

THE JOHN–NIRENBERG CONSTANT OF BMO^p , $p > 2$

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ABSTRACT. This paper is a continuation of earlier work by the first author who determined the John–Nirenberg constant of $BMO^p((0,1))$ for the range $1 \leq p \leq 2$. Here, that constant is computed for $p > 2$. As before, the main results rely on Bellman functions for the L^p norms of the logarithms of A_∞ weights, but for $p > 2$ these functions turn out to have a significantly more complicated structure than for $1 \leq p \leq 2$.

§1. PRELIMINARIES AND MAIN RESULTS

For a finite interval J and a function $\varphi \in L^1(J)$, let $\langle \varphi \rangle_J$ denote the average of φ over J with respect to Lebesgue measure, $\langle \varphi \rangle_J = \frac{1}{|J|} \int_J \varphi$. Take an interval Q and $p > 0$, and let $BMO(Q)$ be the (factor-)space

$$(1.1) \quad BMO(Q) = \{\varphi \in L^1(Q) : \|\varphi\|_{BMO^p(Q)} := \sup_{\text{interval } J \subset Q} \langle |\varphi - \langle \varphi \rangle_J|^p \rangle_J^{1/p} < \infty\}.$$

The classical fact that all p -based (quasi)norms are equivalent justifies omitting the index p on the left-hand side.

A weight is a function that is positive almost everywhere. We say that a weight w belongs to $A_\infty(Q)$, $w \in A_\infty(Q)$, if both w and $\log w$ are integrable on Q and the following condition is fulfilled:

$$[w]_{A_\infty(Q)} := \sup_{\text{interval } J \subset Q} \langle w \rangle_J e^{-\langle \log w \rangle_J} < \infty.$$

The quantity $[w]_{A_\infty(Q)}$ is called the $A_\infty(Q)$ -characteristic of w . When Q is fixed or not important, we write simply BMO for $BMO(Q)$ and A_∞ for $A_\infty(Q)$.

BMO functions are integrable locally exponentially. We can state this property in the form of the so-called integral John–Nirenberg inequality, which is a version of the classical weak-type inequality proved in [5].

Theorem (John–Nirenberg). *For every $p > 0$, there exists a number $\varepsilon_0(p) > 0$ such that if $\varepsilon \in [0, \varepsilon_0(p))$, Q is an interval, and $\varphi \in BMO(Q)$ with $\|\varphi\|_{BMO^p(Q)} \leq \varepsilon$, then there is a number $C(\varepsilon, p) > 0$ such that for any interval $J \subset Q$ we have*

$$(1.2) \quad \langle e^\varphi \rangle_J \leq C(\varepsilon, p) e^{\langle \varphi \rangle_J}.$$

We shall always use $\varepsilon_0(p)$ to denote the best – largest possible – constant in this theorem and call it the John–Nirenberg constant of BMO^p (on an interval). Similarly, $C(\varepsilon, p)$ will denote the smallest possible constant in (1.2).

Observe that (1.2) means that if $\varphi \in BMO$, then $e^{\varepsilon\varphi} \in A_\infty$ for all sufficiently small $\varepsilon > 0$. For $\varphi \in BMO$, let

$$(1.3) \quad \varepsilon_\varphi = \sup\{\varepsilon : e^{\varepsilon\varphi} \in A_\infty\}.$$

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In fact, it can be shown that

$$\varepsilon_0(p) = \inf\{\varepsilon_\varphi : \|\varphi\|_{\text{BMO}^p} = 1\} = \sup\{\varepsilon : \forall \varphi, \|\varphi\|_{\text{BMO}^p} = 1 \implies e^{\varepsilon\varphi} \in A_\infty\}.$$

In this paper, our goal is to compute $\varepsilon_0(p)$ for the case where $p > 2$. Here are some previous results in that direction: Korenovskii [6] and Lerner [7] computed the analogs of $\varepsilon_0(1)$ and $C(\varepsilon, 1)$, respectively for the weak-type John–Nirenberg inequality; in [9], we determined $\varepsilon_0(2)$ and $C(\varepsilon, 2)$; in [12], the second author and A. Volberg found all constants in the weak-type inequality for $p = 2$; and, finally, in [8], the first author determined $\varepsilon_0(p)$ for $p \in [1, 2]$ (including new proofs for $p = 1$ and $p = 2$) and $C(\varepsilon, p)$ for $p \in (1, 2]$ and sufficiently large ε . This last paper built the framework that we follow here, and we refer the reader to it for an in-depth discussion of the tools involved and the differences between the cases of $p = 2$ and $p \neq 2$.

Let us state the relevant theorem from [8].

Theorem 1.1 ([8]). *For $p \in [1, 2]$,*

$$\varepsilon_0(p) = \left[\frac{p}{e} \left(\Gamma(p) - \int_0^1 t^{p-1} e^t dt \right) + 1 \right]^{1/p}.$$

Furthermore, if $1 < p \leq 2$, then for all $\varepsilon \in [(2-p)\varepsilon_0(p), \varepsilon_0(p))$ we have

$$(1.4) \quad C(\varepsilon, p) = \frac{e^{-\varepsilon/\varepsilon_0(p)}}{1 - \varepsilon/\varepsilon_0(p)};$$

and for all $0 \leq \varepsilon < \frac{2}{e}$ we have

$$(1.5) \quad \frac{e^{-\frac{\varepsilon}{2}}}{1 - \frac{\varepsilon}{2}} \leq C(\varepsilon, 1) \leq \frac{1}{1 - \frac{\varepsilon}{2}}.$$

We can finally complete the picture for all $p \geq 1$. Remarkably, the formula for $\varepsilon_0(p)$ in the case of $p > 2$ is the same as for $1 \leq p \leq 2$, though it takes much more work to show.

Theorem 1.2. *For $p > 2$,*

$$(1.6) \quad \varepsilon_0(p) = \left[\frac{p}{e} \left(\Gamma(p) - \int_0^1 t^{p-1} e^t dt \right) + 1 \right]^{1/p}.$$

In contrast to the case where $1 < p \leq 2$, for $p > 2$ we do not know the exact $C(\varepsilon, p)$ for any ε . While we could estimate this constant in a manner somewhat similar to (1.5), the estimates we currently have seem much too implicit to be useful, so we omit them.

Without entering into details, we mention an important difference between the cases of $p \leq 2$ and $p > 2$. It was shown in [8] that the constant $\varepsilon_0(p)$ is attained in the weak-type John–Nirenberg inequality for $1 < p \leq 2$ (the case where $p = 1$ was treated in [6] and [7], while the case where $p = 2$ had been previously addressed in [12]). However, the method used to show this fact for $p \leq 2$ fails for $p > 2$, and we do not actually know if the constant is attained (though we conjecture that it is).

On the other hand, another interesting result from [8] does go through for $p > 2$. Specifically, we have the following theorem, which extends to $p > 2$ the main result of Corollary 1.5 from [8]. It is a sharp lower estimate for the distance in BMO to L^∞ in the spirit of Garnett and Jones [1].

Theorem 1.3. *If $p > 2$, Q is an interval, and $\varphi \in \text{BMO}(Q)$, then*

$$(1.7) \quad \inf_{f \in L^\infty(Q)} \|\varphi - f\|_{\text{BMO}^p(Q)} \geq \frac{\varepsilon_0(p)}{\min\{\varepsilon_\varphi, \varepsilon_{-\varphi}\}},$$

and this inequality is sharp.

As was explained in [8], the main idea behind computing $\varepsilon_0(p)$ for $p \neq 2$ is to consider the dual problem: instead of estimating the values of $\|\varphi\|_{BMO^p}$ for which the exponential oscillation $\langle e^{\varphi - \langle \varphi \rangle} \rangle$ might become unbounded, one estimates from below the BMO^p oscillations of the logarithms of A_∞ weights and computes their asymptotics as the A_∞ -characteristic goes to infinity. This idea is formalized in the following general theorem.

Fix $p > 0$. For $C \geq 1$, let

$$(1.8) \quad \Omega_C = \{x \in \mathbb{R}^2 : e^{x_1} \leq x_2 \leq C e^{x_1}\}.$$

For an interval Q and every $x = (x_1, x_2) \in \Omega_C$, let

$$(1.9) \quad E_{x,C,Q} = \{\varphi \in L^1(Q) : \langle \varphi \rangle_Q = x_1, \langle e^\varphi \rangle_Q = x_2, [e^\varphi]_{A_\infty(Q)} \leq C\}.$$

The elements of $E_{x,C,Q}$ will be called test functions. Define the following lower Bellman function:

$$(1.10) \quad \mathbf{b}_{p,C}(x) = \inf \{ \langle |\varphi|^p \rangle_Q : \varphi \in E_{x,C,Q} \}.$$

Theorem 1.4 ([8]). *Take $p > 0$. Assume that there exists a family of functions $\{b_C\}_{C \geq 1}$ such that for each C , b_C is defined on Ω_C , $b_C \leq \mathbf{b}_{p,C}$, and $b_C(0, \cdot)$ is continuous on the interval $[1, C]$. Then*

$$(1.11) \quad \varepsilon_0^p(p) \geq \limsup_{C \rightarrow \infty} b_C(0, C).$$

Thus, to estimate $\varepsilon_0(p)$, we need a suitable family $\{b_C\}_C$ of minorants of $\mathbf{b}_{p,C}$. Precisely as was done in [8], we actually find the functions $\mathbf{b}_{p,C}$ themselves, for all $p > 2$ and all sufficiently large C . We proceed as follows: in §2, we construct the so-called Bellman candidate, denoted by $b_{p,C}$. This construction is subtler and more technical than that in [8], and we briefly discuss the challenges involved. The proof that $b_{p,C} \leq \mathbf{b}_{p,C}$ constitutes §3. It is then an easy matter to prove Theorems 1.2 and 1.3, and it is taken up in §4. Finally, in §5, we obtain the reverse inequality by demonstrating explicit test functions that realize the infimum in (1.10).

§2. THE CONSTRUCTION OF THE BELLMAN CANDIDATE

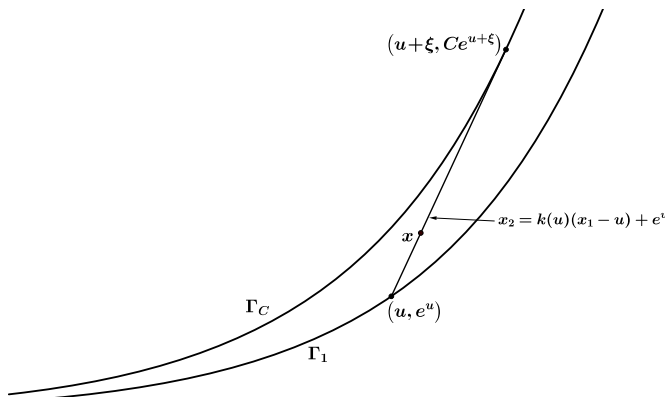
For $R > 0$, let

$$\Gamma_R = \{x \in \mathbb{R}^2 : x_2 = R e^{x_1}\}.$$

Then the domain Ω_C defined in (1.8) is the plane region lying between Γ_1 and Γ_C .

2.1. Discussion and preliminaries. As has been mentioned earlier, the construction of the Bellman candidate given here for $p > 2$ is more involved than those presented in [8] for $p = 1$ and $p \in (1, 2]$. However, our main goal is the same as before: we are building the largest locally convex function b on Ω_C that satisfies the boundary condition $b(x_1, e^{x_1}) = |x_1|^p$.

We briefly explain the similarities and differences between the cases where $p \in (1, 2]$ and $p > 2$ (the case of $p = 1$ is different from both). In all cases, the graph of the candidate b is a convex ruled surface, which means that a straight-line segment contained in the graph passes through each point on the graph. The domain Ω_C then splits into a collection of subdomains with disjoint interiors, $\Omega_C = \bigcup_j R_j$, such that b is twice differentiable and satisfies the homogeneous Monge–Ampère equation $b_{x_1 x_1} b_{x_2 x_2} = b_{x_1 x_2}^2$ in the interior of each R_j . Moreover, for each subdomain R_j , either b is affine in the entire R_j , or R_j is foliated by straight-line segments connecting two points of the boundary $\Gamma_1 \cup \Gamma_C$, and each point $x \in \text{int}(R_j)$ lies on only one such segment. We call such segments Monge–Ampère characteristics of b . Typically, if one knows the characteristics everywhere in Ω_C , one knows the function b .

FIGURE 1. The geometric meaning of $u(x)$ and ξ .

Thus, to construct a candidate one has to understand how to split Ω_C into subdomains and how to foliate each of them so that the resulting function b be locally convex. If this is done and certain compatibility conditions are ensured, then b will almost automatically be the *largest* locally convex function with the given boundary conditions, as desired. However, in general, this is a difficult task, and the situation is further complicated by the fact that the splitting in question is usually different for different C .

Fortunately, at present there is a fairly general theory for constructing such foliations on special nonconvex domains such as ours. Started in [10] in the context of BMO^2 , it was much developed and systematized in [2] and [3] (still for the parabolic strip of BMO^2); now it is adapted to general domains like Ω_C , see [4]. We also mention the recent paper [11], which formalized a theoretical link between the Bellman functions and the smallest locally concave (or largest locally convex, as is our case) functions on the corresponding domains.

A key building block for many Monge–Ampère foliations is the tangential foliation. Let us explain this notion in our setting.

For $C \geq 1$, let $\xi = \xi(C)$ be a unique nonnegative solution of the equation

$$e^{-\xi} = C(1 - \xi), \quad 0 \leq \xi < 1.$$

Note that $\xi(1) = 0$ and that ξ is strictly monotone increasing with $\lim_{C \rightarrow \infty} \xi(C) = 1$. Let

$$(2.1) \quad k(z) = \frac{e^z}{1 - \xi}, \quad z \in \mathbb{R},$$

and define a new function $u = u(x)$ on Ω_C by the implicit formula

$$(2.2) \quad x_2 = k(u)(x_1 - u) + e^u.$$

This function has a simple geometrical meaning illustrated in Figure 1: if one draws the one-sided tangent to Γ_C that passes through x so that the tangency point is to the right of x , then this tangent intersects Γ_1 at the point (u, e^u) , while the tangency point is $(u + \xi, Ce^{u+\xi})$. In particular $u(0, C) = -\xi$. (We note that, in [8], ξ and u were called ξ^+ and u^+ , respectively.)

In the case where $1 < p \leq 2$, for sufficiently large C , all of Ω_C was foliated by the tangents (2.2) for $u \in (-\infty, \infty)$; thus, there was no need to split it into subdomains. However, for $p > 2$, this uniform tangential foliation fails to yield a locally convex function on the entire Ω_C , for any C . What actually happens — and, again, only for sufficiently large C — is shown in Figure 2 later in this section. There we have two

tangentially foliated subdomains, R_1 and R_3 , linked by a special “transition regime” consisting of two more subdomains: one is R_2 , where the candidate is affine and the foliation is thus degenerate, and the second is R_4 , where the characteristics are chords connecting two points of Γ_1 . (In recent Bellman-function literature, these two particular shapes are called “trolleybus” and “cup”, respectively; see [2, 3, 4].) This transition regime shrinks as C grows, but never disappears. To show how all this fits together, we need some technical preparation.

2.2. Technical lemmas.

Lemma 2.1.

(1) If $w > 0$ and $v \in (-w, -w \frac{p-1}{p})$, then

$$(2.3) \quad \frac{w^{p-1} + (-v)^{p-1}}{e^w - e^v} < (p-1)(-v)^{p-2}e^{-v}.$$

(2) If $0 < w \leq \frac{p-2}{p-1}$ and $v \in (-w, 0)$, then

$$(2.4) \quad \frac{w^{p-1} + (-v)^{p-1}}{e^w - e^v} < (p-1)w^{p-2}e^{-w}.$$

Proof. For statement (1), note that $e^{w-v} - 1 \geq w - v > 0$, and so it suffices to check that

$$w^{p-1} + (-v)^{p-1} < (p-1)(-v)^{p-2}(w-v).$$

Put $\theta = -\frac{v}{w}$; then this inequality becomes

$$(p-2)\theta^{p-1} + (p-1)\theta^{p-2} - 1 > 0, \quad \frac{p-1}{p} < \theta < 1.$$

The left-hand side is monotone increasing in θ , and equals $2\frac{(p-1)^p}{p^{p-1}} - 1$ when $\theta = \frac{p-1}{p}$. In its turn, this is a monotone increasing function of p , equal to 0 at $p = 2$.

For (2), observe that, since $w > -v$ and $1 - w \geq 1 - \frac{p-2}{p-1} = \frac{1}{p-1}$, we have

$$1 - e^{-(w-v)} > (w-v) \left[1 - \frac{1}{2}(w-v) \right] > (w-v)(1-w) \geq \frac{w-v}{p-1},$$

and (2.4) follows from the obvious relation $w^{p-1} + (-v)^{p-1} < w^{p-2}(w-v)$. \square

For any $v < 0$ and $w > 0$, put

$$(2.5) \quad r(v, w) = \frac{e^w - e^v}{w - v}, \quad q(v, w) = \frac{w^p - (-v)^p}{w - v}.$$

Lemma 2.2. For each $w \in (0, \frac{p-2}{3p})$, there exists a unique $v \in (-w, -w \frac{p-1}{p})$ such that

$$(2.6) \quad \frac{q(v, w) + p(-v)^{p-1}}{r(v, w) - e^v} = \frac{pw^{p-1} - q(v, w)}{e^w - r(v, w)} = p \frac{w^{p-1} + (-v)^{p-1}}{e^w - e^v}.$$

Proof. Observe that it suffices to show only the first identity in (2.6), because then the second follows by elementary rearrangement. In its turn, the first identity is equivalent to the relation

$$(2.7) \quad F(v, w) := (e^w - e^v)(w^p - (-v)^p - pw^{p-1} - p(-v)^{p-1}) + p(w-v)(w^{p-1}e^v + (-v)^{p-1}e^w) = 0.$$

Assume that $w \in (0, \frac{p-2}{3p})$ and put $\lambda = \frac{p-1}{p}$. To show that there exists $v \in (-w, -\lambda w)$ such that $F(v, w) = 0$, we compare the signs of $F(-w, w)$ and $F(-\lambda w, w)$.

Since $F(-w, w) = 4pw^{p-1}(w \cosh w - \sinh w) > 0$, we want to check that $F(-\lambda w, w) < 0$. Since

$$F(-\lambda w, w) = (e^w - e^{-\lambda w})[(1 - \lambda^p)w^p - p(1 + \lambda^{p-1})w^{p-1}] + p(1 + \lambda)w^p[e^{-\lambda w} + \lambda^{p-1}e^w],$$

the inequality $F(-\lambda w, w) < 0$ is equivalent to

$$\frac{(1+\lambda)w}{e^{(1+\lambda)w}-1} - 1 + \frac{1+p\lambda^{p-1}+(p-1)\lambda^p}{p(1+\lambda^{p-1})}w < 0.$$

Let

$$\psi(t) = \frac{t}{e^t-1} - 1 + \frac{1+p\lambda^{p-1}+(p-1)\lambda^p}{p(1+\lambda)(1+\lambda^{p-1})}t.$$

We shall show that $\psi(t) < 0$ for $t \in (0, \frac{p-2}{3p}(1+\lambda))$. Note that, for $t > 0$,

$$e^t > 1+t+\frac{1}{2}t^2 \implies \frac{t}{e^t-1} - 1 < -\frac{t}{t+2}.$$

Therefore, for $t > 0$ we have

$$\psi(t) < -t \left[\frac{1}{t+2} - \frac{1+p\lambda^{p-1}+(p-1)\lambda^p}{p(1+\lambda)(1+\lambda^{p-1})} \right],$$

and it suffices to check that

$$\frac{p-2}{3p}(1+\lambda) < \frac{p(1+\lambda)(1+\lambda^{p-1})}{1+p\lambda^{p-1}+(p-1)\lambda^p} - 2 = \frac{(p-2)(1-\lambda^p)+p\lambda(1-\lambda^{p-2})}{1+p\lambda^{p-1}+(p-1)\lambda^p}.$$

Since $(p-2)(1-\lambda^p)+p\lambda(1-\lambda^{p-2}) > (p-2)(1+\lambda)(1-\lambda^{p-1})$ and $\lambda^p < \lambda^{p-1}$, it suffices to verify that

$$3 > \frac{1/p + (2-1/p)\lambda^{p-1}}{1-\lambda^{p-1}} = \frac{2}{1-\lambda^{p-1}} - 2 + 1/p.$$

This is true because the right-hand side is monotone decreasing in p and equals $\frac{5}{2}$ when $p=2$. This proves that the desired v exists for each w .

To show that v is unique, we differentiate the function F with respect to v . This derivative can be written as follows:

$$\begin{aligned} F_v(v, w) &= (e^w(w-v) - e^w + e^v)e^v \left(\frac{pw^{p-1}(w-v) - w^p + (-v)^p}{e^w(w-v) - e^w + e^v} - p(p-1)(-v)^{p-2}e^{-v} \right) \\ &= (e^w(w-v) - e^w + e^v)e^v \left(p \frac{w^{p-1} + (-v)^{p-1}}{e^w - e^v} - p(p-1)(-v)^{p-2}e^{-v} \right), \end{aligned}$$

where we have used the second identity in (2.6). Now, the first factor is positive, because the function $t \mapsto e^t$ is strictly convex, while the last factor is negative by (2.3). Therefore, $F_v(v, w)$ is negative for any root v of the equation $F(v, w) = 0$ that lies in the interval $(-w, -w\frac{p-1}{p})$, which is possible only when such a root is unique. \square

From now on, when we use v and w , it is always presumed that $w \in (0, \frac{p-2}{3p})$, $v \in (-w, -w\frac{p-1}{p})$, and the pair $\{v, w\}$ satisfies (2.6). For such v and w , each of the three equal quantities in (2.6) is a function of w , and it is convenient to give them a common name. Let

$$(2.8) \quad D(w) = \frac{q(v, w) + p(-v)^{p-1}}{r(v, w) - e^v} = \frac{pw^{p-1} - q(v, w)}{e^w - r(v, w)} = p \frac{w^{p-1} + (-v)^{p-1}}{e^w - e^v};$$

then D is a function of w defined on the interval $(0, \frac{p-2}{3p})$. We list some of its properties.

Lemma 2.3. *We have*

$$(2.9) \quad D(w) < p(p-1)(-v)^{p-2}e^{-v}$$

and

$$(2.10) \quad D(w) < p(p-1)w^{p-2}e^{-w}.$$

Furthermore, $D' > 0$ on $(0, \frac{p-2}{3p})$.

Proof. Inequalities (2.9) and (2.10) come directly from (2.3) and (2.4), respectively (note that $\frac{p-2}{3p} < \frac{p-2}{p-1}$, so (2.4) applies).

To check the sign of D' , we shall treat q and r as functions of w and use the prime sign to indicate the total derivative with respect to w . Thus, $q' = q_w + q_v v_w$ and $r' = r_w + r_v v_w$, where v_w can be computed with the help of (2.7). Also, we denote $f(w) := w^p$ and $g(w) := e^w$.

We need the following simple but key fact: equation (2.6) can be written as

$$(2.11) \quad \frac{q'}{r'} = \frac{q - f'}{r - g'} = D.$$

Using this identity, we see that

$$\begin{aligned} D' &= \left(\frac{q - f'}{r - g'} \right)' = \frac{(q' - f'')(r - g') - (q - f')(r' - g'')}{(r - g')^2} \\ &= \frac{g''(q - f') - f''(r - g')}{(r - g')^2} = \frac{g''}{r - g'} \left(\frac{q - f'}{r - g'} - \frac{f''}{g''} \right). \end{aligned}$$

Since g is strictly convex, we have $g'' > 0$ and $r - g' < 0$. On the other hand, the expression in parentheses is negative by (2.10). \square

For $p > 2$, let

$$(2.12) \quad \xi_0(p) = 1 - \frac{1}{3^{p+2}\Gamma(p)}, \quad C_0(p) = \frac{e^{-\xi_0(p)}}{1 - \xi_0(p)}.$$

Lemma 2.4. *Assume that $\xi > \xi_0(p)$. Let*

$$c_1 = \xi [e(1 - \xi)\Gamma(p - 1)]^{1/(p-2)}, \quad c_2 = \xi [2e(1 - \xi)\Gamma(p)]^{1/(p-2)}.$$

Then the equation

$$(2.13) \quad \left(\frac{1}{\xi} - 1 \right) \int_w^\infty s^{p-2} e^{-s/\xi} ds - w^{p-2} e^{-w/\xi} = 0$$

has a unique solution w_ in the interval $(0, c_1)$.*

Furthermore, the equation

$$(2.14) \quad \left(\frac{1}{\xi} - 1 \right) p(p - 1) e^{w(1/\xi - 1)} \int_w^\infty s^{p-2} e^{-s/\xi} ds = D(w)$$

has a unique solution \bar{w} in the interval (w_, c_2) .*

Proof. First, observe that $c_1 < c_2 < \xi$. The first inequality is trivial, while the second is equivalent to $\xi > 1 - \frac{1}{2e\Gamma(p)}$, which is clearly satisfied under the assumption $\xi > \xi_0(p)$. Second, we have $c_2 < \frac{p-2}{3p}$. Indeed, this inequality is equivalent to

$$\xi > 1 - \frac{\left(1 - \frac{2}{p}\right)^{p-2}}{2e(3\xi)^{p-2}\Gamma(p)}.$$

Since $\xi < 1$ and $\left(1 - \frac{2}{p}\right)^{p-2} > e^{-2}$, this inequality is weaker than $\xi > 1 - \frac{9}{2e^3 3^p \Gamma(p)}$, which is in its turn weaker than $\xi > \xi_0(p)$.

Consider equation (2.13). When $w = 0$, the left-hand side of (2.13) is positive. For $w = c_1$, we have

$$\begin{aligned} \left(\frac{1}{\xi} - 1 \right) \int_{c_1}^\infty s^{p-2} e^{-s/\xi} ds - c_1^{p-2} e^{-c_1/\xi} &= (1 - \xi) \xi^{p-2} \int_{c_1/\xi}^\infty s^{p-2} e^{-s} ds - c_1^{p-2} e^{-c_1/\xi} \\ &< (1 - \xi) \xi^{p-2} \Gamma(p - 1) - c_1^{p-2} e^{-c_1/\xi} = \xi^{p-2} (1 - \xi) \Gamma(p - 1) (1 - e^{1-c_1/\xi}) < 0, \end{aligned}$$

because $c_1 < \xi$. Thus, a solution $w_* \in (0, c_1)$ exists. To prove that it is unique, we note that the left-hand side of (2.13) is monotone decreasing in w for $w \in (0, p-2)$, and that $c_1 < p-2$.

Turning now to (2.14), for $w = w_*$ we have

$$\left(\frac{1}{\xi} - 1\right)p(p-1)e^{w_*(1/\xi-1)} \int_{w_*}^{\infty} s^{p-2} e^{-s/\xi} ds = p(p-1)w_*^{p-2} e^{-w_*} > D(w_*)$$

by (2.13) and (2.10). Observe that, for any w , since $1 - e^{v-w} < 1 - e^{-2w} < 2w$, it follows that

$$D(w) = p \frac{w^{p-1} + (-v)^{p-1}}{e^w - e^v} > p e^{-w} \frac{w^{p-1}}{1 - e^{v-w}} > \frac{1}{2} p e^{-w} w^{p-2}.$$

Therefore, putting $w = c_2$ on the left-hand side of (2.14), we get

$$\left(\frac{1}{\xi} - 1\right)p(p-1)e^{c_2(1/\xi-1)} \int_{c_2}^{\infty} s^{p-2} e^{-s/\xi} ds < (1-\xi)\xi^{p-2} p \Gamma(p) e^{1-c_2} = \frac{1}{2} p e^{-c_2} c_2^{p-2} < D(c_2),$$

which implies the existence of a solution $\bar{w} \in (w_*, c_2)$. To prove uniqueness, observe that the derivative of the left-hand side of (2.14) is a positive multiple of the left-hand side of (2.13); thus, it equals zero at $w = w_*$ and is monotone decreasing for $w \in (0, p-2)$; in particular, it is negative for $w > w_*$. Therefore, the left-hand side of (2.14) is monotone decreasing in w for $w \geq w_*$, while the right-hand side is monotone increasing by Lemma 2.3. \square

Remark 2.5. In what follows, in addition to \bar{w} , we shall also use \bar{v} , which is a unique solution of the equation $F(v, \bar{w}) = 0$ guaranteed by Lemma 2.2.

2.3. The Bellman candidate. As was mentioned earlier, we split the domain Ω_C into four subdomains, $\Omega_C = \bigcup_{j=1}^4 R_j$. Besides the numbers \bar{v} and \bar{w} given by Lemma 2.14 and Remark 2.5, in the definition below we use the function k defined in (2.1) and the function r defined in (2.5). The splitting is pictured in Figure 2.

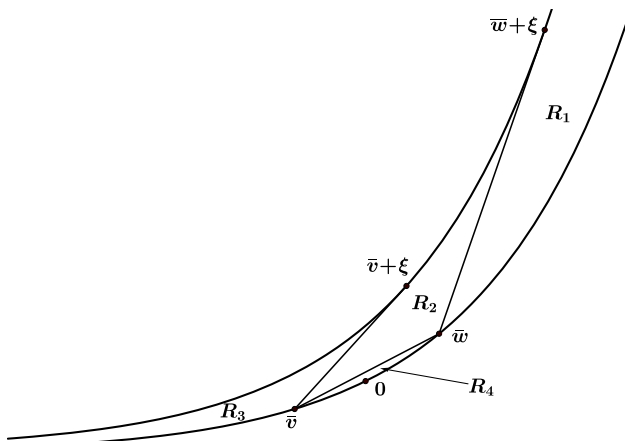


FIGURE 2. The splitting of Ω_C for sufficiently large C .

$$\begin{aligned}
R_1 &= \{x \in \Omega_C : x_2 \leq k(\bar{w})(x_1 - \bar{w}) + e^{\bar{w}}\} \cup \{x \in \Omega_C : x_1 \geq \bar{w} + \xi\}; \\
R_2 &= \{x \in \Omega_C : x_2 \leq k(\bar{v})(x_1 - \bar{v}) + e^{\bar{v}}, x_2 \geq r(\bar{v}, \bar{w})(x_1 - \bar{w}) + e^{\bar{w}}, \\
&\quad x_2 \geq k(\bar{w})(x_1 - \bar{w}) + e^{\bar{w}}\} \\
(2.15) \quad &\cup \{x \in \Omega_C : \bar{v} + \xi \leq x_1 \leq \bar{w} + \xi, x_2 \geq k(\bar{v})(x_1 - \bar{v}) + e^{\bar{v}}, \\
&\quad x_2 \geq k(\bar{w})(x_1 - \bar{w}) + e^{\bar{w}}\}; \\
R_3 &= \{x \in \Omega_C : x_1 \leq \bar{v} + \xi, x_2 \geq k(\bar{v})(x_1 - \bar{v}) + e^{\bar{v}}\}; \\
R_4 &= \{x \in \Omega_C : x_2 \leq r(\bar{v}, \bar{w})(x_1 - \bar{w}) + e^{\bar{w}}\}.
\end{aligned}$$

Our Bellman candidate will have a different expression in each of the four subdomains, and will require several auxiliary objects. For $z \in \mathbb{R}$, we put

$$(2.16) \quad m_1(z) = \frac{p}{\xi} e^{z/\xi} \int_z^\infty s |s|^{p-2} e^{-s/\xi} ds,$$

and, for $z < \bar{v}$,

$$(2.17) \quad m_3(z) = -\frac{p}{\xi} e^{z/\xi} \int_z^{\bar{v}} (-s)^{p-1} e^{-s/\xi} ds + e^{(z-\bar{v})/\xi} (e^{\bar{v}-\bar{w}} (m_1(\bar{w}) - p\bar{w}^{p-1}) - p(-\bar{v})^{p-1}).$$

The following clear lemma, whose simple proof is left to the reader, gives rise to two new functions on R_4 .

Lemma 2.6. *For each $x = (x_1, x_2) \in R_4$ there exists a unique pair $\{v, w\}$ satisfying (2.7) and such that the line segment connecting the points (w, e^w) and (v, e^v) passes through x . Thus,*

$$x_2 = r(v, w)(x_1 - w) + e^w.$$

In the special case where $x = (0, 1)$, this segment degenerates to a point: $v = w = 0$.

From here on, we reserve the symbols w and v for the two functions on R_4 given by this lemma: $w = w(x)$ and $v = v(x)$; see Figure 3.

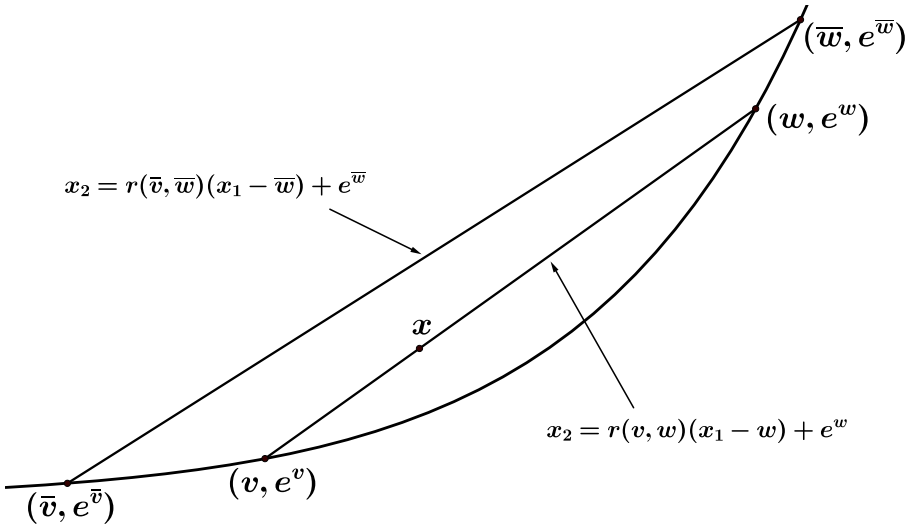


FIGURE 3. The subdomain R_4 and the geometric meaning of $v(x)$ and $w(x)$.

Finally, here is our complete Bellman candidate. For $p > 2$ and $C > C_0(p)$, let

$$(2.18) \quad b_{p,C}(x) = \begin{cases} m_1(u)(x_1 - u) + u^p, & x \in R_1, \\ q(\bar{v}, \bar{w})(x_1 - \bar{w}) + \bar{w}^p \\ \quad + \frac{m_1(\bar{w}) - q(\bar{v}, \bar{w})}{k(\bar{w}) - r(\bar{v}, \bar{w})} (x_2 - r(\bar{v}, \bar{w})(x_1 - \bar{w}) - e^{\bar{w}}), & x \in R_2, \\ m_3(u)(x_1 - u) + (-u)^p, & x \in R_3, \\ q(v, w)(x_1 - w) + w^p, & x \in R_4. \end{cases}$$

Recall that here $u = u(x)$ is given by (2.2); $v = v(x)$ and $w = w(x)$ were defined in Lemma 2.6; k is given by (2.1); r and q are given by (2.5); m_1 is given by (2.16); and m_3 is given by (2.17). In addition, \bar{w} was defined in Lemma 2.4 as the solution of equation (2.14), while \bar{v} was defined in Remark 2.5 as a unique solution of equation $F(v, \bar{w}) = 0$ with F given by (2.7).

In the next section we present the main theorem relating the candidate $b_{p,C}$ and the Bellman function $\mathbf{b}_{p,C}$ as defined in (1.10).

§3. THE MAIN BELLMAN THEOREM AND THE PROOF OF THE LOWER ESTIMATE

The following result is the principal ingredient in the proofs of Theorems 1.2 and 1.3.

Theorem 3.1. *If $p > 2$ and $C > C_0(p)$, then*

$$(3.1) \quad \mathbf{b}_{p,C} = b_{p,C} \quad \text{in} \quad \Omega_C.$$

As is common, we split the proof of Theorem 3.1 into two parts: the so-called direct inequality $\mathbf{b}_{p,C} \geq b_{p,C}$ and its reverse.

Lemma 3.2. *If $p > 2$ and $C > C_0(p)$, then*

$$(3.2) \quad \mathbf{b}_{p,C} \geq b_{p,C} \quad \text{in} \quad \Omega_C.$$

Lemma 3.3. *If $p > 2$ and $C > C_0(p)$, then*

$$(3.3) \quad \mathbf{b}_{p,C} \leq b_{p,C} \quad \text{in} \quad \Omega_C.$$

The proofs of Theorems 1.2 and 1.3 involve only Lemma 3.2, which we prove in this section. For the sake of completeness, we shall also show that the infimum in the definition of the Bellman function is attained at every point in Ω_C , and our candidate is in fact the Bellman function. This is done in §5, where we prove Lemma 3.3.

An analog of Lemma 3.2 for $p \in [1, 2]$ was proved in §5 of [8]. In fact, the proof given there did not depend on the specific range of p . Rather, its main ingredient was showing that $b_{p,C}$ is locally convex in Ω_C , i.e., convex along every line segment contained in Ω_C . More precisely, the main result of Lemma 5.1 in [8] can be restated as follows.

Lemma 3.4 ([8]). *Fix $p > 0$ and assume that for some $C(p) \geq 1$ there is a family of functions $\{b_{p,C}\}_{C \geq C(p)}$ satisfying the following conditions for each C :*

- (1) $b_{p,C}$ is locally convex in Ω_C ;
- (2) $b_{p,C}$ is continuous in Ω_C ;
- (3) for each $x \in \Omega_C$, we have

$$\lim_{c \searrow C} b_{p,c}(x) = b_{p,C}(x);$$

- (4) for each $s \in \mathbb{R}$, we have $b_{p,C}(s, e^s) = |s|^p$.

Then for all $C \geq C(p)$ we have

$$\mathbf{b}_{p,C} \geq b_{p,C} \quad \text{in} \quad \Omega_C.$$

It is routine to check that conditions (2)–(4) are satisfied for the $b_{p,C}$ defined in (2.18). Therefore, Lemma 3.2 will be proved once we have established the following result.

Lemma 3.5. *For $p > 2$ and $C > C_0(p)$, the function $b_{p,C}$ is locally convex in Ω_C .*

Let $p > 2$ and $C > C_0(p)$ be fixed; till the end of this section we write simply b for $b_{p,C}$. Before proving Lemma 3.5, we collect several useful facts from earlier work. First, as was explained in [10] and [8], showing that b is locally convex in Ω_C is the same as showing that it is locally convex in each subdomain R_k and that b_{x_2} is monotone increasing in x_2 across the shared boundaries between subdomains. Second, in R_1 and R_3 the function b has the form

$$b(x) = m(u)(x_1 - u) + |u|^p,$$

where m stands for m_1 or m_3 , respectively, and in each case satisfies the differential equation

$$(3.4) \quad m'(u) = \frac{1}{\xi}(m(u) - pu|u|^{p-2}),$$

while $u = u(x)$ is given by (2.2). As was shown in [8], then we have

$$(3.5) \quad b_{x_2} = m'(u)e^{-u}(1 - \xi)$$

and also

$$(3.6) \quad b_{x_1 x_1} b_{x_2 x_2} = b_{x_1 x_2}^2, \quad \text{sgn } b_{x_2 x_2} = \text{sgn } (m'(u) - m''(u)).$$

Therefore, to show that b is locally convex in R_1 and R_3 , we simply need to check that $m'_1(u) - m''_1(u) > 0$ in R_1 and $m'_3(u) - m''_3(u) > 0$ in R_3 .

Proof of Lemma 3.5. First, we show the local convexity of b in each subdomain R_k .

In R_1 , a direct computation gives

$$\frac{\xi^2}{p(p-1)}(m'_1(u) - m''_1(u))e^{-u/\xi} = \xi u^{p-2}e^{-u/\xi} - (1 - \xi) \int_u^\infty e^{-s/\xi} s^{p-2} ds =: H_1(u),$$

where $u \geq \bar{w}$. We have

$$H'_1(u) = \xi u^{p-3}e^{-u/\xi}(p - 2 - u).$$

Therefore, H_1 is monotone increasing for $u \in (0, p - 2)$ and decreasing for $u > p - 2$. Since $H_1(u) \rightarrow 0$ as $u \rightarrow \infty$, to show that $H_1(u) > 0$ for $u \geq \bar{w}$, it suffices to show that $H_1(\bar{w}) > 0$. This immediately follows by applying first (2.14) and then (2.10) with $w = \bar{w}$:

$$(1 - \xi) \int_{\bar{w}}^\infty e^{-s/\xi} s^{p-2} ds = \frac{\xi e^{\bar{w}(1-1/\xi)}}{p(p-1)} D(\bar{w}) < \xi \bar{w}^{p-2} e^{-\bar{w}/\xi}.$$

Therefore, $b_{x_2 x_2} > 0$ in R_1 , so that b is locally convex in this subdomain.

In R_2 , b is affine and thus locally convex.

In R_3 , we compute

$$\begin{aligned} & \frac{\xi^2}{p(p-1)}(m'_3(u) - m''_3(u))e^{-u/\xi} \\ &= \xi(-u)^{p-2}e^{-u/\xi} - (1 - \xi) \left(\int_u^{\bar{v}} e^{-s/\xi} (-s)^{p-2} ds + e^{(\bar{w}-\bar{v})(1/\xi-1)} \int_{\bar{w}}^\infty e^{-s/\xi} s^{p-2} ds \right) \\ &=: H_3(u), \end{aligned}$$

where $u \leq \bar{v}$. We have

$$H'_3(u) = \xi(-u)^{p-3}e^{-u/\xi}(u - p + 2) < 0,$$

and so to show that $H_3(u) > 0$, it suffices to show that $H_3(\bar{v}) > 0$. Like in the case of H_1 , this follows by applying (2.14) and then (2.9) with $v = \bar{v}$:

$$(1 - \xi)e^{(\bar{w} - \bar{v})(1/\xi - 1)} \int_{\bar{w}}^{\infty} e^{-s/\xi} s^{p-2} ds = \frac{\xi e^{-\bar{v}(1/\xi - 1)}}{p(p-1)} D(\bar{w}) < \xi(-\bar{v})^{p-2} e^{-\bar{v}/\xi}.$$

Thus, b is locally convex in R_3 .

We state the result for R_4 separately.

Lemma 3.6. *b is convex in R_4 .*

Proof. In R_4 , b is given by

$$b(x) = q(v, w)(x_1 - w) + f(w), \quad x_2 = r(v, w)(x_1 - w) + g(w),$$

where, as in the proof of Lemma 2.3, we write $f(w) = w^p$ and $g(w) = e^w$. Again, as we did there, we shall use the prime sign to indicate the total derivative with respect to w .

To show that b is convex, we check that $b_{x_1 x_1} b_{x_2 x_2} = b_{x_1 x_2}^2$ and $b_{x_2 x_2} > 0$ in the interior of R_4 . Differentiation gives

$$w_{x_1} = \frac{-r}{r'(x_1 - w) - r + g'}, \quad w_{x_2} = \frac{1}{r'(x_1 - w) - r + g'},$$

and

$$b_{x_1} = [q'(x_1 - w) - q + f']w_{x_1} + q = -r \frac{q'(x_1 - w) - q + f'}{r'(x_1 - w) - r + g'} + q = -rD + q,$$

where we have used (2.11). Similarly,

$$(3.7) \quad b_{x_2} = [q'(x_1 - w) - q + f']w_{x_2} = \frac{q'(x_1 - w) - q + f'}{r'(x_1 - w) - r + g'} = D.$$

Therefore,

$$b_{x_1 x_1} = -rD'w_{x_1}, \quad b_{x_1 x_2} = -rD'w_{x_2}, \quad b_{x_2 x_2} = D'w_{x_2},$$

and, since, $w_{x_1} = -rw_{x_2}$, we see that $b_{x_1 x_1} b_{x_2 x_2} = b_{x_1 x_2}^2$.

Furthermore, since $D' > 0$ by Lemma 2.3, and since from geometry it is clear that $w_{x_2} > 0$, we have $b_{x_2 x_2} > 0$, which completes the proof. \square

To finish the proof of Lemma 3.5, we need to verify that b_{x_2} is monotone increasing in x_2 across the boundaries between subdomains. We can write this requirement symbolically as follows:

$$b_{x_2}|_{R_1, u=\bar{w}} \leq b_{x_2}|_{R_2}, \quad b_{x_2}|_{R_4, w=\bar{w}} \leq b_{x_2}|_{R_2}, \quad b_{x_2}|_{R_2} \leq b_{x_2}|_{R_3, u=\bar{v}}.$$

In fact, all three statements hold with equality (which implies that b is of class C^1 in the interior of Ω_C , though we shall not use this fact).

By (3.7), we have $b_{x_2}|_{R_4, w=\bar{w}} = D(\bar{w})$. Now we use consecutively relations (3.5), (3.4), and (2.16), integration by parts, and (2.14) to show that

$$(3.8) \quad b_{x_2}|_{R_1, u=\bar{w}} = m'_1(\bar{w})e^{-\bar{w}}(1-\xi) = \frac{1}{\xi}(1-\xi)p(p-1)e^{\bar{w}(1/\xi-1)} \int_{\bar{w}}^{\infty} s^{p-2} e^{-s/\xi} ds = D(\bar{w}).$$

A very similar calculation, but with the use of (2.17) in place of (2.16), gives $b_{x_2}|_{R_3, u=\bar{v}} = D(\bar{w})$.

Finally,

$$b_{x_2}|_{R_2} = \frac{m_1(\bar{w}) - q(\bar{v}, \bar{w})}{k(\bar{w}) - r(\bar{v}, \bar{w})}.$$

By (3.4) and (3.8),

$$m_1(\bar{w}) = \xi m'_1(\bar{w}) + p\bar{w}^{p-1} = \frac{\xi}{1-\xi} e^{\bar{w}} D(\bar{w}) + p\bar{w}^{p-1}.$$

Therefore,

$$b_{x_2}|_{R_2} = \frac{\frac{\xi}{1-\xi} e^{\bar{w}} D(\bar{w}) + p\bar{w}^{p-1} - q(\bar{v}, \bar{w})}{\frac{1}{1-\xi} e^{\bar{w}} - r(\bar{v}, \bar{w})} = \frac{\frac{\xi}{1-\xi} e^{\bar{w}} D(\bar{w}) + (e^{\bar{w}} - r(\bar{v}, \bar{w})) D(\bar{w})}{\frac{1}{1-\xi} e^{\bar{w}} - r(\bar{v}, \bar{w})} = D(\bar{w}).$$

The proof is complete. \square

Now we are in a position to prove the main theorems stated in §1.

§4. PROOFS OF THEOREMS 1.2 AND 1.3

We need two auxiliary results proved in [8].

For $p > 0$, let

$$\omega(p) = \left[\frac{p}{e} \left(\Gamma(p) - \int_0^1 t^{p-1} e^t dt \right) + 1 \right]^{1/p}.$$

Lemma 4.1 ([8]). *Let $\varphi_0(t) = \log(1/t)$, $t \in (0, 1)$. Then*

$$(4.1) \quad \varepsilon_{\varphi_0} = 1, \quad \varepsilon_{-\varphi_0} = \infty.$$

If $p \geq 1$, then

$$(4.2) \quad \|\varphi_0\|_{BMO^p((0,1))} = \omega(p).$$

Consequently,

$$(4.3) \quad \varepsilon_0(p) \leq \omega(p).$$

Lemma 4.2 ([8]). *Let φ be a nonconstant BMO function. For $\varepsilon \in [0, \varepsilon_\varphi)$, let $F(\varepsilon) = [e^{\varepsilon\varphi}]_{A_\infty}$. Then F is a strictly monotone increasing and continuous function on $[0, \varepsilon_\varphi)$, and $\lim_{\varepsilon \rightarrow \varepsilon_\varphi} F(\varepsilon) = \infty$.*

Proof of Theorem 1.2. We use Theorem 1.4 with $b_C = b_{p,C}$ given by (2.18). In view of Lemma 3.2, $b_C \leq \mathbf{b}_{p,C}$, as required.

We need to compute $b_{p,C}(0, C)$. Note that $\bar{w} < \xi$ by Lemma 2.4, and, thus, $\bar{v} > -\bar{w} > -\xi$ by Lemma 2.2. Therefore, the point $(0, C)$ is in the subdomain R_3 and, since $u(0, C) = -\xi$, we have

$$b_{p,C}(0, C) = m_3(-\xi)\xi + \xi^p.$$

The quantity $m_3(-\xi)$ is given by (2.17):

$$m_3(-\xi) = -\frac{p}{\xi} e^{-1} \int_{-\xi}^{\bar{v}} (-s)^{p-1} e^{-s/\xi} ds + e^{(-\xi-\bar{v})/\xi} (e^{\bar{v}-\bar{w}} (m_1(\bar{w}) - p\bar{w}^{p-1}) - p(-\bar{v})^{p-1}).$$

By Lemma 2.4, $\bar{w} \in (0, c_2)$ with $c_2 \rightarrow 0$ as $\xi \rightarrow 1$. By Lemma 2.2, $\bar{v} \in (-\bar{w}, 0)$. Therefore,

$$\lim_{\xi \rightarrow 1} \bar{v} = \lim_{\xi \rightarrow 1} \bar{w} = 0$$

and

$$\lim_{C \rightarrow \infty} b_{p,C}(0, C) = \lim_{\xi \rightarrow 1} (m_3(-\xi)\xi + \xi^p) = -\frac{p}{e} \int_{-1}^0 (-s)^{p-1} e^{-s} ds + e^{-1} m_1(0) + 1 = \omega^p(p),$$

where we have used (2.16), which implies $m_1(0) = p\Gamma(p)$.

Hence, by Theorem 1.4, we have $\varepsilon_0(p) \geq \omega(p)$, and application of Lemma 4.1 finishes the proof. \square

The proof of Theorem 1.3 below is a version of the argument used in the proof of Corollary 1.5 in [8]; the proof of sharpness, which involves the function φ_0 occurring in Lemma 4.1, is exactly the same and we omit it.

Proof of Theorem 1.3. Take any $\varphi \in \text{BMO}(Q)$. Without loss of generality, assume that $\varepsilon_\varphi < \infty$. For $\varepsilon \in [0, \varepsilon_\varphi)$, let $F(\varepsilon) = [e^{\varepsilon\varphi}]_{A_\infty(Q)}$. By Lemma 4.2, for sufficiently large ε we have $F(\varepsilon) \geq C_0(p)$. Therefore, for any subinterval J of Q ,

$$\langle |\varepsilon\varphi - \langle \varepsilon\varphi \rangle_J|^p \rangle_J \geq \mathbf{b}_{p,F(\varepsilon)}(0, \langle e^{\varepsilon\varphi - \langle \varepsilon\varphi \rangle_J} \rangle_J) \geq b_{p,F(\varepsilon)}(0, \langle e^{\varepsilon\varphi - \langle \varepsilon\varphi \rangle_J} \rangle_J).$$

Take a sequence $\{J_n\}$ such that $\langle e^{\varepsilon\varphi - \langle \varepsilon\varphi \rangle_{J_n}} \rangle_{J_n} \rightarrow F(\varepsilon)$. Since the left-hand side is bounded from above by $\varepsilon^p \|\varphi\|_{\text{BMO}^p(Q)}^p$, we have

$$\varepsilon^p \|\varphi\|_{\text{BMO}^p(Q)}^p \geq b_{p,F(\varepsilon)}(0, F(\varepsilon)).$$

Now, we let $\varepsilon \rightarrow \varepsilon_\varphi$ (and, thus, $F(\varepsilon) \rightarrow \infty$). This gives

$$\varepsilon_\varphi^p \|\varphi\|_{\text{BMO}^p(Q)}^p \geq \varepsilon_0^p(p).$$

If $f \in L^\infty(Q)$, then $\varepsilon_{\varphi-f} = \varepsilon_\varphi$. Thus, we can replace φ with $\varphi - f$ above, which gives

$$\|\varphi - f\|_{\text{BMO}^p(Q)} \geq \frac{\varepsilon_0(p)}{\varepsilon_\varphi}.$$

The same inequality holds true with φ replaced by $-\varphi$, and it remains to take the infimum over $f \in L^\infty(Q)$ on the left. \square

§5. OPTIMIZERS AND THE REVERSE INEQUALITY

In this section, we complete the proof of Theorem 3.1 by proving Lemma 3.3. For this, we present a set of special test functions on the interval $(0, 1)$ that realize the infimum in definition (1.10) of the Bellman function $\mathbf{b}_{p,C}$.

Without loss of generality we may assume that $C > 1$. Let $Q = (0, 1)$. Recall the Bellman candidate $b_{p,C}$ given by formula (2.18). For $x \in \Omega_C$, we say that a function φ_x on Q is an optimizer for $b_{p,C}$ at x if

$$(5.1) \quad \varphi_x \in E_{x,C,Q} \quad \text{and} \quad \langle |\varphi_x|^p \rangle_Q = b_{p,C}(x),$$

where the set of test functions $E_{x,C,Q}$ is defined by (1.9). Observe that if we have such a function φ_x for all $x \in \Omega_C$, then

$$b_{p,C}(x) = \langle |\varphi|^p \rangle_Q \geq \mathbf{b}_{p,C}(x),$$

which is the claim of Lemma 3.3.

Our optimizers φ_x will have different forms depending on the location of x in Ω_C . Specifically, we shall have a different optimizer for each of the four subdomains R_k of Ω_C defined by formula (1.8) and pictured in Figure 2. We do not discuss the construction of these optimizers, but simply give formulas for them. A reader interested in where they come from is invited to consult the papers [10] and [3], where a number of similar constructions were carried out in the context of BMO^2 .

For each $x \in R_1$, let

$$(5.2) \quad \varphi_x(t) = u + \xi \log\left(\frac{\alpha}{t}\right) \chi_{(0,\alpha)}(t),$$

where $u = u(x)$ is defined by (2.2) and

$$(5.3) \quad \alpha = \frac{x_1 - u}{\xi}.$$

(This optimizer was defined in §5 of [8] under the name φ_x^+ .)

Now consider the subdomain R_2 . Let us give names to its four corners, clockwise from top right:

$$X = (\bar{w} + \xi, e^{\bar{w} + \xi}), \quad Y = (\bar{w}, e^{\bar{w}}), \quad Z = (\bar{v}, e^{\bar{v}}), \quad W = (\bar{v} + \xi, e^{\bar{v} + \xi}).$$

We already know the optimizers for the points X , Y , and Z : the first comes from formula (5.2) (which applies because $X \in R_1 \cap R_2$) with $\alpha = 1$; the other two are trivial, because for each $x \in \Gamma_1$ the set $E_{x,C,Q}$ contains only one element, namely, the constant function $\varphi(t) = x_1$. Therefore, for all $t \in Q$, we define

$$\varphi_X(t) = \xi \log\left(\frac{1}{t}\right), \quad \varphi_Y(t) = \bar{w}, \quad \varphi_Z(t) = \bar{v}.$$

Now we use these three optimizers to define φ_x for every $x \in R_2$. Observe that R_2 is contained in the triangle with the vertices X, Y, Z . This means that every x in R_2 has a unique representation as a convex combination of these three points. Thus, there exist nonnegative numbers α_1, α_2 , and α_3 such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and

$$(5.4) \quad x = \alpha_1 X + \alpha_2 Y + \alpha_3 Z.$$

To obtain φ_x , we concatenate φ_X, φ_Y , and φ_Z in the appropriate proportion:

$$\varphi_x(t) = \varphi_X\left(\frac{t}{\alpha_1}\right)\chi_{(0,\alpha_1)}(t) + \varphi_Y\left(\frac{t-\alpha_1}{\alpha_2}\right)\chi_{(\alpha_1,\alpha_1+\alpha_2)}(t) + \varphi_Z\left(\frac{t-\alpha_1-\alpha_2}{\alpha_3}\right)\chi_{(\alpha_1+\alpha_2,1)}(t),$$

or, equivalently,

$$(5.5) \quad \varphi_x(t) = \bar{w} \chi_{(0,\alpha_1+\alpha_2)}(t) + \xi \log\left(\frac{\alpha_1}{t}\right)\chi_{(0,\alpha_1)}(t) + \bar{v} \chi_{(\alpha_1+\alpha_2,1)}(t),$$

with $\alpha_k = \alpha_k(x)$ defined by (5.4).

This formula applies, in particular, to the fourth corner of R_2 , i.e., the point W . That point also lies in the subdomain R_3 , and is the key to defining the optimizer for all other points in that subdomain. Specifically, with the knowledge of φ_W , we define the optimizer φ_x for an arbitrary point $x \in R_3$ by the formula

$$(5.6) \quad \varphi_x(t) = \varphi_W\left(\frac{t}{\tau\alpha}\right)\chi_{(0,\tau\alpha)}(t) + \xi \log\left(\frac{\alpha}{t}\right)\chi_{(\tau\alpha,\alpha)}(t) + u\chi_{(\tau\alpha,1)}(t).$$

Here, u is given by (2.2), α is as in (5.3), and we also set

$$(5.7) \quad \tau = e^{(u-\bar{v})/\xi}.$$

It remains to define φ_x for $x \in R_4$. Recall the two auxiliary functions $v = v(x)$ and $w = w(x)$ defined by Lemma 2.6 (see Figure 3). Every point $x \in R_4 \setminus \Gamma_1$ lies on the line segment connecting the points (v, e^v) and (w, e^w) . Accordingly, we define φ_x to be the appropriate concatenation of the two constant optimizers corresponding to those points:

$$(5.8) \quad \varphi_x(t) = w \chi_{(0,\beta)}(t) + v \chi_{(\beta,1)}(t),$$

where

$$(5.9) \quad \beta = \frac{x_1 - v}{w - v}.$$

The following lemma immediately yields Lemma 3.3.

Lemma 5.1. *Let φ_x be defined by (5.2) and (5.3) for $x \in R_1$; by (5.4) and (5.5) for $x \in R_2$; by (5.6), (5.3), and (5.7) for $x \in R_3$; and by (5.8) and (5.9) for $x \in R_4$. Then φ_x is an optimizer for $b_{p,C}$ at every $x \in \Omega_C$.*

Remark 5.2. If a point x lies on a boundary shared by two subdomains, φ_x seems to be defined by two different formulas. However, as is easy to check, in all cases above, such two formulas give exactly the same function.

The proof of this lemma is very similar to that of Lemma 5.2 in [8], and we leave it to the reader.

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