Алгебра и анализ
Том 29 (2017), № 1

St. Petersburg Math. J.
Vol. 29 (2018), No. 1, Pages 33–41
http://dx.doi.org/10.1090/spmj/1480
Article electronically published on December 27, 2017

A SURVIVAL GUIDE FOR FEEBLE FISH

D. BURAGO, S. IVANOV, AND A. NOVIKOV

Dedicated to Yu. D. Burago
on the occasion of his 80th birthday

Abstract. As avid anglers, the authors are interested in the survival chances of fish in turbulent oceans. This paper addresses this question mathematically. It is shown that a fish with bounded aquatic locomotion speed can reach any point in the ocean if the fluid velocity is incompressible, bounded, and has small mean drift.

§1. Introduction

Suppose a locally Lipschitz vector field $V(x)$ and a set of bounded controls

$$\mathcal{A}_t = \{\alpha \in L^\infty([0,t];\mathbb{R}^d) \ : \ \|\alpha\|_\infty \le 1\}$$

are given. For each $x \in \mathbb{R}^d$ and a control $\alpha \in \mathcal{A}_t$, the function $X_x^\alpha(s)$ is defined as a unique solution of

$$\frac{d}{ds}X_x^\alpha(s) = V(X_x^\alpha(s)) + \alpha(s), \quad s \in [0,t], \quad X_x^\alpha(0) = x. \tag{1}$$

The *travel time* from $x \in \mathbb{R}^d$ to $y \in \mathbb{R}^d$ is how long it takes to reach y from x with optimal control:

$$\tau(x,y) = \inf\left\{t \ge 0 \mid X_x^\alpha(t) = y, \text{ for some } \alpha \in \mathcal{A}_t\right\}. \tag{2}$$

If $\tau(x,y)$ is finite for any $x, y \in \mathbb{R}^d$, then $\tau(x,y)$ can be viewed as a "non-symmetric metric" on $\mathbb{R}^d$. For us the value of the control $\alpha(s)$ is precisely the strength of aquatic locomotion of fish at time s, and $V(x)$ is the velocity field of the ambient ocean. The finiteness of travel time between any x and y guarantees fish can travel anywhere it wants. Naturally, fish can rely on its strength in calm water, that is when $\|V\|_{L^\infty} < 1$, but what happens in violent storms? Here we intend to clarify the situation in this less obvious regime $\|V\|_{L^\infty} \gg 1$.

There are two natural obstructions to the finiteness of $\tau(x,y)$. The first one is compressibility. If V has a sink at x_0, then, depending on the strength of the flow near x_0, we may have $\tau(x_0,y) = \infty$ for all $y \ne x_0$. The second obstruction is a strong mean drift of V. Indeed, if V is simply a large constant vector, then $\tau(x,y) = \infty$ for many x and y. For example, if $V = (v_1, 0, \dots, 0)$, $v_1 \gg 1$, then $\tau(x,y) = \infty$ for points x and y if the first coordinate of x is larger than the first coordinate of y. In order to rule out these two natural obstructions, we assume V is incompressible,

$$\operatorname{div} V = 0, \tag{3}$$

2010 *Mathematics Subject Classification.* Primary 49K15.

Key words and phrases. G-equation, small controls, incompressible flow, reachability.

The first author was partially supported by NSF grant DMS-1205597. The second author was partially supported by RFBR grant 14-01-00062. The third author was partially supported by NSF grant DMS-1515187.

and it has small mean drift:

$$\lim_{L\to\infty} \sup_{x\in\mathbb{R}^d} \left\| \frac{1}{L^d} \int_{[0,L]^d} V(x+y)\,dy \right\| = 0. \tag{4}$$

We also assume that V is bounded:

$$\|V\|_{L^\infty(\mathbb{R}^d)} \le M < \infty. \tag{5}$$

Our main result is the following.

Theorem 1.1. *Suppose V is locally Lipschitz, bounded* (5)*, incompressible* (3)*, and has small mean drift* (4)*. Then $\tau(x,y) < \infty$ for all $x, y \in \mathbb{R}^d$.*

If the vector field is globally Lipschitz, we have the following estimate on travel times.

Theorem 1.2. *Suppose V is incompressible* (3)*, has small mean drift* (4)*, and $V \in$ Lip(K), i.e.,*

$$\|V(x) - V(y)\| \le K|x-y|, \quad \forall x, y \in \mathbb{R}^d. \tag{6}$$

Then the travel-time estimate

$$\tau(x,y) \le C_1\|x-y\| + C_2, \tag{7}$$

holds with some C_1 and C_2 that depend on V only.

In the two-dimensional case, estimate (7) was obtained in [3] under slightly different assumptions on V. This estimate was used in [3] to characterize effective behavior of solutions of the random two-dimensional G-equation, a certain Hamilton–Jacobi partial differential equation that arises in modeling propagation of flame fronts in turbulent media [4, 5]. Subsequently, effective behavior of solutions of the random G-equation was characterized in [1] for any dimension. The approach in [1] is different from that in [3]. In particular, it relies on ergodic properties of the flow V instead of its geometry.

Throughout the rest of the paper we assume that $d \ge 3$. The case of $d = 2$ easily follows by adding an auxiliary coordinate.

Remark 1.3. We do not know whether the Lipschitz condition (6) in Theorem 1.2 is necessary. More precisely, we do not know if it can be replaced by the conditions of local Lipschitz continuity and boundedness (5). Note that the conditions of Lipschitz continuity (6) and small mean drift (4) imply the boundedness (5).

One possible approach is to sacrifice a part of the available control and use this part to push V into some compact class of vector fields. Then we can use an argument similar to that in the proof of Theorem 1.2.

Remark 1.4. One may wonder whether uniform convergence is essential in (4). That is, we ask if Theorem 1.1 remains valid if we replace (4) by the following assumption:

$$\lim_{L\to\infty} \left\| \frac{1}{(2L)^d} \int_{[-L,L]^d} V(x+y)\,dy \right\| = 0 \tag{8}$$

for every $x \in \mathbb{R}^d$. The answer is no. A counter-example exists already in two dimensions. Consider the vector field V on $\mathbb{R}^2$ given by

$$V(x_1, x_2) = 10 \cdot (\operatorname{sgn}(x_2), \operatorname{sgn}(x_1)).$$

It is discontinuous, but this can be fixed by convolution with a suitable mollifier. The resulting field is bounded, incompressible, and it satisfies (8). On the other hand, the fish cannot leave the region $\{x_1 \ge 1, x_2 \ge 1\}$.

Remark 1.5. As could be seen from the proof, the small mean drift assumption (4) in Theorem 1.1 can be replaced by a more technical quantitative assumption. Namely it suffices to assume there is $L_0 > 0$ such that

$$\left\| \frac{1}{L^d} \int_{[0,L]^d} V(x+y)\, dy \right\| \le \varepsilon = \varepsilon(d, M)$$

for all $L > L_0$. Similarly, the parameters determining the constants in Theorem 1.2 are the uniform bound M, the Lipschitz constant K, and the above-mentioned L_0.

Motivating ideas of the proof. Let us start with some motivations. These are not proofs but they indicate what brought us to a relatively technical argument presented below.

A feasible counter-example to Theorem 1.1 would look like this. Consider a hypersurface $S \subset \mathbb{R}^d$ dividing the space into two regions $\mathcal{R}$ and $\mathcal{R}'$. Suppose that for every $x \in S$, the vector $V(x)$ is directed inward $\mathcal{R}$ and its normal component with respect to S is greater than 1. Then, if the fish starts at a point in $\mathcal{R}$, it can never cross S outwards and hence cannot leave $\mathcal{R}$.

In fact, *any* counter-example should look like this. Indeed, let $\mathcal{R}$ be the set of all points that our fish can reach from its initial position. Then a simple technical argument (see §2) shows that the boundary $\partial\mathcal{R}$ is locally the graph of a Lipschitz function. And since the fish cannot leave $\mathcal{R}$, the flow at the boundary is directed inward $\mathcal{R}$ with the normal component bounded away from zero. With a little help of the Geometric Measure Theory, one sees that Lipschitz surfaces are as good as smooth ones for our purposes. Thus, Theorem 1.1 is equivalent to the non-existence of a hypersurface S with the above properties.

Now, let us consider some simple cases. First of all, if the reachable set $\mathcal{R}$ is compact, then the contradiction is obvious. Since the flow field at the boundary points strictly inwards, the flux through the boundary is nonzero, and this contradicts the incompressibility condition.

A more interesting situation to consider is the case where $\mathcal{R}$ is a tube (a neighborhood of a straight line) and there is a parallel tube with opposite flow to cancel the mean drift. Since the normal component of the flow field on the boundary is bounded away from zero, the total flux though the boundary is unbounded. This and incompressibility imply that V is unbounded, contrary to our assumptions.

Things are however more complicated since *a priori* the tubes can branch or widen or have more complicate structure. The main part of our strategy is to show that this "branching" must be exponential and there is not enough room for this in the Euclidean space.

Remark 1.6. Our proof is not constructive. It does not give us an actual trajectory that the fish can follow to reach a given point. The optimal trajectory can be found by studying how the reachable set evolves in time. This could be done by solving the G-equation, since a certain spatial level set of its solution at a fixed time t is precisely the boundary of the set of all points that our fish can reach before t.

Our motivating ideas are formulated in a very informal way so far. Let us proceed with actual proofs of the theorems.

§2. Local geometry of reachable sets

Throughout the rest of the paper we assume that V is a vector field in $\mathbb{R}^d$ satisfying the assumptions of Theorem 1.1. The letters C and c denote various positive constants depending on V. The same letter C may denote different constants, even within one formula. We fix the notation M for the bound on $\|V\|$, see (5).

We denote by $\overline{S}$ the closure of a set $S \subset \mathbb{R}^d$. We refer to [2] for basic properties of rectifiable sets in $\mathbb{R}^d$. For a k-dimensional rectifiable set $S \subset \mathbb{R}^d$, $0 \le k \le d$, we denote by $|S|$ its k-dimensional volume. For a $(d-1)$-rectifiable co-oriented set S we denote by $\operatorname{flux}(V, S)$ the flux of V through S. (Recall that co-orientation of a hypersurface is a choice of one of the two normal directions.)

In this section we establish basic properties of the set of points reachable from a fixed point $x \in \mathbb{R}^d$. For technical reasons, we prefer to work with open reachable sets, defined as follows.

Definition 2.1. For $x \in \mathbb{R}^d$ and $\tau > 0$, we denote by $\mathcal{R}_x^\tau$ the set of points reachable from x in positive time less than τ using controls strictly smaller than 1. That is,

$$\mathcal{R}_x^\tau = \{ y \in \mathbb{R}^d \mid y = X_\alpha^t(x) \text{ for some } t \in (0, \tau) \text{ and } \alpha \in \mathcal{A}_t \text{ such that } \|\alpha\|_\infty < 1 \}.$$

We define the *reachable set* $\mathcal{R}_x$ of x by $\mathcal{R}_x = \bigcup_{\tau > 0} \mathcal{R}_x^\tau$.

Clearly $\mathcal{R}_x^\tau$ and $\mathcal{R}_x$ are open sets. We are going to show that the boundary $\partial\mathcal{R}_x$ enjoys some regularity properties.

Definition 2.2. Given a point $y \in \mathbb{R}^d$, a vector $v \in \mathbb{R}^d$, and a parameter $\lambda \in (0, 1)$, we define an open cone

$$C_y^\lambda(v) = \{ y + tw \mid t > 0, \ w \in \mathbb{R}^d, \ \|w - v\| < \lambda \}.$$

If $\|v\| > \lambda$, then $C_y^\lambda(v)$ is an open "round cone" with its apex at y. If $\|v\| < \lambda$, then $C_y^\lambda(v) = \mathbb{R}^d$. If $\|v\| = \lambda$, then $C_y^\lambda(v)$ is an open half-space. For fixed v and λ, the cones $C_{y_1}^\lambda(v)$ and $C_{y_2}^\lambda(v)$ are parallel translates of each other.

Lemma 2.3. *Let $\mathcal{R} = \mathcal{R}_x$ and $\lambda \in (0, 1)$. Then for every $y_0 \in \partial\mathcal{R}$ there exists a neighborhood U of y_0 such that for every $y \in \overline{\mathcal{R}} \cap U$ one has $C_y^\lambda(v_0) \cap U \subset \mathcal{R}$, where $v_0 = V(y_0)$.*

Proof. Let $U \ni y_0$ be a convex neighborhood so small that $\|V(y) - v_0\| < 1 - \lambda$ for all $y \in U$. First, consider $y \in \mathcal{R} \cap U$. Starting at x and using controls strictly bounded by 1, our fish can reach y and then follow any path $t \mapsto y + tw$, $\|w - v_0\| < \lambda$, until it leaves U. Hence, $C_y^\lambda(v_0) \cap U \subset \mathcal{R}$ for any $y \in \mathcal{R} \cap U$.

If $y \in \partial\mathcal{R} \cap U$, the cone $C_y^\lambda(v_0)$ is the limit of cones $C_{y'}^\lambda(v_0)$, $y' \in \mathcal{R} \cap U$, $y' \to y$. More precisely, for every $z \in C_y^\lambda(v_0)$ we have $z \in C_{y'}^\lambda(v_0)$ for all $y' \in \mathcal{R}$ sufficiently close to y. Since the desired property is already verified for $y' \in \mathcal{R}$, it follows that it holds for $y \in \partial\mathcal{R}$. □

Lemma 2.4. *$\partial\mathcal{R}_x$ is a locally Lipschitz hypersurface, and $\mathcal{R}_x$ locally lies to one side of $\partial\mathcal{R}_x$.*

Proof. Let $\mathcal{R} = \mathcal{R}_x$ and $y_0 \in \partial\mathcal{R}$. Fix $\lambda = \frac{1}{2}$ and let U be a neighborhood of y_0 constructed in Lemma 2.3. Since y_0 is a boundary point of $\mathcal{R}$, Lemma 2.3 implies that $C_y^\lambda(v_0) \ne \mathbb{R}^d$ for any y. Choose a Cartesian coordinate system in $\mathbb{R}^d$ such that the vector $v_0 = V(y_0)$ is nonnegatively proportional to the last coordinate vector. In these coordinates, every cone $C_y^\lambda(v_0)$ is the epigraph of the function $F_y \colon \mathbb{R}^{d-1} \to \mathbb{R}$ given by $F_y(u) = y_n + C\,\|u - u_0\|$, where $C = \sqrt{\|v\|^2 - 1}$, y_n is the last coordinate of y, and u_0 is the projection of y to the first coordinate hyperplane. This fact and Lemma 2.3 imply that $\partial\mathcal{R} \cap U$ is the graph of a C-Lipschitz function and the set $R \cap U$ lies above this graph. The lemma follows. □

Lemma 2.4 implies that $\partial\mathcal{R}_x$ is a $(d-1)$-dimensional locally rectifiable set and it has a tangent hyperplane at almost every point. We equip $\partial\mathcal{R}_x$ with a co-orientation determined by the choice of the normal pointing inwards $\mathcal{R}_x$.

Lemma 2.5. *For every measurable set $S \subset \partial\mathcal{R}_x$, $\operatorname{flux}(V,S) \geq |S|$.*

Proof. It suffices to verify that, for every $y_0 \in \partial\mathcal{R}_x$ such that $\partial\mathcal{R}_x$ has a tangent hyperplane at y_0, we have $\langle v_0, n\rangle \geq 1$, where $v_0 = V(y_0)$ and n is the inner normal to $\partial\mathcal{R}_x$ at y_0. Suppose the contrary and fix λ between $\langle v_0, n\rangle$ and 1. By Lemma 2.3, $\mathcal{R}_x$ contains a set $C^\lambda_{y_0}(v_0) \cap U$, where U is a neighborhood of y_0. The cone $C^\lambda_{y_0}(v_0)$ contains the ray $\{y_0 + tw \mid t > 0\}$, where $w = v_0 - n\langle v_0, n\rangle$. The vector w is orthogonal to n and hence belongs to the tangent hyperplane to $\partial\mathcal{R}_x$ at y_0. Thus, the tangent hyperplane has a nonempty intersection with $C^\lambda_{y_0}(v_0)$, a contradiction. □

§3. Flux estimates

First we show that the average flux trough a large $(d-1)$-dimensional cube is small. This is the only place in the proof where we use the small mean drift assumption.

Lemma 3.1. *For every $\varepsilon > 0$ there exists $A_0 > 0$ such that the following holds. If F is a $(d-1)$-dimensional cube with edge length $A > A_0$, then*

$$|\operatorname{flux}(V,F)| \leq \varepsilon A^{d-1}. \tag{9}$$

Proof. We may assume that $F = \{0\} \times [0,A]^{d-1}$. By the small mean drift property (4), there exists L_0 such that

$$\left\| \frac{1}{L^d} \int_{[0,L]^d} V(x+y)\, dy \right\| < \varepsilon/2 \tag{10}$$

for every $L > L_0$ and all $x \in \mathbb{R}^d$. If $A > L_0$, then $A = mL$, where $m \in \mathbb{Z}$ and $L_0 \leq L \leq 2L_0$. Consider the layer

$$Q = [0,L] \times [0,A]^{d-1}.$$

It can be partitioned into cubes with edge length L, hence (10) implies that

$$\frac{1}{LA^{d-1}} \left\| \int_Q V(x)\, dx \right\| < \varepsilon/2.$$

The mean value theorem implies that there is $t \in [0,L]$ such that a similar inequality holds for the slice $F_t = \{t\} \times [0,L]^{d-1}$ of Q by the hyperplane $\{x_1 = t\}$. Namely

$$\frac{1}{LA^{d-1}} |\operatorname{flux}(V,F_t)| = \frac{1}{A^{d-1}} \left| \int_{F_t} \langle V(x), \mathbf{e}_1\rangle\, dx \right| < \varepsilon/2,$$

where integration on the right-hand side is taken against the $(d-1)$-dimensional volume. Let $Q_t = [0,t] \times [0,A]^{d-1}$. Incompressibility implies that the flux of V through the boundary ∂Q_t is zero. This boundary contains two "large" cubic faces F and F_t, and the area of the remaining part is "small":

$$|\partial Q_t \setminus (F \cup F_t)| = (2d-2)tA^{d-2} \leq CLA^{d-2}.$$

Hence,

$$|\operatorname{flux}(V,F)| \leq \frac{\varepsilon}{2} A^{d-1} + CLMA^{d-2}.$$

Choosing A sufficiently large we obtain (9). □

Denote by I_t the n-dimensional cube with edge length $2t$ centered at zero: $I_t = [-t,t]^d$. The following two lemmas concern estimates on the flux of V through subsets of ∂I_t.

Lemma 3.2. *Let D be a subset of ∂I_t with a $(d-2)$-rectifiable boundary ∂D. Then*

$$|\operatorname{flux}(V, D)| \le C|\partial D|^{(d-1)/(d-2)}, \tag{11}$$

where $C = C(d, M)$.

Proof. By incompressibility,

$$|\operatorname{flux}(V, D)| = |\operatorname{flux}(V, \partial I_t \setminus D)|.$$

Hence,

$$|\operatorname{flux}(V, D)| \le M \min\big\{|D|, |\partial I_t \setminus D|\big\}.$$

By the isoperimetric inequality,

$$|\partial D| \ge C \min\big\{|D|, |\partial I_t \setminus D|\big\}^{(d-2)/(d-1)}$$

for some $C = C(d)$. The last two inequalities imply (11). □

Lemma 3.3. *For every $\varepsilon > 0$ there exist $A_0 > 0$ and $C_0 = C_0(\varepsilon, V) > 0$ such that for every $t > A_0$ the following holds. If D is a subset of ∂I_t with a $(d-2)$-rectifiable boundary ∂D, then*

$$|\operatorname{flux}(V, D)| \le C_0|\partial D| + \varepsilon t^{d-1}. \tag{12}$$

Proof. By Lemma 3.1, there exists $A_0 > 0$ such that the flux of V through every $(d-1)$-dimensional cube with edge length $A > A_0$ is bounded by $\frac{\varepsilon}{2d}A^{d-1}$. If $t > A_0$, then $t = mA$, where $m \in \mathbb{Z}$ and $A_0 \le A \le 2A_0$. We divide ∂I_t into $(d-1)$-dimensional cubes Q_i, $i = 1, 2, \dots, 2dm^{d-1}$, with edge length A. For each i, define $P_i = |\partial D \cap Q_i|$ and $S_i = \min\{|Q_i \cap D|, |Q_i \setminus D|\}$. By the isoperimetric inequality,

$$S_i \le CP_i^{(d-1)/(d-2)}$$

for some $C = C(d)$. And, trivially,

$$S_i \le |Q_i| = A^{d-1}.$$

Combining these two inequalities, we obtain

$$S_i = S_i^{(d-2)/(d-1)} S_i^{1/(d-1)} \le CP_iA. \tag{13}$$

By the choice of A_0 we have

$$|\operatorname{flux}(V, Q_i)| \le \frac{\varepsilon}{2d}|Q_i|.$$

Hence,

$$\big||\operatorname{flux}(V, Q_i \cap D)| - |\operatorname{flux}(V, Q_i \setminus D)|\big| \le \frac{\varepsilon}{2d}|Q_i|.$$

Since at least one of the terms $|\operatorname{flux}(V, Q_i \cap D)|$ and $|\operatorname{flux}(V, Q_i \setminus D)|$ is bounded by MS_i, it follows that

$$|\operatorname{flux}(V, Q_i \cap D)| \le MS_i + \frac{\varepsilon}{2d}|Q_i| \le CP_iA + \frac{\varepsilon}{2d}|Q_i|.$$

Summing up over all i and setting $C_0 = 2CA_0$ yields (12). □

§4. Proof of Theorem 1.1

Theorem 1.1 is an immediate corollary to the following lemma.

Lemma 4.1. *For every $x \in \mathbb{R}^d$, the reachable set $\mathcal{R}_x$ is the entire $\mathbb{R}^d$.*

Proof. Arguing by contradiction, assume that $\mathcal{R}_x \neq \mathbb{R}^d$ for some $x \in \mathbb{R}^d$. Lemma 2.4 implies that $\partial\overline{\mathcal{R}_x} = \partial\mathcal{R}_x$ and hence $\overline{\mathcal{R}_x} \neq \mathbb{R}^d$. For $t > 0$ denote

$$D_t = \overline{\mathcal{R}_x} \cap \partial I_t, \quad S_t = \partial\mathcal{R}_x \cap I_t, \quad L_t = \partial\mathcal{R}_x \cap \partial I_t,$$

where I_t is the cube defined in §3. Since $\partial\mathcal{R}_x$ is a nonempty locally Lipschitz hypersurface, we have $|S_t| > 0$ for all $t > t_0$, where t_0 is the distance from 0 to $\partial\mathcal{R}_x$ and $|S_t|$ is the $(d-1)$-dimensional volume of S_t.

The sets L_t are slices of $\partial\mathcal{R}_x$ by level sets of the 1-Lipschitz function $x \mapsto \max|x_i|$. Hence, L_t is a $(d-2)$-rectifiable set for almost every t. In the sequel we consider only those values $t > t_0$ for which L_t is $(d-2)$-rectifiable. In particular, the $(d-1)$-dimensional volume of L_t is zero. This implies that $L_t = \partial D_t$, where ∂D_t denotes the boundary of D_t in ∂I_t. Let $P(t) = |L_t| = |\partial D_t|$ denote the $(d-2)$-dimensional volume of this set. By the co-area inequality,

$$|S_t| \geq A(t) := \int_0^t P(s)\,ds.$$

Observe that the union $D_t \cup S_t$ is the boundary of the set $\overline{\mathcal{R}_x} \cap I_t$. In addition, $D_t \cap S_t = L_t$, which is $(d-2)$-dimensional. Hence, by the incompressibility condition (3),

$$|\operatorname{flux}(V, D_t)| = |\operatorname{flux}(V, S_t)| \geq |S_t|,$$

where the last inequality follows from Lemma 2.5. Thus,

$$|\operatorname{flux}(V, D_t)| \geq |S_t| \geq A(t). \tag{14}$$

By Lemma 3.2,

$$|\operatorname{flux}(V, D_t)| \leq CP(t)^{(d-1)/(d-2)}.$$

In particular, $P(t) > 0$ for $t > t_0$. It follows that $A(t) > 0$ for all $t > t_0$ and

$$\tfrac{d}{dt}A(t) = P(t) \geq c\,A(t)^{(d-2)/(d-1)}$$

for some $c > 0$ and almost every $t > t_0$. Therefore, $A(t) \geq c_0(t-t_0)^{d-1}$ and hence

$$|\operatorname{flux}(V, D_t)| \geq c_0(t-t_0)^{d-1},$$

where $c_0 > 0$ is some constant depending on V. Now we apply Lemma 3.3 to $\varepsilon = c_0/3$ and obtain

$$c_0(t-t_0)^{d-1} \leq |\operatorname{flux}(V, D_t)| \leq CP(t) + \frac{c_0}{3}t^{d-1}. \tag{15}$$

For all sufficiently large t we have $(t-t_0)^{d-1} > \frac{2}{3}t^{d-1}$ and therefore $CP(t) + \frac{c_0}{3}t^{d-1} \geq \frac{2c_0}{3}t^{d-1}$. Hence, $\frac{c_0}{3}t^{d-1} \leq CP(t)$. Thus, (15) implies

$$|\operatorname{flux}(V, D_t)| \leq CP(t)$$

for all sufficiently large t. Combining this with (14) yields $\frac{d}{dt}A(t) \geq cA(t)$ for all sufficiently large t. Hence, $A(t)$ grows exponentially.

On the other hand,

$$|\operatorname{flux}(V, D_t)| \leq M|\partial I_t| \leq Ct^{d-1}.$$

This and (14) imply that $A(t)$ grows at most polynomially. This contradiction proves Lemma 4.1 and Theorem 1.1. ☐

Corollary 4.2. *For every compact set $B \subset \mathbb{R}^d$ there exists $\tau_0 = \tau_0(V, B) > 0$ such that $\tau(x, y) \leq \tau_0$ for all $x, y \in B$.*

Proof. Fix $x_0 \in B$. Recall that $\mathcal{R}_{x_0}$ is the union of nested open sets $\mathcal{R}^{\tau}_{x_0}$, $\tau > 0$. Since $\mathcal{R}_{x_0} = \mathbb{R}^d$ and B is compact, it follows that $B \subset \mathcal{R}^{\tau_1}_{x_0}$ for some $\tau_1 > 0$. Thus, $\tau(x_0, y) \le \tau_1$ for all $y \in B$.

To complete the proof, we show that x_0 can be reached from any point $x \in B$ in a uniformly bounded time. This is equivalent to reaching x from x_0 in the flow defined by the opposite vector field $-V$. Applying the above argument to $-V$ yields that there is $\tau_2 > 0$ such that $\tau(x, x_0) \le \tau_2$ for all $x \in B$. Hence, $\tau(x, y) \le \tau_0 := \tau_1 + \tau_2$ for all $x, y \in B$. □

§5. Proof of Theorem 1.2

Now we assume that V satisfies the assumptions of Theorem 1.2. To prove Theorem 1.2 it suffices to verify the following: there is a constant $C > 0$ such that $\tau(x, y) \le C$ for all $x, y \in \mathbb{R}^d$ satisfying $\|x - y\| \le 1$.

Suppose the contrary. Then there exist two sequences of points $\{x_n\}$ and $\{y_n\}$ in $\mathbb{R}^d$ such that $\|x_n - y_n\| \le 1$ and $\tau(x_n, y_n) > n$ for all n. Consider the shifted vector fields V_n given by

$$V_n(x) = V(x - x_n).$$

By the Arzela–Ascoli theorem, there exists a subsequence $\{V_{n_i}\}$ that converges to a vector field V_0 uniformly on compact sets. The vector field V_0 inherits boundedness, Lipschitz continuity, and the small mean drift property (4) from V.

Thus, we can apply Theorem 1.1 and Corollary 4.2 to V_0 in place of V. By means of rescaling, this works even if the control is $\frac{1}{2}$-bounded, that is if we consider the travel times

$$\widetilde{\tau}(x, y) = \inf\big\{t \ge 0 \mid X_x^{\alpha/2}(t) = y \text{ for some } \alpha \in \mathcal{A}_t\big\}. \tag{16}$$

By Corollary 4.2, there is τ_0 such that $\widetilde{\tau}(x, y) \le \tau_0$ for all x, y from the unit ball centered at 0, where $\widetilde{\tau}$ is defined by (16) for the vector field V_0. In particular, the travel times $\widetilde{\tau}(0, y_n - x_n)$ are bounded by τ_0. Since our vector field is bounded, all trajectories realizing these travel times are confined to some ball $B_R(0)$. For n_i large enough we have $\|V_{n_i} - V_0\| < \frac{1}{2}$ on $B_R(0)$, and hence all such trajectories are also trajectories for V_{n_i} with 1-bounded control. Thus, $\tau(x_n, y_n) \le \widetilde{\tau}(0, y_n - x_n) \le \tau_0$, a contradiction.

References

[1] P. Cardaliaguet and P. E. Souganidis, *Homogenization and enhancement of the G-equation in random environments*, Comm. Pure Appl. Math. **66** (2013), no. 10, 1582–1628. MR3084699

[2] H. Federer, *Geometric measure theory*, Grundlehren Math. Wiss., Bd. 153, Springer-Verlag, New York, 1969. MR0257325

[3] J. Nolen and A. Novikov, *Homogenization of the G-equation with incompressible random drift*, Commun. Math. Sci. **9** (2011), no. 2, 561–582. MR2815685

[4] N. Peters, *Turbulent combustion*, Cambridge Monogr. Mech., Cambridge Univ. Press, Cambridge, 2000. MR1792350

[5] F. A. Williams, *Turbulent combustion*, In: J. D. Buckmaster (Ed.), The Mathematics of combustion, SIAM, Philadelphia, 1985, pp. 97–131. MR806548

The Pennsylvania State University, Department of Mathematics, University Park, PA 16802, USA

Email address: burago@math.psu.edu

St. Petersburg Branch, Steklov Institute of Mathematics, Russian Academy of Sciences, Fontanka 27, 191023 St. Petersburg, Russia

Email address: svivanov@pdmi.ras.ru

The Pennsylvania State University, Department of Mathematics, University Park, PA 16802, USA

Email address: anovikov@math.psu.edu

Received 15/MAY/2016

Originally published in English