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ON OPERATORS COMMUTING WITH A POMMIEZ TYPE OPERATOR IN WEIGHTED SPACES OF ENTIRE FUNCTIONS

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Abstract. A description is presented for continuous linear operators defined on a countable inductive limit of weighted Fréchet spaces of entire functions and commuting with a Pommiez type operator.

Introduction

In [20, 21, 22] Pommiez investigated the consecutive remainders for the Taylor series of functions analytic in the disk centered at zero. With the help of this remainders and in terms of them, he studied series expansions of such functions with respect to a sequence of special polynomials and proved uniqueness theorems; he also obtained certain conditions for such remainders to be univalent. In [23], generalized Newton series expansions for analytic functions were treated. In all the papers [20, 21, 22, 23], the following difference operator was used:

$$D_z(f)(t) := \begin{cases} \frac{f(t)-f(z)}{t-z}, & t \neq z, \\ f'(z), & t = z. \end{cases}$$

Observe that if $f(t) = \sum_{k=0}^{\infty} a_k t^k$, $|t| < R$, then $D_0^n(f)(t) = \sum_{k=0}^{\infty} a_{k+n} t^k$ for all integers $n \geq 0$. After the papers [20, 21, 22, 23], it has become usual to refer to the operators D_z as Pommiez operators. It should be noted that they had been employed and studied even before, moreover, the theory of finite differences for functions of a complex variable dates back to as an early time as the 30s of the last century (see, for example, Gelfond's monograph [1]). Khaplanov (see [9]) found a sufficient condition for the completeness of the system of remainders $f_n(t) = \sum_{k=0}^{\infty} a_{k+n} t^k$, $n \geq 0$ (i.e., of the sequence $(D_0^n(f))_{n\geq 0}$) for the Taylor series of a function $f(t) = \sum_{k=0}^{\infty} a_k t^k$ in the disk $|t| < R$. Kaz′min (see [4]) established a completeness criteria for the system $(D_0^n(f))_{n\geq 0}$ in a simply connected domain $G \subseteq \mathbb{C}$ containing zero, and investigated the completeness of systems of the form $(D_{\alpha_n}(f))_{n\in\mathbb{N}}$.

Subsequently, fairly many papers devoted to D_z appeared. We mention some of them. Commutants and cyclic elements for D_0 were studied by N. Linchuk in [7], by Dimovski and Hristov in [15], by Yu. Linchuk in [19]. The equivalence of Pommiez operators in spaces of analytic functions was treated, in particular, by S. Linchuk and Nagnibida in [8]. Korobeĭnik (see [6]) and Sherstyukov (see [11]) used the Pommiez operator to expand analytic functions in partial fractions series.

In [16], cyclic elements and invariant subspaces were studied for the operator D_0 in the Hardy space H^2 in the unit disk (in that paper, D_0 was called the backward shift). In [17], a backward shift operator in a Banach space was introduced, and its commutants were studied along with more general properties of its cyclic elements. We also mention

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a series of papers by Korobeĭnik and his students and followers about the description of the commutators of the direct and backward shift operators and their generalizations in spaces of numerical sequences (see the bibliography in [5]). In [2] and [3], the operator D_z (more generally, the operator D_{z,g_0}; see below) was used to construct an abstract version of Leont′ev's interpolating function; this function is widely used in the theory of series of exponentials and convolution operators.

In the present paper, an operator *of Pommiez type* will be discussed. For $z \in \mathbb{C}$, this operator is defined as follows:

$$D_{z,g_0}(f)(t) := \begin{cases} \frac{f(t)-g_0(t-z)f(z)}{t-z}, & t \neq z, \\ f'(z) - g_0'(0)f(z), & t = z. \end{cases}$$

Here f belongs to a certain countable inductive limit E of weighted Fréchet spaces of entire functions on $\mathbb{C}$, and g_0 is a fixed function in E such that $g_0(0) = 1$. If g_0 is identically equal to 1, the operator D_{z,g_0} coincides with D_z. Clearly, D_{0,g_0} maps E to E linearly and continuously. The class of spaces E in question includes the most part of the standard weighted spaces of entire functions that occur in Fourier analysis and arise under realization of various function spaces and their duals with the help of the Fourier–Laplace transformation and its analogs. Not every such space E contains the polynomials, so that the standard operator D_0, which corresponds to the case where $g_0 \equiv 1$, may fail to act in E. Thus, the passage to a function $g_0 \in E$ with $g_0(0) = 1$ proves to be quite natural.

The principal result of the paper is Theorem 15 about the general form of the commutants for D_{0,g_0} in E. The first result of this sort was obtained by N. Linchuk in [7] for the space $A(G)$ of functions analytic in a domain $G \subseteq \mathbb{C}$ that contains 0, and for $g_0 \equiv 1$. Yu. Linchuk (see [19]) proved the corresponding result for an arbitrary linear left inverse to the operator of multiplication by the independent variable that is continuous in $A(G)$. Both in E and in $A(G)$, the set of such operators coincides with the set of all operators D_{0,g_0} (see Theorem 2 and Remark 3). It should be noted that the exposition in [7] and [19] involved in an essential way Köthe's theory (see [18]) of characteristic functions of continuous linear operators in $A(G)$, which is specific for $A(G)$. Dimovski and Hristov (see [15]) gave a proof of the commutant representation theorem for D_0 in $A(G)$ (for a simply connected domain G) that differs from the proof in [7]. In [15], the density of the set of polynomials in $A(G)$ was used critically, along with the fact that the *shift operators* T_z corresponding to D_0 act on polynomials in a simple way. The crux of the method used in [15] is in the proof of the implication *"commutation with* D_0 implies *commutation with an arbitrary shift operator* T_z". The new setting explored here has brought about some difficulties related to lifting the assumptions mentioned above and fulfilled automatically for $A(G)$ in the case where $g_0 \equiv 1$. These difficulties are overcome on the basis of a quasianalytic nature of E. Specifically, we use the fact that the functionals $f \mapsto f^{(n)}(0)$, $n \geq 0$, $f \in E$, constitute a dense subset of the dual E' for E in the Mackey topology, which implies a due weighted approximability of the shift operators T_z and of the commutants for D_{0,g_0} by polynomials of D_{0,g_0}.

The paper is organized as follows. In §1 we introduce a Pommiez type operator D_{0,g_0} and describe its properties. In particular, we show that the set of all continuous linear left inverses to the operator of multiplication by the independent variable in E consists precisely of the operators D_{0,g_0}. In §2 some auxiliary statements are proved. There we introduce and explore a binary operation (convolution) $\otimes$ in E' generated by a shift for a Pommiez type operator. In §3 we prove the principal Theorem 15 and its consequences. It is shown that E' with the operation $\otimes$ is isomorphic to the algebra of all continuous and linear operators in E that commute with D_{0,g_0}.

§1. Pommiez type operator and its properties

In [3], a Pommiez type operator in a weighted (LF)-space of entire functions was studied. We present the necessary information from [3]. Given a continuous function $v\colon \mathbb{C} \to \mathbb{R}$, for a function $f\colon \mathbb{C} \to \mathbb{C}$ we put

$$p_v(f) := \sup_{z\in\mathbb{C}} \frac{|f(z)|}{\exp v(z)}.$$

Suppose that continuous functions $v_{n,k}\colon \mathbb{C} \to \mathbb{R}$ satisfy

$$v_{n,k+1} \le v_{n,k} \le v_{n+1,k}, \quad n, k \in \mathbb{N},$$

on $\mathbb{C}$. As usual, the symbol $A(\mathbb{C})$ denotes the space of all entire functions (on $\mathbb{C}$). We introduce the weighted Fréchet spaces

$$E_n := \{f \in A(\mathbb{C}) \,:\, p_{v_{n,k}}(f) < +\infty \quad \text{for all} \quad k \in \mathbb{N}\}, \quad n \in \mathbb{N}.$$

Note that E_n is embedded in E_{n+1} continuously for every $n \in \mathbb{N}$. Put

$$E := \operatorname*{ind}_{n\to} E_n.$$

In the sequel, we assume that the double sequence $(v_{n,k})_{n,k\in\mathbb{N}}$ satisfies the following condition:

$$(1.1)\ \forall n\ \exists m\ \forall k\ \exists s\ \exists C \ge 0 \,:\, \sup_{|t-z|\le 1} v_{n,s}(t) + \ln(1+|z|) \le \inf_{|t-z|\le 1} v_{m,k}(t) + C, \quad z \in \mathbb{C}.$$

Then E is invariant under differentiation and translation, and for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that every bounded set in E_n is relatively compact in E_m, see [3, Remark 1]. Moreover, E is invariant under multiplication by the independent variable.

We assume that E contains a function that is not identically zero. Since for every $f \in E$ and $z \in \mathbb{C}$ with $f(z) = 0$ the function $\frac{f(t)}{t-z}$ also belongs to E, we see that there exists a function $g_0 \in E$ with $g_0(0) = 1$.

Let $g_0 \in E$ satisfy $g_0(0) = 1$. An operator *of Pommiez type* D_{z,g_0}, $z \in \mathbb{C}$, is defined on E as follows: for $f \in E$, we put

$$D_{z,g_0}(f)(t) := \begin{cases} \frac{f(t)-g_0(t-z)f(z)}{t-z}, & t \ne z, \\ f'(z) - g_0'(0)f(z), & t = z. \end{cases}$$

The operator D_{z,g_0} maps E to E linearly and continuously, see [3, Lemma 6].

We denote by $\mathcal{L}(E)$ the space of all continuous linear operators on E.

Let $M(f)(z) := zf(z)$, $f \in E$, $z \in \mathbb{C}$, be the operator of multiplication by the independent variable. Clearly, $M \in \mathcal{L}(E)$. We show that every operator D_{0,g_0} is a continuous linear left inverse to $M\colon E \to E$ (and *vice versa*). For the space $A(G)$, this result was mentioned in [19] (see Remark 3).

Let $\operatorname{Ker} L$ stand for the kernel of the operator $L \in \mathcal{L}(E)$.

Lemma 1. *Let $L \in \mathcal{L}(E)$ be a continuous linear left inverse to $M\colon E \to E$. Then there exists a function $f \in \operatorname{Ker} L$ with $f(0) = 1$.*

Proof. We show that $\operatorname{Ker} L \ne \{0\}$. Let $\operatorname{Ker} L = \{0\}$. Since is $L\colon E \to E$ a surjection, we see that L is an algebraic isomorphism of E onto E. Then M is also an algebraic isomorphism of E onto E. This contradicts the fact that $M(E) \ne E$.

Let $h \in \operatorname{Ker} L$, and let $h \ne 0$. Suppose that $h(0) = 0$. Then

$$h_0(z) := \frac{h(z)}{z} \in E$$

and

$$0 = L(h) = L(M(h_0)) = h_0,$$

which contradicts the fact that $h \ne 0$. It remains to put $f := \frac{h}{h(0)}$. □

Theorem 2. *The following conditions are equivalent*:

(i) *L is a continuous linear left inverse to $M\colon E \to E$;*

(ii) *there exists a function $g_0 \in E$ such that $g_0(0) = 1$ and $L = D_{0,g_0}$.*

Proof. The implication (ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (ii): By Lemma 1, there exists a function $g_0 \in \operatorname{Ker} L$ with $g_0(0) = 1$. Then for every $f \in E$ and every $z \in \mathbb{C} \setminus \{0\}$ we have

$$\begin{aligned} L(f)(z) &= L_t\big(tD_{0,g_0}(t) + f(0)g_0(t)\big)(z) \\ &= \frac{f(z) - f(0)g_0(z)}{z} + f(0)L(g_0)(z) = D_{0,g_0}(f)(z). \end{aligned}$$

Thus, $L = D_{0,g_0}$. □

Remark 3. In [19, p. 384], Yu. Linchuk observed that, in the Fréchet space $A(G)$ of functions analytic in a domain $G \subseteq \mathbb{C}$ containing 0, the general form of continuous linear left inverses to M is given by the formula

$$A(f) = D_0(f) + f(0)\varphi, \quad f \in A(G), \tag{1.2}$$

where φ is a function in $A(G)$. The operator A in (1.2) is the operator D_{0,g_0} of Pommiez type for $g_0(t) = 1-t\varphi(t)$. Reciprocally, any operator D_{0,g_0} is A as above if $\varphi(t) = \frac{1-g_0(t)}{t}$. This is true both for $A(G)$ and for E.

§2. Auxiliary results

2.1. The shift operator for a Pommiez type operator. Its properties. Fix a function $g_0 \in E$ with $g_0(0) = 1$. As in [14, 15], for $z \in \mathbb{C}$ we introduce the operator

$$T_z(f)(t) := \begin{cases} \frac{tf(t)g_0(z) - zf(z)g_0(t)}{t-z}, & t \neq z, \\ zg_0(z)f'(z) - zf(z)g_0'(z) + f(z)g_0(z), & t = z, \end{cases}$$

$f \in E$. (By analogy with [14], it is natural to call T_z the *shift operator for the Pommiez type operator*.)

Lemma 4. (i) $\forall n\ \exists m\ \forall k, l\ \exists s\ \exists C \geq 0\ :\ \forall f \in E_n$

$$|T_z(f)(t)| \leq Cp_{v_{n,s}}(f) \exp v_{m,k}(t) \exp v_{m,l}(z), \quad t, z \in \mathbb{C}. \tag{2.1}$$

(ii) *For every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that T_z maps E_n to E_m continuously and linearly for every $z \in \mathbb{C}$; consequently, it maps E to E.*

(iii) *For every $f \in E$ there exists $m \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ the set $V_k(f) := \big\{\exp(-v_{m,k}(t))T_t(f)\ :\ t \in \mathbb{C}\big\}$ is bounded in E_m.*

Proof. (i) This is a consequence of (1.1).

(ii) Formula (2.1) shows that $\forall n\ \exists m\ \forall k\ \exists s\ \exists C \geq 0$: $\forall f \in E_n$

$$p_{v_{m,k}}(T_z(f)) \leq C \exp v_{m,k}(z) p_{v_{n,s}}(f), \quad z \in \mathbb{C}.$$

Therefore, the operator $T_z\colon E_n \to E_m$ is continuous for every $z \in \mathbb{C}$.

(iii) This follows from (2.1) if we recall that

$$T_z(f)(t) = T_t(f)(z),\ t, z \in \mathbb{C}.$$

□

For $z \in \mathbb{C}$, we introduce the delta-functions δ_z in a standard way:

$$\delta_z(f) = f(z), \quad f \in E.$$

Clearly, $\delta_z \in E'$ for every $z \in \mathbb{C}$. Here E' stands for the topological dual to E. Observe the obvious identity

$$\delta_0 T_z = \delta_z, \quad z \in \mathbb{C}.$$

The following lemma is immediate.

Lemma 5. *For every $z \in \mathbb{C}$ we have*

$$T_z D_{0,g_0} = D_{0,g_0} T_z.$$

Let $\mathcal{F}_z = T_z(f)$, $f \in E$, $z \in \mathbb{C}$.

Lemma 6. *For all integers $n \geq 0$ and all $z \in \mathbb{C}$, we have*

$$\big(D_{0,g_0}(\mathcal{F}_z)\big)^{(n)}(0) = \frac{1}{n+1}(\mathcal{F}_z)^{(n+1)}(0) - \frac{g_0^{(n+1)}(0)}{n+1}\mathcal{F}_z(0). \tag{2.2}$$

Proof. We verify (2.2) for $n = 0$. Since for every $t \neq z$, $t \neq 0$ we have

$$D_{0,g_0}(\mathcal{F}_z)(t) = \frac{\frac{tf(t)g_0(z)-zf(z)g_0(t)}{t-z} - f(z)g_0(t)}{t} = \frac{f(t)g_0(z) - f(z)g_0(t)}{t-z},$$

it follows that for every $t \neq z$ we have the identity

$$D_{0,g_0}(\mathcal{F}_z)(t) = \frac{f(t)g_0(z) - f(z)g_0(t)}{t-z}. \tag{2.3}$$

Thus, for $z \neq 0$ we obtain

$$D_{0,g_0}(\mathcal{F}_z)(0) = \frac{f(z) - g_0(z)f(0)}{z} = D_{0,g_0}(f)(z).$$

Consequently,

$$D_{0,g_0}(\mathcal{F}_z)(0) = D_{0,g_0}(f)(z)$$

for all $z \in \mathbb{C}$. On the other hand, if $z \neq 0$, then

$$\begin{aligned}(\mathcal{F}_z)'(0) &= \frac{(f(0)g_0(z) - zf(z)g_0'(0))(-z) + zf(z)}{z^2}\\ &= \frac{f(z) - f(0)g_0(z) + zg_0'(0)f(z)}{z} = D_{0,g_0}(f)(z) + g_0'(0)f(z)\\ &= D_{0,g_0}(f)(z) + g_0'(0)\mathcal{F}_z(0).\end{aligned}$$

Therefore,

$$D_{0,g_0}(\mathcal{F}_z)(0) = (\mathcal{F}_z)'(0) - g_0'(0)\mathcal{F}_z(0), \quad z \in \mathbb{C}.$$

Now let $n \geq 1$, $t \neq z$. By (2.3), differentiation in t yields

$$\begin{aligned}\big(D_{0,g_0}(\mathcal{F}_z)\big)^{(n)}(t) &= \left(\frac{f(t)g_0(z)}{t-z}\right)^{(n)} - \left(\frac{f(z)g_0(t)}{t-z}\right)^{(n)}\\ &= g_0(z)\sum_{k=0}^{n} C_n^k f^{(n-k)}(t)\frac{(-1)^k k!}{(t-z)^{k+1}} - f(z)\sum_{k=0}^{n} C_n^k g_0^{(n-k)}(t)\frac{(-1)^k k!}{(t-z)^{k+1}}.\end{aligned}$$

So, for $t = 0$ and $z \neq 0$ we have

$$\begin{aligned}\big(D_{0,g_0}(\mathcal{F}_z)\big)^{(n)}(0) &= f(z)\sum_{k=0}^{n} C_n^k g_0^{(n-k)}(0)\frac{k!}{z^{k+1}} - g_0(z)\sum_{k=0}^{n} C_n^k f^{(n-k)}(0)\frac{k!}{z^{k+1}}\\ &= f(z)\sum_{k=0}^{n} \frac{n!}{(n-k)!}\frac{g_0^{(n-k)}(0)}{z^{k+1}} - g_0(z)\sum_{k=0}^{n}\frac{n!}{(n-k)!}\frac{f^{(n-k)}(0)}{z^{k+1}}.\end{aligned} \tag{2.4}$$

At the same time, for $t \neq z$ we have

$$(\mathcal{F}_z)^{(n+1)}(t) = g_0(z)\left(\frac{tf(t)}{t-z}\right)^{(n+1)} - zf(z)\left(\frac{g_0(t)}{t-z}\right)^{(n+1)}$$
$$= g_0(z)\sum_{k=0}^{n+1} C_{n+1}^k (tf(t))^{(n+1-k)}\frac{(-1)^k k!}{(t-z)^{k+1}} - zf(z)\sum_{k=0}^{n+1} C_{n+1}^k g_0^{(n+1-k)}(t)\frac{(-1)^k k!}{(t-z)^{k+1}},$$

whence for $z \neq 0$ we obtain

$$\begin{aligned}(\mathcal{F}_z)^{(n+1)}(0) = &\ -g_0(z)\sum_{k=0}^{n} C_{n+1}^k (n+1-k) f^{(n-k)}(0)\frac{k!}{z^{k+1}}\\ &+ zf(z)\sum_{k=0}^{n+1} C_{n+1}^k g_0^{(n+1-k)}(0)\frac{k!}{z^{k+1}} = g_0^{(n+1)}(0)f(z)\\ &+ f(z)\sum_{k=1}^{n+1}\frac{(n+1)!}{(n+1-k)!}\frac{g_0^{(n+1-k)}(0)}{z^k}\\ &- g_0(z)\sum_{k=0}^{n}\frac{(n+1)!}{(n+1-k)!}(n+1-k)\frac{f^{(n-k)}(0)}{z^{k+1}}\\ = g_0^{(n+1)}(0)f(z) &+ (n+1)f(z)\sum_{k=0}^{n}\frac{n!}{(n-k)!}\frac{g_0^{(n-k)}(0)}{z^{k+1}}\\ &- (n+1)g_0(z)\sum_{k=0}^{n}\frac{n!}{(n-k)!}\frac{f^{(n-k)}(0)}{z^{k+1}}.\end{aligned} \tag{2.5}$$

Now, (2.4) and (2.5) imply (2.2) for $n \geq 1$, $z \neq 0$. Clearly, (2.2) is true for all $z \in \mathbb{C}$. □

Below we prove Lemma 7, which is similar to Lemma 2 in [19]. That lemma was proved in [19] for the operators A^n acting in the Fréchet space $A(G)$ of functions analytic in a domain $G \subseteq \mathbb{C}$ (see Remark 3). The proof of Lemma 2 in [19] involved critically the characteristic functions of the operators A^n. The characteristic function techniques for continuous linear operators on $A(G)$ is specific for $A(G)$ and cannot be used in the setting treated here. (Before, such characteristic functions had been employed in [7] for the operator $A = D_0$ that acts in $A(G)$.) Here Lemma 7 is proved in a different, more elementary and general way (without use of specific features of E). This method is also applicable in the setting of [19].

Lemma 7. *For every integer $n \geq 0$, we have*

$$D_{0,g_0}^n(f)(z) = \varphi_n(T_z(f)), \quad f \in E,\ z \in \mathbb{C}, \tag{2.6}$$

where $\varphi_0 = \delta_0$; if $n \in \mathbb{N}$, then there exist $c_{k,n} \in \mathbb{C}$, $0 \leq k \leq n-1$, with

$$\varphi_n(f) = \frac{1}{n!}f^{(n)}(0) + \sum_{k=0}^{n-1} c_{k,n} f^{(k)}(0), \quad f \in E.$$

Proof. For $n = 0$, identity (2.6) is obvious. For $n = 1$, we have

$$D_{0,g_0}(f)(z) := \frac{f(z) - g_0(z)f(0)}{z}, \quad z \neq 0.$$

On the other hand (the differentiations are in the variable t), for $z \neq 0$ we have

$$\left(\frac{tf(t)g_0(z) - zf(z)g_0(t)}{t-z}\right)'\Big|_{t=0} - g_0'(0)\left(\frac{tf(t)g_0(z) - zf(z)g_0(t)}{t-z}\right)\Big|_{t=0}$$
$$= \frac{-zf(0)g_0(z) + z^2 f(z)g_0'(0) + zf(z) - g_0'(0)f(z)z^2}{z^2} = \frac{f(z) - g_0(z)f(0)}{z}.$$

It follows that

$$D_{0,g_0}(f) = \varphi_1(T_z(f)), \quad f \in E,$$

where $\varphi_1(f) := f'(0) - g_0'(0)f(0)$.

Suppose that for $n \geq 1$ there exist numbers $c_{k,n}$, $0 \leq k \leq n-1$, such that (2.6) is true. Taking Lemmas 5 and 6 into account, for $f \in E$ and $z \in \mathbb{C}$ we obtain

$$\begin{aligned}
D_{0,g_0}^{n+1}(f)(z) &= D_{0,g_0}^n\big(D_{0,g_0}(f)\big)(z) \\
&= \varphi_n\big(T_z(D_{0,g_0}(f))\big) = \varphi_n\big(D_{0,g_0}(T_z(f))\big) = \varphi_n\big(D_{0,g_0}(\mathcal{F}_z)\big) \\
&= \frac{1}{n!}\big(D_{0,g_0}(\mathcal{F}_z)\big)^{(n)}(0) + \sum_{k=0}^{n-1} c_{k,n}\big(D_{0,g_0}(\mathcal{F}_z)\big)^{(k)}(0) = \frac{1}{(n+1)!}\mathcal{F}_z^{(n+1)}(0) \\
&\quad + \sum_{k=0}^{n-1} c_{k,n}\Big(\frac{1}{k+1}(\mathcal{F}_z)^{(k+1)}(0) - \frac{g_0^{(k+1)}(0)}{k+1}\mathcal{F}_z(0)\Big) - \frac{g_0^{(n+1)}(0)}{(n+1)!}\mathcal{F}_z(0) \\
&= \frac{1}{(n+1)!}\mathcal{F}_z^{(n+1)}(0) + \sum_{k=1}^{n} c_{k-1,n}\Big(\frac{1}{k}(\mathcal{F}_z)^{(k)}(0) - \frac{g_0^{(k)}(0)}{k}\mathcal{F}_z(0)\Big) \\
&\quad - \frac{g_0^{(n+1)}(0)}{(n+1)!}\mathcal{F}_z(0) = \frac{1}{(n+1)!}(\mathcal{F}_z)^{(n+1)}(0) + \sum_{k=1}^{n} \frac{c_{k-1,n}}{k}(\mathcal{F}_z)^{(k)}(0) \\
&\quad - \Big(\sum_{k=1}^{n} c_{k-1,n}\frac{g_0^{(k)}(0)}{k}\Big)\mathcal{F}_z(0) - \frac{g_0^{(n+1)}(0)}{(n+1)!}\mathcal{F}_z(0) \\
&= \frac{1}{(n+1)!}(\mathcal{F}_z)^{(n+1)}(0) + \sum_{k=0}^{n} c_{k,n+1}(\mathcal{F}_z)^{(k)}(0),
\end{aligned}$$

where

$$\begin{aligned}
c_{k,n+1} &= \frac{c_{k-1,n}}{k}, \quad 1 \leq k \leq n, \\
c_{0,n+1} &= -\sum_{k=1}^{n} c_{k-1,n}\frac{g_0^{(k)}(0)}{k} - \frac{g_0^{(n+1)}(0)}{(n+1)!}.
\end{aligned}$$

□

For $f \in E$, $z, t \in \mathbb{C}$, put

$$\widetilde{T}_z(f)(t) := \begin{cases} \frac{f(t)g_0(z) - f(z)g_0(t)}{t-z}, & t \neq z, \\ g_0(z)f'(z) - f(z)g_0'(z), & t = z; \end{cases}$$
$$\tau_z(f)(t) := f(t+z).$$

Condition (1.1) shows that the linear operators $\widetilde{T}_z$ and τ_z, $z \in \mathbb{C}$, map E into itself continuously (and an analog of Lemma 4 is valid for $\widetilde{T}_z$).

To prove that for $\varphi \in E'$ and $f \in E$ the function $z \mapsto \varphi(T_z(f))$, $z \in \mathbb{C}$, is entire, we shall need certain identities for the differences $\widetilde{T}_\mu(f) - \widetilde{T}_z(f)$ and the differentiability of the vector-valued mapping $z \in \mathbb{C} \mapsto \widetilde{T}_z(f) \in E$, which follows from these identities.

Remark 8. Formula (2.6) is also true for all functions $f \in A(\mathbb{C})$.

Lemma 9. (i) *For $f \in E$, $\mu, z \in \mathbb{C}$, we have*

$$\begin{aligned}\widetilde{T}_\mu(f) - \widetilde{T}_z(f) = g_0(\mu)\big((\mu - z)D_{\mu,g_0}(D_{z,g_0}(f)) + f(z)D_{\mu,g_0}(\tau_{-z}(g_0))\big)\\ - f(\mu)\big((\mu - z)D_{\mu,g_0}(D_{z,g_0}(g_0)) + g_0(z)D_{\mu,g_0}(\tau_{-z}(g_0))\big)\\ + (g_0(\mu) - g_0(z))D_{z,g_0}(f) - (f(\mu) - f(z))D_{z,g_0}(g_0).\end{aligned}$$

(ii) *For every $f \in E$, $z \in \mathbb{C}$ there exists $r \in \mathbb{N}$ such that the following formula is true in E_r:*

$$\begin{aligned}\lim_{\mu\to z} \frac{\widetilde{T}_\mu(f) - \widetilde{T}_z(f)}{\mu - z} &= g_0(z)\big(D^2_{z,g_0}(f) + f(z)D_{z,g_0}(\tau_{-z}(g_0'))\big)\\ &- f(z)\big(D^2_{z,g_0}(g_0) + g_0(z)D_{z,g_0}(\tau_{-z}(g_0'))\big) + g_0'(z)D_{z,g_0}(f) - f'(z)D_{z,g_0}(g_0).\end{aligned}$$

(iii) *For every $\varphi \in E'$ and $f \in E$, the function $\varphi(T_z(f))$ (of z) belongs to E.*

Proof. (i) Since

$$\begin{aligned}\widetilde{T}_z(f)(t) &= \frac{f(t)g_0(z) - f(z)g_0(t)}{t - z}\\ &= g_0(z)\frac{f(t) - g_0(t - z)f(z)}{t - z} - f(z)\frac{g_0(t) - g_0(t - z)g_0(z)}{t - z}\\ &= g_0(z)D_{z,g_0}(f)(t) - f(z)D_{z,g_0}(g_0)(t)\end{aligned}$$

for $t \neq z$, it follows that

$$\widetilde{T}_z(f) = g_0(z)D_{z,g_0}(f) - f(z)D_{z,g_0}(g_0).$$

Taking into account the relation

$$D_{\mu,g_0}(f) - D_{z,g_0}(f) = (\mu - z)D_{\mu,g_0}\big(D_{z,g_0}(f)\big) + f(z)D_\mu(\tau_{-z}(g_0)),$$

see [3, Lemma 4], for $\mu, z \in \mathbb{C}$ we obtain

$$\begin{aligned}\widetilde{T}_\mu(f) - \widetilde{T}_z(f) &= g_0(\mu)\big(D_{\mu,g_0}(f) - D_{z,g_0}(f)\big) + \big(g_0(\mu) - g_0(z)\big)D_{z,g_0}(f)\\ &\quad - f(\mu)\big(D_{\mu,g_0}(g_0) - D_{z,g_0}(g_0)\big) - \big(f(\mu) - f(z)\big)D_{z,g_0}(g_0)\\ &= g_0(\mu)\big((\mu - z)D_{\mu,g_0}(D_{z,g_0}(f)) + f(z)D_{\mu,g_0}(\tau_{-z}(g_0))\big)\\ &\quad - f(\mu)\big((\mu - z)D_{\mu,g_0}(D_{z,g_0}(g_0)) + g_0(z)D_{\mu,g_0}(\tau_{-z}(g_0))\big)\\ &\quad + \big(g_0(\mu) - g_0(z)\big)D_{z,g_0}(f) - \big(f(\mu) - f(z)\big)D_{z,g_0}(g_0).\end{aligned}$$

(ii) Statement (i) shows that

$$\begin{aligned}\frac{\widetilde{T}_\mu(f) - \widetilde{T}_z(f)}{\mu - z} &= g_0(\mu)\left(D_{\mu,g_0}(D_{z,g_0}(f)) + f(z)\frac{D_{\mu,g_0}(\tau_{-z}(g_0))}{\mu - z}\right)\\ &\quad - f(\mu)\left(D_{\mu,g_0}(D_{z,g_0}(g_0)) + g_0(z)\frac{D_{\mu,g_0}(\tau_{-z}(g_0))}{\mu - z}\right)\\ &\quad + \frac{g_0(\mu) - g_0(z)}{\mu - z}D_{z,g_0}(f) - \frac{f(\mu) - f(z)}{\mu - z}D_{z,g_0}(g_0)\end{aligned}$$

for $\mu \neq z$. Now, (ii) follows from [3, Lemma 7 (iii), (iv)].

(iii) Since $M(f) \in E$ (M is the operator of multiplication by the independent variable), the function $\varphi(T_z(f)) = \varphi(\widetilde{T}_z(M(f)))$, $z \in \mathbb{C}$, is entire by (ii).

Let $M(f) \in E_n$, and let m be chosen by n as in inequality (2.1) of Lemma 4. Since $\varphi \in E'$, the functional φ is continuous on E_m; therefore, there exists k and $B \geq 0$ such that

$$|\varphi(h)| \leq Bp_{v_{m,k}}(h), \quad h \in E_m. \tag{2.7}$$

By (2.7) and inequality (2.1) in Lemma 4, for every l there exists $s \in \mathbb{N}$ and $C \geq 0$ such that

$$\big|\varphi(T_z(f))\big| \leq Bp_{v_{m,k}}(T_z(f)) \leq BCp_{v_{n,s}}(f)\exp v_{m,l}(z), \quad z \in \mathbb{C}. \tag{2.8}$$

Consequently, the function $\varphi(T_z(f))$, $z \in \mathbb{C}$, belongs to E_m. □

Remark 10. Let φ_n, $n \geq 0$, be functionals as in Lemma 7. The system $\{\varphi_n : n \geq 0\}$ is complete in E' in every topology λ compatible with the natural duality between E' and E (that is, the closure of the linear hull of the set $\{\varphi_n : n \geq 0\}$ in (E', λ) coincides with E').

Proof. Let $f \in E$ satisfy $\varphi_n(f) = 0$, $n \geq 0$. Then $f^{(n)}(0) = 0$ for every $n \geq 0$, whence $f \equiv 0$. Therefore, the system $\{\varphi_n : n \geq 0\}$ is complete in E' in an arbitrary topology compatible with the natural duality between E' and E, see [13, Chapter 2, §2.3, Theorem 2.3.1]. □

2.2. The convolution in E' generated by the shift operator. We denote by $\tau := \tau(E', E)$ the Mackey topology in E', i.e., the topology of uniform convergence on the family of all absolutely convex $\sigma(E, E')$-compact subsets of E, see [13, Chapter 8, 8.3.3]. Here $\sigma(E, E')$ is the weak topology in E determined by the natural duality between E and E'.

For $f \in E$, we put

$$\widetilde{f}(t, z) := T_z(f)(t), \quad t, z \in \mathbb{C}.$$

Lemma 11. *Suppose that a net $\Psi_\mu \in E'$, $\mu \in \Delta$, converges to $\psi \in E'$ in (E', τ). Then for every function $f \in E$ there exists $m \in \mathbb{N}$ such that in E_m (consequently, also in E) we have*

$$(\Psi_\mu)_z(\widetilde{f}(\cdot, z)) \underset{\mu\in\Delta}{\longrightarrow} \psi_z(\widetilde{f}(\cdot, z)).$$

Proof. Take $f \in E$. By Lemma 4, there exists $m \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ the set

$$V_k(f) = \{\exp(-v_{m,k}(t))\widetilde{f}(t, \cdot) : t \in \mathbb{C}\}$$

is bounded in E_m. Fixing $k \in \mathbb{N}$, we denote by $\mathrm{ac}(V_k(f))$ the absolutely convex hull of $V_k(f)$ in E_m. The set $\mathrm{ac}(V_k(f))$ is also bounded in E_m. By [3, Remark 1], there exists $s \in \mathbb{N}$ such that $\mathrm{ac}(V_k(f))$ is relatively compact in E_s. The closure $\overline{\mathrm{ac}(V_k(f))}$ of $\mathrm{ac}(V_k(f))$ in E_s is compact. Since E_s embeds continuously in $(E, \sigma(E, E'))$, the absolutely convex set $\overline{\mathrm{ac}(V_k(f))}$ is also $\sigma(E, E')$-compact. Therefore, $\Psi_\mu \underset{\mu\in\Delta}{\to} \psi$ uniformly on $\overline{\mathrm{ac}(V_k(f))}$ and, *a fortiori*, on $V_k(f)$. Consequently,

$$\sup_{t\in\mathbb{C}}\big|\exp(-v_{m,k}(t))(\Psi_\mu - \psi)_z(\widetilde{f}(t, z))\big| \underset{\mu\in\Delta}{\longrightarrow} 0.$$

Thus,

$$p_{v_{m,k}}\big((\Psi_\mu - \psi)_z(\widetilde{f}(\cdot, z))\big) \underset{\mu\in\Delta}{\longrightarrow} 0.$$

So, we have

$$(\Psi_\mu)_z(\widetilde{f}(\cdot, z)) \underset{\mu\in\Delta}{\longrightarrow} \psi_z(\widetilde{f}(\cdot, z))$$

in E_m. □

We introduce a binary operation on E': for φ, $\psi \in E'$ and $f \in E$ put

$$(\varphi \otimes \psi)(f) := \varphi_t\big(\psi_z(\widetilde{f}(t, z))\big).$$

Lemma 9 (iii) shows that the operation $\otimes$ is well defined. Observe that for every $\psi, \varphi \in E'$ and $f \in E$ we have

$$(\varphi \otimes \psi)(f) = \varphi_t\big(\psi(T_t(f))\big). \tag{2.9}$$

It is natural to call the operation $\otimes$ the *convolution generated by a Pommiez type shift operator*. The space E' with the operation $\otimes$ in the role of multiplication is an algebra. Below we shall show that the operation $\otimes$ is commutative.

Lemma 12. *Let φ_n, $n \geq 0$, be functionals as in Lemma 7. For every entire function h on $\mathbb{C}^2$ and every integers $j, k \geq 0$ we have*

$$(\varphi_j)_z\big((\varphi_k)_t(h(t,z))\big) = (\varphi_k)_t\big((\varphi_j)_z(h(t,z))\big).$$

This result is a consequence of the independence of the partial derivatives for h of the differentiation order (see also Remark 8).

Lemma 13. *We have $\varphi \otimes \psi = \psi \otimes \varphi$ for every $\varphi, \psi \in E'$.*

Proof. Let $\varphi_n \in E'$, $n \geq 0$, be functionals as in Lemma 7. By Remark 10, the system $\{\varphi_n : n \geq 0\}$ is complete in (E', τ). Consequently, there exist nets $\big\{\Phi_\lambda := \sum_{j=0}^{m_\lambda} a_{j\lambda}\varphi_j : \lambda \in \Lambda\big\}$ and $\big\{\Psi_\mu := \sum_{k=0}^{n_\mu} b_{k\mu}\varphi_k : \mu \in \Delta\big\}$ ($a_{j\lambda}, b_{k\mu} \in \mathbb{C}$, $m_\lambda, n_\mu \in \mathbb{N} \cup \{0\}$) convergent in (E', τ) to φ and ψ, respectively.

Take $f \in E$. Invoking Lemma 11 and the fact that, by Lemma 12,

$$(\Phi_\lambda)_t\big((\Psi_\mu)_z(\widetilde{f}(t,z))\big) = (\Psi_\mu)_z\big((\Phi_\lambda)_t(\widetilde{f}(t,z))\big), \quad \lambda \in \Lambda, \ \mu \in \Delta,$$

we obtain

$$\begin{aligned}
(\varphi \otimes \psi)(f) &= \varphi_t\big(\psi_z(\widetilde{f}(t,z))\big) = \varphi_t\big(\lim_{\mu \in \Delta}(\Psi_\mu)_z\big(\widetilde{f}(t,z)\big)\big) \\
&= \lim_{\mu \in \Delta} \varphi_t\big((\Psi_\mu)_z(\widetilde{f}(t,z))\big) = \lim_{\mu \in \Delta}\big(\lim_{\lambda \in \Lambda}(\Phi_\lambda)_t\big((\Psi_\mu)_z(\widetilde{f}(t,z))\big)\big) \\
&= \lim_{\mu \in \Delta}\big(\lim_{\lambda \in \Lambda}(\Psi_\mu)_z\big((\Phi_\lambda)_t(\widetilde{f}(t,z))\big)\big) = \lim_{\mu \in \Delta}\big((\Psi_\mu)_z\big(\lim_{\lambda \in \Lambda}(\Phi_\lambda)_t(\widetilde{f}(t,z))\big)\big) \\
&= \lim_{\mu \in \Delta}(\Psi_\mu)_z\big(\varphi_t(\widetilde{f}(t,z))\big) = \psi_z\big(\varphi_t(\widetilde{f}(t,z))\big) = (\psi \otimes \varphi)(f).
\end{aligned}$$

□

Theorem 14. *If $B \in \mathcal{L}(E)$ and $BD_{0,g_0} = D_{0,g_0}B$ on E, then*

$$BT_z = T_zB$$

for every $z \in \mathbb{C}$.

Proof. Fixing $z \in \mathbb{C}$, we observe that (here $\mathcal{F}_t := T_t(f)$)

$$T_z(f)(t) = \delta_z(\mathcal{F}_t).$$

Let φ_n, $n \geq 0$, be functionals as in Lemma 7. By Remark 10, there exists a net $\big\{\Phi_\alpha := \sum_{j=0}^{n_\alpha} b_{j\alpha}\varphi_j : \alpha \in \Lambda\big\}$ ($b_{j\alpha} \in \mathbb{C}, n_\alpha \in \mathbb{N} \cup \{0\}$) convergent to δ_z in (E', τ). We show that

$$\lim_{\alpha \in \Lambda} \sum_{j=0}^{n_\alpha} b_{j\alpha} D_{0,g_0}^j(f) = T_z(f)$$

in E for every function $f \in E$. Take $f \in E$. As in the proof of Lemma 11, we deduce the existence of $m \in \mathbb{N}$ such that in E_m (with respect to t) we have

$$\Phi_\alpha(\mathcal{F}_t) \underset{\alpha \in \Lambda}{\longrightarrow} \delta_z(\mathcal{F}_t).$$

Since

$$\Phi_\alpha(\mathcal{F}_t) = \sum_{j=0}^{n_\alpha} b_{j\alpha}\varphi_j(\mathcal{F}_t) = \sum_{j=0}^{n_\alpha} b_{j\alpha} D_{0,g_0}^j(f)(t),$$

the following is true in E_m (consequently, in E):

$$\sum_{j=0}^{n_\alpha} b_{j\alpha} D_{0,g_0}^j(f) \underset{\alpha \in \Lambda}{\longrightarrow} T_z(f).$$

Thus, for $f \in E$ we obtain

$$B(T_z(f)) = B\Big(\lim_{\alpha\in\Lambda}\sum_{j=0}^{n_\alpha} b_{j\alpha} D^j_{0,g_0}(f)\Big) = \lim_{\alpha\in\Lambda}\sum_{j=0}^{n_\alpha} b_{j\alpha} B\big(D^j_{0,g_0}(f)\big)$$
$$= \lim_{\alpha\in\Lambda}\sum_{j=0}^{n_\alpha} b_{j\alpha} D^j_{0,g_0}(B(f)) = T_z(B(f)).$$

□

§3. Principal result

3.1. A general form of the operators commuting with D_{0,g_0}, and consequences.

Theorem 15. *The following statements are equivalent*:

(i) $B \in \mathcal{L}(E)$ *and* $BD_{0,g_0} = D_{0,g_0}B$ *on* E;

(ii) *there exists* $\varphi \in E'$ *with*

$$B(f)(z) = \varphi(T_z(f)), \quad f \in E, \ z \in \mathbb{C}.$$

Proof. (i) $\Rightarrow$ (ii). The proof of this implication is analogous to the proof of a similar implication in Theorem 1.8 in [15].

By Theorem 14, $BT_z = T_zB$ on E for every $z \in \mathbb{C}$. Since

$$T_\xi(f)(z) = T_z(f)(\xi),$$

we have

$$B(T_z(f))(\xi) = T_z(B(f))(\xi) = T_\xi(B(f))(z). \tag{3.1}$$

Put $\varphi(f) := B(f)(0)$, $f \in E$. Then $\varphi \in E'$. Formula (3.1) with $\xi = 0$ implies

$$T_0(B(f))(z) = B(T_z(f))(0),$$

i.e. (because T_0 is the identity operator),

$$B(f)(z) = \varphi(T_z(f)), \quad f \in E, \ z \in \mathbb{C}. \tag{3.2}$$

(ii) $\Rightarrow$ (i). By Lemma 9 (iii), the linear operator B maps E to E. Since the graph of B is closed in $E \times E$, we have $B \in \mathcal{L}(E)$ by the closed graph theorem, see [13, Chapter 6, §6.7, Theorem 6.7.1] (the continuity of $B : E \to E$ can also be deduced from inequality (2.8)). The relation $BD_{0,g_0} = D_{0,g_0}B$ on E is proved by direct inspection. □

Remark 16. (a) If an operator $B \in \mathcal{L}(E)$ commutes with D_{0,g_0} on E, the functional $\varphi \in E'$ in (ii) is unique.

Indeed, formula (3.2) with $z = 0$ shows that

$$\varphi(f) = B(f)(0), \quad f \in E.$$

(b) The operators commuting with A (as in Remark 3; *then they also commute with* D_{0,g_0}) in $A(G)$ were described by Yu. Linchuk, see [19]. It was the paper [19] that suggested us the formula in (ii) in the case under study. In Linchuk's proof, the Köthe theory of characteristic functions $l(\lambda, z)$ for the operator $A \in \mathcal{L}(A(G))$ was used substantially:

$$l(\lambda, z) := A_t\Big(\frac{1}{\lambda - t}\Big)(z), \quad \lambda \notin G, \ z \in G.$$

A similar result for $\psi \equiv 0$ (see (1.2)), i.e., for the usual Pommiez operator D_0, had been obtained by N. Linchuk by the same method even before, see [7].

(c) Using a different method, Dimovski and Hristov (see [15]) described the commutants for the usual Pommiez operator D_0 on $A(G)$ for a simply connected domain $G \subseteq \mathbb{C}$. We employ the idea of Dimovski and Hristov, which reduces the corresponding description to Theorem 14. The implementation of this idea in the present setting has met considerable difficulties related to the absence of certain important properties intrinsic for $A(G)$, specifically, the simple way in which T_z acts on polynomials (for $g_0 \equiv 1$) and the density of the set of polynomials on $A(G)$.

We denote by $\mathcal{K}(D_{0,g_0})$ the set of all commutants for D_{0,g_0} in E. Note that $\mathcal{K}(D_{0,g_0})$ is a subspace of $\mathcal{L}(E)$ and is an algebra with composition of operators in the role of multiplication.

For $\varphi \in E'$, we put

$$\kappa(\varphi)(f)(z) := \varphi(T_z(f)), \quad f \in E, \ z \in \mathbb{C}.$$

By Theorem 15, κ takes E' onto $\mathcal{K}(D_{0,g_0})$ bijectively.

Lemma 17. *For every $\varphi, \psi \in E'$ we have*

$$\kappa(\varphi \otimes \psi) = \kappa(\varphi)\kappa(\psi).$$

Proof. Take $\varphi, \psi \in E'$. On the one hand, for every $f \in E$, $z \in \mathbb{C}$ we can use (2.9) to obtain

$$\kappa(\varphi \otimes \psi)(f)(z) = (\varphi \otimes \psi)(T_z(f)) = \varphi_t(\psi(T_t(T_z(f)))).$$

On the other hand, since the operators $\kappa(\psi)$ and T_z commute by Theorems 14 and 15, for $f \in E$ and $z \in \mathbb{C}$ we arrive at

$$\begin{aligned}\kappa(\varphi)\kappa(\psi)(f)(z) &= \kappa(\varphi)\big(\kappa(\psi)(f)\big)(z) = \varphi\big(T_z(\kappa(\psi)(f))\big)\\ &= \varphi\big(\kappa(\psi)(T_z(f))\big) = \varphi_t\big(\psi\big(T_t(T_z(f))\big)\big).\end{aligned}$$

Hence, $\kappa(\varphi \otimes \psi) = \kappa(\varphi)\kappa(\psi)$. $\square$

Corollary 18. *The mapping $\kappa\colon (E', \otimes) \to \mathcal{K}(D_{0,g_0})$ is an isomorphism of algebras.*

Proof. By Theorem 15, the (linear) mapping

$$\kappa\colon E' \to \mathcal{K}(D_{0,g_0})$$

is bijective. By Lemma 17, $\kappa\colon E' \to \mathcal{K}(D_{0,g_0})$ is an isomorphism of algebras. $\square$

Remark 19. Corollary 18 and Lemma 13 show that the commutative operation $\otimes$ is also associative, and the composition of operators in $\mathcal{K}(D_{0,g_0})$ is not only associative, but also commutative. Therefore, $(E', \otimes)$ and $\mathcal{K}(D_{0,g_0})$ are associative and commutative algebras.

Let $\mathcal{P}(D_{0,g_0})$ be the set of all polynomials of the operator D_{0,g_0}, i.e., the set of all operators of the form

$$P(D_{0,g_0}) = \sum_{j=0}^{n} a_j D_{0,g_0}^j, \quad a_j \in \mathbb{C}, \ n \geq 0.$$

Clearly, $\mathcal{P}(D_{0,g_0})$ is a subspace of $\mathcal{K}(D_{0,g_0})$.

Corollary 20. *$\mathcal{K}(D_{0,g_0})$ coincides with the closure of $\mathcal{P}(D_{0,g_0})$ in $\mathcal{L}(E)$ with the topology of pointwise convergence* (*see* [12, Chapter III, §3, example 4(a)]).

Proof. Let $B \in \mathcal{K}(D_{0,g_0})$. By Theorem 15, there exists $\varphi \in E'$ such that

$$B(f)(z) = \varphi(T_z(f)), \quad f \in E, \ z \in \mathbb{C}.$$

By Remark 10, there exists a net $\big\{\Phi_\alpha := \sum_{j=0}^{n_\alpha} c_{j\alpha}\varphi_j \ : \ \alpha \in \Lambda\big\}$ ($c_{j\alpha} \in \mathbb{C}, n_\alpha \in \mathbb{N} \cup \{0\}$) convergent to φ in (E', τ).

Take $f \in E$. Arguing as in the proof of Lemma 11, we arrive at the existence of $m \in \mathbb{N}$ such that

$$\sup_{h \in W_k(f)} |(\varphi - \Phi_\alpha)(h)| \underset{\alpha \in \Lambda}{\longrightarrow} 0 \tag{3.3}$$

for every $k \in \mathbb{N}$. (Here $W_k(f) := \{\exp(-v_{m,k}(t)) T_t(f) : t \in \mathbb{C}\}$.) Therefore, for every $k \in \mathbb{N}$, using Lemma 7 and (3.3), we obtain the following relation for the polynomials $P_\alpha(z) := \sum_{j=0}^{n_\alpha} c_{j\alpha} z^j$:

$$\begin{aligned} p_{v_{m,k}}\big((B - P_\alpha(D_{0,g_0}))(f)\big) &= \sup_{t \in \mathbb{C}} \bigg(\exp(-v_{m,k}(t)) \bigg| \varphi(T_t(f)) - \sum_{j=0}^{n_\alpha} c_{j\alpha} D_{0,g_0}^j(f)(t) \bigg| \bigg) \\ &= \sup_{t \in \mathbb{C}} \bigg(\exp(-v_{m,k}(t)) \bigg| \varphi(T_t(f)) - \sum_{j=0}^{n_\alpha} c_{j\alpha} \varphi_j(T_t(f)) \bigg| \bigg) \\ &= \sup_{h \in W_k(f)} \big| \varphi(h) - \Phi_\alpha(h) \big| \underset{\alpha \in \Lambda}{\longrightarrow} 0. \end{aligned}$$

Thus, $P_\alpha(D_{0,g_0})(f) \underset{\alpha \in \Lambda}{\to} B(f)$ in E for every $f \in E$. □

3.2. Example. Consider the following situation. Let G be a convex domain in $\mathbb{C}$ containing 0, let $(K_n)_{n \in \mathbb{N}}$ be a sequence of compact convex subsets of $\mathbb{C}$ with $K_n \subset \operatorname{int} K_{n+1}$, $n \in \mathbb{N}$, and let $G = \bigcup_{n \in \mathbb{N}} K_n$ ($\operatorname{int} \Omega$ denotes the interior of a subset Ω of $\mathbb{C}$). We denote by H_n the support function for K_n:

$$H_n(z) := \sup_{t \in K_n} \operatorname{Re}(tz), \quad z \in \mathbb{C}, \ n \in \mathbb{N}.$$

Put $v_{n,k} := H_n$, $n, k \in \mathbb{N}$. The functions $v_{n,k}$ (they do not depend on k) satisfy (1.1). In the case in question, E is an (LB)-space: $E = \operatorname{ind}_{n \to} E_n$, where the weighted Banach spaces E_n are defined as follows:

$$E_n := \bigg\{ f \in A(\mathbb{C}) : \|f\|_n := \sup_{z \in \mathbb{C}} \frac{|f(z)|}{\exp H_n(z)} < +\infty \bigg\}.$$

For the role of g_0, we take the function $g_0 \equiv 1$, which belongs to E. We shall write D_z in place of D_{z,g_0}, $z \in \mathbb{C}$. Let $e_z(t) := \exp(tz)$, $t, z \in \mathbb{C}$.

As is well known, in the present setting the space E' can be identified with the Fréchet space of all functions analytic in G. More precisely, the transformation

$$\Phi \colon E' \to A(G), \quad \varphi \mapsto \varphi(e_z), \ z \in G,$$

is an algebraic isomorphism (which is also topological if we endow E' with the strong topology of the dual to E). Since the structure of Φ will be used in what follows, we include the proof of this fact. For a locally convex space H, we denote by H'_b the strong dual to H. By [10, Theorem 4.5.3], the Laplace transformation

$$\mathcal{F}(\psi)(\lambda) := \psi(e_\lambda), \quad \psi \in A(G)', \ \lambda \in \mathbb{C},$$

is a linear topological isomorphism of $A(G)'_b$ onto E. The space $A(G)$ is reflexive, whence we see that the canonical mapping (see [12, Chapter 4, §5, p. 182])

$$\mathcal{J} \colon A(G) \to A(G)'', \ f \mapsto \mathcal{J}(f) : \mathcal{J}(f)(\psi) = \psi(f), \quad \psi \in A(G)',$$

is bijective (and continuous if $A(G)''$ is endowed with the topology of the second strong dual). Consequently, the mapping $\mathcal{J}^{-1}\mathcal{F}'$ is a linear topological isomorphism of E'_b onto $A(G)$. Moreover, for $\varphi \in E'$ and $z \in G$ we have

$$\begin{aligned} (\mathcal{J}^{-1}\mathcal{F}')(\varphi)(z) &= \delta_z\big((\mathcal{J}^{-1}\mathcal{F}')(\varphi)\big) = \mathcal{J}\big((\mathcal{J}^{-1}\mathcal{F}')(\varphi)\big)(\delta_z) \\ &= \mathcal{F}'(\varphi)(\delta_z) = \varphi(\mathcal{F}(\delta_z)) = \varphi(e_z). \end{aligned}$$

Therefore, $\Phi = \mathcal{J}^{-1}\mathcal{F}'$, and Φ is a linear topological isomorphism of E'_b onto $A(G)$.

The natural duality between E' and E induces a duality between $A(G)$ and E. The latter is given by the bilinear form

$$\langle h, f\rangle = \Phi^{-1}(h)(f), \quad h \in A(G),\ f \in E.$$

Since $A(G)$ is reflexive, for $h \in A(G)$ and $f \in E$ we obtain

$$\Phi^{-1}(h)(f) = ((\mathcal{F}')^{-1}\mathcal{J})(h)(f) = ((\mathcal{F}^{-1})'\mathcal{J})(h)(f) = \mathcal{J}(h)(\mathcal{F}^{-1}(f)) = \mathcal{F}^{-1}(f)(h). \tag{3.4}$$

By (3.4) and [10, Chapter IV, 4.5, p. 137–139], we have

$$\langle h, f\rangle = \frac{1}{2\pi i}\int_C \gamma_f(t)h(t)dt, \tag{3.5}$$

where γ_f is the Borel transform of $f \in E$, and C is a closed rectifiable Jordan curve lying both in G and in the domain of analyticity of γ_f.

It should be noted that

$$\langle e_\lambda, f\rangle = f(\lambda), \quad f \in E,\ \lambda \in \mathbb{C}$$

by (3.5). Moreover,

$$\langle h, e_z\rangle = \Phi^{-1}(h)(e_z) = \Phi(\Phi^{-1}(h))(z), \quad h \in A(G),\ z \in G.$$

Lemma 21. *Let $D'_z\colon A(G) \to A(G)$, $z \in \mathbb{C}$, and $M'\colon A(G) \to A(G)$ be the adjoints to $D_z\colon E \to E$ and $M\colon E \to E$ (M is multiplication by the independent variable) when the topological dual to E is identified with $A(G)$. Then:*

(i) *we have the identity*

$$D'_z(h)(t) = \int_0^t e^{\xi z}h(t-\xi)d\xi, \quad h \in A(G),\ z \in G, \tag{3.6}$$

where the integral is taken over the interval $[0, t]$;

(ii) *M' is the differentiation operator on $A(G)$.*

Proof. (i) For $f \in E$ and $h \in A(G)$ we have

$$\langle D'_z(h), f\rangle = \langle h, D_z(f)\rangle.$$

If $h = e_\lambda$, $\lambda \in \mathbb{C}$, then for $\lambda \neq z$ we obtain

$$\langle D'_z(e_\lambda), f\rangle = \langle e_\lambda, D_z(f)\rangle = D_z(f)(\lambda) = \frac{f(\lambda) - f(z)}{\lambda - z} = \Big\langle \frac{e_\lambda - e_z}{\lambda - z}, f\Big\rangle.$$

Therefore, if $\lambda \neq z$, then

$$D'_z(e_\lambda) = \frac{e_\lambda - e_z}{\lambda - z},$$

i.e.,

$$D'_z(e_\lambda)(t) = \int_0^t e^{\xi z}e^{\lambda(t-\xi)}\,d\xi, \quad \lambda, z \in \mathbb{C},\ t \in G.$$

Since D'_z is continuous and linear on $A(G)$ and the set $\{e_\lambda : \lambda \in \mathbb{C}\}$ is complete in $A(G)$, identity (3.6) is fulfilled for all $h \in A(G)$.

(ii) Suppose that $D(h) := h'$ and $h \in A(G)$. For $\lambda \in \mathbb{C}$ and $f \in E$ we have

$$\langle D(e_\lambda), f\rangle = \langle \lambda e_\lambda, f\rangle = \lambda f(\lambda) = M(f)(\lambda) = \langle e_\lambda, M(f)\rangle = \langle M'(e_\lambda), f\rangle.$$

Thus, $M'(e_\lambda) = D(e_\lambda)$, $\lambda \in \mathbb{C}$. Since the set $\{e_\lambda : \lambda \in \mathbb{C}\}$ is complete in $A(G)$, we see that $M' = D$ on $A(G)$. □

Since $g_0 \equiv 1$, we have $T_z = D_zM$. Therefore,

$$\varphi(T_z(f)) = \varphi\big(D_z(M(f))\big)$$

for $\varphi \in E'$, $z \in \mathbb{C}$, and $f \in E$. For $\varphi \in E'$, we put $\widehat{\varphi} := \Phi(\varphi)$.

Applying Lemma 21 and formula (3.5), for $\varphi \in E'$, $f \in E$, and $z \in \mathbb{C}$ we obtain

$$\varphi(T_z(f)) = \langle \widehat{\varphi}, D_z(M(f))\rangle = \langle D_z'(\widehat{\varphi}), M(f)\rangle = \frac{1}{2\pi i}\int_C \gamma_{M(f)}(t)\bigg(\int_0^t e^{z\xi}\widehat{\varphi}(t-\xi)\,d\xi\bigg)dt,$$

where C is a closed rectifiable Jordan curve contained in G and in the domain where γ_f (and $\gamma_{M(f)}$) are analytic. Since $\gamma_{M(f)} = -\gamma_f'$, integration by parts yields

$$\begin{aligned}\varphi(T_z(f)) &= \frac{1}{2\pi i}\int_C \gamma_f(t)\bigg(e^{zt}\widehat{\varphi}(0) + \int_0^t e^{z\xi}(\widehat{\varphi})'(t-\xi)\,d\xi\bigg)dt \\ &= \widehat{\varphi}(0)f(z) + \frac{1}{2\pi i}\int_C \gamma_f(t)\bigg(\int_0^t e^{z\xi}(\widehat{\varphi})'(t-\xi)\,d\xi\bigg)dt.\end{aligned}$$

By Theorem 15 and isomorphism between E' and $A(G)$, we have proved the following result.

Theorem 22. (i) *If an operator $B \in \mathcal{L}(E)$ commutes with D_0 in E, then there exists a unique function $v \in A(G)$ such that*

$$B(f)(z) = v(0)f(z) + \frac{1}{2\pi i}\int_C \gamma_f(t)\bigg(\int_0^t e^{z\xi}v'(t-\xi)\,d\xi\bigg)dt, \quad f \in E,\ z \in \mathbb{C}, \tag{3.7}$$

where γ_f is Borel associated with f, and C is a closed rectifiable Jordan curve in G that lies in the domain where γ_f is analytic.

(ii) *If B is defined by* (3.7), *where $v \in A(G)$, then B is continuous and linear on E, and we have $BD_0 = D_0B$ on E.*

In conclusion, we explain how the binary operation $\otimes$ is realized if the dual to E is identified with $A(G)$. For $\varphi, \psi \in E'$ and $\lambda \in G$, we have

$$\begin{aligned}\widehat{\varphi\otimes\psi}(\lambda) &= (\varphi\otimes\psi)(e_\lambda) = \varphi_t(\psi(T_t(e_\lambda)) = \varphi_t\big(\psi\big(D_t(M(e_\lambda))\big)\big) \\ &= \varphi_t\big(\langle\widehat{\psi}, D_t(M(e_\lambda))\rangle\big) = \varphi_t\big(\langle D_t'(\widehat{\psi}), M(e_\lambda)\rangle\big) = \varphi_t\big(\langle M'D_t'(\widehat{\psi}), e_\lambda\rangle\big).\end{aligned} \tag{3.8}$$

Next, for $\tau \in G$ we have

$$M'\big(D_t'(\widehat{\psi})\big)(\tau) = \frac{d}{d\tau}\bigg(\int_0^\tau e^{\xi t}\widehat{\psi}(\tau-\xi)\,d\xi\bigg)(\tau) = \widehat{\psi}(0)e^{t\tau} + \int_0^\tau e^{\xi t}(\widehat{\psi})'(\tau-\xi)\,d\xi.$$

Thus, formula (3.8) implies

$$\begin{aligned}\widehat{\varphi\otimes\psi}(z) &= \bigg\langle\bigg(\widehat{\psi}(0)\widehat{\varphi}(\tau) + \int_0^\tau \widehat{\varphi}(\xi)(\widehat{\psi})'(\tau-\xi)\,d\xi\bigg)_\tau, e_z\bigg\rangle \\ &= \widehat{\psi}(0)\widehat{\varphi}(z) + \int_0^z \widehat{\varphi}(\xi)(\widehat{\psi})'(z-\xi)\,d\xi, \quad z \in G.\end{aligned}$$

So, the operation $\otimes$ is realized in terms of the Duhamel integral (convolution).

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