))}) \right. \\ & \quad \left. + \|e^{\beta t} \vartheta_t^{(1)}\|_{W_2^{l+1,0}(I^{\pm} \times (0,T))}) \right). \end{aligned}$$

By applying the Laplace transformation and eliminating $\vartheta^{(2)}$, we convert (4.20) into the Sturm–Liouville type problem similar to (2.41):

$$(4.21) \quad \begin{cases} R(s)\tilde{u}_2 - \frac{d^2 \tilde{u}_2}{dy_3^2} = \tilde{G}, & y_3 \in I^{\pm}, \\ [\tilde{u}_2] = 0, & \left[\lambda(s) \frac{d\tilde{u}_2}{dy_3} \right] = \tilde{A}, \\ u_2 = 0, & y_3 = \pm 2d_0, \end{cases}$$

where $\lambda^{\pm}(s) = \frac{a^{\pm}s+b}{a_0s+b}$, $a_0 > 0$,

$$R(s) = R^{\pm}(s) = \frac{\bar{\rho}s^2}{a^{\pm}s+b}, \quad a^{\pm} = 2\mu^{\pm} + \mu_1^{\pm}, \quad b = \bar{\rho}p'(\bar{\rho}),$$

and G and A are functions satisfying

$$\begin{aligned} & \sum_{\pm} \|e^{\beta t} G\|_{W_2^{l,l/2}(I^{\pm} \times (0,T))}^2 + \|e^{\beta t} A\|_{W_2^{1/2+l/2}(0,T)} \\ & \leq c \left(\sum_{\pm} (\|e^{\beta t} g_3^{(1)}\|_{W_2^{l,l/2}(I^{\pm} \times (0,T))}^2 + \|e^{\beta t} e^{(1)}\|_{W_2^{l+1,0}(I^{\pm} \times (0,T))}) + \|e^{\beta t} a_3^{(1)}\|_{W_2^{l/2+1/4}(0,T)} \right). \end{aligned}$$

The solution of (4.21) can be expressed as the sum $\tilde{u}_2 = \tilde{u}_3 + \tilde{u}_4$, with $\tilde{u}_3$, $\tilde{u}_4$ defined by

$$(4.22) \quad R(s)\tilde{u}_3 - \frac{d^2 \tilde{u}_3}{dy_3^2} = \tilde{G}, \quad y_3 \in I^{\pm}, \quad \tilde{u}_3 = 0, \quad y_3 = 0, \pm 2d_0,$$

and

$$(4.23) \quad \begin{cases} R(s)\tilde{u}_4 - \frac{d^2 \tilde{u}_4}{dy_3^2} = 0, & y_3 \in I^{\pm}, \\ \tilde{u}_4 = 0, & y_3 = 0, \pm 2d_0, \\ [\tilde{u}_4] = 0, & \left[\lambda(s) \frac{d\tilde{u}_4}{dy_3} \right] = \tilde{A}^{(1)} = \tilde{A} - \left[\lambda(s) \frac{d\tilde{u}_3}{dy_3} \right], \quad y_3 = 0. \end{cases}$$

The functions $\tilde{u}_3^{\pm}$ are given by a formula similar to (2.43), and

$$(4.24) \quad \begin{aligned} \tilde{u}_4(y_3) &= C_1(e^{r^+(2d_0-y_3)} - e^{-r^+(2d_0-y_3)}), \quad y_3 \in I^+, \\ \tilde{u}_4(y_3) &= C_2(e^{-r^-(2d_0+y_3)} - e^{r^-(2d_0+y_3)}), \quad y_3 \in I^-, \\ C_1 &= -D^{-1}(e^{2d_0r^-} - e^{-2d_0r^-})\tilde{A}^{(1)}, \\ C_2 &= -D^{-1}(e^{2d_0r^+} - e^{-2d_0r^+})\tilde{A}^{(1)}, \end{aligned}$$

where $r^{\pm}(s) = \pm\sqrt{R(s)}$ and

$$D = \lambda^- r^- (e^{-2d_0r^-} + e^{2d_0r^-}) (e^{2d_0r^+} - e^{-2d_0r^+}) + \lambda^+ r^+ (e^{2d_0r^-} - e^{-2d_0r^-}) (e^{2d_0r^+} + e^{-2d_0r^+}).$$

It can be shown that

$$\sum_{\pm} \|e^{\beta t} u_3\|_{W_2^{2+l,1+l/2}(I^{\pm} \times (0,T))}^2 \leq c \sum_{\pm} \|e^{\beta t} G\|_{W_2^{l,l/2}(I^{\pm} \times (0,T))}^2,$$

$$\sum_{\pm} \|e^{\beta t} u_4\|_{W_2^{2+l,1+l/2}(I^{\pm} \times (0,T))}^2 \leq c \left(\|e^{\beta t} A\|_{W_2^{1/4+l/2}(0,T)}^2 + \sum_{\pm} \|e^{\beta t} u_3\|_{W_2^{2+l,1+l/2}(I^{\pm} \times (0,T))}^2 \right).$$

The functions w_{α} are estimated in a similar way, so finally we arrive at the inequality

$$\begin{aligned} \sum_{\pm} \left(\|e^{\beta t} \mathbf{w}_k\|_{W_2^{2+l,1+l/2}(\Omega_T^{\pm})}^2 + \|e^{\beta t} \vartheta_k\|_{W_2^{1+l,0}(\Omega_T^{\pm})}^2 + \|e^{\beta t} \vartheta_{k,t}\|_{W_2^{1+l,0}(\Omega_T^{\pm})}^2 \right) \\ \leq \sum_{\pm} \left(\|e^{\beta t} \mathbf{g}_k\|_{W_2^{l,l/2}(\Omega_T^{\pm})}^2 + \|e^{\beta t} e_k\|_{W_2^{1+l,0}(\Omega_T^{\pm})}^2 + \|e^{\beta t} \mathbf{a}_k\|_{W_2^{l+1/2,l/2+1/4}(\Omega_T^{\pm})}^2 \right. \\ \left. + \|\mathbf{w}_k(\cdot, 0)\|_{W_2^{l+1}(\Omega^{\pm})}^2 + \|\vartheta_k(\cdot, 0)\|_{W_2^{l+1}(\Omega^{\pm})}^2 \right) \end{aligned}$$

for the solution $(\mathbf{w}_k, \vartheta_k)$ of (4.18), (4.19). Taking account of the interior estimates of $(\mathbf{w}_k, \vartheta_k)$ and estimates near S , we complete the proof of (4.9).

We restrict ourselves to the above comments concerning Theorem 4. Because of the lack of space, we omit the arguments related to the estimates of nonlinear terms (4.4) and construction of the solution of problem (4.3). They are similar to the corresponding arguments in §3. As a result, we prove the following theorem.

Theorem 5. Suppose $S, \Gamma_0 \in W_2^{3/2+l}$, $1/2 < l < 1$, $p \in C^2(\bar{\rho}/2, 3\bar{\rho}/2)$, $\bar{\rho}, p'(\bar{\rho}) > 0$. For arbitrary $\mathbf{v}_0 \in W_2^{l+1}(\Omega_0^{\pm})$ and $\rho_0 \in W_2^{l+1}(\Omega_0^{\pm})$ satisfying the conditions

$$(4.25) \quad \begin{aligned} \mathbf{v}_0(x, 0) = 0, \quad x \in S, \quad \int_{\Omega_0} \theta_0(x) dx = 0, \\ [\mathbf{v}_0] = 0, \quad [-p(\rho_0)\mathbf{n}_0 + T(\mathbf{u}_0)\mathbf{n}_0] = 0, \quad x \in \Gamma_0, \\ |\rho_0| \geq c_0 > 0, \end{aligned}$$

$$(4.26) \quad \sum_{\pm} \|\mathbf{v}_0\|_{W_2^{l+1}(\Omega^{\pm})} + \sum_{\pm} \|\theta_0^{\pm}\|_{W_2^{l+1}(\Omega^{\pm})} \leq \epsilon \ll 1,$$

problem (4.3) has a unique solution defined for $t \geq 0$, and

$$(4.27) \quad \begin{aligned} \sum_{\pm} \left(\|e^{\beta t} \mathbf{v}\|_{W_2^{l+2,l/2+1}(Q_{\infty}^{\pm})} + \|e^{\beta t} \theta^{\pm}\|_{W_2^{l+1,0}(Q_{\infty}^{\pm})} + \|e^{\beta t} \theta_t^{\pm}\|_{W_2^{l+1,0}(Q_{\infty}^{\pm})} \right) \\ \leq c \sum_{\pm} \left(\|\mathbf{v}_0\|_{W_2^{l+1}(\Omega^{\pm})} + \|\theta_0\|_{W_2^{l+1}(\Omega^{\pm})} \right). \end{aligned}$$

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Received 2/DEC/2014
 Originally published in English