Алгебра и анализ
Том 28 (2016), № 6

St. Petersburg Math. J.
Vol. 28 (2017), No. 6, Pages 773–782
http://dx.doi.org/10.1090/spmj/1472
Article electronically published on October 2, 2017

NUMERICALLY DETECTABLE HIDDEN SPECTRUM OF CERTAIN INTEGRATION OPERATORS

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Abstract. It is shown that the critical constant for effective inversions in operator algebras alg(V) generated by the Volterra integration $Jf = \int_0^x f\,dt$ in the spaces $L^1(0,1)$ and $L^2(0,1)$ are different: respectively, $\delta_1 = 1/2$ (i.e., the effective inversion is possible only for polynomials $T = p(J)$ with a small condition number $r(T^{-1})\|T\| < 2$, $r(\cdot)$ being the spectral radius), and $\delta_1 = 1$ (no norm control of inverses). For more general integration operator $J_\mu f = \int_{[0,x\rangle} f\,d\mu$ on the space $L^2([0,1],\mu)$ with respect to an arbitrary finite measure μ, the following $0-1$ law holds: either $\delta_1 = 0$ (and this happens if and only if μ is a purely discrete measure whose set of point masses $\mu(\{x\})$ is a finite union of geometrically decreasing sequences), or $\delta_1 = 1$.

§1. Introduction

1.1. What are effective inversions? Let A be a Banach algebra of bounded operators on a Banach (or Hilbert) space; often $A = \mathrm{alg}(T)$, the algebra generated by an operator T (norm closure of the polynomials in T). Given $a \in A$, $\sigma_V(a)$ denotes a "visible part" of the spectrum $\sigma(a)$ (often, the set of eigenvalues, but sometimes simply the entire spectrum $\sigma(a)$). "Constructive", or effective approach to the inversion problem in A consists in studying the function

$$c_1(\delta, A) = \sup\left\{\|a^{-1}\|_A \,:\, \delta \le m_a \le \|a\|_A \le 1\right\}, \quad 0 < \delta \le 1,$$

where $m_a = \inf\{|\lambda| \,:\, \lambda \in \sigma_V(a)\}$, which is the best possible upper estimate of inverses in terms of the lower bound δ of the "visible" spectrum.

An important quantity is also the so-called *"critical constant"*

$$\delta_1(A) = \delta_1(A, \sigma_V) = \inf\{\delta \,:\, c_1(\delta, A) < \infty\}.$$

Thus, for $a \in A$ with $\delta_1 < m_a \le \|a\|_A \le 1$ there is an estimate for $\|a^{-1}\|_A$ in terms of m_a, $\|a^{-1}\|_A \le c_1(m_a, A)$, but for a with $m_a < \delta_1$ there is no such estimate, $c_1(m_a, A) = \infty$. We refer to [Nik1999, ENZ1999, Nik2001, GMN2008] for more explanations and examples (and to [Bj1972, Ol2001, AD2006] for a different approach to norm control of inverses). See also §5 for comments.

1.2. Algebras generated by an integration operator. Given a Banach space X and a (linear) bounded operator $J\colon X \to X$, we let

$$A = \mathrm{alg}(J\colon X \to X)$$

be the norm closure of polynomials in J (assuming, by definition, $1(J) = \mathrm{id}$).

2010 *Mathematics Subject Classification.* Primary 47C05.
Key words and phrases. Effective inversion, visible spectrum, integration operator.
This research is supported by the project "Spaces of analytic functions and singular integrals", RSF grant 14-41-00010.

In this paper, we deal with compact operators J only, and so it is natural to take as the "visible spectrum" $\sigma_V(J)$ the whole spectrum $\sigma(J)$, which reduces to the eigenvalues of J, plus the point $\{0\}$ if $\dim X = \infty$. The "visible" part of the spectrum of a polynomial $T = p(J)$, $p = \sum_{k=0}^{n} c_k z^k$, will be $\sigma_V(T) = p(\sigma_V(J))$, by the spectral mapping theorem.

Below, the generating operator J will be one of the following integration operators:

$$J_\mu f(x) = \int_{[0,x\rangle} f\, d\mu,$$

or

$$\widetilde{J}_\mu f(x) = \int_{\langle x,1]} f\, d\mu \quad (0 < x < 1)$$

on one of the spaces $L^p([0,1],\mu)$, where μ stands for a finite (nonnegative) Borel measure, and

$$\int_{[0,x\rangle} f\, d\mu = \int_{[0,x)} f\, d\mu + \frac{1}{2}\mu(\{x\})f(x), \quad x \in [0,1],$$

and

$$\int_{\langle x,1]} f\, d\mu = \int_{(x,1]} f\, d\mu + \frac{1}{2}\mu(\{x\})f(x), \quad x \in [0,1].$$

In order to explain this choice of integration, recall that the measure μ is the sum of a discrete and a continuous component,

$$\mu = \mu_d + \mu_c,$$

where

$$\mu_d = \sum_{y\in[0,1]} \mu(\{y\})\delta_y$$

and δ_y stands for a Dirac unit mass at y. The integration against μ_c over $[0,x)$ and $[0,x]$ is the same, so

$$\int_{[0,x\rangle} f\, d\mu_c = \int_{[0,x)} f\, d\mu_c.$$

It is easily seen that the adjoint operator $(J_{\mu_c})^*$ is given by $(J_{\mu_c})^* f = \int_{[x,1]} f\, d\mu_c$, so that $J_{\mu_c} + (J_{\mu_c})^*$ is a rank 1 operator, and hence iJ_{μ_c} is a rank 1 perturbation of a selfadjoint operator. But for J_{μ_d}, there will be an extra-term — a diagonal operator on $L^2(\mu_d)$. In order to avoid this unsymmetry, we "equidistribute" the diagonal term between J_μ and $(J_\mu)^*$, which leads exactly to the above definition. It follows that

$$\Big(\frac{1}{i}J_\mu - \Big(\frac{1}{i}J_\mu\Big)^*\Big)f = \int_{[0,1]} f\, d\mu, \quad f \in L^2(\mu),$$

which anew implies that $\frac{1}{i}J_\mu$ is a rank 1 perturbation of a selfadjoint operator (and similarly for $\widetilde{J}_\mu$). We refer to [NV1998] for these conclusions and detailed computations.

In this paper, we view J_μ and $\widetilde{J}_\mu$ as operators on $L^p(\mu)$ spaces for $p = 1, 2, \infty$, and consider the corresponding algebras

$$\mathrm{alg}_{L^p(\mu)}(J_\mu), \quad \mathrm{alg}_{L^p(\mu)}(\widetilde{J}_\mu).$$

In §2 below (see §2(6), §2(7)), we show that $\widetilde{J}_\mu$ is unitarily equivalent to $J_{\widetilde{\mu}}$, and so we can reduce the discussion to one of them, say J_μ. Notice that, for every p, $1 \le p \le \infty$, $J_\mu\colon L^p(\mu) \to L^p(\mu)$ is a compact operator whose spectrum $\sigma(J_\mu\colon L^p(\mu) \to L^p(\mu))$ does not depend on p and consists of $\{0\}$ and the eigenvalues $\frac{1}{2}\mu(\{y\})$, $y \in [0,1]$ (which can be arranged in a sequence tending to 0 because $\sum_{y\in[0,1]} \mu(\{y\}) < \infty$), see §2 for the proof. However, the "effective inversion behavior" of J_μ heavily depends on p — we present below two results supporting this claim, for $p = 1, \infty$ and $p = 2$. Namely, the following theorems hold; for the proofs, see §3 below.

1.3. Theorem. *Let μ be a continuous measure ($\mu_d = 0$, i.e., $\mu(\{x\}) = 0$ for all $x \in [0,1]$), and $A = \mathrm{alg}_{L^1(\mu)}(J_\mu)$ or $A = \mathrm{alg}_{L^\infty(\mu)}(J_\mu)$. Then, $\sigma(J_\mu) = \{0\}$ (and hence $\sigma(p(J_\mu)) = \{p(0)\}$ for all p), and*

$$\delta_1(A) = 1/2, \quad c_1(\delta, A) = \frac{1}{2\delta - 1} \quad \text{for} \quad 1/2 < \delta \le 1.$$

Similar claims are valid for $\widetilde{A} = \mathrm{alg}_{L^1(\mu)}(\widetilde{J}_\mu)$ or $\widetilde{A} = \mathrm{alg}_{L^\infty(\mu)}(\widetilde{J}_\mu)$.

For $p = 2$, we are able to treat the case of a general measure, for which we need the following terminology. A sequence $(a_j)_{j\ge1}$ of positive real numbers ($a_j > 0$) is said to *decrease geometrically* if

$$\sup_{j\ge1} \frac{a_{j+1}}{a_j} < 1.$$

We say that J_μ *has a purely discrete geometric spectrum* if $\mu_c = 0$ and the set $\{y \in [0,1] : \mu(\{y\}) > 0\}$ is a finite union of sequences, say $(y_{j,k})_{j\ge1}$, $k = 1, \dots, N$, for which every $(\mu(\{y_{j,k}\}))_{j\ge1}$ decreases geometrically.

1.4. Theorem. *Let μ be a finite measure on $[0,1]$, and let $A = \mathrm{alg}_{L^2(\mu)}(J_\mu)$ or $A = \mathrm{alg}_{L^2(\mu)}(\widetilde{J}_\mu)$. The following alternative holds.*

(1) *Either J_μ has purely discrete geometric spectrum, and then $\delta_1(A) = 0$ and*

$$c_1(\delta, A) \le a \frac{\log \frac{1}{\delta}}{\delta^{2N}}, \quad 0 < \delta < 1,$$

where N is a number from the definition of the geometric spectrum, and $a > 0$ depends on N and the ratios of geometric sequences in $\sigma(J_\mu)$; or

(2) *this is not the case, and then $\delta_1(A) = 1$ (so that $c_1(\delta, A) = \infty$ for every $0 < \delta < 1$).*

Notice that for $p \ne 1, 2, \infty$, the question on effective inversions in $\mathrm{alg}_{L^p(\mu)}(J_\mu)$ should be more involved because even in the simplest case when μ is the Lebesgue measure ($d\mu(x) = dx$), the open problem on characterization of the L^p convolutions (multipliers) $f \longmapsto f * S$ is implicitely present. On the other hand, Yuri Tomilov (Institute of Mathematics of Polish Academy) attracted my attention to Yu. Lyubich's paper [Lyu2010] (and to many others quoted in that paper) from which the case of *the classical Volterra operator $J = J_\mu$, $d\mu(x) = dx$ on the space* $L^2(0,1)$ easily follows, as the following argument shows.

Since $\|(I + J)^{-n}\| \le 1$ for every $n \ge 1$ (obvious from J. von Neumann's inequality) and $\sigma((I+J)^{-1}) = \{1\}$, we get $\lim_n \|(I + J)^n\| = \infty$, which is also obvious from the Gelfand–Hille's old (and simple) lemma saying that an operator T with one point spectrum $\sigma(T) = \{1\}$ and bounded powers $\sup_{n\in\mathbb{Z}} \|T^n\| < \infty$ is the identity, $T = \mathrm{id}$. Therefore, $\delta_1(A) = 1$ for $A = \mathrm{alg}_{L^2(dx)}(J)$.

In fact, much more on the behavior of $\|f(J)^n\|$ is known for various functions f (see [Lyu2010] and references therein), and in particular, it is shown — contrary to the property used above — that for $T = f(J)$, $f(1) = 1$, on the spaces $L^p(0,1)$, $p \ne 2$, the behavior of $\|T^n\|$ and $\|T^{-n}\|$ is rather symmetric (as $n \to \infty$). After normalization $T^n/\|T^n\|$, this implies only the inequality

$$c_1\left(\frac{r(T)^n}{\|T^n\|}, \mathrm{alg}(J)\right) \ge \|T^n\| \cdot \|T^{-n}\|,$$

whose value depends on concrete growing rates of $\|T^n\|$ and $\|T^{-n}\|$. The author supposes to return elsewhere to the analysis of these and other known results on integral operators.

A few more comments on the above results 1.3–1.4 are given below, see §5.

§2. Preliminaries on J_μ

Estimates in algebras $\mathrm{alg}_{L^p(\mu)}(J_\mu)$ depend on the spectral properties of J_μ. Here we list some of them for the reader's convenience (although, some of these properties — or maybe all of them — are known to the experts, see for example [Lyu2010]).

(1) The operator $J_\mu\colon L^p(\mu)\to L^p(\mu)$ *is compact for every* p, $1\le p\le\infty$; *moreover,* $J_\mu L^1(\mu)\subset L^\infty(\mu)$.

Indeed, clearly $J_\mu f\in L^\infty(\mu)$ for every $f\in L^1(\mu)$. For compactness, it suffices to show that both $J_\mu\colon L^1(\mu)\to L^1(\mu)$ and $J_\mu\colon L^\infty(\mu)\to L^\infty(\mu)$ are compact. We have $J_\mu = T_k + J_\mu^d$, where T_k stands for the integral operators

$$T_k f = \int_{[0,1]} k(x,y) f(y)\, d\mu(y), \quad x\in[0,1],$$

with the L^∞ kernel $k(x,y)=\chi_{[0,x)}(y)$, and $J_\mu^d\colon L^p(\mu_d)\to L^p(\mu_d)$ is the multiplication operator $J_\mu^d f(x)=\frac12\mu(\{x\})f(x)$ by the sequence $\{\frac12\mu(\{x\})\}$ tending to 0. The operator J_μ^d is obviously compact on any sequence space $L^p(\mu)$, $1\le p\le\infty$, whereas the former one, T_k, has the norm $\|T_k\colon L^1(\mu)\to L^1(\mu)\| = \sup_y\int_{[0,1]}|k(x,y)|\,d\mu(x)$, and hence can be norm approximated by operators with degenerate kernels (so, finite rank operators), and similarly for $T_k\colon L^\infty(\mu)\to L^\infty(\mu)$. The result follows by the Riesz–Torin L^p interpolation.

(2) The case where $p=2$**.** First, we introduce the following notation, referring for all definitions to the textbooks on Hardy spaces, for example, to [Gar1981, Nik2002, Nik2012]. H^2 stands for the Hardy space of the disk $\mathbb{D}=\{z : z\in\mathbb{C}, |z|<1\}$, and, given an inner function θ, $K_\theta = H^2\ominus\theta H^2$ is the backward shift invariant "model space" corresponding to θ. With an operator $J_\mu\colon L^2(\mu)\to L^2(\mu)$ we associate the inner function θ_μ,

$$\theta_\mu(z)=\prod_{k\ge1} b_{\lambda_k}(z)\cdot\exp\Big(-\mu_c([0,1])\frac{1+z}{1-z}\Big),$$

where

$$\lambda_k=\frac{1-\mu(\{x_k\})/2}{1+\mu(\{x_k\})/2}$$

((x_k) is an enumeration of the set $\{x\in[0,1] : \mu(\{x\})>0\}$) and $b_{\lambda_k}(z)=\frac{\lambda_k-z}{1-\overline{\lambda}_k z}$) is an elementary Blaschke factor. The model operator M_θ is defined by

$$M_\theta f = P_\theta(zf)(f\in K_\theta),$$

where P_θ stands for the orthoprojection onto K_θ.

In this notation, the following statement was proved in [NV1998].

The operator $iJ_\mu\colon L^2(\mu)\to L^2(\mu)$ *is dissipative,* $2\,\mathrm{Im}(J_\mu)f=(f,1)_{L^2(\mu)}1$, $f\in L^2(\mu)$ (*hence,* $\mathrm{Im}(J_\mu)\ge0$), *and its Cayley transform*

$$C_\mu =: (I-J_\mu)(I+J_\mu)^{-1}$$

is a contraction unitarily equivalent to the model operator $M_{\theta_\mu}\colon K_{\theta_\mu}\to K_{\theta_\mu}$.

(3) The spectrum $\sigma(J_\mu\colon L^p(\mu)\to L^p(\mu))$ *does not depend on* p *and consists of* $\{0\}$ *and the eigenvalues* $\frac12\mu(\{y\})>0$, $y\in[0,1]$; *if a number* $\lambda>0$ *is an eigenvalue of* J_μ, *the dimension of the Jordan block corresponding to* λ *is* $\mathrm{card}\{y\in[0,1] : \lambda=\frac12\mu(\{y\})\}$, *i.e.,*

$$\dim\bigcup_{k\ge1}\mathrm{Ker}(J_\mu-\lambda I)^k=\mathrm{card}\Big\{y\in[0,1] : \lambda=\frac12\mu(\{y\})\Big\}.$$

Indeed, since $\mathrm{Re}(J_\mu) \geq 0$, a number $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \geq 0$ is an eigenvalue of J_μ if and only if $\frac{1-\lambda}{1+\lambda}$ is an eigenvalue of C_μ. Now, for $p = 2$, the point spectra of the operators C_μ and M_{θ_μ} are the same, and for the latter one, we know (see [Nik2002], for example) that, on the space K_{θ_μ}, its point spectrum coincides with the zeros of the Blaschke factor in θ_μ, and the size of a Jordan block corresponding to a number λ is exactly $\mathrm{card}\{k : \lambda_k = \lambda\}$ (we use the notation of (2) above). This implies the claimed description for $\sigma(J_\mu \colon L^2(\mu) \to L^2(\mu))$.

To settle the case of all other p, $1 \leq p \leq \infty$, it suffices to observe that

$$\mathrm{Ker}((J_\mu | L^1(\mu)) - \lambda I)^k \subset L^\infty(\mu)$$

for every $\lambda \neq 0$ and $k \geq 1$ (i.e., every eigen- or associate-vector of J_μ in $L^1(\mu)$ is, in fact, in $L^\infty(\mu)$); the last inclusion follows from the identity $(z-\lambda)^k = (-\lambda)^k + zq(z)$, where q is a polynomial, and the inclusion $J_\mu L^1(\mu) \subset L^\infty(\mu)$ from (1) above. Now, the claim is proved.

(4) Continuous measures μ and the standard Volterra operator. By the standard Volterra operator J we mean $J_\mu \colon L^p(\mu) \to L^p(\mu)$ with $d\mu(x) = dx$, so that

$$Jf(y) = \int_0^y f\,dx, \quad J\colon L^p(0,1) \to L^p(0,1).$$

The following property should be known but we cannot localize a reference.

Let μ be a continuous probability measure on $[0,1]$ (i.e., $\mu([0,1]) = 1$ and $\mu_d = 0$), *and let $\varphi(x) = \mu((0,x))$, $0 \leq x \leq 1$. Then, the composition $C_\varphi f =: f \circ \varphi$ is a surjective isometry $C_\varphi \colon L^p(0,1) \to L^p((0,1),\mu)$ and*

$$J_\mu C_\varphi = C_\varphi J.$$

Indeed, φ is a continuous monotone function and $\varphi(0) = 0$, $\varphi(1) = 1$, so that for every interval $[a,b] \subset [0,1]$ we have $\varphi^{-1}([a,b]) = [\alpha,\beta]$ and $\varphi(\alpha) = a$, $\varphi(\beta) = b$ ($\varphi^{-1}(A)$ stands for the preimage of A). Taking $f = \chi_{[a,b]}$, we obtain $\int f \circ \varphi\,d\mu = \int \chi_{[\alpha,\beta]}\,d\mu = \varphi(\beta) - \varphi(\alpha) = b - a = \int f\,dx$, and hence the same identity

$$\int f \circ \varphi\,d\mu = \int f\,dx$$

is valid for all $f \in L^1(0,1)$. Applying it to $\int |f|^p\,dx$, we see that the map

$$C_\varphi \colon L^p(0,1) \to L^p((0,1),\mu)$$

is a linear isometry. It is onto, because its range is dense, containing any indicator function $\chi_{[\alpha,\beta]}$ due to the relation $\chi_{[\alpha,\beta]} = \chi_{\varphi^{-1}(\varphi[\alpha,\beta])}$, which is fulfilled in the space $L^p(\mu)$ (because $\mu(\varphi^{-1}([\alpha,\beta]) \setminus [\alpha,\beta]) = 0$). The last argument also implies that, given $y \in [0,1]$, we have $\chi_{[0,y]}(t) = \chi_{[0,\varphi(y)]}(\varphi(t))$ for μ-a.e. $t \in [0,1]$, whence

$$\begin{aligned} J_\mu C_\varphi f(y) &= \int \chi_{[0,y]}(t) f(\varphi(t))\,d\mu(t) = \int \chi_{[0,\varphi(y)]}(\varphi(t)) f(\varphi(t))\,d\mu(t) \\ &= \int \chi_{[0,\varphi(y)]} f\,dx = (C_\varphi J f)(y) \end{aligned}$$

for every $f \in L^1(0,1)$. Therefore, $J_\mu C_\varphi = C_\varphi J$.

(5) The Volterra algebra $A = \mathrm{alg}_{L^1(0,1)}(J)$. The following gives a description of the above algebra as a convolution algebra $L^1(0,1)$ with an identity added.

For any complex polynomial p, we have

$$\|p(J) \colon L^1(0,1) \to L^1(0,1)\| = |p(0)| + \|p - p(0)\|_{L^1(0,1)},$$

whence $\mathrm{alg}_{L^1(0,1)}(J)$ *is a convolution algebra,*

$$A = \mathrm{alg}_{L^1(0,1)}(J) = \delta_0 \cdot \mathbb{C} + L^1(0,1),$$

with the measure norm $\|\lambda\delta_0 + f\|_A = |\lambda| + \|f\|_{L^1(0,1)}$.

Indeed, $Jh = \chi * h|[0,1]$, where $\chi = \chi_{[0,\infty)}$ and $*$ stands for the convolution on $\mathbb{R}$:

$$\chi * h(x) = \int_{\mathbb{R}} h(t)\chi(x-t)\,dt = \int_0^x h(t)\,dt, \quad x \in [0,1],$$

so that $J^n h = \chi_n * h|[0,1]$, $\chi_n(x) = x^{n-1}/(n-1)!$ $(n = 1, 2, \dots)$. Therefore, $p(J) = \sum_{k=0}^n c_k J^k$ is a convolution with the measure $S = c_0\delta_0 + \sum_1^n c_k\chi_k$, and

$$\|p(J)\| = \|S|[0,1]\| = |c_0| + \left\| \sum_1^n c_k\chi_k|[0,1] \right\|_1$$

(the upper estimate $\le$ is obvious, and the lower one $\ge$ follows after considering the approximate identity $h_\epsilon = \frac{1}{\epsilon}\chi_{[0,\epsilon]} \in L^1(0,1)$ as $\epsilon \to 0$).

Since polynomials are dense in $L^1(0,1)$, the claim follows.

(6) Adjoint operator J_μ^*. *Given* p, $1 \le p < \infty$, *we have*

$$(J_\mu \colon L^p(\mu) \to L^p(\mu))^* = \widetilde{J}_\mu \colon L^{p'}(\mu) \to L^{p'}(\mu),$$

where $\frac{1}{p'} + \frac{1}{p} = 1$.

Indeed, $J_\mu f(x) = T_k f(x) =: \int_{[0,1]} k(x,y) f(y)\,d\mu(y)$, where $k(x,y) = \chi_{[0,x)}(y) + \frac{1}{2}\chi_{\{x\}}(y)$, and hence $J_\mu^* = T_{k_*} \colon L^{p'}(\mu) \to L^{p'}(\mu)$,

$$k_*(x,y) = \chi_{[0,y)}(x) + \frac{1}{2}\chi_{\{y\}}(x),$$

so that

$$\begin{aligned} J_\mu^* f(x) &= \int_{[0,1]} k_*(x,y) f(y)\,d\mu(y) \\ &= \int_{(x,1]} f(y)\,d\mu(y) + \frac{1}{2}\mu(\{x\}) f(x) = \widetilde{J}_\mu f(x), \quad x \in [0,1]. \end{aligned}$$

(7) Unitary equivalence. *The operator* $\widetilde{J}_\mu \colon L^p(\mu) \to L^p(\mu)$ *is unitarily equivalent to* $J_{\widetilde{\mu}} \colon L^p(\widetilde{\mu}) \to L^p(\widetilde{\mu})$, *where* $\widetilde{\mu}(B) = \mu(1-B)$, $1 - B = \{1 - x : x \in B\}$, $B \subset [0,1]$.

Indeed, let $Vf(x) = f(1-x)$, $x \in [0,1]$. Clearly, $V^2 = \mathrm{id}$ and the mappings $V \colon L^p(\mu) \to L^p(\widetilde{\mu})$ and $V \colon L^p(\widetilde{\mu}) \to L^p(\mu)$ are unitary ($\equiv$ isometric isomorphisms). Moreover, by a staightforward verification,

$$\widetilde{J}_\mu V = V J_{\widetilde{\mu}}.$$

§3. Proof of Theorem 1.3

Since, by 2(4), J_μ is unitarily equivalent to J, it suffices to prove the claim for the Volterra algebra $A = \mathrm{alg}_{L^1(0,1)}(J)$ of Subsection 2(5). Let

$$1/2 < \delta \le |\lambda| \le \|\lambda\delta_0 + f\|_A \le 1;$$

writing $\lambda\delta_0 + f = \lambda(\delta_0 + f/\lambda)$, we have $\|\lambda\delta_0 + f\|_A = |\lambda| + \|f\| \le 1$ and $\|f/\lambda\| \le 1/|\lambda| - 1 < 1$, so that

$$\|(\lambda\delta_0 + f)^{-1}\|_A = |\lambda|^{-1}\|(\delta_0 + f/\lambda)^{-1}\|_A \le |\lambda|^{-1}(1 - (1/|\lambda| - 1))^{-1} = \frac{1}{2|\lambda| - 1},$$

which gives $c_1(\delta, A) \le \frac{1}{2\delta - 1}$ for $1/2 < \delta \le 1$.

In order to prove the reverse (lower) estimate for $c_1(\delta, A)$, we use the following lemma from [Nik1999].

If, for every $\epsilon > 0$ and every $N \in \mathbb{N}$, there exists an element $a \in A$ such that $|\widehat{a}| < \epsilon\|a\|$ ($\widehat{a}$ stands for the Gelfand transform of a) and the system $(a^k/\|a\|^k)_{0\le k\le N}$ is $(1+\epsilon)$-equivalent to the unit basis in l^1_{N+1}, i.e.,

$$(1+\epsilon)\Big\|\sum_{0\le k\le N} c_k a^k/\|a\|^k\Big\| \ge \sum_{0\le k\le N} c_k \quad \textit{for every} \quad c_k \ge 0,$$

then $c_1(\delta, A) \ge \frac{1}{2\delta-1}$ for every $1/2 < \delta \le 1$.

In our case $A = \delta_0 \cdot \mathbb{C} + L^1(0,1)$, and we take

$$a = \delta^{-1}\chi_\Delta \quad \text{where} \quad \Delta = [1/2N, \delta + 1/2N] \quad \text{with} \quad \delta < 1/2N^2.$$

Then $a^k = a * a * \cdots * a$ is supported on the interval $\Delta_k = [k/2N, k(\delta + 1/2N)]$, so that $\Delta_k \cap \Delta_l = \varnothing$ for $1 \le k \ne l \le N$. It is also clear that $\|a^k\|_{L^1(0,1)} = 1$ for $0 \le k \le N$, and hence the needed property follows with $\epsilon = 0$. By the lemma quoted, $c_1(A, \delta) \ge \frac{1}{2\delta-1}$ for every $1/2 < \delta \le 1$, and the claim on $J_\mu\colon L^1(\mu) \to L^1(\mu)$ follows.

For $\widetilde{J}_\mu\colon L^1(\mu) \to L^1(\mu)$, we use Subsection 2(7), which shows that

$$V^{-1}\widetilde{J}_\mu V = J_{\widetilde{\mu}}\colon L^1(\widetilde{\mu}) \to L^1(\widetilde{\mu}).$$

Since $(\widetilde{\mu})_c = (\mu_c)$, and since a unitary equivalence preserves the polynomial calculus $V^{-1}p(\widetilde{J}_\mu)V = p(J_{\widetilde{\mu}})$, the norm $\|p(\widetilde{J}_\mu)\| = \|p(J_{\widetilde{\mu}})\|$, and the spectrum $\sigma(p(\widetilde{J}_\mu)) = \sigma(p(J_{\widetilde{\mu}}))$, we can extend it to the algebras,

$$V \,:\, \widetilde{A}(\mu) =: \mathrm{alg}_{L^1(\mu)}(\widetilde{J}_\mu) \to A(\widetilde{\mu}) =: \mathrm{alg}_{L^1(\widetilde{\mu})}(J_{\widetilde{\mu}}),$$

obtaining $c_1(\delta, A(\widetilde{\mu})) = c_1(\delta, \widetilde{A}(\mu))$, $\delta_1(A(\widetilde{\mu})) = \delta_1(\widetilde{A}(\mu))$. Now, the result for $\widetilde{A}(\mu)$ follows from that for $A(\widetilde{\mu})$.

It is also clear that the functions $c_1(\delta)$ and the constants δ_1 coincide for the algebras $A = \mathrm{alg}_X(T)$ and $A_* = \mathrm{alg}_{X^*}(T^*) = \{S^* : S \in A\}$ (because $\|S\| = \|S^*\|$ and $\sigma(S) = \sigma(S^*)$, for a bilinear duality). Applying this to

$$(J_\mu\colon L^1(\mu) \to L^1(\mu))^* = \widetilde{J}_\mu\colon L^\infty(\mu) \to L^\infty(\mu)$$

and using already proved assertions for $A = \mathrm{alg}_{L^1(\mu)}(J_\mu)$ and $\widetilde{A} = \mathrm{alg}_{L^1(\mu)}(\widetilde{J}_\mu)$, we finish the proof.

§4. Proof of Theorem 1.4

First, we consider the algebra $A = \mathrm{alg}_{L^2(\mu)}(J_\mu)$, and start with proving that the algebras $\mathrm{alg}(J_\mu)$ and $\mathrm{alg}(C_\mu)$ generated, respectively, by J_μ and its Cayley transform $C_\mu = (I - J_\mu)(I + J_\mu)^{-1}$, coincide. From Subsection 2(2), it follows that $\mathrm{Re}(J_\mu) \ge 0$, and so $(I + J_\mu)^{-1}$ is bounded, and moreover $(I + J_\mu)^{-1} \in \mathrm{alg}(J_\mu)$ (because all resolvent values $(\lambda I - J_\mu)^{-1}$ for $\lambda \in \mathbb{C}$ in the unbounded connected component Ω of $\mathbb{C} \setminus \sigma(J_\mu)$ are in $\mathrm{alg}(J_\mu)$; in our case, $\Omega = \mathbb{C} \setminus \sigma(J_\mu)$). Therefore, $\mathrm{alg}(C_\mu) \subset \mathrm{alg}(J_\mu)$.

Conversely, from the statement in Subsection 2(2) it follows that $\sigma(C_\mu) = \{1\} \cup \{\lambda_k : k \ge 1\}$ (with the notation of 2(2)), so that, $(I + C_\mu)^{-1}$ is bounded, and for the same reason as above, $(I + C_\mu)^{-1} \in \mathrm{alg}(C_\mu)$ and $J_\mu = (I - C_\mu)(I + C_\mu)^{-1} \in \mathrm{alg}(C_\mu)$, whence $\mathrm{alg}(J_\mu) \subset \mathrm{alg}(C_\mu)$.

So, it is proved that $\mathrm{alg}(C_\mu) = \mathrm{alg}(J_\mu)$, and moreover, from the previous arguments it is clear that the Gelfand transform f (on $\sigma(J_\mu)$) of an element $T \in \mathrm{alg}(J_\mu)$ coincides with the Gelfand transform of T regarded as an element of the algebra $\mathrm{alg}(C_\mu)$ (and defined on $\sigma(C_\mu)$) up to the change of variables $f \longmapsto f \circ \omega$, $\omega(z) = (1 - z)(1 + z)^{-1}$.

Conclusion: the algebras $\mathrm{alg}(J_\mu)$ and $\mathrm{alg}(C_\mu)$ — and due to Subsection 2(2), the algebra $\mathrm{alg}(M_\theta)$ (M_θ and $\theta = \theta_\mu$ are defined in Subsection 2(2) above) — have the same values of δ_1 and $c_1(\delta)$.

From the Sarason commutant lifting theorem, we know that

$$\|f(M_\theta)\| = \|f\|_{H^\infty/\theta H^\infty} = \min\{\|f + \theta h\|_\infty : h \in H^\infty\}$$

for every polynomial f (and, in fact, for every $f \in H^\infty$), so that $\mathrm{alg}(M_\theta)$ is isometrically isomorphic to the closure of polynomials $\mathrm{clos}(\mathcal{P}_+/\theta H^\infty)$ in the quotient algebra $H^\infty/\theta H^\infty$. It is known (see [GMN2008] for the details) that in our case (where the set $\sigma(M_\theta) \cap \mathbb{T}$ is a singleton), the last closure is the image $C_a(\mathbb{D})/\theta H^\infty$ of the *disk algebra* $C_a(\mathbb{D}) = \mathrm{clos}(\mathcal{P}_+)$ for the quotient map.

Now, let $\theta_\mu = B$ be a Blaschke product (i.e., $\mu_c = 0$). For the algebras

$$\mathcal{A} = H^\infty/BH^\infty \quad \text{and} \quad A = C_a(\mathbb{D})/BH^\infty,$$

where B is a Blaschke product and the "visible spectrum" is defined as the point spectrum $\sigma_p(M_B)$ (i.e., the zeros of the product B), the quantities δ_1 and $c_1(\delta)$ were found in [GMN2008]. For our case (B is a Blaschke product with real zeros λ_k defined in Subsection 2(2) above and tending to 1), the results of [GMN2008] can be summarized as follows.

(a) $\delta_1(\mathcal{A}) = \delta_1(A)$ and $c_1(\delta, \mathcal{A}) = c_1(\delta, A)$ for every $0 < \delta < 1$, see [GMN2008, Theorem 4.2].

(b) $\delta_1(A) = 0 \Leftrightarrow$ the sequence $\sigma = (\lambda_k)$ of eigenvalues of M_θ, $-1 < \lambda_k < 1$, defined in Subsection 2(2) above is a Newman–Carleson sequence, i.e., $\nu =: \sum_k (1-\lambda_k)\delta_{\lambda_k}$ is a Carleson measure ($H^2|\sigma \subset L^2(\nu)$) (see Theorem 3.3 and Proposition (P7) in §3 of [GMN2008]).

It is well known that a sequence (λ_k) lying on the diameter $(-1, 1)$ and having $\lim_k \lambda_k = 1$, is Newman–Carleson if and only if it is a finite union of sequences (λ_{k_j}) tending to 1 at least geometrically, i.e. $\sup_j \frac{1-\lambda_{k_{j+1}}}{1-\lambda_{k_j}} < 1$ (for example, see [Nik2002]; C.3.7.2, items (c) and (f), or [Gar1981]).

(c) In the case where $\sigma = (\lambda_k)$ is a finite union (say, N) of sequences (λ_{k_j}) tending to 1 at least geometrically, we have the following estimate:

$$c_1(\delta, A) \le a \frac{\log \frac{1}{\delta}}{\delta^{2N}}, \quad 0 < \delta < 1,$$

(see [GMN2008, Corollary 3.6]); the constant $a > 0$ depends on N and the ratios of geometric sequences in σ.

It remains to show that if (λ_k) is not Newman–Carleson, or $\mu_c \ne 0$, then $\delta_1(A) = 1$. For this, we make use of pseudohyperbolic geometry of sequences in the unit disk, in the same spirit as in [GMN2008] (for general properties of pseudohyperbolic metrics, see [Gar1981], or [Nik1986, Nik2002]).

First, suppose $\mu_c \ne 0$, that is $\theta_\mu = BS$, where B stands for the Blaschke product $B = \prod_k b_{\lambda_k}$ and $S = \exp(-a\frac{1+z}{1-z})$, $a = \mu_c([0,1]) > 0$. Given $0 < \delta < 1$, there exists a straight horde γ of the circle $\mathbb{T} = \{|z| = 1\}$ passing by 1 and so close to $\mathbb{T}$ that $|b_\lambda(z)| > \delta$ for every $\lambda \in (-1, 1)$ and $z \in \gamma$. Since $\lim_{z\in\gamma, z\to 1} \theta_\mu(z) = 0$, we obtain $|b_z(\lambda_k)| > \delta$ for every k and $z \in \gamma$, and on the other hand

$$\lim_{\substack{z\in\gamma\\ z\to 1}} \Big(\inf_{w\in\mathbb{D}} (|b_z(w)| + |\theta_\mu(w)|) \Big) \le \lim_{\substack{z\in\gamma\\ z\to 1}} |\theta_\mu(z)| = 0,$$

which means that $\lim_{\substack{z\in\gamma\\ z\to 1}} \|b_z^{-1}\|_{C_a/\theta_m H^\infty} = \infty$, and hence

$$c_1(\delta, C_a/\theta_m H^\infty) = \infty.$$

This implies $\delta_1(A) = 1$.

Now, we assume that the sequence (λ_k) is not Newman–Carleson and, given $0 < \delta < 1$, use the same horde γ as before. There exists $0 < a < 1$ so close to 1 that the disk $\{w : |b_{\lambda_k}(w)| < a\}$ contains a point z of γ, let $z = z_k$. Since (λ_k) is not Newman–Carleson and lies on a diameter, the sets $A_{k,\epsilon} = \{j : |b_{\lambda_k}(\lambda_j)| < \epsilon\}$ are arbitrarily large for every $\epsilon > 0$:

$$\limsup_{k\to 0} N(k,\epsilon) = \infty, \quad \text{where} \quad N(k,\epsilon) = \operatorname{card}(A_{k,\epsilon}).$$

Now, let $a + \epsilon < 1$. Then, $|b_{z_k}(\lambda_j)| > \delta$ for every k and j, but

$$\liminf_k \Big(\inf_{w\in\mathbb{D}} (|b_{z_k}(w)| + |\theta_\mu(w)|) \Big) \le \liminf_k |B(z_k)| = 0$$

because $|b_{\lambda_j}(z_k)| \le |b_{\lambda_j}(\lambda_k)| + |b_{\lambda_k}(z_k)| < \epsilon + a$ ($|b_\lambda(z)|$ is a metric, see [Gar1981]) whence

$$|B(z_k)| \le \prod_{j\in A_{k,\epsilon}} |b_{\lambda_j}(z_k)| \le (\epsilon + a)^{N(k,\epsilon)}.$$

As before, this means that

$$\limsup_k \|b_{z_k}^{-1}\|_{C_a/\theta_\mu H^\infty} = \infty,$$

and hence $c_1(\delta, C_a/\theta_m H^\infty) = \infty$ for every $0 < \delta < 1$, which implies $\delta_1(A) = 1$. So, all is proved for the algebra $A = \operatorname{alg}_{L^2(\mu)}(J_\mu)$.

The case of the algebra $\widetilde{A} = \operatorname{alg}_{L^2(\mu)}(\widetilde{J}_\mu)$ reduces to the preceding one (with μ replaced by $\widetilde{\mu}$, for which $\{\mu(\{x\}) : x \in [0,1]\} = \{\widetilde{\mu}(\{x\}) : x \in [0,1]\}$) by using the same argument as at the end of the proof in §3.

§5. Conclusion

Given a Banach algebra A with fixed "visible" spectrum $\sigma_V(a)$ of its elements, one can distinguish the following two phenomenons on the spectral behavior.

(1) *A Wiener–Pitt type phenomenon of "invisible spectrum".* This is the case when there exists $a \in A$ such that

$$\sigma(a) \ne \operatorname{clos}(\sigma_V(a)).$$

The very first appearance of this phenomenon is for A to be the convolution algebra of (complex) measures $A = \mathcal{M}(\mathbb{R})$, where $\sigma_V(\mu) = \widehat{\mu}(\mathbb{R})$ is the range of the Fourier transform of $\mu \in A$, see [WP1938]. In this case, if for example $0 \in \sigma(a) \setminus \operatorname{clos}(\sigma_V(a))$, one gets $m_a > 0$, but a is not invertible, and, moreover, $c_1(m_a) = \infty$ and $\delta_1(A) \ge m_a$. The reasons for the appearance of an "invisible spectrum" vary dramatically from algebra to algebra (generalized characters measurable with respect to singular σ-subalgebras of the σ-algebra of Borel subsets of $\mathbb{R}$ for $A = \mathcal{M}(\mathbb{R})$; a forced holomorphic extension for $A = Mult(L^p(\mathbb{T}, w))$ from [Nik2009] and [NVer2015]; boundary fiber homomorphisms of $H^\infty/\theta H^\infty$ that are invisible but numerically detectable from $C_a/\theta H^\infty$, see [GMN2008, NV2011])…, so that, for the moment, it seems impossible to find a common point between them.

(2) *No "invisible spectra", but there is a numerically detectable "invisible spectrum".* This is a more refined phenomenon, which happens in an algebra A where $m_a > 0$ always implies that $a \in A$ is invertible, but there is no estimate of the form $\|a^{-1}\| \le \varphi(m_a)$ (assuming the normalization $\|a\| \le 1$; without normalization, such an estimate entails already that the norm $\|\cdot\|_A$ is equivalent to a uniform norm, see [Nik1999], which case is trivial for the efficient inversions problem). The algebras A considered in this paper are exactly of this type; in order to treat them we introduced the quantities $c_1(\delta, A)$,

$\delta_1(A)$, etc. One can observe that this refined phenomenon often occurs for algebras A whose "weak completion" $\overline{A}$ has already a type (1) "invisible spectrum".

Acknowledgments

I am indebted to Dmitry Yakubovich who convinced me (Madrid, 2014) in an interest and value of the spectral analysis in singly generated operator algebras. I am also grateful to Yuri Tomilov for attracting my attention to the paper [Lyu2010], as well as for interesting and inspiring discussions on the subject of this paper.

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Received 25/JUN/2016

Originally published in English