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A PANORAMIC GLIMPSE OF MANIFOLDS WITH SECTIONAL CURVATURE BOUNDED FROM BELOW

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Dedicated to Yu. D. Burago
on his 80th birthday

Abstract. Rather than providing a comprehensive survey on manifolds curved from below, the paper is aimed at exhibiting and discussing some of the main ideas and tools that have been developed over decades. For the same reason, only a relatively small sample of results is presented to illustrate this development and in doing this, simplicity is emphasized over generality. In the same vein, at most a glimpse of an idea or strategy of a proof is given.

The search for relations between *geometry* and *topology* of Riemannian manifolds has been at the center of *global Riemannian geometry* since its beginning. On the geometric side, a large part of this line of investigations has been dominated by the role of *curvature* in various forms. For natural and geometrically appealing reasons, early efforts were focussed on manifolds whose curvature does not change sign, i.e., with lower or upper curvature bound 0, and in particular with positive or negative curvature. It took several decades for ideas and tools to emerge breaking the ground for dealing with manifolds with curvatures of both signs.

In this survey we will focus our attention on (closed) Riemannian manifolds with a given *lower* (sectional) *curvature bound*. In addition to the basic global *Toponogov distance comparison theorem*, and the *Bishop–Gromov volume comparison theorem*, the key tools are *critical point theory for distance functions*, the *Gromov–Hausdorff topology*, and *Alexandrov geometry*. Our aim is to discuss these tools as well as more classical tools such the *Morse theory of geodesics*, and in the case of nonnegative curvature also *convexity*. The insights gained from using these tools are typically expressed in *finiteness, structure, recognition*, and *rigidity results*.

The importance of Alexandrov geometry to Riemannian geometry stems from the fact that there are several natural geometric operations that are closed in Alexandrov geometry but not in Riemannian geometry. These include *taking Gromov–Hausdorff limits*, *taking quotients* by isometric group actions, and *forming joins* of positively curved spaces. In particular, limits (or quotients) of Riemannian manifolds with a lower (sectional) curvature bound are Alexandrov spaces, and only rarely Riemannian manifolds. Analyzing limits frequently involves blow ups leading to spaces with *nonnegative curvature* as, e.g., in Perelman's deep work on the geometrization conjecture. Also the infinitesimal structure of an Alexandrov space, which is expressed via its "tangent spaces" (blow ups at

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points), are cones on positively curved spaces. Hence, the collection of all compact positively curved spaces (up to scaling) agrees with the class of all possible so-called *spaces of directions*, in Alexandrov spaces. So, spaces of positive curvature play the same role in Alexandrov geometry as round spheres do in Riemannian geometry.

In addition to positively, and nonnegatively curved spaces, yet another class of spaces has emerged in the general context of convergence, under a lower curvature bound, namely, the *almost nonnegatively curved* spaces. These are spaces allowing metrics with diameter say 1, and lower curvature bound arbitrarily close to 0. They are expected to play a role among spaces with a lower curvature bound, analogous to that almost flat spaces play for spaces with bounded curvature. In summary, the following classes of spaces play essential roles in the study of spaces $\mathcal{M}_k$ with a lower curvature bound k:

$$\mathcal{M}_+ \subset \mathcal{M}_0 \subset \mathcal{M}_{0^-}$$

corresponding to the *positively curved*, *nonnegatively curved*, and *almost nonnegatively curved* spaces. Among all manifolds, these form "the tip of the iceberg". Yet, aside from being *nilpotent spaces* (up to finite covers) and having a priori *bounded topology* in terms of generators for homology and fundamental groups, only a few general obstructions are known, and none in the simply connected case. Moreover, so far only obstructions on fundamental groups distinguish the three classes.

Although Alexandrov geometry enjoys a rich and continually evolving life on its own, our discussion and treatment here is guided by its role in Riemannian geometry. This is clearly illustrated by *equivariant Riemannian geometry*, where (groups of) symmetries are present (or more generally, but yet not fully pursued, *singular Riemannian foliations*). In this case, the Alexandrov spaces that are used and arise naturally are orbits spaces/spaces of leaves (or closely related to such), and their structure is much simpler than general Alexandrov spaces, yet rich enough to illuminate many general phenomena. When sufficiently many or special symmetries are present, clear distinctions emerge between the basic classes $\mathcal{M}_+ \subset \mathcal{M}_0 \subset \mathcal{M}_{0^-}$. In addition to the tools already mentioned, here *Lie theory*, *representation theory*, as well as the theory of *buildings* pioneered and developed by Tits, play important roles.

Our treatment is divided into the following nine sections: *Characterizations and impact of lower curvature bounds*, *Convexity in non-negative curvature*, *Critical point theory for distance functions*, *Gromov–Hausdorff metric and Alexandrov geometry*, *Join operation applications*, *Collapse and almost non-negative curvature*, *Constructions and examples*, *The presence of symmetries*, and *Complete open manifolds*.

For basic sources we refer to [Pe1, CE, BBI] and [BGP]. We also recommend several surveys on various topics treated in much more depth than here [Pl, Pt, Zi, W6] and [Ro].

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§1. Characterizations and impact of lower curvature bounds

Throughout, all Riemannian manifolds M considered will be complete, in fact usually compact.

The basic Alexandrov–Toponogov *triangle comparison* theorem (see bullets below) can be expressed in a number of equivalent useful ways. To ease their formulation we agree that a *triangle* has minimal geodesic sides unless otherwise stated. Also a *comparison triangle* is a triangle in the simply connected space form S^n_k of constant curvature k whose sidelengths are the same as those it is compared with. The *comparison angles* are the angles of the comparison triangle. Similarly, a *hinge* is a triangle with one side missing, and a *comparison hinge* is a hinge in constant curvature with the same angle as the one it is compared with. Note, that with the exception of (degenerate) triangles with one side of length πr in the sphere of radius r, i.e., curvature $k = 1/r^2$, all comparison triangles are unique up to isometry, and only depend on the distances between the vertices of the triangle in question. We therefore also talk about comparison triangles of triangles all of whose sides are missing, i.e. of *virtual triangles*, or simply *three tuples* of points. Similarly we refer to four tuples as *virtual tetrahedra*.

The condition, $\sec M \geq k$, that M has *sectional curvature* bounded from below by k, i.e., the sectional curvature of any 2-plane in its tangent bundle TM is at least k, *can be characterized geometrically* by any of the following five equivalent conditions.

- *Angle version*: The angles in any triangle in M are at least as big as the corresponding angles in a comparison triangle in S^2_k.
- *Hinge version*: The distance between the endpoints of a hinge in M is at most as big as the corresponding distance for a comparison hinge in S^2_k.
- *Distance version*: The distance between a vertex and a point on the opposite side of a triangle in M is at least as big as the corresponding distance in a comparison triangle in S^2_k.
- 4-*point version*: For any virtual tetrahedron in M, the sum of all three comparison angles in S^2_k corresponding to any vertex is at most 2π.
- *Embedding version*: Any virtual tetrahedron in M admits an isometric embedding into $\mathrm{S}^3_{k'}$ for some $k' \geq k$.

The first three conditions have important rigidity versions (best expressed in the second and third) when equality holds. We also note that the last three versions do not a priori assume the existence of angles, and the last two not even the existence of geodesics. The embedding version has its roots in the work of Wald [Wd], and was developed by Berestovskii, as was the 4-point version [Bs]. Since curvature is usually local in nature, the natural assumptions would be that the above conditions hold locally in the space in question. As it turns out, this leads nontrivially to the above global statements under fairly general conditions [BGP, P2]. The latter two make sense for arbitrary metric spaces X including discrete spaces, and when they are satisfied, we use the notation $\operatorname{curv} X \geq k$.

Note also that the hinge version has the following nice geometric interpretation. Let the *segment domain* at p be the subset $\operatorname{seg}(p) \subset T_pM$ consisting of those v for which the geodesic $c_v\colon [0,1] \to M$ with initial velocity v is minimal. In other words, $\operatorname{seg}(p)$ is the closure of the largest open starshaped domain ("centered" at the origin) on which the *exponential map* $\exp\colon T_pM \to M$ defined by $\exp(v) = c_v(1)$ is injective. Observe also that $\exp(\operatorname{seg}(p)) = M$. Now equip T_pM (or the open ball, $\mathrm{B}_p(\pi/\sqrt{k})$ when $k > 0$) with a metric of constant curvature k obtained from the Euclidean metric by radial warping. With the exception of S_k, $k > 0$ (Toponogov's *diameter rigidity theorem*), we can then view the segment domain as a subset $\operatorname{seg}(p)_k \subset \mathrm{S}^n_k$, and the hinge version simply states that

$$\exp\colon \operatorname{seg}(p)_k \to M \text{ is distance nonincreasing.} \tag{1.1}$$

A trivial but powerful consequence of this is the following general *volume comparison*:

$$\operatorname{vol}(\exp H) \leq \operatorname{vol}(H) \text{ for measurable sets } H \subset \operatorname{seg}(p)_k. \tag{1.2}$$

For starshaped sets, already the weaker curvature condition

$$\operatorname{ric} M^n \geq (n-1)k$$

yields volume comparison. The important so-called Bishop–Gromov *relative volume comparison* theorem states the following.

Theorem 1.3 (relative volume comparison). *Let M be a Riemannian n-manifold with* $\sec M \geq k$*, or more generally, with Ricci curvature* $\operatorname{ric} M^n \geq (n-1)k$*. Then for every* $p \in M$ *the function*

$$V(r) = \operatorname{vol} D(p,r)/\operatorname{vol} D_k^n(r)$$

is nonincreasing.

This comparison result for volumes of closed balls in M relative to those in S_k^n has the following simple but crucial consequence.

Lemma 1.4 (covering lemma). *Let M be a Riemannian n-manifold with* $\sec M \geq k$*, or more generally,* $\operatorname{ric} M^n \geq (n-1)k$*. For any $R > \epsilon > 0$ there is $N = N(n,k,R,\epsilon)$ such that any R-ball in M can be covered by N or fewer ϵ-balls.*

To see this, pick a maximal *ϵ-separating* set in $D(p,R)$ (the closed ball centered at p with radius R), i.e., a maximal set of points where any two have distance at least ϵ. Being maximal implies that the ϵ-neighborhood of it covers $D(p,R)$, i.e., the set is an ϵ-net, and the relative volume comparison yields an upper estimate for the number of these balls since the $\epsilon/2$-balls are disjoint.

This lemma is the germ of *finiteness and compactness results* in Riemannian geometry (see [G3] for a survey).

The condition $\sec M \geq 0$ (respectively, $\sec M > 0$) can also be characterized by invoking *parallel transport* as follows:

$$I(X,X) \leq 0 \text{ (respectively, } < 0\text{) for any parallel field orthogonal to any geodesic } c\colon [0,1] \to M \tag{1.5}$$

where

$$I(X,X) := \int_0^1 \langle X', X'\rangle - \langle R(X,c')c', X\rangle = \frac{1}{L(c)}\frac{d^2}{ds^2}L(c_s), \tag{1.6}$$

is the *index form*, expressed via the second derivative of the length functional L, and $c_s := \exp sX$ is the exponential variation of c corresponding to the variation field X.

This simple fact by itself has rather immediate and strong consequences in the case where $\sec M > 0$, as pioneered by Synge (cf. also [Pe2]):

- If M is orientable and $\dim M$ is even, then M is simply connected (Synge).
  For such M any orientation preserving isometry A has fixed points (Weinstein).
- If $\dim M$ is odd, then M is orientable (Synge).
  For such M any orientation reversing isometry A has fixed points (Weinstein).
- If $V, W \subset M$ are totally geodesic and $\dim V + \dim W \geq \dim M$, then $V \cap W \neq \varnothing$ (Frankel).
- If $V \subset M$ is totally geodesic, the inclusion $V^{n-k} \subset M^n$ is $(n-2k+1)-$ connected (Wilking).

For the latter, recall that a map $f\colon X \to Y$ is ℓ-connected if the induced map $f_q\colon \pi_q(X) \to \pi_q(Y)$ on homotopy groups is onto for $q = \ell$ and an isomorphism for $q < \ell$.

The result by Weinstein is actually for the more general case where A is a conformal diffeomorphism [We]. All but Wilking's result can be viewed as the rudimentary fact of *Morse theory* that the index at a minimum must be zero. However, no knowledge of Morse theory is needed. The point of Wilking's result [W3] is an estimate for the index of the first non-minimal critical point, "as" in Bott's original proof of his *periodicity theorem.* The relevant spaces of paths for these statements are: the space of closed curves (Synge), the space of A-invariant curves (Weinstein), the space of paths joining V and W (Frankel), and the space of paths from V to itself (Wilking).

A small variation of this idea (multiply X by a function) also yields the classical Bonnet theorem that $\operatorname{diam} M \leq \pi$, when $\sec M \geq 1$, and also eventually leads to the distance comparison theorem. Here Toponogov used his comparison theorem to show that in fact M is isometric the the unit sphere if $\operatorname{diam} M = \pi$.

§2. Convexity in nonnegative curvature

In general it is known from [CG] that any closed (locally) convex set $C \subset M$ has the structure of a topological manifold possibly with nonempty boundary ∂C and smooth totally geodesic interior.

One of the key observations by Cheeger and Gromoll was that when $\sec M \geq 0$, and $C \subset M$ is a compact convex subset with nonempty boundary ∂C, then

- the distance function $\operatorname{dist}(\partial C, \cdot)\colon C \to \mathbb{R}$ is concave,

which follows from (1.5) in conjunction with the distance comparison theorem.

This immediately implies that the sublevel sets

$$C^a = \{x \in C \mid \operatorname{dist}(\partial C, x) \geq a\} \subset C$$

are also (locally) convex, and one proves that $\dim C^{\max} < \dim C$. Thus after finitely many iterations of this process (if necessary), one then arrives at a (locally) convex set without boundary, i.e., a totally geodesic submanifold, the soul $S \subset C$. Moreover, C has the structure of a disk bundle over S (see also [BZ]).

Much of this applies in a much more general context, e.g., that of orbit spaces, with strong applications (see the last section).

We also point out that if $\sec M > 0$, then $\operatorname{dist}(\partial C, \cdot)\colon C \to \mathbb{R}$ is strictly concave along all geodesics in C that are not minimal to the boundary ∂C. In particular, in this case the soul S is a point reached in one step.

As might be expected, the existence of large topologically nontrivial convex sets in manifolds of nonnegative curvature is rare. Nonetheless, they do appear naturally in some important contexts, such as in complete, noncompact, nonnegatively curved manifolds, and where group actions are present. In the latter case, there is actually an abundance of them in particular among orbit spaces.

In the first scenario just alluded to, one has the following celebrated theorem.

Theorem 2.1 (soul theorem)**.** *Any complete noncompact manifold M with $\sec M \geq 0$ is diffeomorphic to the normal bundle $S^\perp$ of a compact totally geodesic submanifold S in M.*

The key issue in the proof is the construction of an exhaustion of M by compact convex sets C_s, where each C_s, $s > 0$, has top dimension, and $C_t = C_s^{(s-t)}$ when $0 < t < s$. This starts with the observation that for any *ray*, i.e., a minimal geodesic $c\colon [0, \infty) \to M$, the associated *half-space* $H_c = M - \cup_t B(c(t), t)$ is *totally convex* (by a distance comparison argument). Here a set is said to be totally convex if any geodesic between points of it lies entirely in it. Now consider all rays c emanating from a fixed point $p \in M$. It follows that M is exhausted by the compact totally convex subsets $C_s = \cap_c H_{c^s}$, $s \geq 0$, where $c^s(t) = c(s + t)$. Moreover, for any $s > 0$ we have $\dim C_s = \dim M$, and in particular $\partial C_s \neq \varnothing$.

From the construction and convexity it now follows that $\operatorname{dist}(S, \cdot)\colon M \to \mathbb{R}$ has no critical points outside S (cf. the next section), and the theorem follows.

Note that if $\sec M > 0$ in the above theorem, then $C^{\max}$ and hence S is a point. In particular, such a manifold is diffeomorphic to Euclidean space.

More generally, as conjectured by Cheeger and Gromoll, in [Pr1] it was shown that the soul is a point if all curvatures at just one point are positive (see also [Bu]). The observation due to Perelman was in fact a general strong rigidity property for noncompact manifolds M with $\sec M \geq 0$. In particular, there is exactly one distance nonincreasing retraction $s\colon M \to S$ (one was constructed by Sharafutdinov [Sf]) characterized by $s(\exp(v_p)) = p$ for any $v_p \in S^{\perp}$, and it is a *submetry*, i.e., $s(B(q,r)) = B(s(q),r)$ for all $q \in M$ and all $r > 0$. Submetries between Riemannian manifolds are always of class at least $C^{1,1}$ [BG]. From the general work by Wilking on duality among Riemannian foliations in nonnegative curvature it follows that the Sharafutdinov retraction s is in fact smooth, improving [Gu1]. This duality also provides further rigidity for noncompact manifolds of nonnegative curvature (see [W5]).

§3. Critical point theory for distance functions

Let $A \subset M$ be a closed subset of M. Although the distance function $\operatorname{dist}(A,.)$ is not smooth, there is a good notion for points being regular/critical. Specifically,

- A point $q \in M$ is called a *regular point* for $\operatorname{dist}(A, .\,)$ if there is a vector $v \in T_qM$, such that the angle between v and any minimal geodesics from q to A exceeds $\pi/2$.

Such vectors v are said to be regular.

- If q is not regular it is *critical.*

As in smooth critical point theory, one has the following crucial statement.

Lemma 3.1 (isotopy lemma)**.** *If* $\operatorname{dist}(A, \cdot\,)$ *has only regular values in the interval* $[r,R]$*, then the set* $D(A,R) = \{q \in M \mid \operatorname{dist}(A,q) \leq R\}$ *deformation retracts to* $D(A,r)$*. In fact,* $D(A,R) - B(A,r)$ *is homeomorphic to* $S(A,r) \times [r,R]$*, where* $S(A,r) = D(A,r) - B(A,r)$ *is the boundary of the* r*-neighborhood of* A*.*

This is a simple consequence of the existence of a smooth "gradient like" vector field on the "annular" region, and a first variation argument. Such a vector field is constructed by partition of unity, based on the simple fact that the condition of being a regular vector is open and convex.

A fully fledged "Morse Theory" for such functions, including what happens when passing a critical value remains to be developed. For ideas about "index" in this context, see [HW] though.

Nonetheless, we proceed to indicate the importance of the isotopy lemma in the proofs of the diameter sphere and rigidity theorem, the Betti number theorem and the homotopy finiteness theorem.

The observation that if p_0 and p_1 are points at maximal distance in a manifold M with $\sec M \geq 1$ and $\operatorname{diam} M > \pi/2$, then all $q \in M - \{p_0, p_1\}$ are regular for $d(p_i, \cdot\,)$, yields the following [GSh].

Theorem 3.2 (diameter sphere theorem)**.** *Any manifold* M^n *with* $\sec M \geq 1$ *and* $\operatorname{diam} M > \pi/2$ *is homeomorphic to* $\mathbb{S}^n$*.*

This result fails only in a rigid fashion when the diameter is allowed to be $\pi/2$.

Theorem 3.3 (diameter rigidity)**.** *A manifold* M *with* $\sec M \geq 1$ *and* $\operatorname{diam} M = \pi/2$ *is either homeomorphic to a sphere, or its universal cover* $\widetilde{M}$ *is isometric to a rank one symmetric space.*

If in this theorem M is not simply connected, M is either isometric to a unique $\mathbb{Z}_2$-quotient of a complex odd-dimensional projective space, or to a space form $\mathbb{S}^n/\Gamma$, where Γ acts reducibly on $\mathbb{R}^{n+1}$.

In the proof of the diameter rigidity theorem, the points p_0 and p_1 are replaced by sets A_0 and A_1 each of which is the set of points at distance $\pi/2$ from the other and the distance between any pair of points in A_0 and A_1 is $\pi/2$. It is a direct consequence of the rigid version of the distance comparison theorem that the A_i are convex subsets of M. If M is not a sphere, it turns out that both convex sets A_i are totally geodesic submanifolds (one of them possibly a point), and any unit normal vector to A_0 determines a minimal geodesic from A_0 to A_1 and *vice versa*. The resulting maps $\mathbb{S}^{\perp}_{p_0} \to A_1$ are Riemannian submersions. When A_1 is simply connected, such a map is congruent to a Hopf fibration by [GG2] and [W1], and the rigidity theorem follows [GG1].

A general and deep application of critical point theory is Gromov's amazing [Gr1].

Theorem 3.4 (Betti number theorem). *For each $n \in \mathsf{N}, k \in \mathbb{R}$, and $D > 0$, there is a constant $C = C(n, k, D) > 0$ such that the homology and the fundamental group of any M^n with $\sec M \geq k$ and $\operatorname{diam} M \leq D$ can be generated by C or fewer elements.*

The main ideas of the proof are present already in the case where $\sec M \geq 0$. Here, the germ of one of the geometric ideas comes from the observation that one does not need any convexity arguments to see that a complete non-negatively curved manifold M has *finite topological type*. In fact, Gromov showed that the distance function $\operatorname{dist}(p, \cdot)\colon M \to \mathbb{R}$ from some (in fact any) point $p \in M$ has no critical points outside some ball $B(p, R(p)) \subset M$. Indeed, in this case M is diffeomorphic to $N \supset B(p, R)$, where ∂N is a smooth approximation of $\partial B(p, R)$. If on the contrary there is no such R, it would be possible to find a sequence of critical points $\{q_i\}$ for $\operatorname{dist}(p, \cdot)$ with, say, $\operatorname{dist}(p, q_{i+1}) \geq 2 \operatorname{dist}(p, q_i)$ for all i. A simple consequence of the Toponogov triangle comparison theorem, referred to as the *criticality principle* in [G2], then yields a uniform lower bound θ for the angle at p between any pair of minimal geodesics from p to q_i and from p to q_j, independent of i and j. This is obviously impossible by compactness of the unit sphere S_pM.

It is important to note that the angle bound θ exhibited above only depends on the choice of the factor $d = 2 > 1$ used in separating critical values of $\operatorname{dist}(p, \cdot)$. A simple packing argument for balls of radius $\theta/2$ on the sphere S_pM, yields an explicit upper bound for the number of such critical values, which in addition only depends on n. Summarizing, we have the following.

Lemma 3.5 (critical values lemma). *There an explicit $C' = C'(n, d), d > 1$, such that $\operatorname{dist}(p, \cdot)$ has at most C' critical values $\{v_i\}$ with $v_{i+1} \geq dv_i$ for any $p \in M$ and any n-dimensional Riemannian manifold with $\sec M \geq 0$.*

Combined with the covering Lemma 1.4, this is the main geometric ingredient in the proof.

As yet another application of critical point theory in conjunction with distance comparison via curvature, we mention the following (see [GP1])

Theorem 3.6 (homotopy finiteness theorem). *For each $n \in \mathsf{N}$, $k \in \mathbb{R}$, D, and $v \in \mathbb{R}_+$, there is a constant $C = C(n, k, D, v) > 0$ such that there are at most C different homotopy types among manifolds M^n with $\sec M \geq k$, $\operatorname{diam} M \leq D$, and $\operatorname{vol} M \geq v$.*

As in the Betti number theorem, the constant here can be estimated explicitly. Also here, the covering lemma plays an essential role, but the most important additional geometric input in the proof comes from the observation that there is an *a priori* $\epsilon = \epsilon(n, k, D, v)$ such that the distance function $\operatorname{dist}(\Delta M, \cdot)\colon M \times M \to \mathbb{R}$ has no critical points other than ΔM in an ϵ-neighborhood of the diagonal ΔM.

Note that for closed manifolds, the invariant $\min(\sec M)\big(\frac{\operatorname{diam}(M)}{\pi}\big)^2$ is scale invariant and takes its maximal value 1 at the round sphere (Toponogov's rigidity theorem). Alternatively, we may *scale all closed manifolds* $\mathcal{M}$ to have *diameter* π and consider the function

$$\min\sec\colon \mathcal{M} \to \mathbb{R} \quad \text{with range} \quad (-\infty, 1].$$

With this terminology, the above results can be stated as follows:

- Any $M \in \mathcal{M}$ with $\min\sec M > 1/4$ is a topological sphere,

and its rigidity version states what happens otherwise when $\min\sec M = 1/4$.

- Given $n \geq 2$, and $k \leq 1$, there is a number $C = C(n,k)$ such that $H(M)$ is generated by at most C elements for any n-manifold $M \in \mathcal{M}$ with $\min\sec M \geq k$.
- Given $n \geq 2$, $k \leq 1$ and $v > 0$, there is a number $C = C(n,k,v)$ so that there are at most C homotopy types of n-manifold $M \in \mathcal{M}$ with $\min\sec M \geq k$ and $\mathrm{vol}(M) \geq v$.

To gain further information, and in particular to achieve the finiteness of topological types in the last theorem above, one needs additional insights gained from convergence techniques to be discussed next.

§4. Gromov–Hausdorff metric and Alexandrov geometry

In this section we shall describe the so-called *Gromov–Hausdorff metric* on the space of isometry classes of compact metric spaces, introduced by Gromov in connection with his work on groups of polynomial growth [Gr2], and discuss some of the important insights one gains from being able to take and analyze limit spaces. In the context of manifolds with lower sectional curvature bounds this naturally leads to the notion of *Alexandrov spaces*. These spaces are interesting in their own right and have applications to Riemannian geometry beyond convergence.

Recall that the Hausdorff distance between two closed subsets A and B of a compact metric space Z is given by

$$d_H^Z(A,B) = \inf\{\epsilon > 0 | D(A,\epsilon) \supset B \text{ and } D(B,\epsilon) \supset A\}.$$

For arbitrary compact metric spaces X and Y, the Gromov–Hausdorff distance is defined by

$$d_{GH}(X,Y) = \inf\{d_H^Z(X,Y) | X, Y \subset Z\},$$

where the infimum is taken over all possible isometric embeddings of X and Y into all possible compact metric spaces Z. It actually suffices to take $Z = X \coprod Y$ with metrics extending the metrics on X and on Y, i.e.,

$$d_{GH}(X,Y) = \inf_{Z = X \coprod Y} \{d_H^Z(X,Y)\}.$$

One shows that $d_{GH}(X,Y) = 0$ if and only if X and Y are isometric, and that this indeed defines a metric on the isometry classes of compact metric spaces.

It is obvious that relative to this metric any compact metric space can be approximated arbitrarily well by finite metric spaces. In other words, the induced topology is extremely *coarse* in general. At the same time this indicates that geometry of *finite metric spaces* may well have substantial impact on other geometries, including Riemannian geometry. Moreover, we have already hinted at one of the starting points in Gromov's theory:

Lemma 4.1 (precompactness lemma)**.** *A class $\mathcal{C}$ of compact metric spaces is precompact if and only if this class has a* covering function $N(\epsilon)$*, i.e., for any $\epsilon > 0$, each $X \in \mathcal{C}$ can be covered by $N(\epsilon)$ or fewer ϵ-balls.*

This already yields some sort of "finiteness result": For each ϵ, there are finitely many n-manifolds $\{M_i\}$ with $\mathrm{ric}\, M_i \geq (n-1)k$ and $\mathrm{diam}\, M_i \leq D$ so that any other manifold satisfying these bounds is within Gromov–Hausdorff distance ϵ to one of $\{M_i\}$'s. The problem is, however, that essentially no common features among manifolds from this class, which are close to one another, are known. One reason for this is that in general one has rather limited control about spaces in the closure of this class (cf. [Co] though).

For these ideas to be useful, it is important that interesting properties be preserved by the process of taking Gromov–Hausdorff limits. To analyze this, it is a useful fact that if X is the Gromov–Hausdorff limit of a sequence $\{X_i\}$, then

$$\text{there is a metric on } Z = X \coprod_i X_i \quad \text{such that} \quad \lim d_H^Z(X, X_i) = 0.$$

This basically implies that the *invariants* that can be *described in purely metric terms* are preserved in the limit. In particular, since a complete space being a *length space* (the distance between any pair of points is the infimum of lengths of curves joining them) is equivalent to having ϵ-almost midpoints for every $\epsilon > 0$, it follows that

- *The Gromov–Hausdorff limit of length spaces is a length space.*

Similarly, from the tetrahedral version of the distance comparison theorem it follows that this is also preserved in the limit, i.e.,

- *The Gromov–Hausdorff limit of spaces with curvature at least k, has curvature at least k.*

In particular, the Gromov–Hausdorff limit of a sequence of Riemannian n-manifolds with a uniform lower bound on sectional curvatures, is a length space with a lower curvature bound. In addition, it is important that the (Hausdorff) dimension of such a limit is at most n (cf., e.g., the precursor [GP2], to Alexandrov spaces).

In contrast, we point out that upper curvature bounds are not in general preserved by the process of taking Gromov–Hausdorff limits. Although the condition $\sec \leq K$ can be expressed locally as in the distance comparison theorem with all inequalities reversed, these geometric descriptions fail globally in general. In particular, an upper bound for the curvature is only preserved by the process of taking Gromov–Hausdorff limits if the distance comparison holds uniformly on all balls of a fixed size. Simply connected manifolds with nonpositive curvature form an important class of examples where the comparisons hold globally. Another class with this property is the class of manifolds M^n with $k \leq \sec M \leq K, \operatorname{diam} M \leq D$, and $\operatorname{vol} M \geq v$. The reason for this is Cheeger's *a priory* injectivity/convexity radius estimate for this class [C1].

We now turn our attention to Alexandrov spaces (with a lower curvature bound). By definition,

- An *Alexandrov space* is a length space X with $\operatorname{curv} X \geq k \in \mathbb{R}$, and $\dim_H X < \infty$.

It turns out that the Hausdorff dimension $\dim_H$ of such a space is equal to its topological dimension, and in particular that it is an integer [BGP]. These spaces have a surprisingly "simple" structure which we shall describe below.

First, however, we give some *examples and constructions*:

- $X = \partial C$, where $C \subset \mathbb{R}^n$ is a convex body.
- $X = \lim M_i^n$, where M_i are Riemannian manifolds with $\sec M_i \geq k$.
- $X = C_0 Y$, the *Euclidean cone* on an Alexandrov space Y with $\operatorname{curv} Y \geq 1$.
- $X = \Sigma_1 Y$, the *spherical suspension* on an Alexandrov space Y with $\operatorname{curv} Y \geq 1$.
- $X = Y_1 *_1 Y_2$, the *spherical join* of two Alexandrov spaces Y_i with $\operatorname{curv} Y_i \geq 1$.
- $X = M/G$, where G is a compact group of isometries and $\sec M \geq k$.
- $X = Y_1 \cup_\partial Y_2$, where Y_i are Alexandrov spaces with isometric boundary ∂ (see below).
- Special branched covers $\widetilde{X}$ of an Alexandrov space X (instrumental in [GWi] and [HS]).

The metric on the Euclidean cone $X = C_0Y = Y \times [0,\infty)/Y \times \{0\}$ is defined by the requirement that the distance between (u,s) and (v,t) be the distance in the Euclidean plane between the endpoints of a hinge with sides of lengths s and t and angle $\operatorname{dist}(u,v)$ in Y. The metric on the spherical suspension and more generally spherical join is defined similarly using the unit sphere in place of the Euclidean space.

These examples already suggest reasons for the utility of Alexandrov spaces. The fact that the operations of taking Gromov–Hausdorff limits, of taking spherical joins, and of taking isometric quotients are closed in Alexandrov geometry all have in fact applications to Riemannian geometry. In this section we shall illustrate the first of these operations. The second will be illustrated in the section about collapse, and the third will be the subject of the last section.

Since Alexandrov spaces are locally compact [BGP], any pair of points is joined by a minimal geodesic. The curvature condition implies that geodesics in such a space cannot bifurcate, and are uniquely determined by their germs. A germ of (normal) geodesics emanating at a point $p \in X$ is called a *geodesic direction* at p. Again the curvature condition allows to define the *angle* of a hinge at p as a limit of comparison angles. This yields a metric on the space of geodesic directions at p, and the closure of this set is called the *space of directions* at p and is denoted by S_pX. The *tangent space* at p is now defined as $T_pX = C_0(S_pX)$, i.e., as the *Euclidean cone* of the space of directions at p. Alternatively, T_pX is the pointed Gromov–Hausdorff limit relative to p of scaled copies λX of X as $\lambda \to \infty$. From this description it is easily seen that $\operatorname{curv} T_pX \geq 0$, $\operatorname{curv} S_pX \geq 1$, and $\dim S_pX = \dim X - 1$.

The fact that $\dim S_pX = \dim X - 1$ allows for an inductive definition of the *boundary* ∂X of an Alexandrov space X, based on the fact that the only compact 1-dimensional Alexandrov spaces are circles and intervals:

- A point x belongs to ∂X if and only if $\partial S_p \neq \varnothing$.

For the local structure of an Alexandrov space, the following result is due to Perelman [Pr2].

Theorem 4.2 (structure theorem). *For any point p in an Alexandrov space X, there is $\epsilon = \epsilon(p) > 0$ such that the open ball $B(p,\epsilon)$ is homeomorphic to T_pX.*

From this result it follows that X globally admits a stratification into topological manifolds, where the m-strata consist of those points $p \in X$ where T_pX topologically splits off an $\mathbb{R}^m$-factor and not an $\mathbb{R}^{m+1}$-factor.

The proof of the local structure theorem relies on a highly nontrivial extension of *critical point theory* for distance functions discussed in the Riemannian setting in the previous section. As there, $q \in X$ is said to be a *regular point* for the distance function $\operatorname{dist}(A,.)$ from a closed subset $A \subset X$ if there is a direction $v \in S_qX$ such that the angle between v and any minimal geodesics from q to the set A exceeds $\pi/2$. If q is not regular it is *critical.* The statement of the isotopy lemma carries over verbatim to this general situation, but its proof is much deeper in the context of n-dimensional Alexandrov spaces. To achieve it, one proves in general that a "submersion" of this kind to $\mathbb{R}^k$, $1 \leq k \leq n$, (like k "independent distance functions") is a locally trivial bundle. This is proved by inverse induction on k, i.e., starting with $k = n$. The induction start is in fact fairly simple in contrast to the highly nontrivial induction step.

The local structure theorem is also instrumental in Perelman's far reaching *topological stability* result [Pr3], a published proof of which is provided in [Ka1].

Theorem 4.3 (stability theorem). *For any compact Alexandrov space X with* $\operatorname{curv} X \geq k$, *there is $\epsilon = \epsilon(X) > 0$ such that any Alexandrov space Y with* $\operatorname{curv} Y \geq k$ *and $d_{GH}(X,Y) < \epsilon$ is homeomorphic to X if* $\dim Y = \dim X$.

From this and what we have seen above, the homotopy finiteness theorem can be improved (although non quantitatively) to the following (cf. also [GPW]).

Theorem 4.4 (homeomorphism finiteness theorem)**.** *For given n, k, D and v, there are at most finitely many topological types among the n-manifolds M with $\sec M \geq k$, $\operatorname{diam} M \leq D$, and $\operatorname{vol} M \geq v$.*

The first two conditions assure that the Gromov–Hausdorff closure of the class of manifolds considered is compact and consists of Alexandrov spaces X with $\operatorname{curv} X \geq k$ and $\operatorname{diam} X \leq D$. The last condition assures that no collapse occurs and hence $\dim X = n$, so that the stability theorem applies. We point out that it is a general fact that in all dimensions but four, there are only finitely many diffeomorphism types for a given manifold M. In particular, the above result yields a *diffeomorphism finiteness* theorem in all dimensions but four.

In the above proof one might wonder what it takes to ensure that all manifolds sufficiently close to an $X := \lim M_i$ are actually diffeomorphic to one another. We refer to this as the *differentiable stability question/problem.*

The situation in dimension four is not yet completely settled although a promising plan of attack was offered by Curtis Pro and Fred Wilhelm. They obtained a general *diffeomorphism stability theorem* for limit objects whose singular set is reasonably controlled and of codimension at least four [PrW].

A special situation where [PrW] applies is the recognition of manifolds with given radius, a lower curvature bound, and near extremal volume. To describe this,

let $\mathcal{M}^\pi(n)$ denote the collection of all closed manifolds with *radius* π,

and in addition to $\min\sec\colon \mathcal{M}^\pi \to (-\infty, 1]$ consider the volume function

$$\operatorname{vol}\colon \mathcal{M}^\pi(n) \to \mathbb{R}_+.$$

Define $V_n\colon (-\infty, 1] \to \mathbb{R}_+$ by

$$V_n(k) = \begin{cases} \operatorname{vol} \mathsf{D}^n_k(\pi), & \text{when } k \leq 1/4,\\ 2\operatorname{vol}\left(\frac{1}{\sqrt{k}}\left(\mathsf{S}^{n-2}_1 * [0, \pi\sqrt{k}]\right)\right), & \text{when } 1/4 \leq k \leq 1. \end{cases}$$

From (1.2), $\operatorname{vol} M \leq \operatorname{vol} \mathsf{D}^n_k(\pi)$ for any $M \in \mathcal{M}^\pi_k(n)$, where equality occurs only when M is isometric to the unit sphere S^n_1 or the projective space $\mathsf{S}^n_{1/4}/\langle -\mathrm{Id}\rangle$. It was proved in [GP3], that for $k \leq 1/4$ this volume estimate is optimal, whereas for $k \in (1/4, 1)$ the optimal estimate is $\sqrt{k}\ 2\operatorname{vol}\left(\frac{1}{\sqrt{k}}\left(\mathsf{S}^{n-2}_1 * [0, \pi\sqrt{k}]\right)\right)$. Moreover, it was proved that the corresponding Gromov–Hausdorff limit objects are $D^n_k(\pi)$ with boundary points identified by either the antipodal map, or by a reflection when $k \leq 1/4$, and the singular sphere $\frac{1}{\sqrt{k}}\left(\mathsf{S}^{n-2}_1 * \sqrt{k}\mathsf{S}^1_1\right)$ for $k \in (1/4, 1)$. In particular, from Perelman's stability theorem one concludes that manifolds of a given radius, with lower curvature bound and almost optimal volume are topologically either a sphere or a real projective space.

In fact, invoking [PrW] (or the earlier paper [PSW]), one has the following statement.

Theorem 4.5 (differentiable recognition theorem)**.** *For each $k \leq 1$ and an integer n at least two, there is $\epsilon = \epsilon(k, n)$ such that any $M \in \mathcal{M}^\pi_k(n)$ with $V(k) - \operatorname{vol}(M) \leq \epsilon$ is diffeomorphic to the n-sphere or the real projective n-space.*

When $k > 1/4$ this also follows from the packing radius sphere theorem (see next section).

§5. Join operation applications

As was indicated in the previous section, it is often the case that manifolds with a lower curvature bound and additional almost optimal values for various geometric invariants are recognized topologically or even sometimes up to diffeomorphism (see also [GM]).

In view of the *soft* diameter sphere theorem and the corresponding *maximal diameter rigidity* theorem, it is natural to wonder what can be said if, say, $\sec M \geq 1$ and $\operatorname{diam} M$ is close to the extreme value π.

Here one notes that there are metrics on the sphere that are arbitrarily Gromov–Hausdorff close to the interval $[0, \pi]$, illustrating the phenomenon of collapse to be discussed in the next section. On the other hand, if on M^n with $\sec M \geq 1$ one has $n+2$ points any two of which have distance larger than $\pi/2$ to one another (think of the vertices of an $(n+1)$ simplex), then it is not hard to see that no collapse occurs. Moreover, Riemannian geometric tools suffice to rule out exotic spheres in this case.

We point out that no exotic spheres M with $\sec M \geq 1$ and $\operatorname{diam} M > \pi/2$ are known.

On the other hand, the following generalization of the diameter sphere theorem was proved in [GW1, GW2].

Theorem 5.1 (handlebody)**.** *Any Riemannian manifold M^n with $\sec M \geq 1$, and $q+1$ points at pairwise distances exceeding $\pi/2$ is diffeomorphic to a handlebody $\mathbb{D}^q \times \mathbb{S}^{n-q} \cup_f \mathbb{S}^{q-1} \times \mathbb{D}^{n-q+1}$.*

Note that of course $q \leq n$ in this theorem, and when $q = n+1$ is maximal, M is diffeomorphic to $\mathbb{S}^n$ as claimed above. The same conclusion follows from the above theorem when $q \geq n-3$, where Hatcher's theorem [Hat] that $\operatorname{Diff}_+(\mathbb{S}^3)$ has $\mathsf{SO}(4)$ as a strong deformation retract is invoked.

Corollary 5.2. *Any Riemannian manifold M^n with $\sec M \geq 1$, and $n-2$ points at pairwise distances exceeding $\pi/2$ is diffeomorphic to $\mathbb{S}^n$.*

The key idea in the proof of the handlebody decomposition theorem is to change the *global* Riemannian geometric problem to a *local* Alexandrov geometric problem. Specifically, one looks at the $(n+1)$-dimensional Alexandrov space $X := \Sigma M$, the spherical suspension of M, which has $\operatorname{curv} X \geq 1$ and is smooth everywhere except at the north/south suspension points. By appealing to the techniques developed in [GS] and [GWu], the existence of $q+1$ points at pairwise distances larger than $\pi/2$ in M (placed slightly below the equator M of X) allows for the construction of q concave functions near the north vertex (smooth away from the vertex), together forming a submersion. An analysis of an annular region around the north vertex based on this observation then leads to the decomposition of a sphere (hence M) centered at the vertex.

As another application of this geometric setup, in [GW2] it was also proved that if $\sec M^n \geq 1$ and only $\operatorname{diam} M > \pi/2$, then there exists an n-dimensional Alexandrov space Y which is the simultaneous Gromov–Hausdorff limit of Riemannian metrics on M as well as on the standard sphere, all with sectional curvature at least 1. In particular,

Theorem 5.3. *If the differentiable stability problem has a positive solution, then any M^n with $\sec M \geq 1$ and $\operatorname{diam} M > \pi/2$ is diffeomorphic to $\mathbb{S}^n$.*

Note that for the conclusion one needs stability "only" for the particular setup described above.

Yet another application of the same setup is due to Burkhard Wilking [W6].

Theorem 5.4. *If there is an exotic sphere M with $\sec M \geq 1$ and $\operatorname{diam} M > \pi/2$, then its diameter can be chosen arbitrarily close to π.*

§6. Collapse and almost nonnegative curvature

Except for the differentiable stability problem, the relationship between elements in $\mathcal{M}_k(n)$ and n-dimensional limit objects, i.e., the noncollapsing case is rather well understood (cf. also [Ka2] where it was shown that such limits have significantly more structure than n-dimensional Alexandrov manifolds).

The situation is dramatically different in the case of *collapse*, i.e., when a limit object $X = \lim M_i$, $M_i \in \mathcal{M}_k(n)$ has dimension $m < n$. Relatively little is known in general, except for the immensely important case of $n = 3$, where understanding the structure of collapse ([SY1, SY2]) is instrumental in Perelman's solution of Thurston's *geometrization conjecture* (for $n = 4$ see [Ya4]).

Another exception is the special case where X is a Riemannian manifold. Here, for i sufficiently large, there is an *almost Riemannian submersion* $F_i\colon M_i \to X$, see [Ya1].

In general, when the limit is not a Riemannian manifold, one might wonder if almost Riemannian submersion may be replaced by *almost submetry* (possibly after perturbing metrics on domain and target) and what can be said about such maps restricted to inverse images of natural singular strata of X. The depth and immense difficulties surrounding this phenomenon may be appreciated from the deceivingly "simple" case where X is an interval!

What is apparent, however, is that the manifolds M that support metrics collapsing to a point with a lower curvature bound play a pivotal role in understanding collapse in general. Such manifolds are said to be *almost nonnegatively curved*, because after scaling (to, say, fixed diameter), limits with nonnegative curvature will arise. In fact, an essential tool for analyzing such manifolds is via scalings/blowups, eventually leading to a noncollapsed situation.

After the pioneering work of Yamaguchi and Fukaya (some of whose results are also valid for manifolds in $\mathcal{M}_\epsilon$ with ϵ sufficiently close to 0), the most far reaching work so far in this direction is due to Kapovitch, Petrunin, and Tuschmann [KPT]; among other things they proved the following.

Theorem 6.1. *The fundamental group $\pi_1(M)$ of any closed n-manifold $M \in \mathcal{M}_{0^-}$ contains a nilpotent subgroup of index at most $C(n)$.*

Theorem 6.2. *Any closed $M \in \mathcal{M}_{0^-}$ has a finite cover that is nilpotent.*

Recall here that a connected CW complex is said to be *nilpotent* if its fundamental group is nilpotent, and its action on the higher homotopy groups is nilpotent.

The essential additional new tool introduced here is that of a *gradient push* in Alexandrov spaces, the idea behind which is somewhat reminiscent of the Sharafutdinov retraction of convex sets in nonnegative curvature.

When combined with the Cheeger–Gromoll structure result for fundamental groups of nonnegatively curved manifolds (they contain a finite index free Abelian subgroup), it is a simple matter to exhibit manifolds showing that the inclusions $\mathcal{M}_+ \subset \mathcal{M}_0 \subset \mathcal{M}_{0^-} \subset \mathcal{M}_k$, $k < 0$, are all strict.

Although the above results say nothing for simply connected M, it is conceivable that the ideas and tools developed in [KPT] will lead to obstructions to almost nonnegative curvature in this case, where so far the only known one is Gromov's Betti number theorem.

We pause to mention that there are natural weaker as well as stronger notions of *almost nonnegative curvature*, the weaker one being called *almost non-negative curvature in a generalized sense* (cf. [Ya1, KPT]) and the stronger stemming from *continuous collapse to a point* with a lower curvature bound. The latter obviously holds for positively and for nonnegatively curved manifolds simply by scaling.

Another natural and important example of continuous collapse occurs in the context of isometric group actions $\mathsf{G} \times M \to M$ by a compact Lie group. This is the so-called *Cheeger deformation* [C2] that provides a continuous collapse of M to its orbits space M/G.

Remark 6.3. In general, the collection of Alexandrov spaces X that a given manifold M can collapse to under a lower curvature bound obviously provides much information about M. That said, it is not clear how to measure, quantify, or approach this. But even, say, the possible dimensions of such X have significance.

Just how little is known is illustrated by the following basic problem.

Problem 6.4. Does any $M \in \mathcal{M}_{0^-}$ also admit collapse to an Alexandrov space X with $0 < \dim X < \dim M$?

This question is potentially related to the next conjecture.

Conjecture (Bott)**.** *The Betti numbers of the loop space of any* (*closed simply connected*) $M \in \mathcal{M}_0$ *grow at most polynomially.*

The natural approach to this would be via classical Morse theory (cf. the work on positively pinched manifolds in [BBo]). In fact, it would suffice to control the number of geodesics and their indices for some pair of points in M (possibly after altering the metric). The work in [BP], however, makes this rather unpromising. Over the rationals, the above property of the loop space is equivalent to $\dim(\pi_*(M) \otimes \mathbb{Q}) < \infty$. Such M are said to be *rationally elliptic.* This property alone is very strong and leads via rational homotopy theory to additional important restrictions, including Gromov's optimal conjecture for $\dim H_*(M)$ to be that of the n-torus (cf., e.g., [GH]).

There are interesting cases where rational ellipticity can be proved via a combination of collapse and Morse theory including manifolds of cohomogeneity one and nonnegatively curved manifolds of cohomogeneity two [Ye] (see also [GZ2]). The common feature in these cases is that M projects to the orbits space X (limit of the collapse) and this Alexandrov space is *geometrically elliptic* in the sense that there are points $q_0, q_1 \in X$ between which the number of geodesic (billiards) joining them grows at most polynomially as a function of length. This property is shared by geodesics in M orthogonal to the preimage P_1 of q_1 from a point p_0 in the preimage of q_0. Those are exactly the critical points for the energy integral of the homotopy fiber of the inclusion of $P_1 \subset M$. In these examples, where P_1 is homogeneous, P_1 is rationally elliptic. Combining these facts leads to the claim that M is rationally elliptic.

The following is a much harder question than Problem 6.4 above.

Problem 6.5. Does any $M \in \mathcal{M}_{0^-}$ also have an Alexandrov limit space X with elliptic geometry?

If so, this might lead to a proof by induction on the dimension (along the lines alluded to in the group action cases above) that $M \in \mathcal{M}_{0^-}$ are rationally elliptic. Note that in such a strategy it is necessary to work with the larger class $\mathcal{M}_{0^-}$ and not just $\mathcal{M}_+$ or $\mathcal{M}_0$ as in Bott's conjecture.

Although we do not treat manifolds with bounded curvature here, we point out that in sharp contrast to our discussion above, there is a well developed structure theory for collapse with bounded curvature, developed by Fukaya, Cheeger, and Gromov culminating in [CFG]. This has had tremendous impact on the subject including the celebrated finiteness theorem for 2-connected manifolds with bounded curvature and diameter by Petrunin, Tuschmann, Fang, and Rong [PT, FR1].

We conclude this section with a brief discussion of the *structure at infinity*, $M(\infty)$ of complete open manifolds M of nonnegative curvature, this being closely related to collapse as well. On the one hand, this can be defined as the *space of rays* from an arbitrary point $p \in M$ under suitable equivalence. Alternatively, it can be described via the pointed limit of *blowdowns* at $p \in M$. This Gromov–Hausdorff limit, $\lim(\lambda M, p)$ as λ approaches 0, is the Euclidean cone $C_0(M(\infty))$.

In particular, $M(\infty)$ is an Alexandrov space with curv ≥ 1. As an example $\mathbb{R}^{n+1}(\infty) = \mathbb{S}^n$, and a capped off cylinder $\mathbb{S}^n \times [0, \infty)$ has ideal boundary a point.

Problem 6.6. Which Alexandrov spaces X can occur as the ideal boundary $M(\infty)$ of a complete manifolds M with nonnegative curvature?

In [GuK] it was shown that if $X = M(\infty)$ is a Riemannian manifold, then it is the base of a fibration from a sphere. When simply connected, this in particular implies by work of W. Brouwder that X topologically looks like a rank one symmetric space. In particular, the class of positively curved spaces that can appear as $M(\infty)$ is rather small. Among non-manifolds, it does however include all spherical orbit spaces $\mathbb{S}/\,\mathsf{G}$.

One might wonder if the above problem is related to the following.

Problem 6.7. Which Alexandrov spaces X can occur as spaces of directions of (collapsing) limits of Riemannian manifolds with a lower sectional curvature bound?

§7. Constructions and examples

So far we have discussed a sample of general results for the classes $\mathcal{M}_+ \subset \mathcal{M}_0 \subset \mathcal{M}_{0^-} \subset \mathcal{M}_k$ obtained via comparison tools, critical point theory, Gromov–Hausdorff convergence, and Alexandrov geometry.

Our focus in this section is to exhibit examples from the gem of manifolds $\mathcal{M}_+ \subset \mathcal{M}_0 \subset \mathcal{M}_{0^-}$, and present constructions leading to them. To a large extent this is closely related to the role of symmetries to be discussed in the last section. In fact, the efforts spent on deriving general results in the presence of symmetries shed new light and pave the way towards the discovery and construction of examples. As this collection of examples grows and is closely examined geometrically as well as (differential) topologically, existing conjectures are either supported or refuted, and new insights are gained, possibly leading to new conjectures.

At the core of this is the fact that any Lie group G with a biinvariant metric belongs to $\mathcal{M}_0$.

Projections. It is an important simple fact that the *Riemannian submersions* $M \to N$ are sectional curvature nondecreasing on *horizontal planes* (precise formulas provided by [ON] and [Gr]). In particular,

- the base N of a Riemannian submersion $M \to N$ is in $\mathcal{M}_0$ when $M \in \mathcal{M}_0$.

Given that any compact Lie group $\mathsf{G} \in \mathcal{M}_0$, it follows that all *homogeneous* manifolds G/H and more generally *biquotients* $\mathsf{G}/\!/\mathsf{H}$ are in $\mathcal{M}_0$. Here, in the latter case $\mathsf{H} \subset \mathsf{G} \times \mathsf{G}$ acts from both sides of G, and when this action is free, $\mathsf{G}/\!/\mathsf{H}$ is a manifold.

Much work has been done towards analyzing which G/H and $\mathsf{G}/\!/\mathsf{H}$ support metrics in $\mathcal{M}_+$.

In fact, there is a complete classification of homogeneous $M = \mathsf{G}/\mathsf{H} \in \mathcal{M}_+$; the list provided by Berger [Be], Wallach [Wa], and Aloff–Wallach [AW] was shown to be exhaustive by Berard–Bergery [BB] and Wilking–Ziller [WZ]. In the simply connected case, aside from the CROSS, there is one example in each of the dimensions $6, 12, 13$, and 24 and an infinite family in dimension 7.

So far there is no classification of $M = \mathsf{G}/\!/\mathsf{H} \in \mathcal{M}_+$, but there is one example in dimension 6, infinitely many in dimension 7 all due to Eschenburg [E1], and infinitely many in dimension 13 due to Bazaikin [Ba1].

To illustrate how important it is to further analyze properties of this list of examples, we mention that an analysis of their isometry groups led to the discovery [Sh] of several examples in $\mathcal{M}_+$ with Abelian but noncyclic fundamental group (answering in the negative a question by Chern). For further examples, see [GS, Ba2].

Bundles. Taking products obviously preserves the classes $\mathcal{M}_0 \subset \mathcal{M}_{0^-}$. In contrast, no known example in $\mathcal{M}_+$ is topologically a product! (compare this with the classical Hopf conjecture that no metric on $\mathbb{S}^2 \times \mathbb{S}^2$ puts it in $\mathcal{M}_+$).

In general, assume $P \to N$ is a principal G-bundle with $P \in \mathcal{M}_0$ and G acting freely by isometries. Moreover, assume G acts isometrically on $F \in \mathcal{M}_0$. Then, since taking products and quotients preserves $\mathcal{M}_0$,

- the total space $M = P \times_{\mathsf{G}} F := (P \times F)/\,\mathsf{G}$ of the associated bundle is in $\mathcal{M}_0$.

The same holds with the class $\mathcal{M}_0$ replaced by the larger class $\mathcal{M}_{0^-}$, with G acting isometrically for a defining family of metrics. The utility of this construction of course depends on the extent to which the desired principal bundles can be constructed (see below). For the class $\mathcal{M}_0$ this is a very hard problem in general. For the class $\mathcal{M}_{0^-}$, however, we have the following most desirable result due to Fukaya and Yamaguchi [FY].

Theorem 7.1 (bundle)**.** *Let $P \to N$ be a principal G-bundle with G compact. If $N \in \mathcal{M}_{0^-}$, then there is a G-invariant metric on $P \in \mathcal{M}_{0^-}$.*

This gives rise to lots of examples, including all exotic 7 (respectively, 15) spheres that fiber over $\mathbb{S}^4$ (respectively, $\mathbb{S}^8$) with fiber $\mathbb{S}^3$ (respectively, $\mathbb{S}^7$). A much more general result was obtained by Searle and Wilhelm in [SW], in particular yielding that all exotic 15 spheres that bound parallelizable manifolds are in $\mathcal{M}_{0^-}$.

Gluing constructions. In general, gluing constructions that yield manifolds in $\mathcal{M}_+$ are expected to be hard, and depending on its meaning one could even argue that none exist yet (see below though).

The reason for this is that the simplest type of gluings (preserving curvature bounds) "require" totally geodesic boundaries isometric to one another (and even better with product metric near the boundary). In $\mathcal{M}_0$, this means convexity. Nonetheless, this is exactly what has been accomplished in many interesting cases starting with [C2], where it was shown that

- the connected sum of any two projective spaces is in $\mathcal{M}_0$.

This is achieved by showing that the disk bundle of the canonical line bundle over a projective space can be given a metric with nonnegative curvature, and totally geodesic boundary isometric to the standard sphere, and product near the boundary (see also [BM]).

In general, any cohomogeneity one G-manifold M with orbit space an interval is the union of two disk bundles, namely the normal disk bundles over the two nonprincipal orbits. In [GZ1] it was shown that

- a cohomogeneity one manifold with singular orbits of codimension two has a G-invariant metric in $\mathcal{M}_0$.

Indeed, one constructs such a metric on each disk bundle with the property that it is product near the boundary and the two sphere bundles (a principal orbit) are isometric (this is where the codimension is crucial).

Part of the utility of this is based on a *principal bundle construction* over cohomogeneity one manifolds whose total space is a cohomogeneity one manifold with singular orbits of the same codimension as those on the base [GZ1]. In particular, it turns out that this yields numerous principal bundles $P \to M$ with $P \in \mathcal{M}_0$, and thus numerous associated bundles as well. In particular,

• the total space M of any vector or sphere bundle over $\mathbb{S}^4$ is in $\mathcal{M}_0$.

This class contains all the exotic Milnor spheres. We point out that by extending the cohomogeneity one method above to biquotiens, just very recently Goette, Kerin, and Shankar [GKS] have succeed in proving the following remarkable fact.

Theorem 7.2. *All exotic* 7*-spheres belong to* $\mathcal{M}_0$.

The result above for vector bundles over $\mathbb{S}^4$ above also addresses the following *converse of the soul theorem* question.

Problem 7.3. Which vector bundles E over $S \in \mathcal{M}_0$ can have total space $E \in \mathcal{M}_0$?

Originally this was only formulated by Cheeger and Gromoll for $S = \mathbb{S}^n$, where it is still unknown in general except for $n \leq 5$ (cf. [GZ1]). For all the compact rank one symmetric spaces as base (as well as all known simply connected 4-manifolds in $\mathcal{M}_0$) it is known that stably, i.e., by possibly adding a sufficiently large trivial bundle, the conclusion holds [Ri, GA] ([GZ1]).

In general, the answer is in the negative, as was first shown by Özaidin and Walschap [OW]. Since then numerous topological obstructions have been found by Belgradek and Kapovitch when S has infinite fundamental group [BK].

In all the gluing constructions discussed so far, the manifold M was exhibited as the union of two nonnegatively curved disk bundles, product near the boundary and isometric boundaries.

It is interesting that, in fact, any complete open manifold of nonnegative curvature (and the interior of any compact convex subset in a nonnegatively curved manifold) has such a structure, as was proved by Guijarro [Gu]. From this it follows immediately that if M and N are nonnegatively curved manifolds with convex boundaries ∂M and ∂N, then also

• the boundary $\partial(M \times N)$ is in $\mathcal{M}_0$.

To see this, simply use Guijarro's observation combined with

$$\partial(M \times N) = M \times \partial N \cup \partial M \times N$$

and the fact that $\partial(M \times \partial N) = \partial M \times \partial N = \partial(\partial M \times N)$.

A very special case of this is what was referred to as an *open book* in [FG1], where, say, $M = \mathrm{D}(\nu)$ is a nonnegatively curved convex disk bundle, and $N = \mathbb{D}^k$. Here, the resulting nonnegatively curved manifold $M_{\nu,k}$ can be thought of as an open book whose pages $\mathrm{D}(\nu)$ are parametrized by the sphere $\mathbb{S}^{k-1}$ and the common binding is $\mathrm{S}(\nu)$. Alternatively, $M_{\nu,k} = \mathrm{S}(\nu \oplus \epsilon^k)$, the sphere bundle of the sum of ν with a trivial k-dimensional bundle.

This type of manifold $M \in \mathcal{M}_0$ arises naturally in the presence of reflection groups, when the generating reflections have a common mirror (and the representation of the reflection group orthogonal to the intersection of the mirrors is irreducible, cf. [FG1]).

In this context, the construction needs to be generalized and iterated (when the representation of the reflection group orthogonal to the intersection of the mirrors is reducible, cf. [FG1]). For this, the building block is not a convex nonnegatively curved manifold with boundary, but rather a convex nonnegatively curved manifold

$$\mathrm{D}(\nu) = \mathrm{D}(\nu_1) \oplus \mathrm{D}(\nu_2) \oplus \cdots \oplus \mathrm{D}(\nu_\ell)$$

with corners. Here the total boundary consists of faces any two of which meet perpendicularly, and the metric near each face is a product. Moreover, each $\mathrm{D}(\nu_i)$ is totally geodesic in $\mathrm{D}(\nu)$ all meeting orthogonally at the soul S. Inductively, each step of the open book construction applied here reduces the number of faces by one, so after ℓ open book constructions we arrive at a nonnegatively curved $M_{\nu_1,\ldots,\nu_\ell;k_1,\ldots,k_\ell}$. Alternatively,

$M_{\nu_1,\dots,\nu_\ell;k_1,\dots,k_\ell}$ is the pull back by the diagonal $S \to S \times \cdots \times S$ of the product of sphere bundles $\mathrm{S}(\nu_i \oplus \epsilon^{k_i})$, $i = 1, \dots \ell$.

It is possible to generalize the above construction for the metric on $\partial(M \times N) = M \times \partial N \cup \partial M \times N$ to involve several choices of manifolds with corners, as in the special case of iterated open books above, and one wonders what kind of manifolds this leads to.

We conclude our discussion about gluing methods to a case technically very far from what was discussed above and indeed more subtle. Note that in positive curvature it is impossible to glue two convex sets together (they are disks) without the resulting manifold being a sphere topologically. In particular, all non-sphere cohomogeneity one manifolds in $\mathcal{M}_+$ are glued together by two disk bundles whose common boundary is not totally geodesic.

The work towards the classification of cohomogeneity one manifolds in $\mathcal{M}_+$ [GWZ] (see the next section) yielded an explicit exhaustive list, including two infinite families P_k and Q_k, where at the time none other than $P_1 = \mathbb{S}^7$ and $Q_1 = A^7_{1,1}$ (the normal homogeneous Aloff–Wallach space [AW, W2]) were known to carry an (invariant) metric in $\mathcal{M}_+$. Since then, explicit constructions [GVZ, De], unlike all previous ones in $\mathcal{M}_+$, led to the following.

Theorem 7.4. *The cohomogeneity one $\mathsf{SO}(4)$-manifold $P_2 = \mathrm{T}_1\mathbb{S}^4 \# \Sigma^7$ admits an invariant metric in $\mathcal{M}_+$.*

Here the description of P_2 as an exotic smooth structure on the unit tangent bundle $\mathrm{T}_1\mathbb{S}^4$ of $\mathbb{S}^4$ is due to Goethe [Go].

Deformations. One of the most natural attempts in the search for metrics with desired properties is via deformations of a starting metric.

From the point of view of what we have discussed here related to Alexandrov geometry, the so-called *Cheeger deformation* (whose origin stems from the construction of *Berger spheres*) stands out. For G a closed connected Lie group, this is a deformation that naturally shrinks the orbits of an isometric action $\mathsf{G} \times M \to M$ to points, providing a deformation of M to M/G under a lower curvature bound.

Conformal changes play significant roles in various problems, often involving PDE methods not treated here. Of course, in dimension two the classical *uniformization theorem* stands out. In higher dimensions, important cases where curvature bounds like those considered here are dealt with include the work in [PW] and [SW].

Despite the spectacular applications the *Ricci flow* has had, so far no major applications have been found in the primary context of lower (and no upper) sectional curvature bounds. This is of course not surprising given that lower sectional curvature bounds are not preserved.

It would be interesting to produce new examples of manifolds in, say, $\mathcal{M}_+ \subset \mathcal{M}_0$ by deforming Alexandrov manifolds with nonnegative or positive curvature (cf. [Dy] and [Sp] though).

§8. The presence of symmetries

Although a generic Riemannian metric has trivial isometry group, the "nicest", "most optimal" ones typically have lots of symmetries. Of course, this is to some extent a matter of taste, but most would agree that for example the exclusive, yet rich class of *symmetric spaces* are indeed models of perfection among Riemannian manifolds.

Many problems in geometry are motivated by a search for "optimal/nicest" metrics on a given manifold, often involving curvature properties such as constant *scalar -, ricci -*, or *sectional* curvature, or perhaps *holonomy* restrictions. Here we will add *symmetry* to the mix.

The presence of a group G of isometries on a Riemannian manifold M has strong immediate impact for the following reasons.

- The fixed point set M^{G} of G consists of totally geodesic submanifolds of M.
- The orbit space M/G is an Alexandrov space (at least as curved as M).

In addition, the *isotropy/slice representations* of all *isotropy* subgroups have rather direct impact as well. For example, each *orbit stratum* (collection of orbits of the same type) in M is a minimal submanifold, and in M/G it is totally geodesic. Note, that all of this of course also applies to any subgroup $\mathsf{K} \subset \mathsf{G}$.

In this context, it is natural to investigate what can be said about the classes $\mathcal{M}_+ \subset \mathcal{M}_0 \subset \mathcal{M}_{0^-}$ in the presence of *large and/or special* groups of isometries. Such investigations have led to several strong results including *classification* type theorems, as well as to the discovery and construction of one new member of $\mathcal{M}_+$ and numerous members of $\mathcal{M}_0$ as discussed in the previous section.

Cohomogeneity. The cohomogeneity of an isometric group action $\mathsf{G} \times M \to M$ is the codimension of its principal orbits, or equivalently in our case (where G and M are compact), the dimension of its orbit space M/G.

As already pointed out, *all homogeneous manifolds* $M = \mathsf{G}/\mathsf{H}$, i.e., manifolds of cohomogeneity zero, admit a G-invariant metric with $M \in \mathcal{M}_0$, and those with $M \in \mathcal{M}_+$ are classified ([BB] and [WZ]).

An application of the Cheeger deformation shows that *all cohomogeneity one manifolds* admit invariant metrics in $\mathcal{M}_{0^-}$ [ST].

However, even among simply connected closed cohomogeneity one manifolds (*all rationally elliptic* [GH]), there is a distinction among the classes $\mathcal{M}_+ \subset \mathcal{M}_0 \subset \mathcal{M}_{0^-}$.
In particular:

- No exotic spheres admit cohomogeneity one metrics in $\mathcal{M}_0$ [GVWZ, HR].
- All cohomogeneity one manifolds with both singular orbits of codimension 2 admit invariant metrics in $\mathcal{M}_0$ [GZ1].
- Only a few of the latter admit invariant metrics in $\mathcal{M}_+$ [GWZ].

Note that any G-manifold M with M/G a circle has a G-invariant metric in $\mathcal{M}_0$. However, the following is a difficult problem.

Problem 8.1. Determine all closed simply connected cohomogeneity one manifolds in $\mathcal{M}_0$.

For positive curvature, there is a complete classification in dimensions $\neq 7$ (see [V1, V2] and [GWZ]), which contains infinite subfamilies of the *Eschenburg* examples in dimension 7 as well as an infinite subfamily of the *Bazaikin* examples in dimension 13. In the remaining dimensions, the following is true.

Theorem 8.2. *A simply connected cohomogeneity one G-manifold $M^n \in \mathcal{M}_+$ is G-equivariantly diffeomorphic to a rank one symmetric space, as long as $n \neq 7, 13$.*

Recall that an exhaustive list of simply connected cohomogeneity one manifolds admitting positive curvature was derived in [GWZ] (an exceptional one of which was later shown not to support such a metric [VZ]).

For any cohomogeneity, one has the following remarkable "finiteness analog" of the classification of homogeneous manifolds in $\mathcal{M}_+$ (see [W4]).

Theorem 8.3 (Wilking)**.** *Any simply connected cohomogeneity $k \geq 1$ manifold $M^n \in \mathcal{M}_+$ with $n \geq 18(k+1)^2$ is a rank one symmetric space up to tangential homotopy equivalence.*

So far, even in cohomogeneity two, we have limited knowledge about restrictions for $M \in \mathcal{M}_0 \subset \mathcal{M}_{0-}$. However,

Theorem 8.4 (Yeager). *Any simply connected cohomogeneity* 2 *manifold* $M \in \mathcal{M}_0$ *is rationally elliptic.*

We note that it is possible to extend the proof in [Ye] to the class $\mathcal{M}_{0-}$, and there are numerous cohomogeneity two manifolds not in $\mathcal{M}_{0-}$. The latter can also be seen from the work of Yamaguchi on collapse of 4-manifolds, see [Ya4], combined with the following simple fact.

• The connected sum of *any number* of $\pm\mathbb{CP}^2$ supports a T^2-action of cohomogeneity two.

In fact the actions are *polar* (see [GZ2] and below), with section $\mathbb{RP}^2$ (respectively, $\mathsf{T}^2/\mathbb{Z}_2$) with one (respectively, two) projective spaces are involved. When more than three projective spaces are involved, the section is hyperbolic, with hyperbolic polygon as orbit space.

Torus Symmetry. Torus actions play a particular role for various reasons. For example, from the Cheeger–Fukaya–Gromov theory if follows that collapse with bounded curvature of simply connected manifolds is given in terms of such actions (see also [Ro]). Secondly, for manifold in $\mathcal{M}_+$ the method of Synge implies that an isometric action by a torus T has fixed points in even dimensions, and either fixed points or circle orbits in odd dimensions (cf. §1). In particular, there are isotropy groups of maximal rank in even dimension and at most corank 1 in odd dimensions.

Since the principal isotropy group of an effective T^k-action on M^n is trivial, it follows that $k \leq n$, with equality if an only if $M = \mathsf{T}^n$. Also observe that if T^{n-1} acts effectively on M^n, and hence by cohomogeneity one, it follows that $\pi_1(M)$ is infinite unless $n = 2, 3$. However, there are cohomogeneity two actions by T^{n-2} on simply connected n-manifolds for all $n \geq 3$ [KMP] (and free actions by T^{n-4} on simply connected n-manifolds for all $n \geq 5$ [KMP]).

If, however, M^n is rationally elliptic (but still under no curvature assumption), in [GKR] it was proved that $k \leq \lfloor \frac{2n}{3} \rfloor$ for any effective T^k-action on M. Moreover, $k \leq \lfloor \frac{n}{3} \rfloor$ if the action is almost free.

In view of the Bott conjecture, this supports the following so-called *maximal symmetry rank conjecture* proposed by B. Wilking and independently by F. Galaz-Garcia and C. Searle.

Conjecture. *If* T^k *acts isometrically on a simply connected* n*-manifold* $M \in \mathcal{M}_0$, *then* $k \leq \lfloor \frac{2n}{3} \rfloor$.

The conjecture also includes a classification up to equivariant diffeomorphism in case of equality.

It was proved in dimensions ≤ 9 by Galaz-Garcia and Searle [GaS1], and recently, under an additional assumption, in general by Escher and Searle [ES]. The additional assumption is that the smallest dimension of a T^k-orbit is either $2k - n$ or $2k - n + 1$.

Following the above conjecture, we say that a simply connected $M^n \in \mathcal{M}_0$ has *maximal* (respectively, *almost maximal symmetry rank*) if the rank of the isometry group of M is $\lfloor \frac{2n}{3} \rfloor$ (respectively, $\lfloor \frac{2n}{3} \rfloor - 1$).

In particular, for a 4-manifold, almost maximal symmetry rank means that M admits an isometric circle action. To describe the optimal classification for this case we mention the remarkable fact that any T^2-action on any of $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{CP}^2 \pm \mathbb{CP}^2$ is induced from the standard product action by $\mathsf{T}^4 = \mathsf{T}^2 \times \mathsf{T}^2$ on $\mathbb{S}^3 \times \mathbb{S}^3$ via a free T^2-subaction, see [GaK]. (For a more general result for *torus manifolds* in $\mathcal{M}_0$, see [Wm].)

Theorem 8.5. *A closed simply connected* S^1*-manifold* $M^4 \in \mathcal{M}_0$ *is equivariantly diffeomeorphic to*

- *a linear* S^1*-action on* $\mathbb{S}^4$*, or* $\mathbb{CP}^2$ *(the only possibility when* $M \in \mathcal{M}_+$*),*

or to

- *an* S^1*-subaction of a* T^2*-action on one of* $\mathbb{S}^2 \times \mathbb{S}^2$*,* $\mathbb{CP}^2 \pm \mathbb{CP}^2$*.*

The assertions as well as the proofs have evolved over time, as can be seen in the contributions [HK, Kl, SY, GS1, GaG, GWi, Sp]:

- The topological classification hinged on deep topology (Freedman's classification of simply connected topological 4-manifolds) and beautiful equivariant comparison geometry, showing that $\chi(M^{\mathsf{S}^1}) \leq 4$ (at most 3 when $M \in \mathcal{M}_+$).
- Using convexity and critical point theory, the geometric arguments became more transparent and with stronger conclusions when the Alexandrov geometry of the orbit space M/S^1 was utilized. In particular, equivariant diffeomorphism was first proved in the case where the S^1-action has a fixed point component of dimension two, and hence acts transitively on its normal sphere. (For a classification of such *fixed point homogeneous manifolds* in $\mathcal{M}_+$ in general, see [GS2].)
- When the S^1-action only has isolated fixed points, M/S^1 is a simply connected topological 3-manifold, and hence $\mathbb{S}^3$ by the Poincaré conjecture. In this case one can obtain the diffeomorphism classification by invoking work of Fintushel and Pao on smooth circle actions on closed simply connected 4-manifolds.
- To obtain the assertion up to equivariant diffeomorphism, however, one needs to know that the singular set in M/S^1 does not contain a knotted circle. This follows by showing that the canonical 2-fold branched cover along such a circle has fundamental group of order at least 3 if the curve is knotted, and that in our case it is an Alexandrov space and hence its universal cover would have too many very singular points (lifts of fixed points).
- The dependence on the Poincaré conjecture can be avoided by doing more geometry (smoothing out singularities) and appealing to Hamilton's classification of 3-manifolds with positive Ricci curvature.

The theorem above of course provides support for the classical Hopf conjecture that there is no metric on $\mathbb{S}^2 \times \mathbb{S}^2$ with positive curvature. Indeed if there is one, it can have at most a finite isometry group. One can speculate that on the list in the theorem we have all the closed simply connected 4-manifolds in $\mathcal{M}_0$. Topologically, this would follow from the ellipticity conjecture.

Problem 8.6. Does the conclusion of the above result hold for S^1-manifolds in $\mathcal{M}_{0-}$?

We point out that there is also a classification of closed simply connected T^2-manifolds in $\mathcal{M}_0$ in dimension five due to F. Galaz-Garcia and C. Searle [GaS2].

In general, the *symmetry rank* symrank(M) of a Riemannian manifold M is the rank of its isometry group. In positive curvature, there are naturally much stronger restrictions on the symmetry rank expressed in the following rigidity and "pinching" theorems due to Grove–Searle [GS1] and Wilking [W3] respectively.

Theorem 8.7 (rank rigidity)**.** *Any* $M^n \in \mathcal{M}_+$ *has* symrank$(M) \leq \lfloor \frac{n+1}{2} \rfloor$ *with equality if and only if* M *is diffeomorphic to either* $\mathbb{S}^n$*,* $\mathbb{CP}^{n/2}$*, or a lense space* $\mathbb{S}^n/\mathbb{Z}_k$*.*

Moreover, the actions are also known (standard on the model spaces).

Theorem 8.8 (rank pinching)**.** *Any simply connected n-manifold* $M \in \mathcal{M}_+$ *such that* symrank$(M) \geq n/4 + 1$ *and* $n \neq 7$ *is homotopy equivalent to a CROSS.*

In this (not strongest, but uniform) formulation, results have been combined: it is due to Wilking for $n \geq 10$ and to Fang–Rong [FR2] for $7 < n < 10$, and is covered by the rigidity theorem in dimensions $n < 7$. As, e.g, the Aloff–Wallach examples $A_{p,q} = \mathsf{SU}(3)/\mathsf{S}^1_{p,q}$ all have isometry group of rank 3, the conclusion fails in dimension 7.

When $\mathrm{symrank}(M) = \lfloor \frac{n+1}{2} \rfloor - 1$, we say that M has *almost maximal* symmetry rank. Here one has classification in all dimensions but 6 and 7.

Problem 8.9. Classify $M^n \in \mathcal{M}_+$ with almost maximal symmetry rank when $n = 6$ and 7.

Recall that the *soft* symmetry rank pinching theorem asserts that $M \in \mathcal{M}_+$ with linear symmetry rank growth (slope roughly 1/4, cf. Theorem 8.8) up to homotopy equivalence are rank 1 symmetric space.

In intriguing work pioneered by L. Kennard [Ke1, Ke2, AKe] it was discovered that even a logarithmic growth on the symmetry rank for manifolds $M^n \in \mathcal{M}_+$ yields strong restrictions along the lines of the *classical Hopf conjectures*, including positivity and bounds on the *Euler characteristic*, *obstructions for products* to have positive curvature, etc. Here the main tools are Wilking's connectivity theorem combined with algebraic topology.

Very recently, an in depth analysis of fixed point sets or torus actions in positive curvature has led to the following Corollary by Kennard and Wilking [KW]: an even-dimensional manifold of positive curvature of symmetry rank 5 has Euler characteristic at least 2.

Even the weakest symmetry rank assumption (having an isometric S^1-action) yields obstructions in support of the Chern conjecture in high dimensions [Ro1, Ro2].

Theorem 8.10 (Rong)**.** *Given n, there is $w(n)$ such for any S^1-manifold $M^n \in \mathcal{M}_+$, $\pi_1(M)$ has a cyclic subgroup of index at most $w(n)$. In particular, if $\mathbb{Z}_p \times \mathbb{Z}_p \subset \pi_1(M)$ for some prime p, then $p \leq w(n)$.*

Special isometric group actions. An isometric action $\mathsf{G} \times M \to M$ is said to be *polar* if it admits a so-called *section*, i.e., an immersed manifold $\sigma : \Sigma \to M$ whose image intersects all G-orbits orthogonally. Extreme cases include transitive actions, and actions by a discrete group. In the first case Σ is a point, and in the latter $\Sigma = M$. Unless otherwise stated, we assume throughout that we are not in these extreme cases. More interestingly, any cohomogeneity one isometric action is polar.

An exclusive yet rich class of examples are provided by *isotropy representations* and *-actions* of symmetric spaces. This includes the adjoint action of a compact Lie group, where the section is a maximal torus.

The basic general theory for polar actions was developed independently by Szentze [Sz] and Palais–Terng [PTe]. It is important that sections are totally geodesic, and that all isotropy representations are polar representations. Moreover, the stabilizer group of any section modulo its kernel is a discrete group Π and $\Sigma/\Pi = M/\mathsf{G}$.

When M is simply connected, it turns out ([Al, AT], see also [GZ2]) that Π is a *reflection group*, W, i.e., it is generated by *reflections*. Here a reflection is simply an isometric involution r with a fixed point component of codimension one. We refer to such a component as a *mirror* for r. A connected component c of the complement of all mirrors in Σ is called an *open chamber* of Σ, and its closure C is simply a *chamber*. Again assuming that M is simply connected and G is connected, W will act simply transitively on the set of all (open) chambers, and $M/\mathsf{G} = \Sigma/\mathsf{W} = C$. Here the boundary ∂C of C is the union of its faces, each face being the intersection of a mirror with C of codimension one.

In the setting above, where W acts simply transitively on the set of all chambers, one can *reconstruct* M together with its G action from its *polar data*, i.e., from a chamber C and the isotropy groups and representations along C, see [GZ2]. Based on this, a classification of all closed simply connected polar manifolds of dimension at most five was carried out in [Go].

Alternatively, one can also view M together with its polar G action *combinatorially* via its so-called associated homogeneous *chamber system* $\mathcal{C}(M;\mathsf{G}) = \cup_{\mathsf{g}\in\mathsf{G}}\, \mathsf{g}\, C$ consisting of all chambers in all sections of M. Here two chambers $\mathsf{g}\, C$ and $\mathsf{h}\, C$ are said to be *adjacent* if and only if they have a common face, or equivalently, $\mathsf{g}\,\mathsf{h}^{-1}$ is in a face isotropy group of C. It is a crucial fact that in this case all so-called *residues* of $\mathcal{C}(M;\mathsf{G})$ are canonically identified with an associated chamber system for a slice representation, which in turn all are *spherical Tits buildings*.

In this combinatorial setup, there is a natural metric on the chamber system $\mathcal{C}(M;\mathsf{G})$, namely, the induced length metric. Its induced topology is called the *thin* topology. There is also a combinatorial notion of *chamber system covers*, which in special important cases coincide with topological covers relative to the thin topology. The fundamental group of this space is typically enormous, also when the chambers are simple, e.g., contractible. The Hausdorff topology on the set of closed subsets of M will often give rise to a natural topology on the chamber system's universal cover. Together with the thin topology, this will then be an interesting object related to M with its topology. When this all works (e.g., when $M \in \mathcal{M}_+ \subset \mathcal{M}_0$), the universal cover $\widetilde{\mathcal{C}}(M;\mathsf{G})$ of $\mathcal{C}(M;\mathsf{G})$ with its new *thick* topology will be a principal bundle P over M.

Remark 8.11. In general, although quite special as G-manifolds, polar actions and manifolds constitute a rich and for many problems manageable class, which is likely to play an important future role in a variety of geometric problems.

In the context of lower curvature bounds, the examples of T^2-actions we have discussed in dimension 4 already indicate that all lower curvature bounds (fixing a diameter) are needed. However, the following problem is wide open.

Problem 8.12. Describe closed simply connected polar manifolds in $\mathcal{M}_{0^-}$ of cohomogeneity at least two.

Although the same question for the class $\mathcal{M}_0$ is still far from solved, much progress has been made and a clear goal has emerged (see [Go] for dimensions at most five though).

For the class $\mathcal{M}_+$, the following complete answer was obtained in [FGT1, FGT2].

Theorem 8.13 (smooth polar rigidity)**.** *A closed simply connected polar* G*-manifold* $M \in \mathcal{M}_+$ *of cohomogeneity at least two is equivariantly diffeomorphic to a rank one symmetric space with a polar* G*-action.*

In all cases, the natural point of departure is to understand the chambers, section(s), and the associated reflection group W. From convexity and critical point theory one obtains the following classification result [FGT1].

Theorem 8.14. *A closed manifold* $\Sigma \in \mathcal{M}_+$ *with a reflection group* W *admits a* W*-invariant metric of constant curvature* 1*.*

In particular, Σ is either $\mathbb{S}^n$ or the real projective space, and W (or its lift to $\mathbb{S}^n$) is a finite Coveter group.

Unlike the case of positive curvature, when the polar manifold $M \in \mathcal{M}_0$ the section Σ need not be closed, but at least the action by W is co-compact. In this case, one has the following (see [FG1]).

Theorem 8.15. *Let* $\widetilde{\Sigma}$ *be the universal cover of a* $\Sigma \in \mathcal{M}_0$ *with a co-compact reflection group* W*, and let* $\widehat{\mathsf{W}}$ *be the lifted reflection group on* $\widetilde{\Sigma}$*. Then* W *is a product of Coxeter* *groups*

$$\widehat{\mathsf{W}} = \widehat{\mathsf{W}}_0 \times \prod_{i=1}^{\ell-1} \widehat{\mathsf{W}}_i \times \ \widehat{\mathsf{W}}_\ell,$$

where $\widehat{\mathsf{W}}_0$ *is affine, and the remaining factors are spherical. Correspondingly,* $\widetilde{\Sigma}$ *splits isometrically and* $\widehat{\mathsf{W}}$*-invariantly as*

$$\widetilde{\Sigma} = \mathbb{R}^k \times \prod_{i=1}^{\ell-1} \mathbb{S}^{k_i} \times \Theta_\ell \times N,$$

where $\mathbb{S}^{k_i} \in \mathcal{M}_0$ *is a standard sphere with a linear* $\widehat{\mathsf{W}}_i$*-action,* $\Theta_\ell \in \mathcal{M}_0$ *is a compact simply connected* (iterated) open book*, and* N *can be any simply connected compact manifold in* $\mathcal{M}_0$ *on which all* $\widehat{\mathsf{W}}_i$ *act trivially.*

This splitting theorem stems from the corresponding metric splitting of the chamber $C = \Sigma/\mathsf{W} = M/\,\mathsf{G}$ into *Euclidean simplices* (including intervals), *spherical simplices* (of dimension at least two), and (iterated) *open book chamber* and any closed non-negatively curved manifold (the latter corresponds to taking the product with a manifold on which G acts trivially).

We say that a simply connected polar manifold in $\mathcal{M}_0$ is *indecomposable* if its orbitspace has only one factor, i.e., an interval (cohomogeneity one), a Euclidean or spherical simplex of dimension at least two, or an (iterated) open book chamber.

Surprisingly, it seems now almost certain that any such M is the base of a principal bundle whose total space splits as a product accordingly, when no book chamber is present (cf. [FG2]). Moreover, the general case reduces to the non-book case in the sense that it fibers over a non-book polar manifold with book polar fibers. Also perhaps surprisingly, book polar manifolds of non-negative curvature are in a sense well understood. This then reduces the efforts of understanding polar manifolds in $\mathcal{M}_0$ to the *indecomposable* ones. The indecomposable manifolds with spherical simplex as orbit space are understood via [FGT1] and those with Euclidean simplex as orbit space are expected to be closely related (i.e., up to principal bundles) via *affine Bruhat–Tits buildings* to symmetric spaces with polar actions.

§9. Complete open manifolds

With the exception of Riemannian manifolds M with $\sec M \geq 0$, we have throughout confined our treatment to closed, i.e., compact manifolds without boundary.

The tools and to a large extent the methods discussed here can equally well be used in the setting of complete open Riemannian manifolds M with $\sec M \geq k$, allowing $k < 0$ (noting that $k > 0$ implies that M is compact). So far, however, the author is not aware of any work in this general direction.

In particular, the following appears to be a (simple or hard?) open basic

Problem 9.1. Does any n-manifold M with $n \geq 2$ admit a complete Riemannian metric with $\sec M \geq -1$? If not, determine obstructions and describe those who do admit such metrics.

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