

NEW ALGORITHMS FOR SOLVING TROPICAL LINEAR SYSTEMS

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ABSTRACT. The problem of solving tropical linear systems, a natural problem of tropical mathematics, has already proved to be very interesting from the algorithmic point of view: it is known to be in $NP \cap coNP$, but no polynomial time algorithm is known, although counterexamples for existing pseudopolynomial algorithms are (and must be) very complex.

In this work, the study of algorithms for solving tropical linear systems is continued. First, a new reformulation of Grigoriev’s algorithm is presented, which brings it closer to the algorithm of Akian, Gaubert, and Guterman; this makes it possible to formulate a whole family of new algorithms, and, for some algorithms in this family, none of the known superpolynomial counterexamples work. Second, a family of algorithms for solving overdetermined tropical systems is presented.

An explicit algorithm is exhibited in the paper that can solve a tropical linear system determined by an $(m \times n)$ -matrix with maximal element M in time $\Theta(\binom{m}{n} \text{poly}(m, n, \log M))$, and this time matches the complexity of the best of previously known algorithms for feasibility testing.

§1. INTRODUCTION

1.1. Tropical mathematics and tropical linear algebra. *Tropical mathematics* unites three closely related fields of study: tropical algebra, tropical analysis, and tropical geometry. The term is usually taken to mean mathematics obtained from classical mathematics by replacing the addition and multiplication operations with minimum and addition (respectively), hence the term *min-plus* algebra. Sometimes maximum is used instead of minimum, with perfectly symmetrical results, so in what follows we always use the minimum operation. Taking the minimum in tropical context is usually denoted by \oplus ; addition is denoted by \otimes . The \oplus operation is idempotent, i.e., $a \oplus a = a$, so tropical mathematics is in fact a part of idempotent mathematics, although lately these notions have often been identified so that the term “tropical” is sometimes applied to any mathematical constructions with an idempotent operation.

Tropical algebra was the first section of tropical mathematics to appear. The term originates from French mathematicians and first appeared in the 1980s. Although different authors attribute the term to different researchers [31, 28, 21], all sources agree that the term came into general use in the honor of one of the founders of this field, a Brazilian mathematician Imre Simon, so the term “tropical” simply means how French mathematicians viewed Brazil. Simon himself used the term already in the paper [30] that laid out the foundations of tropical algebra, attributing the term to Christian Choffrut.

Initially, the term “tropical” was used for the discrete version of $(\min, +)$ algebra, but at present the terminology has shifted, and tropical algebra is usually meant to be an algebra over the semifield \mathbb{R}_{\min} (as stated above) or even, sometimes, an arbitrary

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algebra with an idempotent operation, e.g., (\min, \max) algebra. Both Simon and his French colleagues used tropical algebra for the study of finite state machines.

Although a systematic study of the tropical semiring began only after the works of Simon, we should note that the $(\min, +)$ semiring had appeared before in optimization problems. For instance, Floyd's algorithm for finding shortest paths in a graph that was proposed in [9] in the 1960s can be regarded as taking a tropical power of the distance matrix (tropical exponentiation is similar to classical with the difference that we replace addition with \oplus and multiplication with \otimes). Speaking about idempotent algebra as a whole, the first work to make serious use of an algebra over an idempotent ring (apart from Boolean fields) was the work of Kleene [18] that studied nerve nets in the context of finite state machines. At present, there is a host of literature on matrices with idempotent coefficients and their applications, e.g., [5, 19, 22].

Tropical linear algebra is a subfield of tropical algebraic geometry that works with systems of tropical linear equations. Unfortunately, definitions remain a major problem in this subfield. Many notions of linear algebra have several equivalent definitions, but after tropicalization equivalence disappears, and we are left with several different definitions. One striking example of this phenomenon is that there may be several conflicting definitions for the root of a tropical polynomial.

In the case of the root of a polynomial, researchers finally settled on the definition of O. Viro, and now the set of roots of a polynomial is the set of this polynomial's nonsmoothness points. This definition prevailed because it preserves such important properties as, for instance, the fundamental theorem of algebra (that a polynomial of degree n has exactly n roots, counting multiplicities).

The problem of solving linear systems was formulated right after the definition of a root for a tropical polynomial was given, but the first paper [7] actually devoted to tropical linear algebra appeared only as late as in 2005. At present, this field is primarily being developed in France (Akian, Gaubert, Grigoriev and others), sometimes in collaboration with researchers from other countries (Izhakian, Guterman).

Since there are no known efficient algorithms for the main problems of tropical linear algebra, it is currently little used in practice. Nevertheless, there are practical problems that would benefit from developments in this field. For instance, Noel, Grigoriev, Vakulenko, and Radulescu have recently proposed a way to use algorithms for solving tropical linear systems to study stable states of reaction networks in biology [27, 26]. Thus, problems of tropical linear algebra are important from both theoretical and practical points of view.

It is worth noting that many definitions and theorems of tropical geometry (including, in particular, tropical linear algebra) had appeared much earlier than tropical geometry itself took shape as a field of study. In these cases, tropical geometry serves as a language in which it is convenient to state theorems that have already been proved from a different viewpoint. Although at first glance this serves a little useful purpose, sometimes such a translation may lead to new results. For instance, a translation of Viro's patchworking method to the language of tropical geometry has led Mikhalkin to an algorithm for computing Gromov–Witten invariants; see [25].

1.2. Tropical linear systems. After researchers had agreed on the definitions of a tropical polynomial and tropical root, one of the first problems that they tried to solve was constructing an algorithm for testing the feasibility of a linear tropical system. However, unlike the classical case the tropical problem turned out to be much harder, and despite the fact that it was put forward five years ago, no efficient solution is known to date.

Similar to the classical case, a tropical linear system can be conveniently defined with a matrix. One way to test the feasibility of a classical linear system is to test whether

its determinant is zero. Therefore, due to the idempotent correspondence principle, one could expect something similar in the tropical case as well. A formula for the tropical determinant was first proposed by Izhakian in 2008 [13]. The definition of the determinant was completely similar to the classical determinant with the sole difference that one uses \oplus instead of addition, \otimes instead of multiplication, and there is no $(-1)^n$ factor (i.e., this construction corresponds to the determinant and the permanent at the same time). Izhakian also showed that a tropical system defined by a square matrix has a solution if and only if its coefficients are not roots of the tropical determinant (i.e., the determinant of this matrix is minimized only at a single monomial). Such square matrices were called *tropically singular*. Interestingly, 14 years before Izhakian, Butcovič already defined tropical singularity under the name of strong regularity [4]. However, it was done before tropical geometry appeared in earnest, and Butcovič did not establish any relationship between strong regularity and feasibility of tropical systems.

It is important to note that Butcovič proposed an efficient algorithm for testing a tropical matrix for singularity [4]. One can also note that the singularity condition is equivalent to the existence of a unique minimal weight matching and can therefore be efficiently tested with, e.g., the Hungarian method [20]. Thus, the feasibility problem for tropical linear systems determined by square matrices was efficiently solved by Izhakian in 2008. However, for other matrices even pseudopolynomial algorithms (that would work in time polynomial of the numerical value of the input rather than its size) were not known.

In 2009, Izhakian generalized his definition to rectangular matrices [14] and showed that a system determined by a rectangular matrix is infeasible if and only if it contains a singular submatrix of maximal width (i.e., of width equal to that of the matrix). Note that Izhakian's results imply that, as in the classical case, a tropical system with fewer equations than variables is always feasible. However, since a rectangular matrix may contain an exponential number of square submatrices of maximal width (in the case when the height is much larger than the width), this result implied no efficient algorithm for solving tropical linear systems determined by rectangular matrices. We should also note that in the case of finite coefficients Izhakian's results are in fact a simple consequence of the theorem that establishes that the Kapranov rank and the tropical rank of a matrix are maximized simultaneously, a theorem proved by Develin, Santos, and Sturmfels in 2005, see [7].

In the case when the number of equations and the number of variables coincide, it is possible not only to efficiently test the system for feasibility but also to solve it (it is important to note that, in the generic case, finding a solution of a system can be much more difficult than feasibility testing). For this, Grigoriev proposed [10] to drop the equation that intersects with minimal matchings in two cells and apply Cramer's tropical rule to the rest of the matrix [29]. Thus, the problem of solving a tropical system with a square matrix can be solved in polynomial time.

The first pseudopolynomial algorithm for solving rectangular matrices was presented in [1] by Akian, Gaubert, and Guterman in 2010. They showed that the feasibility problem for a tropical linear system has a polynomial reduction to the problem of finding the winner in mean payoff games. Mean payoff games were proposed in [8] by Ehrenfeucht and Mycielski in 1979 and have already been quite comprehensively studied by 2010. For instance, in 1993 Karzanov and Lebedev, using the results of Karp [15], showed that the problem of finding the winner in a mean payoff game lies in the intersection of the complexity classes NP and $coNP$, see [17]; in 1995, Zwick and Paterson proposed a pseudopolynomial algorithm for solving this problem [32]. Several times, there appeared

algorithms that claimed to find the winner of a mean payoff game in polynomial time [16, 23, 24], but so far all of them turned out to contain mistakes [32].

Thus, in 2010 it was shown that the feasibility problem for a tropical system lies in the intersection of NP and $coNP$; this is an interesting complexity class as there are few problems known to be in $NP \cap coNP$ but not known to be in P . The feasibility problem also got a pseudopolynomial algorithm. However, it is easy to construct an example of a matrix that results, by the algorithm of Akian, Gaubert, and Guterman, in a mean payoff game with no known efficient algorithm. Therefore, the problem of finding an efficient algorithm for testing feasibility of a tropical linear system remained open.

In 2011, Grigoriev proposed a different pseudopolynomial algorithm [11] similar to the Gram–Schmidt process that was designed to immediately solve the feasibility problem for a tropical system. Apart from its relative simplicity, an important feature of Grigoriev’s algorithm was that apart from the pseudopolynomial estimate it immediately provided an exponential complexity bound that does not depend on the numbers that occur in the matrix. Later, Davydow showed [6] that if one fixes any one of the three parameters (width, height, and maximal coefficient in the matrix), the algorithm will work in time polynomial with respect to the other two parameters. These were optimistic results: they meant that should there be examples of inputs on which Grigoriev’s algorithm is not polynomial, they would have to be very complex because all three parameters would have to change simultaneously. Unfortunately, Davydow found such a series [6], and by the time Grigoriev’s work was published it was already known that this algorithm is also not efficient in the general case.

As we have already mentioned, in 2010 Akian et al. [1] showed that the feasibility problem for a tropical linear system can be polynomially reduced to the problem of finding the winner in mean payoff games. In 2012, Grigoriev and Podolskii [12] found the inverse reduction, showing that the feasibility problem for a tropical linear system and the winner problem in a mean payoff game are equivalent. They constructed a reduction to the problem of $(\min, +)$ systems (systems of equations of the form $Ax = Bx$, where matrices are multiplied by vectors in the tropical sense), and the equivalence of the problem of $(\min, +)$ systems to the problem of mean payoff games was proved by Bezem, Nieuwenhuis, and Rodríguez-Carbonell already in 2010 [3]. At the same time, a result similar to [12] was directly obtained by Akian, Gaubert, and Guterman [2].

On the one hand, after Grigoriev and Podolskii showed that testing feasibility for a tropical system is as hard as finding the winner in a mean payoff game, it became clear that finding an efficient algorithm is rather unlikely. On the other hand, they showed that this problem is interesting not only as an independent problem but also as a completely new approach to mean payoff games, and that this problem deserves an even more detailed scrutiny.

Thus, at present there exist efficient feasibility testing algorithms for systems where the number of equations exceeds the number of variables by a predefined constant; we shall call such systems *weakly overdetermined*. In this case, it suffices to enumerate all square submatrices of maximal width for the system’s matrix and test each of them for singularity with the Hungarian method. Then, Izhakian’s results imply that a system is feasible if and only if all resulting matrices are nonsingular. For systems with an unbounded number of equations, only pseudopolynomial algorithms are known (the algorithm of Akian–Gaubert–Guterman and Grigoriev’s algorithm). Note that in the case of systems with bounded number of variables Grigoriev’s algorithm works in polynomial time.

1.3. Our contributions. In this work, we present two main results related to new algorithms for solving tropical linear systems. First, we present a new description for

Grigoriev's algorithm for solving tropical linear systems that allows us to generalize Grigoriev's algorithm and another well known algorithm of Akian, Gaubert, and Guterman and consider a whole family of algorithms that differ in the lifting operation. Even a simple straightforward combination of these two algorithms can already solve hard counterexamples for both Grigoriev's and Akian–Gaubert–Guterman algorithms, and we leave devising hard counterexamples for these new algorithms as an interesting open problem.

Second, we present a new algorithm for overdetermined tropical linear systems, i.e., systems that have more equations than variables. We reduce solving an overdetermined tropical system to solving several of its subsystems. This leads to a general algorithm that works on all tropical systems and runs in time $\Theta\left(\binom{m}{n} \text{poly}(m, n, \log M)\right)$. Moreover, we show that weakly overdetermined tropical systems (where equations outnumber variables only by a predefined constant) admit a polynomial time solution.

The paper is organized as follows. In §2, we give a formal definition of the feasibility problem for a tropical system and introduce the notation used throughout the paper. §3 is devoted to optimizing Grigoriev's algorithm for solving linear tropical systems; our optimization leads to a unified approach for Grigoriev's algorithm together with the algorithm of Akian, Gaubert, and Guterman, which in turn lets us combine the two algorithms, getting an algorithm with no known counterexamples where it would have to work for superpolynomial time. Finally, as we have already mentioned, it is possible to test feasibility of weakly overdetermined tropical linear systems in polynomial time. §4 presents a novel algorithm that actually solves such systems in polynomial time. §5 concludes the paper.

§2. PROBLEM SETTING

In this section, we give basic definitions regarding tropical linear systems.

Definition 1. A *tropical linear system* is a rectangular matrix of size $m \times n$. A *solution* of a tropical linear system is a row of n elements such that after adding it to each row of the matrix each sum contains no strict minimum, i.e., the minimal element occurs at least twice in every row. A tropical linear system is called *feasible* if there exists a row that is a solution of this system [11].

Example 1. For the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

the row

$$(1 \ 0 \ 1)$$

represents a solution: after adding it to the first row we get the minimal value 2 in the first and second columns; for the second row, we get the minimal value 2 in the second and third columns.

Example 2. The matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

is obviously infeasible.

In this paper we shall consider the problem of finding a solution for *integer-valued* tropical linear systems. First, we note that we can apply some simple transformations to a system's matrix without changing its feasibility status. The following proposition is obvious.

Proposition 1. *The class of feasible tropical linear systems is invariant with respect to adding an arbitrary constant to all numbers in one row or in one column. Moreover, given a solution of the system after such a transformation, one can find a solution of the original system by adding to the solution the difference between first rows of the matrix before and after the transformation.*

Proposition 1 immediately implies the following remark.

Remark 1. Without loss of generality we may assume that all elements of the system's matrix are nonnegative.

In what follows we introduce the following notation for a matrix A :

- $m(A)$, the number of rows in the matrix A ;
- $n(A)$, the number of columns in the matrix A ;
- $k(A) = m(A) - n(A)$;
- $M(A)$, the maximal number in the matrix A ;
- $R(A)$, the set of rows of the matrix A ;
- $a_i(A)$, the i th row of the matrix A .

We will omit the argument in this notation if it is clear from the context what matrix we are talking about.

§3. GRIGORIEV'S ALGORITHM AND ITS MODIFICATIONS

3.1. The original algorithm. One recently proposed algorithm for solving tropical linear systems is Grigoriev's algorithm. As we have already noted, a key feature of this algorithm is that the paper [11] where this algorithm was proposed immediately shows both an upper bound on the algorithm's complexity that polynomially depends on the matrix size (but it is polynomial in M , the largest element of the matrix, rather than $\log M$) and an upper bound that polynomially depends on $\log M$ (but it is not polynomial in matrix size).

We begin with a description of Grigoriev's algorithm. We begin with noting that, due to Proposition 1, we can find not a solution of the matrix but rather a series of transformations that consists of adding a constant to all elements in a row or in a column that would reduce the original matrix to a matrix that has a zero row for a solution. In what follows, we call such a matrix the *solution matrix*; finding it is equivalent to finding a solution.

To solve a system of size $m \times n$, we proceed by induction and assume that we have solved the system of size $(m - 1) \times n$ obtained from the initial system by removing its first row. From this moment on we shall assume that all rows of the matrix, except possibly the first row, contain no strict minima. Next we define the *lifting* operation with Algorithm 1.

Note that although the lifting algorithm does contain some indeterminacy (it is not specified in what order we add columns to the set J), when the first loop ends the set J is defined uniquely, because the maximal by inclusion set of this sort is unique.

Then we transform the matrix in accordance with Algorithm 2. The time complexity of this algorithm was originally shown to be $O(m^2 n^2 M \log M)$ [11]. Later, a different estimate of $O(\log M \cdot m \cdot n^2 \cdot \binom{m+n}{n})$ was shown by Davydow [6]. Moreover, there is a known counterexample for Grigoriev's algorithm: there exists a sequence of matrices (with unbounded growth in the number of rows and columns) such that Grigoriev's algorithm takes $\Omega(n^{\frac{m}{6}} \log M)$ time to process them, where $M = \text{poly}(n^{\frac{m}{6}})$ [6].

Algorithm 1: Matrix lifting in Grigoriev's algorithm

Data: a matrix A and the index i of the column where the first row's minimum is located.

Result: if the lifting is possible, then A is the lifted matrix

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1   $J \leftarrow \{i\}$ 
2  while there exists a row in which exactly one minimum is achieved in column  $j$ 
    such that  $j \notin J$  (there can be other minimum's in column's not in  $J$ ) do
3  |    $J \leftarrow J \cup \{j\}$ 
4  end
5  if  $|J| = n(A)$  then
6  |   output that lifting is impossible
7  else
8  |    $a \leftarrow \infty$ 
9  |   for  $i = 0$  to  $m$  do
10 |      if there are no minimal elements in row with index  $i$  outside of  $J$  then
11 |           $a_i \leftarrow$  maximal number one can add to columns with indices from the set
             $J$  in such a way that minimal elements remain minimal in the row with
            index  $i$ 
12 |           $a \leftarrow \min(a, a_i)$ 
13 |      end
14 |      foreach  $i \in J$  do
15 |          add  $a$  to the elements of column  $i$  ( $a$  could not be equal to  $\infty$  at thirist point
            as in this case the matrix already had at least to minimums in every row
            pre lifting)
16 |      end
17 end

```

Algorithm 2: Grigoriev's algorithm

Data: A , a matrix of the tropical system

Result: if the tropical system defined by A was feasible then its solution matrix, else "infeasible"

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1  Run this algorithm for the matrix  $A'$  resulting from  $A$  by deleting the first row.
2  Reduce the matrix  $A$  to such a form that no row except possibly the first contains
    a strict minimum.
3  while the first row contains a strict minimum do
4  |   if lifting of the matrix  $A$  is impossible then
5  |       return "infeasible";
6  |   else
7  |       lift matrix  $A$ .
8  |   end
9  end
10 return  $A$ 

```

3.2. Properties of the solutions found by Grigoriev's algorithm. We introduce a partial ordering on the solutions of a tropical linear system: we say that one solution is less than another if it is smaller componentwise. Then the following theorem holds.

Theorem 1. *For a matrix with one strict minimum, Grigoriev’s algorithm finds the smallest nonnegative solution. In terms of solution matrices, Grigoriev’s algorithm finds the smallest solution matrix that is greater than the original matrix.*

Proof. First, note that the existence of such a solution follows from the well-known fact that the set of solutions is tropically linear.

We prove this theorem by induction on the number of liftings. Namely, we show that at each step of Grigoriev’s algorithm the matrix does not become greater than the minimal solution matrix among those that are greater than the original matrix. For the induction base, note that the original matrix obviously satisfies this condition.

For the induction step, note that if, during a lifting, to at least one column we add a number smaller than the one added in the algorithm, then the column to which we added the smallest number will have a strict minimum. This means precisely that in the smallest solution matrix among those that are larger than the original matrix we have to add at least as much as Grigoriev’s algorithm adds. \square

3.3. Optimizing Grigoriev’s algorithm. We begin with a simple corollary to Theorem 1.

Corollary 1. *If Grigoriev’s algorithm has changed every column at least once, it will output “infeasible”.*

Proof. In the smallest nonnegative solution, at least one of the elements must be zero; otherwise, one could subtract it from every element and get a smaller solution. \square

Corollary 1 implies our first optimization of Grigoriev’s algorithm: we can interrupt it and output “infeasible” not when lifting is impossible, but rather when each column has been changed at least once, which can happen much earlier.

For a second optimization, we can also do without the recursion on the matrix height: we can simply add all columns with a strict minimum to the set J from the very beginning. Note that while the first optimization obviously cannot hurt Grigoriev’s algorithm, we do not know this for the second idea, although we have failed to find an example where the original version of the algorithm would work faster than this modification.

For this version of Grigoriev’s algorithm, the same upper and lower bounds can be proved in the same way as for the original version. Apart from some simplification, this version of Grigoriev’s algorithm has the advantage that it is now very similar to the Akian–Gaubert–Guterman algorithm; the only difference remains in the lifting operation: the Akian–Gaubert–Guterman algorithm lifts only columns with strict minima, and only for the value needed in order for the minima to cease being strict (see Algorithm 4).

This leads us to considering an entire scheme of algorithms that differ only in the lifting operation; their general scheme is shown in Algorithm 3. All we need from this operation is that after the lifting the matrix does not exceed the minimal solution that is larger than the matrix before the lifting. For instance, at the lifting step we can add to each column the maximum of the numbers that Akian–Gaubert–Guterman algorithm and Grigoriev’s algorithm propose to add to this column. It is easy to construct even better lifting methods, but at present, we do not know a superpolynomial counterexample even to this simple combination of the Akian–Gaubert–Guterman algorithm and Grigoriev’s algorithm, shown in Algorithm 6. Finding such a counterexample remains an interesting open problem that could shed light on important properties of this class of algorithms.

§4. WEAKLY OVERDETERMINED SYSTEMS

In this section, we proceed to the second main result of this work, namely an algorithm for solving overdetermined tropical systems. In the following theorem, we show how to

Algorithm 3: General scheme

Data: A , a matrix of the tropical system**Result:** if the tropical system determined by A was feasible, then its solution matrix, else “infeasible”

```

1 while there exist rows with strict minima, and there exists a column that has not
   been lifted do
2   |  $A \leftarrow \text{Lifting}(A)$ 
3 end
4 if there exist rows with strict minima then
5   | return “infeasible”
6 end
7 else
8   | return  $A$ 
9 end

```

Algorithm 4: Lifting in the Akian–Gaubert–Guterman algorithm

Data: A , a matrix of the tropical system**Result:** lifted matrix

```

1 foreach strict minimum do
2   | find the number to add to the corresponding column such that the minimum
   | ceases to be strict
3 end
4 Add to each column of  $A$  the maximal of all numbers found in the loop.
5 return  $A$ 

```

Algorithm 5: Lifting in Grigoriev’s optimized algorithm algorithm

Data: A , a matrix of the tropical system**Result:** lifted matrix

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1  $J \leftarrow \emptyset$ 
2 while there exist a row such that its minimum is achieved in a single column  $j$ 
   such that  $j \notin J$  do
3   |  $J \leftarrow J \cup j$ 
4 end
5  $a \leftarrow \infty$ 
6 for  $i = 0$  to  $m$  do
7   | if there are no minimal elements in the row with index  $i$  outside of  $J$  then
8     |  $a_i \leftarrow$  maximal number that can be added to columns with indices from  $J$  so
     | that in the row with index  $i$ , minimal elements remain minimal
9     |  $a \leftarrow \min(a, a_i)$ 
10 end
11 foreach  $i \in J$  do
12   | add  $a$  to column  $i$ 
13 end
14 return  $A$ 

```

Algorithm 6: Lifting for combination of Akian–Gaubert–Guterman and Grigoriev’s algorithms

Data: A , a matrix of the tropical system

Result: lifted matrix

- 1 $B \leftarrow A$
 - 2 $C \leftarrow A$
 - 3 Lift B with Akian–Gaubert–Guterman lifting algorithm (Algorithm 4)
 - 4 Lift C with Grigoriev’s optimized lifting algorithm (Algorithm 5)
 - 5 $A \leftarrow \max(B, C)$, where maximum is componentwise
-

reduce solving an overdetermined tropical system of width n to solving $n + 1$ systems corresponding to its submatrices. Throughout the section, we assume that the arithmetic operations with numbers in the matrix take $O(1)$ time.

Theorem 2. *Consider a matrix A and $n(A) + 1$ subsets of its set of rows such that every row of A is covered at least n times by these subsets. Then, if each subset determines a feasible tropical system, A is also feasible. Moreover, if solutions of each of these systems are known, the solution of the matrix A can be found in polynomial time.*

Proof. We begin with constructing a matrix of solutions S with rows s_i , where s_i is the solution for the system determined by the i th subset. Then we find a solution for the tropical system S^\top (S transposed), denoting it by α ; this solution exists and can be found in polynomial time because S^\top is underdetermined: the number of equations is less than the number of variables. Note that $s_i + \alpha_i$ is still a solution for the system determined by the i th subset because multiplication by a tropical constant preserves a solution. Consider $x = \bigoplus_i (s_i + \alpha_i)$. Then x is the solution for the original matrix A .

Indeed, since α is a solution, we can remove any row from this tropical sum, and x will remain unchanged (because every minimum is achieved twice). Since each of the rows a_i is covered at least n times, by dropping the row corresponding to a set that does not cover a_i we get a solution for a_i because the set of solutions is linear. \square

Example 3. Let us consider how this algorithm works on the following example with the overdetermined matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & 2 & 5 \\ 2 & 3 & 1 \end{pmatrix}.$$

We begin with choosing 4 subsets of its rows:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & 2 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 5 \\ 2 & 3 & 1 \end{pmatrix}.$$

First, we construct the matrix of solutions S and its transpose (solutions for the chosen subsets are (212), (323), (212), and (100), respectively):

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}, \quad S^\top = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 3 & 2 \end{pmatrix}.$$

Next we solve S^T , getting the solution matrix

$$\begin{pmatrix} 3 & 2 & 3 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 3 & 2 \end{pmatrix} \text{ and its transpose } \begin{pmatrix} 3 & 2 & 2 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix},$$

and find the tropical sum of its rows:

$$x = (2 \quad 1 \quad 2).$$

The resulting x is a solution for the original matrix.

Remark 2. As it often happens in tropical mathematics, this theorem holds in the classical case as well. Indeed, the $n+1$ vectors that represent solutions for subsets of equations are necessarily linearly dependent. This means that there exists a vector x that can be expressed as a linear combination of the other vectors; by an argument similar to the proof of Theorem 2, this vector will be a solution for the original problem.

The most straightforward way to turn Theorem 2 into an algorithm is to choose, at each step, $n + 1$ subsets with $n + k - 1$ rows each, making sure that for each of the sets the absent rows are different. Thus, the feasibility problem for the original matrix can be reduced to $n + 1$ problems of smaller size. To estimate the complexity of the resulting algorithm in terms of n and k , we denote this complexity by $T(n, k)$. Recall that $T(n, 0) = \text{poly}(n)$. We get the following recurrent relation for $T(n, k)$: $T(n, k) = (n + 1)T(n, k - 1) + \text{poly}(n)$. This means that $T(n, k) = (n + 1)^k \text{poly}(n)$, which is a polynomial for k bounded by a constant, so we have arrived at the following theorem.

Theorem 3. *The problem of solving weakly overdetermined tropical linear systems (systems for which k is bounded by a constant) can be solved in polynomial time.*

One can consider other ways of choosing the subsets. One of the most efficient methods is the following: consider a matrix A for which we need to find a solution. First, we introduce an ordering on the rows of a matrix corresponding to the order of rows in the matrix A ; we shall further assume that the rows in all subsets are ordered in this way. At each step, we choose $n + 1$ subsets as follows: the first n rows, all rows except the first, all rows except the second, and so on, ending with all rows except the n th. To further improve the algorithm’s running time, we use dynamical programming, storing the matrices that appear over the course of the algorithm’s operation and their corresponding solutions in order to reuse them if the same matrix appears for a second time.

After this optimization, all we need to estimate the running time is to estimate the number of submatrices appearing in the algorithm. Note that all submatrices look like this: the first n rows are an arbitrary ordered subset of rows, and the rest are always several consecutive last rows. Thus, the number of matrices can be bounded from above by $m \binom{m}{n}$ (there are $\binom{m}{n}$ ways to choose the first n rows, and the other rows are uniquely determined by their number). As a result, we see that the algorithm for solving weakly overdetermined systems has complexity $\binom{m}{n} \text{poly}(m, n)$, which up to a polynomial coincides with the upper bound for the best known algorithm for feasibility testing in tropical linear systems. This bound is better than the upper bound on Grigoriev’s algorithm but worse than the best known lower bound. With this algorithm, outlined in Algorithm 7, we have finally proved our main result in this section.

Theorem 4. *Any tropical linear system can be solved in time*

$$\Theta \left(\binom{m}{n} \text{poly}(m, n, \log M) \right).$$

Algorithm 7: Algorithm for solving weakly overdetermined systems

Data: A , the matrix of a tropical system
Result: if the tropical system determined by A was feasible then its solution matrix, else “infeasible”

```

// solutions for submatrices are obtained with this algorithm
1 if  $n(A) = m(A)$  then
2   | return a solution for the matrix  $A$  obtained by Grigoriev’s algorithm for
   | solving square tropical matrices
3 end
4 if  $n(A) > m(A)$  then
5   | return a solution for the matrix  $A$  obtained with Cramer’s tropical rule
6 end
7  $S \leftarrow \emptyset$ 
8 if  $\{a_1, a_2, \dots, a_n\}$  is feasible then
9   |  $S \leftarrow S \cup \{\text{solution of } \{a_1, a_2, \dots, a_n\}\}$ 
10 else
11   | return “infeasible”
12 end
13 for  $i = 0$  to  $n$  do
14   | if  $\{a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n\}$  is feasible then
15     |  $S \leftarrow S \cup \{\text{solution } \{a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n\}\}$ 
16   | else
17     | return “infeasible”
18   | end
19 end
20  $B \leftarrow$  matrix of the rows contained in the set  $S$ 
21 Transpose  $B$ 
22  $C \leftarrow$  solution matrix of system  $B$  obtained with Cramer’s tropical rule
23 Transpose  $C$ 
24 return tropical sum of the rows of  $C$ 

```

§5. CONCLUSION

In this paper, we have presented new algorithms for solving tropical linear systems: a modification of Grigoriev’s algorithm that leads to a new family of algorithms with different lifting operations and a novel algorithm for solving overdetermined tropical systems that has the same time complexity as the previously known feasibility testing algorithm.

Tropical linear systems turn out to have very interesting computational properties: the problem lies in $NP \cap coNP$, but no polynomial algorithm is known, and the best algorithm known so far was polynomial in any two of its characteristics out of three (width, height, and the maximal element in the system’s matrix). Further work in this direction may include further improvements of the algorithms proposed in this paper and finding counterexamples for the new algorithms proposed here: while we do claim that we have made Grigoriev’s algorithm significantly faster in practice, we doubt that a simple combination of Grigoriev’s algorithm and that by Akian–Gaubert–Guterman is indeed polynomial, so we expect the search for hard counterexamples to succeed.

Another interesting direction for further study comes from the observation that for every linear set of points there is a minimal tropical prevariety (with respect to inclusion) that contains this set. In low dimensions such a prevariety can be constructed as the closure of the original set under several simple operations, and one can test infeasibility by testing if a prevariety built on a set of vectors contains the entire space. We believe that this approach may lead to new algorithms for solving tropical linear systems.

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