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ON THE STABILIZERS OF FINITE SETS OF NUMBERS IN THE R. THOMPSON GROUP F

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Dedicated to Professor Yuri Burago on the occasion of his 80th birthday

Abstract. The subgroups H_U of the R. Thompson group F that are stabilizers of finite sets U of numbers in the interval $(0,1)$ are studied. The algebraic structure of H_U is described and it is proved that the stabilizer H_U is finitely generated if and only if U consists of rational numbers. It is also shown that such subgroups are isomorphic surprisingly often. In particular, if finite sets $U \subset [0,1]$ and $V \subset [0,1]$ consist of rational numbers that are not finite binary fractions, and $|U| = |V|$, then the stabilizers of U and V are isomorphic. In fact these subgroups are conjugate inside a subgroup $\overline{F} < \mathrm{Homeo}([0,1])$ that is the completion of F with respect to what is called the Hamming metric on F. Moreover the conjugator can be found in a certain subgroup $\mathcal{F} < \overline{F}$ which consists of possibly infinite tree-diagrams with finitely many infinite branches. It is also shown that the group $\mathcal{F}$ is non-amenable.

§1. Introduction

The R. Thompson group F is one of the most interesting infinite finitely generated groups. It is usually defined as a group of piecewise linear increasing homeomorphisms of the unit interval with all break points of derivative finite dyadic fractions (i.e., numbers from $\mathbb{Z}[\frac{1}{2}]$) and all slopes powers of 2. The group has many other descriptions (for some of them see §2). The group F is finitely presented, does not have free noncyclic subgroups, and satisfies many other remarkable properties which are the subject of numerous papers. One of the main questions about F is whether it is amenable (the problem was first mentioned in print by Ross Geoghegan). Incorrect proofs of amenability and non-amenability of F are published quite often, and some of these papers (despite having incorrect proofs) are quite interesting because they show deep connections of F with diverse branches of mathematics. For example, [26] shows why a mathematical physicist would be interested in the R. Thompson group F, paper [22] shows a deep connection with logic and Ramsey theory, and [27] shows a connection with graphs on surfaces. Quite recently Vaughan Jones discovered a striking connection between F and planar algebras, subfactors and the knot theory [21]. It turned out that just like braid groups, elements of F can be used to construct all links. He also considered several linearized permutational representations of F on the Schreier graphs of some subgroups of F including the subgroup $\vec{F}$ defined in terms of the corresponding sets of links. This motivated our renewed interest in subgroups of F [17, 18]. It turned out that Jones' subgroup $\vec{F}$ is quite interesting. For

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example, even though it is not maximal, there are finitely many (exactly two) subgroups of F bigger than $\overrightarrow{F}$, i.e., $\overrightarrow{F}$ is of *quasifinite index* in F [18].

Other subgroups of interest include the stabilizers H_U of finite sets $U \subset (0,1)$ first considered by Savchuk [24, 25] who proved that the Schreier graph of F/H_U is amenable for every finite U. In [18] we showed that each H_U is also of quasifinite index: every subgroup of F containing H_U is of the form H_V, where $V \subseteq U$.

Savchuk [24] noticed that if U contains an irrational number, then H_U is not finitely generated. It is easy to see that if U consists of numbers from $\mathbb{Z}[\frac{1}{2}]$, then H_U is isomorphic to the direct product of $|U|+1$ copies of F and hence is $(2|U|+2)$-generated. Thus if $U \subset \mathbb{Z}[\frac{1}{2}]$, then the isomorphism class of H_U depends only on the size of U. In fact if $|U| = |V|$ and $U, V \subset \mathbb{Z}[\frac{1}{2}]$, then it is easily seen that H_U and H_V are conjugate in F. This leaves open the following.

Problem 1.1. What is the structure of H_U for an arbitrary finite U? When is H_U finitely generated? When are H_U and H_V isomorphic?

In this paper, we continue the study of subgroups H_U and answer the questions from Problem 1.1. Every subset U of $(0,1)$ is naturally subdivided into three subsets $U = U_1 \cup U_2 \cup U_3$, where U_1 consists of numbers from $\mathbb{Z}[\frac{1}{2}]$ (i.e., numbers of the form $.u$ where u is a finite word in $\{0,1\}$), U_2 consists of rational numbers not in $\mathbb{Z}[\frac{1}{2}]$ (i.e., numbers of the form $.ps^{\mathbb{N}}$ where p, s are finite binary words and s contains both digits 0 and 1), and U_3 consists of irrational numbers. We shall call U_1, U_2, U_3 the *natural partition* of U. We show that H_U is finitely generated if and only if U consists of rational numbers, that is, U_3 is empty. In that case we find the minimal number of generators of H_U and classify subgroups H_U up to isomorphism. In particular, we show (Theorem 4.1) that if $U_1 = U_3 = \varnothing$, then, up to isomorphism, H_U depends only on the size $|U|$. For example, $H_{\{\frac{1}{3}\}}$ is isomorphic to $H_{\{\frac{1}{7}\}}$. Moreover if $\tau(U)$ is the *type* of U, i.e., the word in the alphabet $\{1,2,3\}$ where the i letter of $\tau(U)$ is 1 if the ith number in U (with respect to natural order) is from $\mathbb{Z}[\frac{1}{2}]$, 2 if the ith number in U is rational not from $\mathbb{Z}[\frac{1}{2}]$, and 3 if the ith number is irrational, then H_U is isomorphic to H_V provided $\tau(U) \equiv \tau(V)$[1] (Theorem 4.1).

Note that $H_{\{\frac{1}{3}\}}$ and $H_{\{\frac{1}{7}\}}$ are obviously not conjugate in F or even in $\widetilde{\mathrm{PL}}_2(\mathbb{R})$, the group of piecewise linear homeomorphisms of $\mathbb{R}$ with all break points in $\mathbb{Z}[\frac{1}{2}]$ and all absolute values of slopes powers of 2 (and so not in $\mathrm{Aut}(F)$ [6]). We are going to prove (see Theorem 7.7) that if $\tau(U) \equiv \tau(V)$, then H_U and H_V are conjugate in $\mathrm{Homeo}([0,1])$ and, in fact, in a relatively small subgroup $\mathcal{F}$ of $\mathrm{Homeo}([0,1])$. We will also show that one can construct a completion $\overline{F}$ of F with respect to a certain metric (which is similar to the Hamming metric on the symmetric group S_n) and show that the natural embedding $F \to \overline{F}$ extends to an embedding $\mathcal{F} \to \overline{F}$. The groups $\mathcal{F}$ and $\overline{F}$ are interesting on their own. We prove, in particular, that $\mathcal{F}$ contains a non-Abelian free subgroup, so $\overline{F}$ is a non-amenable completion of F.

Remark 1.2. Note that Theorem 4.1 (the isomorphism theorem for some subgroups H_U) follows from Theorem 7.7 (the conjugacy theorem for some subgroups H_U). Nevertheless, we decided to keep Theorem 4.1 because its proof gives additional algebraic information about subgroups H_U.

Here is a combination of several results proved in this paper.

Theorem 1.3. *Let U be a finite set of numbers from $(0,1)$, and let $U = U_1 \cup U_2 \cup U_3$ be the standard partition, $r = |U|$, $m_i = |U_i|$, $i = 1,2,3$. Then*

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[1] $p \equiv q$ denotes letter-by-letter equality of words p, q.

1. (Theorem 3.2) H_U *is isomorphic to a semidirect product*

$$H_U \cong [F,F]^{r+1} \rtimes \mathbb{Z}^{2m_1+m_2+2}.$$

2. (Theorem 5.9) *The subgroup* H_U *is finitely generated if and only if* U_3 *is empty* (*that is,* U *consists of rational numbers*). *In that case the smallest number of generators of* H_U *is* $2m_1+m_2+2$.
3. (Theorem 6.1) *Let* $U_3=\varnothing$. *Then the subgroup* H_U *is undistorted in* F.

In §8, we list some open problems.

Remark 1.4. After the first versions of our paper appeared on arXiv, Ralph Strebel informed us that several results of this paper, in particular Theorem 3.2, can be proved for the generalizations of the R. Thompson groups considered in [3].

§2. Preliminaries on F

2.1. F as a group of homeomorphisms. Recall that F consists of all piecewise-linear increasing self-homeomorphisms of the unit interval with slopes of all linear pieces powers of 2 and all break points of the derivative in $\mathbb{Z}[\frac{1}{2}]$. The group F is generated by two functions x_0 and x_1 defined as follows [10]:

$$x_0(t)=\begin{cases}2t & \text{if } 0\le t\le \frac14,\\ t+\frac14 & \text{if } \frac14\le t\le\frac12,\\ \frac{t}{2}+\frac12 & \text{if } \frac12\le t\le 1,\end{cases}\qquad x_1(t)=\begin{cases} t & \text{if } 0\le t\le\frac12,\\ 2t-\frac12 & \text{if } \frac12\le t\le\frac58,\\ t+\frac18 & \text{if } \frac58\le t\le\frac34,\\ \frac{t}{2}+\frac12 & \text{if } \frac34\le t\le 1.\end{cases}$$

The composition in F is from left to right.

Every element of F is completely determined by how it acts on the set $\mathbb{Z}[\frac12]$. Every number in $(0,1)$ can be described as $.s$, where s is an infinite word in $\{0,1\}$. For each element $g\in F$ there exists a finite collection of pairs of (finite) words (u_i,v_i) in the alphabet $\{0,1\}$ such that every infinite word in $\{0,1\}$ starts with exactly one of the u_i's. The action of F on a number $.s$ is the following: if s starts with u_i, we replace u_i by v_i. For example, x_0 and x_1 are the following functions:

$$x_0(t)=\begin{cases}.0\alpha & \text{if } t=.00\alpha,\\ .10\alpha & \text{if } t=.01\alpha,\\ .11\alpha & \text{if } t=.1\alpha,\end{cases}\qquad x_1(t)=\begin{cases}.0\alpha & \text{if } t=.0\alpha,\\ .10\alpha & \text{if } t=.100\alpha,\\ .110\alpha & \text{if } t=.101\alpha,\\ .111\alpha & \text{if } t=.11\alpha,\end{cases}$$

where α is any infinite binary word.

For the generators x_0,x_1 defined above, the group F has the following finite presentation [10]:

$$F=\langle x_0,x_1 \mid [x_0x_1^{-1},x_1^{x_0}]=1,\ [x_0x_1^{-1},x_1^{x_0^2}]=1\rangle,$$

where a^b denotes $b^{-1}ab$.

Sometimes, it is more convenient to consider an infinite presentation of F. For $i\ge 1$, let $x_{i+1}=x_0^{-i}x_1x_0^i$. In these generators, the group F has the following presentation [10]:

$$\langle x_i, i\ge 0 \mid x_i^{x_j}=x_{i+1} \text{ for every } j<i\rangle.$$

2.2. Elements of F as pairs of binary trees. Often, it is more convenient to describe elements of F using pairs of finite binary trees drawn on a plane. Trees are considered up to isotopies of the plane. Elements of F are pairs of full finite binary trees (T_+, T_-) which have the same number of leaves. Such a pair will sometimes be called a *tree-diagram*.

If T is a (finite or infinite) binary tree, a *branch* in T is a maximal simple path starting from the root. Every non-leaf vertex of T has two outgoing edges: the left edge and the right edge. If every left edge of T is labeled by 0 and every right edge is labeled by 1, then every branch of T is labeled by a (finite or infinite) binary word u. We will usually ignore the distinction between a branch and its label.

Let (T_+, T_-) be a tree-diagram, where T_+ and T_- have n leaves. Let $u_1, \ldots, u_n$ (respectively, $v_1, \ldots, v_n$) be the branches of T_+ (respectively, T_-), ordered from left to right. For each $i = 1, \ldots, n$ we say that the tree-diagram (T_+, T_-) has a *pair of branches* $u_i \to v_i$. The function g in F corresponding to this tree-diagram takes binary fraction $.u_i\alpha$ to $.v_i\alpha$ for every i and every infinite binary word α. We will also say that the element g takes the branch u_i to the branch v_i. The tree-diagrams of the generators of F, x_0 and x_1, appear in Figure 1.

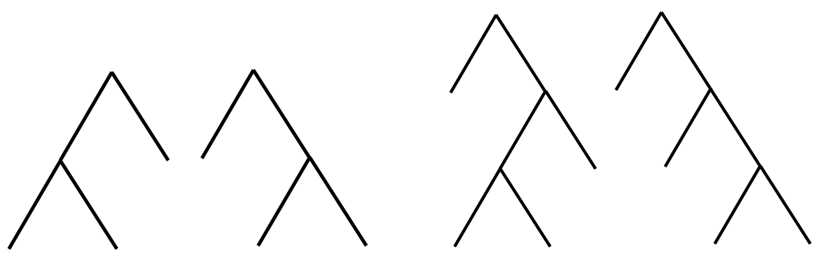

FIGURE 1. (a) The tree-diagram of x_0. (b) The tree-diagram of x_1. In both figures, T_+ is on the left and T_- is on the right.

A *caret* is a binary tree composed of a single vertex with two children. If (T_+, T_-) is a tree-diagram, then attaching a caret to the ith leaf of both T_+ and T_- does not affect the function in F represented by the tree-diagram (T_+, T_-). The inverse action of *reducing* common carets does not affect the function either (the pair (T_+, T_-) has a *common caret* if leaves number i and $i+1$ have a common father in T_+ as well as in T_-). Two pairs of trees (T_+, T_-) and (R_+, R_-) are said to be *equivalent* if one results from the other by a finite sequence of inserting and reducing common carets. If (T_+, T_-) does not have a common caret, then (T_+, T_-) is said to be *reduced*. Every tree-diagram is equivalent to a unique reduced tree-diagram. Thus, the elements of F can be represented uniquely by reduced tree-diagrams [10].

An alternative way of describing the function in F corresponding to a given tree-diagram is the following. For each finite binary word u, we let the *interval associated with* u, denoted by $[u]$, be the interval $[.u, .u1^{\mathbb{N}}]$. If (T_+, T_-) is a tree-diagram for $f \in F$, we let $u_1, \ldots, u_n$ be the branches of T_+ and $v_1, \ldots, v_n$ be the branches of T_-. Then the intervals $[u_1], \ldots, [u_n]$ (respectively, $[v_1], \ldots, [v_n]$) form a subdivision of the interval $[0, 1]$. The function f maps each interval $[u_i]$ linearly onto the interval $[v_i]$.

Below, when we say that a function f has a pair of branches $u_i \to v_i$, the meaning is that some tree-diagram representing f has this pair of branches. In other words, this is equivalent to saying that f maps $[u_i]$ linearly onto $[v_i]$.

Remark 2.1 (see [10]). The tree-diagram where both trees are just singletons plays the role of identity in F. Given a tree-diagram (T_+^1, T_-^1), the inverse tree-diagram is (T_-^1, T_+^1).

If (T_+^2, T_-^2) is another tree-diagram, then the product of (T_+^1, T_-^1) and (T_+^2, T_-^2) is defined as follows. There is a minimal finite binary tree S such that T_-^1 and T_+^2 are rooted subtrees of S (in terms of subdivisions of $[0,1]$, the subdivision corresponding to S is the intersection of the subdivisions corresponding to T_-^1 and T_+^2). Clearly, (T_+^1, T_-^1) is equivalent to a tree-diagram (T_+, S) for some finite binary tree T_+. Similarly, (T_+^2, T_-^2) is equivalent to a tree-diagram (S, T_-). The *product* $(T_+^1, T_-^1) \cdot (T_+^2, T_-^2)$ is (the reduced tree-diagram equivalent to) (T_+, T_-).

Obviously, the mapping of tree-diagrams to functions in F respects operations defined in Remark 2.1.

2.3. Choosing elements in F. In most proofs in this paper, we choose elements with a given set of pairs of branches, or elements which map certain intervals or numbers from $[0,1]$ in a predetermined way. In doing so, we usually apply the next lemma. It follows directly from the proof of [6, Lemma 2.1].

Lemma 2.2. *Let $[a_1, b_1], \dots, [a_m, b_m]$, $[c_1, d_1], \dots, [c_m, d_m]$ be closed subintervals of $[0,1]$ (possibly of length 0, i.e., points) with endpoints from $\mathbb{Z}[\frac{1}{2}]$. Assume that the interiors of the intervals $[a_1, b_1], \dots, [a_m, b_m]$ (respectively, $[c_1, d_1], \dots, [c_m, d_m]$) are pairwise disjoint and that the intervals $[a_1, b_1], \dots, [a_m, b_m]$ (respectively, $[c_1, d_1], \dots, [c_m, d_m]$), considered as sub-intervals of $[0,1]$, are ordered from left to right. Assume in addition that the following conditions are satisfied:*

1) *$[a_i, b_i]$ has empty interior if and only if $[c_i, d_i]$ has empty interior;*
2) *$0 \in [a_1, b_1]$ if and only if $0 \in [c_1, d_1]$;*
3) *$1 \in [a_m, b_m]$ if and only if $1 \in [c_m, d_m]$;*
4) *for all $i = 1, \dots, m-1$, the intervals $[a_i, b_i]$ and $[a_{i+1}, b_{i+1}]$ share a boundary point if and only if the intervals $[c_i, d_i]$ and $[c_{i+1}, d_{i+1}]$ share a boundary point.*

Then there is an element $f \in F$ that maps each interval $[a_i, b_i]$, $i = 1, \dots, m$ onto the interval $[c_i, d_i]$. In addition, if for some i, $[a_i, b_i]$ has positive length and $\frac{b_i - a_i}{d_i - c_i}$ is an integer power of 2 (in particular, if both $[a_i, b_i]$ and $[c_i, d_i]$ are dyadic intervals), then f can be taken to map $[a_i, b_i]$ linearly onto $[c_i, d_i]$.

Remark 2.3. The proof of Lemma 2.1 in [6] also implies (in the notations of Lemma 2.2) that if for each i we choose an element $g_i \in F$ that maps $[a_i, b_i]$ onto $[c_i, d_i]$, then there is an element $f \in F$ such that for all $i \in \{1, \dots, m\}$, the restriction of f to $[a_i, b_i]$ coincides with g_i.

Remark 2.4. Unless explicitly stated otherwise, all closed intervals considered below have positive lengths.

2.4. On branches and fixed points. In this section we consider the relation between the set of fixed points of an element $f \in F$ and a tree-diagram (T_+, T_-) representing it. Let (T_+, T_-) be a tree-diagram of an element $f \in F$. Let $u_i \to v_i$, $i = 1, \dots, n$, be the pairs of branches of (T_+, T_-). Since f is linear on each interval $[u_i]$, if the interval is not fixed (i.e., if the words u_i and v_i are different), then the interval $[u_i]$ contains at most one fixed point. This fixed point can be found as follows.

Let $i \in \{1, \dots, n\}$ and assume that $u_i \not\equiv v_i$, We can assume that $|u_i| \leq |v_i|$ by replacing f by f^{-1} if necessary. If u_i is not a prefix of v_i, then the intervals $[u_i]$ and $[v_i]$ are disjoint and there are no fixed points in $[u_i]$. Otherwise, $v_i \equiv u_i s_i$ for some nonempty suffix s_i. The number $\alpha_i = .u_i s_i^{\mathbb{N}}$ is the unique number fixed in $[u_i]$. Note that if $s_i \equiv 0^k$ or $s_i \equiv 1^k$ for some $k \in \mathbb{N}$, then α_i is from $\mathbb{Z}[\frac{1}{2}]$. Otherwise α_i is a rational (but not in $\mathbb{Z}[\frac{1}{2}]$) fixed point of f.

Corollary 2.5 (Savchuk [24])**.** *Let $f \in F$ and assume that f fixes an irrational number α. Then f fixes a neighborhood $(\alpha - \epsilon, \alpha + \epsilon)$ of α.*

The discussion above also implies the following.

Lemma 2.6. *Let $f \in F$ and assume that f fixes a rational number $\alpha \notin \mathbb{Z}[\frac{1}{2}]$. Let $\alpha = .ps^{\mathbb{N}}$, where s is a minimal period of α. Then f has a pair of branches of the form $ps^{m_1} \to ps^{m_2}$ for some $m_1, m_2 \geq 0$.*

Proof. Let (T_+, T_-) be a tree-diagram for f, and let $u \to v$ be a pair of branches of (T_+, T_-) such that $\alpha \in [u]$. Since α is fixed by f, $\alpha \in [v]$ and we can assume that $|u| \leq |v|$. Similarly, by considering a non reduced form of (T_+, T_-) we can assume that $u = ps^{m_1}$ for some $m_1 \geq 0$. If $v \equiv u$, we are done. Otherwise, let $v \equiv uw$ for some nonempty word w. The unique fixed point of f in $[u]$ is then $.uw^{\mathbb{N}}$. Thus, $\alpha = .ps^{\mathbb{N}} = .us^{\mathbb{N}} = .uw^{\mathbb{N}}$. Since s is a minimal period of α, $w \equiv s^k$ for some $k \in \mathbb{N}$. Thus, $v \equiv ps^{m_1+k}$ and for $m_2 = m_1 + k$ we get the result. □

Lemma 2.6 implies the following.

Corollary 2.7. *Let $\alpha = .ps^{\mathbb{N}}$ be a rational number not in $\mathbb{Z}[\frac{1}{2}]$, and assume that s is a minimal period of α. If $f \in F$ fixes α and has slope 2^a at α, then a is divisible by the length of s.*

Proof. By Lemma 2.6, f has a pair of branches of the form $ps^{m_1} \to ps^{m_2}$ for some $m_1, m_2 \geq 0$. The slope of f on the interval $[ps^{m_1}]$ is $2^{(m_1 - m_2)|s|}$. □

2.5. Natural copies of F. Let f be a function in the Thompson group F. The *support of f*, denoted $\mathrm{Supp}(f)$, is the closure in $[0,1]$ of the subset $\{x \in (0,1) : f(x) \neq x\}$. We say that *$f$ has support in an interval J* if the support of f is contained in J. Note that in this case the endpoints of J are necessarily fixed by f. Hence, the set of all functions from F with support in J is a subgroup of F. We denote this subgroup by F_J.

Let S be a subset of $[0,1]$; the notation $\mathrm{Stab}(S)$ will be used for the pointwise stabilizer of S in F. Thus, if f has support in a closed interval $[a,b]$, then $f \in \mathrm{Stab}([0,a] \cup [b,1])$. We note that if f has support in an interval J and $g \in F$, then f^g has support in the interval $g(J)$. Similarly, $F_J^g = F_{g(J)}$.

The Thompson group F contains many copies of itself (see [7]). The copies of F we will be interested in will be of the following simple form. Let a and b be numbers from $\mathbb{Z}[\frac{1}{2}]$ and consider the subgroup $F_{[a,b]}$. We claim that $F_{[a,b]}$ is isomorphic to F. Note that F can be viewed as a subgroup of $\mathrm{PLF}_2(\mathbb{R})$ of all piecewise linear homeomorphisms of $\mathbb{R}$ with finite number of finite dyadic break points and absolute values of all slopes powers of 2. Let $f \in \mathrm{PLF}_2(\mathbb{R})$ be a function that maps 0 to a and 1 to b (such a function clearly exists). Then F^f is the subgroup of $\mathrm{PLF}_2(\mathbb{R})$ of all orientation preserving homeomorphisms with support in $[a,b]$, that is, $F^f = F_{[a,b]}$.

Let u be a finite binary word and $[u]$ the interval associated with it. The isomorphism between F and $F_{[u]}$ can also be defined by using tree-diagrams. Let g be an element of F represented by a tree-diagram (T_+, T_-). We map g to an element in $F_{[u]}$, denoted by $g_{[u]}$ and referred to as the *copy of g in $F_{[u]}$*. To construct the element $g_{[u]}$ we start with a minimal finite binary tree T that contains the branch u. We take two copies of the tree T. To the first copy, we attach the tree T_+ at the end of the branch u. In the second copy we attach the tree T_- at the end of the branch u. The resulting trees are denoted by R_+ and R_-, respectively. The element $g_{[u]}$ is the one represented by the tree-diagram (R_+, R_-). Note that if g consists of pairs of branches $v_i \to w_i, i = 1, \ldots, k$, and B is the set of branches of T which are not equal to u, then $g_{[u]}$ consists of pairs of branches $uv_i \to uw_i, i = 1, \ldots, k$, and $p \to p, p \in B$.

For example, the copies of the generators x_0, x_1 of F in $F_{[0]}$ are depicted in Figure 2. It is obvious that these copies generate the subgroup $F_{[0]}$.

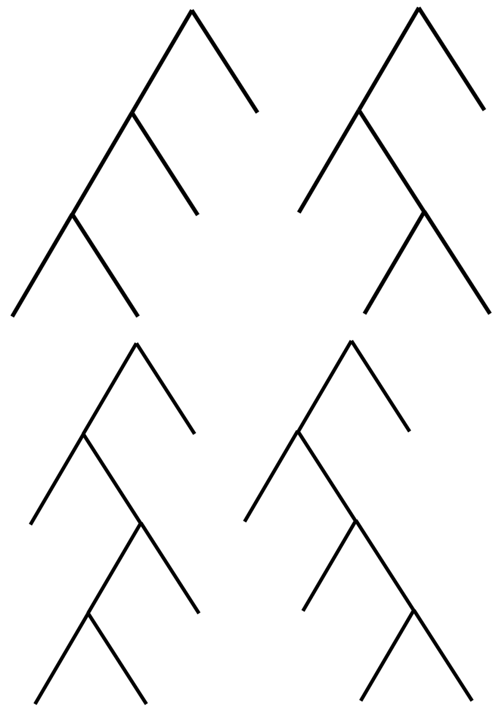

FIGURE 2. (a) The tree-diagram of $(x_0)_{[0]}$. (b) The tree-diagram of $(x_1)_{[0]}$.

The isomorphism above guarantees that if $f, g \in F$, then $f_{[u]}g_{[u]} = (fg)_{[u]}$. Using this isomorphism, we define an addition operation in the Thompson group F as follows. We denote by $\mathbf{1}$ the trivial element in F. We define the sum of an element $g \in F$ with the trivial element $\mathbf{1}$, denoted by $g \oplus \mathbf{1}$, to be the copy of g in $F_{[0]}$. Similarly, the sum of $\mathbf{1}$ and g, denoted by $\mathbf{1} \oplus g$, is the copy of g in $F_{[1]}$. If $g, h \in F$ we define the *sum* of g and h, denoted by $g \oplus h$, to be the product $(g \oplus \mathbf{1})(\mathbf{1} \oplus h)$; i.e., $g \oplus h$ is an element of $Stab(\{\frac{1}{2}\})$ that acts as a copy of g on $[0]$ and as a copy of h on $[1]$. It is easily seen that for $g = \mathbf{1}$ or $h = \mathbf{1}$ this definition coincides with the previous one. Note that $x_1 = \mathbf{1} \oplus x_0$. In particular, x_0 and $\mathbf{1} \oplus x_0$ generate the whole F. If we denote by ζ the function $t \mapsto 1 - t$ from Homeo($[0, 1]$), then $F^\zeta = F$ and $(g \oplus h)^\zeta = h^\zeta \oplus g^\zeta$. One can check that $x_0^\zeta = x_0^{-1}$. Since $(x_0 \oplus \mathbf{1})^\zeta = \mathbf{1} \oplus x_0^{-1} = x_1^{-1}$ and x_0^{-1}, x_1^{-1} clearly generate F, we see that x_0 and $x_0 \oplus \mathbf{1}$ generate F.

Note also that if G is a subgroup of F, then the subgroup $\mathbf{1} \oplus G = \{\mathbf{1} \oplus g : g \in G\}$ is isomorphic to G. Similarly for $G \oplus \mathbf{1}$.

§3. THE STRUCTURE OF STABILIZERS OF FINITE SETS

It is known [10] that the derived subgroup of F is exactly the subgroup $F_{(0,1)}$ of all functions with support in $(0, 1)$. Equivalently, $[F, F]$ is the subgroup of all functions with slope 1 both at 0^+ and at 1^-.

Lemma 3.1. *Let $a < b$ be any two numbers in $[0, 1]$. Then the group $F_{(a,b)}$ of all functions with support in (a, b) is isomorphic to the derived subgroup of F.*

Proof. We prove that $F_{(a,b)}$ is isomorphic to $F_{(0,1)} = [F,F]$. Let $\{a_j\}_{j\in\mathbb{N}}$ and $\{b_j\}_{j\in\mathbb{N}}$ be sequences of numbers from $\mathbb{Z}[\frac{1}{2}]$ such that

(1) $\{a_j\}_{j\in\mathbb{N}}$ is strictly decreasing and converges to a;
(2) $\{b_j\}_{j\in\mathbb{N}}$ is strictly increasing and converges to b; and
(3) $a_1 < b_1$.

Similarly, let $\{c_j\}_{j\in\mathbb{N}}$ and $\{d_j\}_{j\in\mathbb{N}}$ be sequences of numbers from $\mathbb{Z}[\frac{1}{2}]$ such that

(1) $\{c_j\}_{j\in\mathbb{N}}$ is strictly decreasing and converges to 0;
(2) $\{d_j\}_{j\in\mathbb{N}}$ is strictly increasing and converges to 1; and
(3) $c_1 < d_1$.

Recall that $F_{[a_j,b_j]}$ is the subgroup of F of all elements with support in $[a_j,b_j]$. Thus, $F_{(a,b)}$ is the increasing union of subgroups

$$F_{(a,b)} = \bigcup_{j\in\mathbb{N}} F_{[a_j,b_j]}$$

(each of these subgroups is a copy of F, see Subsection 2.5). Similarly,

$$F_{(0,1)} = \bigcup_{j\in\mathbb{N}} F_{[c_j,d_j]}.$$

To prove isomorphism between $F_{(a,b)}$ and $F_{(0,1)}$ it suffices to find a family of compatible isomorphisms $\psi_j\colon F_{[a_j,b_j]} \to F_{[c_j,d_j]}$, $j\in\mathbb{N}$. By *compatible* we mean that for all $j>1$, the restriction of ψ_j to $F_{[a_{j-1},b_{j-1}]}$ coincides with ψ_{j-1}.

We choose elements $g_j \in F$ defined inductively for $j\in\mathbb{N}$, such that

(1) for each j, $g_j(a_j) = c_j$ and $g_j(b_j) = d_j$; and
(2) for each $j>1$, the element g_j coincides with the element g_{j-1} on the interval $[a_{j-1},b_{j-1}]$.

Such a choice is clearly possible by Remark 2.3.

Notice that for each $j\in\mathbb{N}$, ${F_{[a_j,b_j]}}^{g_j} = F_{[c_j,d_j]}$. Similarly, if $j>1$ and $h\in F_{[a_{j-1},b_{j-1}]}$, then $h^{g_j} = h^{g_{j-1}}$. Indeed, that follows from h having support in $[a_{j-1},b_{j-1}]$ and condition (2) in the choice of g_j.

Thus, one can define a compatible family of isomorphisms $\psi_j\colon F_{[a_j,b_j]} \to F_{[c_j,d_j]}$, by taking ψ_j to be the isomorphism of conjugation by g_j. This naturally induces an isomorphism $F_{(a,b)} \to F_{(0,1)}$ because $\bigcup[a_j,b_j] = (a,b)$ and $\bigcup[c_j,d_j] = (0,1)$. □

Theorem 3.2. *Let U be a finite set of numbers in $(0,1)$. Assume that $U = U_1\cup U_2\cup U_3$ is the natural partition of U. Let $m_i = |U_i|$, $i=1,2,3$, $r = |U| = m_1+m_2+m_3$. Then H_U is isomorphic to a semidirect product*

$$H_U \cong [F,F]^{r+1} \rtimes \mathbb{Z}^{2m_1+m_2+2}.$$

Since $[F,F]$ is simple, the rank of the first integral homology group of H_U is $2m_1+m_2+2$.

Proof. For each $\alpha\in U_1$ we choose closed intervals L_α and R_α of positive length with endpoints in $\mathbb{Z}[\frac{1}{2}]$ such that α is the right endpoint of L_α and left endpoint of R_α. We can choose the intervals L_α and R_α to be sufficiently small so that they do not contain points from $U_2\cup U_3$ and the interiors of all these intervals are pairwise disjoint.

For $\alpha\in U_1$, we choose elements g_α and f_α such that g_α has support in L_α and slope 2 at α^- and f_α has support in R_α and slope 2 at α^+.

For each $\beta\in U_2$ we have $\beta = .p_\beta s_\beta^{\mathbb{N}}$ for some finite binary words p_β and s_β, where s_β is a minimal period of β. Let C_β, for $\beta\in U_2$, be pairwise disjoint open intervals with endpoints in $\mathbb{Z}[\frac{1}{2}]$, such that $\beta\in C_\beta$ for all $\beta\in U_2$. Assume also that all C_β are disjoint from the union of all L_α and R_α, $\alpha\in U_1$, and do not contain any numbers from U_3.

For each $\beta \in U_2$ we choose an element h_β such that h_β has support in C_β and has a pair of branches $p_\beta s_\beta^k \to p_\beta s_\beta^{k-1}$ for some $k \in \mathbb{N}$ (the number k can be chosen independently of β). In particular, h_β fixes β and has slope $2^{|s_\beta|}$ at β.

Finally, we choose two additional elements corresponding to the fixed points 0 and 1. Let N_0, N_1 be closed intervals with disjoint interiors containing 0 and 1 (respectively) and having endpoints in $\mathbb{Z}[\frac{1}{2}]$. We assume that N_0 and N_1 are disjoint from all the intervals $L_\alpha, R_\alpha, C_\beta$ chosen above and do not contain any numbers from U_3. We choose elements f and g such that f has support in N_0 and slope 2 at 0^+ and g has support in N_1 and slope 2 at 1^-.

Note that the elements $g_\alpha, f_\alpha (\alpha \in U_1), g_\beta (\beta \in U_2), f, g$ belong to H_U. We let G be the subgroup of H_U generated by these elements. Since the interiors of supports of these elements are pairwise disjoint, they pairwise commute. Thus, G is isomorphic to $\mathbb{Z}^{2n+m+2}$.

Let $\gamma_1, \dots, \gamma_r$ be the elements of U in increasing order. Let $\gamma_0 = 0, \gamma_{r+1} = 1$. Denote by S the group of all elements of F that fix open neighborhoods of each $\gamma_i, i = 0, \dots, r+1$. Clearly $S \leq H_U$.

We claim that H_U is generated by S and G. Indeed, let $h \in H_U$. By Corollary 2.5, h fixes an open neighborhood of each irrational number in U. We claim that one can multiply h from the right by a suitable element $y \in G$ so that the slope of hy at every point $\gamma_i, i = 0, \dots, r+1$ would be 1 (then obviously $hy \in S$ and so $h \in \langle S \cup G \rangle$).

Assume that the slope of h at 0^+ is 2^ℓ for some ℓ. Then $hf^{-\ell}$ has slope 1 at 0^+. Multiplying h by $f^{-\ell}$ does not affect the slope at any point $\gamma_j, j > 0$. Thus, we can replace h by $hf^{-\ell}$. Proceeding in this manner, one can make the slope at each point γ_j be 1 by multiplying from the right by elements of G (we use f_α, g_α for $\alpha \in U_1$, g_β for $\beta \in U_2$ and g for $j = r+1$).

To finish the proof we observe that S is a normal subgroup of H_U and so $H_U = SG$. We claim that $S \cap G$ is trivial. Indeed, the slopes of an element $y \in G$ at (both sides) of the numbers from U determine the element y uniquely. Thus, the only element of G that fixes an open neighborhood around each $\gamma \in U$ is the identity. Thus, $H_U = S \rtimes G$. It remains to note that the group S is isomorphic to the direct product

$$F_{(\gamma_0,\gamma_1)} \times F_{(\gamma_1,\gamma_2)} \times \cdots \times F_{(\gamma_r,\gamma_{r+1})}$$

and by Lemma 3.1 is isomorphic to $[F, F]^{r+1}$. □

Corollary 3.3. *If U and V are finite sets of numbers from $(0,1)$ and $|U| \neq |V|$, then H_U and H_V are not isomorphic.*

Proof. Indeed, by Theorem 3.2 the derived subgroup of H_U is isomorphic to the direct product of $|U|+1$ copies of the simple group $[F, F]$. Thus, it has $2^{|U|+1}$ normal subgroups. So it cannot be isomorphic to a direct power of a different number of simple groups. □

The following is an immediate corollary to the proof of Theorem 3.2 (see [3, Example A12.13]).

Corollary 3.4. *The R. Thompson group F is a semidirect product of the derived subgroup $[F, F]$ and the Abelian subgroup generated by $x_0 \oplus \mathbf{1}$ and $x_1 = \mathbf{1} \oplus x_0$.*

§4. Isomorphism between stabilizers of finite sets

4.1. Isomorphic stabilizers of finite sets. Let $U = \{\gamma_1, \dots, \gamma_n\}$ be a set of numbers from $[0, 1]$ (here and below we assume that $\gamma_1, \dots, \gamma_n$ are listed in increasing order). Let $U = U_1 \cup U_2 \cup U_3$ be the natural partition of U. Then we can define the *type* $\tau(U)$ to be a word in the alphabet $\{1, 2, 3\}$ by taking the word $\gamma_1\gamma_2 \dots \gamma_n$ and replacing each $\gamma_j \in U_i$

by the letter i. Note that, by Corollary 3.3, if $U, V \subseteq (0,1)$ and $|\tau(U)| \neq |\tau(V)|$, then H_U and H_V are not isomorphic.

Theorem 4.1. *If U and V are two finite sets of numbers from $(0,1)$ and $\tau(U) \equiv \tau(V)$, then the subgroups H_U and H_V are isomorphic.*

Note that the converse of Theorem 4.1 does not hold. For example, Lemma 4.10 below implies that if $\tau(U)$ is equal to $\tau(V)$ read backwards, then H_U is isomorphic to H_V. Moreover, in Subsection 4.2 below we show that H_U is a direct product where the factors correspond to subwords of $\tau(U)$. If H_U and H_V are direct products with the same factors, then clearly, $H_U \cong H_V$. Finding a necessary and sufficient condition for H_U and H_V to be isomorphic is still an open problem.

To prove Theorem 4.1, we will realize H_U and H_V as iterated ascending HNN-extensions. Assuming $\tau(U) \equiv \tau(V)$, we will prove that the base groups of the HNN-extensions are isomorphic and the actions of the stable letters commute with the isomorphism between the relevant base groups. That will imply the result.

We need the following three lemmas.

Lemma 4.2. *Let $a, b \in [0,1]$ be such that $a < b$. Let $x, y \in (a,b) \cap \mathbb{Z}[\frac{1}{2}]$ be such that $x < y$. Then $F_{[a,b]}$ is generated by $F_{[a,y]}$ and $F_{[x,b]}$.*

Proof. Let $f \in F_{[a,b]}$. If $f(x) < y$, then $f([a,x]) \subset [a,y)$ and there exists a function $h \in F_{[a,y]}$ such that h coincides with f on the interval $[a,x]$. Then the function fh^{-1} fixes the interval $[a,x]$. In particular, $fh^{-1} \in F_{[x,b]}$ and so $f \in F_{[x,b]}F_{[a,y]}$. If $f(x) \geq y$, then $f(y) > y$. There is a function $g \in F_{[x,b]}$ such that $g(f(y)) = y$. Then y is a fixed point for fg, so $fg \in F_{[a,y]}F_{[y,b]} \subseteq F_{[a,y]}F_{[x,b]}$. ☐

Lemma 4.3. *Let $a, b, c \in (0,1)$ be such that $a < b$ and $a < c$. Let $y \in (a,b) \cap (a,c) \cap \mathbb{Z}[\frac{1}{2}]$. Then there exists an isomorphism $\psi\colon F_{[a,b)} \to F_{[a,c)}$ such that*

(1) *ψ is the identity map on $F_{[a,y]}$; and*
(2) *For any $x \in (a,y)$, $\psi(F_{[x,b)}) = F_{[x,c)}$.*

Proof. We adapt the proof of Lemma 3.1. Let $\{b_j\}_{j\in\mathbb{N}}$ be an increasing sequence of numbers in $[a,b) \cap \mathbb{Z}[\frac{1}{2}]$ that converges to b and is such that $b_1 = y$. Let $\{c_j\}_{j\in\mathbb{N}}$ be an increasing sequence of numbers in $[a,c) \cap \mathbb{Z}[\frac{1}{2}]$ that converges to c, and assume that $c_1 = y$. To define an isomorphism

$$\psi\colon F_{[a,b)} = \bigcup F_{[a,b_j]} \to F_{[a,c)} = \bigcup F_{[a,c_j]},$$

we choose a sequence of elements g_j in a similar way to that in Lemma 3.1. We let g_1 be an element that fixes the interval $[a,y]$. In particular, $g_1(b_1) = c_1$. For each $j > 1$, we let g_j be an element such that $g_j(b_j) = c_j$ and $g_j|_{[a,b_{j-1}]} = g_{j-1}|_{[a,b_{j-1}]}$. The choice of elements g_j determines compatible isomorphisms $\psi_j\colon F_{[a,b_j]} \to F_{[a,c_j]}$, where for each j, ψ_j is the isomorphism of conjugation by g_j. The family of isomorphisms ψ_j gives the required isomorphism ψ.

It suffices to prove that ψ satisfies conditions (1) and (2). If $h \in F_{[a,y]} = F_{[a,b_1]}$, then $\psi(h) = \psi_1(h) = h^{g_1} = h$, where the last equality follows from g_1 fixing the support of h. Thus, condition (1) holds. Let $x \in (a,y)$, and let $f \in F_{[x,b)}$. Then $f \in F_{[x,b_j]}$ for some $j \in \mathbb{N}$ and so $\psi(f) = f^{g_j}$. Notice that $\mathrm{Supp}(\psi(f)) = g_j(\mathrm{Supp}(f)) \subseteq g_j([x,b_j]) = [x,c_j] \subseteq [x,c)$. Thus, $\psi(F_{[x,b)}) \subseteq F_{[x,c)}$. Considering ψ^{-1} instead of ψ gives the reverse inclusion. ☐

The proof of the following lemma is similar to the proof of Lemma 4.3.

Lemma 4.4. *Let $a, b, c \in (0,1)$ be numbers such that $a < b$ and $c < b$. Let $y \in (a,b) \cap (c,b) \cap \mathbb{Z}[\frac{1}{2}]$. Then there exists an isomorphism $\psi\colon F_{(a,b]} \to F_{(c,b]}$ such that*

(1) *ψ is the identity map on $F_{[y,b]}$; and*
(2) *For any $x \in (y,b)$, $\psi(F_{(a,x]}) = F_{(c,x]}$.*

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. We start with replacing H_V by a subgroup H_W that is closer to H_U in some sense. Let $U = U_1 \cup U_2 \cup U_3$ be the natural partition. For each $\beta \in U_2$, let $\beta = .p_\beta s_\beta^{\mathbb{N}}$, where s_β is a minimal period. By replacing the prefixes p_β by longer prefixes $p_\beta s_\beta^k$ if necessary, we can assume that the intervals $[p_\beta]$, $\beta \in U_2$, are pairwise disjoint and that each of these intervals contains exactly one element of $U \cup \{0,1\}$, the number β. (Recall that $[p_\beta] = [.p_\beta, .p_\beta 1^{\mathbb{N}}]$, so $\beta = .p_\beta s_\beta^{\mathbb{N}}$ is always in $[p_\beta]$.)

Similarly, let $V = V_1 \cup V_2 \cup V_3$ be the natural partition of V. For each $\gamma \in V_2$, let $\gamma = .q_\gamma u_\gamma^{\mathbb{N}}$, where u_γ is a minimal period of γ. Assume as above, that the prefixes q_γ are long enough, so that the intervals $[q_\gamma]$, $\gamma \in V_2$, are pairwise disjoint and contain exactly one number from $V \cup \{0,1\}$, the number γ. We claim that H_V is isomorphic to a group H_W (constructed below) such that $\tau(W) \equiv \tau(U) \equiv \tau(V)$ and $W = W_1 \cup W_2 \cup W_3$ (the natural partition of W) satisfies the following conditions.

(1) $W_1 = U_1$.
(2) For each $\delta \in W_2$, $\delta = .p_\beta u_\gamma^{\mathbb{N}}$, where β and γ occupy the same position in the ordered sets U and V (respectively) as δ in W (that is, the two natural order preserving bijections $\eta_{wu}\colon W \to U$ and $\eta_{wv}\colon W \to V$ take δ to β and δ to γ, respectively).

Indeed, the conditions on the intervals $[p_\beta]$ and $[q_\gamma]$ and Lemma 2.2 guarantee that there exists $f \in F$ such that

(1) for each $\gamma \in V_1$, $f(\gamma) \in U_1$; and
(2) for each $\gamma \in V_2$, f has a pair of branches $q_\gamma \to p_{\eta_{vu}(\gamma)} = p_\beta$, where η_{vu} is the natural order preserving bijection from V to U.

Conjugating H_V by f yields a group H_W as described, where $W = f(V)$. Indeed, since $f \in F$, for $i = 1,2,3$ we have $W_i = f(V_i)$. Hence, condition (1) for f guarantees that $W_1 = U_1$. Since f does not change the tail $u_\gamma^{\mathbb{N}}$ of $\gamma \in V_2$, it maps $\gamma = .q_\gamma u_\gamma^{\mathbb{N}}$ to $\delta = .p_\beta u_\gamma^{\mathbb{N}}$.

It will suffice to prove the isomorphism of H_U and H_W. We start by constructing isomorphic subgroups $K_U \leq H_U$ and $K_W \leq H_W$.

Let I be the interval $[0,1]$. We remove from I the endpoints 0 and 1, all the points in U, as well as the entire intervals $[p_\beta]$ for $\beta \in U_2$. The result is a set $J_U \subseteq [0,1]$. If $|U| = n$, then J_U is a union of $n+1$ open intervals (a_i, b_i), $i = 1, \ldots, n+1$, ordered from left to right. We define K_U to be the subgroup generated by all $F_{[a_i,b_i]}$, that is, the direct product of these subgroups:

$$K_U = \prod_{i=1}^{n+1} F_{[a_i,b_i]}.$$

Clearly, K_U fixes all points in U. In particular, $K_U \leq H_U$.

Similarly, removing from I the endpoints 0 and 1, all numbers from W, as well as the intervals $[p_\beta]$ for $\beta \in U_2$ results in a union of $n+1$ open intervals (c_i, d_i) for $i = 1, \ldots, n+1$. We let

$$K_W = \prod_{i=1}^{n+1} F_{[c_i,d_i]}.$$

Note that a_i, b_i, c_i, d_i are endpoints of the removed subintervals or points from $U_1 \cup U_3 \cup W_3$. Hence, each a_i, b_i, c_i, d_i is either in $\mathbb{Z}[\frac{1}{2}]$ or an irrational number. Furthermore, if a_i (respectively, b_i) is in $\mathbb{Z}[\frac{1}{2}]$ then $c_i = a_i$ (respectively, $d_i = b_i$). If a_i (respectively, b_i) is irrational, then c_i (respectively, d_i) is also irrational (this follows from the equality of types $\tau(U)$ and $\tau(W)$). We shall be interested in an isomorphism from K_U to K_W with specific properties. For that, in any interval (a_i, b_i) where exactly one of the endpoints is in $\mathbb{Z}[\frac{1}{2}]$ (and hence exactly one of the endpoints of (c_i, d_i) is in $\mathbb{Z}[\frac{1}{2}]$ and coincides with the finite dyadic endpoint of (a_i, b_i)), we choose a number $y_i \in (a_i, b_i) \cap (c_i, d_i) \cap \mathbb{Z}[\frac{1}{2}]$ (notice that the intersection cannot be empty because the intervals (a_i, b_i) and (c_i, d_i) share an endpoint from $\mathbb{Z}[\frac{1}{2}]$).

Lemma 4.5. *There exists an isomorphism $\phi\colon K_U \to K_W$ such that for each $i = 1, \dots, n+1$, we have the following.*

1. *$\phi(F_{[a_i,b_i]}) = F_{[c_i,d_i]}$.*
2. *If a_i and b_i are in $\mathbb{Z}[\frac{1}{2}]$, then ϕ restricted to $F_{[a_i,b_i]}$ is the identity.*
3. *If a_i is in $\mathbb{Z}[\frac{1}{2}]$ and b_i is irrational, then ϕ restricted to $F_{[a_i,y_i]}$ is the identity and for any $x \in (a_i, y_i)$, $\phi(F_{[x,b_i)}) = F_{[x,d_i)}$.*
4. *If a_i is irrational and b_i is in $\mathbb{Z}[\frac{1}{2}]$ then the restriction of ϕ to $F_{[y_i,b_i]}$ is the identity and for any $x \in (y_i, b_i)$, $\phi(F_{(a_i,x]}) = F_{(c_i,x]}$.*

Proof. To define the isomorphism ϕ it suffices to define for each $i = 1, \dots, n+1$ an isomorphism $\phi_i\colon F_{[a_i,b_i]} \to F_{[c_i,d_i]}$ and let $\phi = \phi_1 \times \cdots \times \phi_{n+1}$. That would guarantee condition (1).

If a_i and b_i are both irrational, then $F_{[a_i,b_i]} = F_{(a_i,b_i)}$, and by Lemma 3.1, it is isomorphic to $F_{(c_i,d_i)} = F_{[c_i,d_i]}$. We let ϕ_i be any isomorphism between the groups.

If a_i and b_i are both in $\mathbb{Z}[\frac{1}{2}]$, then $[a_i, b_i] = [c_i, d_i]$, and we take ϕ_i to be the identity automorphism of $F_{[a_i,b_i]}$. This guarantees that ϕ will satisfy condition (2) of the lemma.

If a_i is in $\mathbb{Z}[\frac{1}{2}]$ and b_i is irrational, then $c_i = a_i$ and d_i is irrational. Thus, $F_{[a_i,b_i]} = F_{[a_i,b_i)}$ and $F_{[c_i,d_i]} = F_{[a_i,d_i)}$. Since $y_i \in (a_i, b_i) \cap (a_i, d_i) \cap \mathbb{Z}[\frac{1}{2}]$, we apply Lemma 4.3 to find an isomorphism ϕ_i so that condition (3) of the lemma would be satisfied.

Similarly, if a_i is irrational and b_i is in $\mathbb{Z}[\frac{1}{2}]$, then c_i is irrational and $d_i = b_i$. Thus, $F_{[a_i,b_i]} = F_{(a_i,b_i]}$ and $F_{[c_i,d_i]} = F_{(c_i,b_i]}$. Since $y_i \in (a_i, b_i) \cap (c_i, b_i) \cap \mathbb{Z}[\frac{1}{2}]$, we apply Lemma 4.4 to find an isomorphism ϕ_i so that condition (4) of the lemma would be satisfied. □

Next, we choose $m = |U_2| = |W_2|$ commuting elements $g_\beta \in H_U$ for $\beta \in U_2$ and m commuting elements $f_\delta \in H_W$ for $\delta \in W_2$. To do so, we first choose a number from $\mathbb{Z}[\frac{1}{2}]$ in each interval $[a_i, b_i]$, where at least one of the endpoints is in $\mathbb{Z}[\frac{1}{2}]$. If a_i is in $\mathbb{Z}[\frac{1}{2}]$ and b_i is irrational, we choose a number $x_i \in \mathbb{Z}[\frac{1}{2}]$ in $(a_i, y_i) = (c_i, y_i)$. Similarly, if a_i is irrational and b_i is in $\mathbb{Z}[\frac{1}{2}]$ we choose a number $x_i \in \mathbb{Z}[\frac{1}{2}]$ in $(y_i, b_i) = (y_i, d_i)$. If a_i and b_i are both in $\mathbb{Z}[\frac{1}{2}]$ we let $x_i = \frac{a_i+b_i}{2}$.

Recall that the elements of U_2 are of the form $\beta = .p_\beta s_\beta^{\mathbb{N}}$, and the elements in W_2 are $\delta = .p_\delta u_\delta^{\mathbb{N}} \in W_2$. Let, again, η_{wu} be the order-preserving bijection $W \to U$. For every $\beta \in U_2$ and $\delta = \eta_{wu}^{-1}(\beta)$, we have $p_\delta = p_\beta$. We choose the element $g_\beta \in H_U$ and $f_\delta \in H_W$ as follows. By construction, β belongs to an interval (b_i, a_{i+1}) for some i, where $b_i = .p_\beta, a_{i+1} = .p_\beta 1^{\mathbb{N}}$ are in $\mathbb{Z}[\frac{1}{2}]$. Let x_i be the number from $\mathbb{Z}[\frac{1}{2}]$ chosen in $[a_i, b_i]$, and let x_{i+1} be the number from $\mathbb{Z}[\frac{1}{2}]$ chosen in $[a_{i+1}, b_{i+1}]$.

We define g_β and f_δ to be functions such that

(1) g_β and f_δ have support in $[x_i, x_{i+1}]$;
(2) g_β has a pair of branches $p_\beta s_\beta \to p_\beta$ and f_δ has a pair of branches $p_\beta u_\delta \to p_\beta$;
(3) g_β and f_δ coincide on the interval $[a_{i+1}, x_{i+1}] = [c_{i+1}, x_{i+1}]$ and map it linearly onto the right half of itself;
(4) g_β and f_δ coincide on the interval $[x_i, b_i] = [x_i, d_i]$ and map it linearly onto the left half of itself.

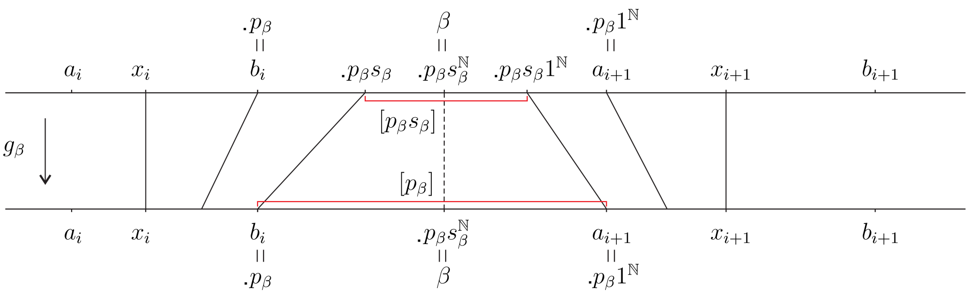

FIGURE 3. The function g_β.

By Lemma 2.2, one can indeed choose elements g_β, f_δ as described (for an illustration of the function g_β, see Figure 3). We note that since the interiors of the intervals $[x_i, x_{i+1}]$ are pairwise disjoint, the elements $g_\beta, \beta \in U_2$, pairwise commute, and the elements $f_\delta, \delta \in W_2$, commute as well. We let $G_U = \langle g_\beta, \beta \in U_2\rangle$ and $G_W = \langle f_\delta, \delta \in W_2\rangle$. Clearly, $G_U \cong G_W \cong \mathbb{Z}^m$.

Lemma 4.6. *H_U is generated by K_U and G_U. Similarly, H_W is generated by K_W and G_W.*

Proof. Notice that the elements $g_\beta, \beta \in U_2$, belong to H_U. Indeed, for each $\beta \in U_2$, g_β has a pair of branches $p_\beta s_\beta \to p_\beta$. As such, it fixes $\beta = .p_\beta s_\beta^{\mathbb{N}}$. It also fixes all other numbers from U because these numbers are not in the support of g_β. Thus, $\langle K_U \cup G_U\rangle \subseteq H_U$.

For the reverse inclusion, recall that $K_U = \prod_{i=1}^{n+1} F_{[a_i,b_i]}$ is the subgroup of F of all functions that fix all points in U as well as all intervals $[p_\beta]$ for $\beta \in U_2$. Now let $h \in H_U$. By Lemma 2.6, for each $\beta \in U_2$, the function h has a pair of branches of the form $p_\beta s_\beta^{\ell_\beta} \to p_\beta s_\beta^{r_\beta}$ for some $\ell_\beta, r_\beta \geq 0$. Let α be the smallest number in U_2 and consider the element $h_\alpha = g_\alpha^{-\ell_\alpha} h g_\alpha^{r_\alpha}$. Since $g_\alpha^{-\ell_\alpha}$ has a pair of branches $p_\alpha \to p_\alpha s_\alpha^{\ell_\alpha}$, h has a pair of branches $p_\alpha s_\alpha^{\ell_\alpha} \to p_\alpha s_\alpha^{r_\alpha}$, and $g_\alpha^{r_\alpha}$ has a pair of branches $p_\alpha s_\alpha^{r_\alpha} \to p_\alpha$, the function h_α has a pair of branches $p_\alpha \to p_\alpha$. In other words, it fixes the interval $[p_\alpha]$. Notice that for all $\beta \in U_2$ such that $\beta \neq \alpha$, the support of g_α is disjoint from $[p_\beta]$. Thus, the functions h and h_α have the same pairs of branches $p_\beta s_\beta^{\ell_\beta} \to p_\beta s_\beta^{r_\beta}$ for $\beta \in U_2 \setminus \{\alpha\}$. Considering other numbers from U_2 one by one, we find an element ghg', where $g, g' \in G_U$, that fixes all intervals $[p_\beta]$ for $\beta \in U_2$. Since ghg' also fixes all points in U, we have that $ghg' \in K_U$. Thus, $h \in G_U K_U G_U$. □

To finish the proof we prove that the group H_U (respectively, H_W) is an iterated ascending HNN-extension of the group K_U (respectively, K_W) with m-stable letters.

For this we are going to use the following simple and well-known fact (see, for example, [14, Lemma 2]).

Lemma 4.7. *Suppose that a group G contains a subgroup K and an element t such that*

(1) *G is generated by K and t;*
(2) *$t^n \notin K$ for all $n > 0$;*
(3) *$K^t \leq K$.*

Then G is isomorphic to an ascending HNN-extension of K with stable letter t.

For each $\beta \in U_2$ we have

$$K_U{}^{g_\beta} \leq K_U.$$

Indeed, K_U is the group of all functions with support in $S = [a_1, b_1] \cup \cdots \cup [a_{n+1}, b_{n+1}]$. Since for every $\beta \in U_2$

$$\mathrm{Supp}(g_\beta) \cap S = [x_i, b_i] \cup [a_{i+1}, x_{i+1}],$$

where $\beta \in (b_i, a_{i+1})$ and g_β maps $[x_i, b_i] \cup [a_{i+1}, x_{i+1}]$ into itself, the support of each function in ${K_U}^{g_\beta}$ is inside S.

Let $U_2 = \{\beta_1, \ldots, \beta_m\}$. Since the $g_\beta, \beta \in U_2$, commute, for each $j = 2, \ldots, m$ we have

$$\langle K_U, g_{\beta_1}, \ldots, g_{\beta_{j-1}} \rangle^{g_{\beta_j}} \leq \langle K_U, g_{\beta_1}, \ldots, g_{\beta_{j-1}} \rangle.$$

Since in addition H_U is generated by K_U and $g_\beta, \beta \in U_2$, and $g_{\beta_j}^k$ does not belong to $\langle K_U, g_{\beta_1}, \ldots, g_{\beta_{j-1}} \rangle$ for every $k > 0$ and every $j \geq 1$, the group H_U is an iterated HNN-extension of K_U with m stable letters $g_\beta, \beta \in U_2$, by Lemma 4.7. Similarly, H_W is an iterated HNN-extension of K_W with m stable letters $f_{\delta_1}, \ldots, f_{\delta_m}$, where $W_2 = \{\delta_1, \ldots, \delta_m\}$.

Theorem 4.1 will follow if we prove that the actions of $g_{\beta_1}, \ldots, g_{\beta_m}$ on K_U and the actions of $f_{\delta_1}, \ldots, f_{\delta_m}$ on K_W commute with the isomorphism $\phi\colon K_U \to K_W$ from Lemma 4.5.

We choose a generating set Y of K_U by fixing a generating set of $F_{[a_i,b_i]}$ for each $i = 1, \ldots, n+1$ and letting Y be the union of these sets. If a_i and b_i are either both from $\mathbb{Z}[\frac{1}{2}]$ or both irrational, we take the entire group $F_{[a_i,b_i]}$ as a generating set of itself. If a_i is from $\mathbb{Z}[\frac{1}{2}]$ and b_i is irrational, recall that there are chosen numbers $x_i < y_i$ in $(a_i, b_i) \cap \mathbb{Z}[\frac{1}{2}]$. We take $F_{[a_i,y_i]} \cup F_{[x_i,b_i]}$ as a generating set of $F_{[a_i,b_i]}$ (notice that by Lemma 4.2, this union generates $F_{[a_i,b_i]}$). Similarly, if a_i is irrational and b_i is from $\mathbb{Z}[\frac{1}{2}]$, then there are chosen numbers $y_i < x_i$ from $\mathbb{Z}[\frac{1}{2}]$ in (a_i, b_i). We take $F_{[a_i,x_i]} \cup F_{[y_i,b_i]}$ as a generating set of $F_{[a_i,b_i]}$.

The following lemma completes the proof of Theorem 4.1.

Lemma 4.8. *For each $h \in Y$ and $\beta \in U_2$ we have*

$$\phi(h^{g_\beta}) = \phi(h)^{f_\delta},$$

where $\delta = \eta_{wu}^{-1}(\beta)$.

Proof. By the construction of Y, $h \in F_{[a_i,b_i]}$ for some i. Suppose that the support of g_β is disjoint from $[a_i, b_i]$. Then $h^{g_\beta} = h$. By Condition (1) in Lemma 4.5, $\phi(h) \in F_{[c_i,d_i]}$. Since the support of f_δ is disjoint from $[c_i, d_i]$, we get $\phi(h)^{f_\delta} = \phi(h) = \phi(h^{g_\beta})$ as required.

Thus, we can assume that the support of g_β intersects $[a_i, b_i]$. This implies that β belongs to either the interval $[b_{i-1}, a_i]$ or to the interval $[b_i, a_{i+1}]$. We only consider the first case, the other case being similar.

Consider the interval (a_i, b_i). The number a_i is from $\mathbb{Z}[\frac{1}{2}]$. If b_i is also from $\mathbb{Z}[\frac{1}{2}]$, then by Condition (2) of Lemma 4.5, the restriction of ϕ to $F_{[a_i,b_i]}$ is the identity. In particular $\phi(h) = h$. Notice that g_β maps the interval $[a_i, b_i]$ into a subinterval of itself. Thus, h^{g_β} also belongs to $F_{[a_i,b_i]}$. Therefore, $\phi(h^{g_\beta}) = h^{g_\beta}$. Thus, it suffices to prove that $h^{g_\beta} = h^{f_\delta}$. That follows immediately from Conditions (1) and (3) in the definition of g_β and f_δ. Indeed, these conditions imply that g_β and f_δ coincide on the support of h.

Next, we assume that b_i is irrational. Since $h \in F_{[a_i,b_i]} \cap Y$, either $h \in F_{[a_i,y_i]}$ or $h \in F_{[x_i,b_i]}$. In the first case, by Condition (3) in Lemma 4.5, we have $\phi(h) = h$. Similarly, since g_β maps the interval $[a_i, y_i]$ onto a subinterval of itself, the conjugate h^{g_β} also belongs to $F_{[a_i,y_i]}$. As such, $\phi(h^{g_\beta}) = h^{g_\beta}$ and it suffices to prove that $h^{g_\beta} = h^{f_\delta}$. That follows as before from the fact that g_β and f_δ coincide on the support of h. If $h \in F_{[x_i,b_i]} = F_{[x_i,b_i)}$, then g_β commutes with h (indeed, their supports have disjoint interiors). By condition (3) from Lemma 4.5, $\phi(h) \in F_{[x_i,c_i)}$, which implies that $\phi(h)$ commutes with f_δ. As such, $\phi(h^{g_\beta}) = \phi(h) = \phi(h)^{f_\delta}$. $\square$

That completes the proof of Theorem 4.1. □

4.2. Algebraic structure of stabilizers of finite sets. The proof of Theorem 4.1 gives us more explicit information about the structure of the stabilizer H_U for a finite set U, as described in the rest of this section. If $U = \{\alpha\}$ for $\alpha \in (0,1)$ such that $\alpha \notin U_2$, then

$$H_U = F_{[0,\alpha]}F_{[\alpha,1]} \cong F_{[0,\alpha]} \times F_{[\alpha,1]}.$$

Indeed, this is obvious for $\alpha \in U_1$. If $\alpha \in U_3$, then every element of H_U fixes an open neighborhood of α and thus belongs to the direct product $F_{[0,\alpha]} \times F_{[\alpha,1]} = F_{[0,\alpha)} \times F_{(\alpha,1]}$. Similarly, let $U = \{\alpha_1, \dots, \alpha_n\} \subseteq (0,1)$. Let $U = U_1 \cup U_2 \cup U_3$ be the natural partition. If $U_2 = \varnothing$, then

$$H_U = F_{[0,\alpha_1]} \times F_{[\alpha_1,\alpha_2]} \times \dots \times F_{[\alpha_{n-1},\alpha_n]} \times F_{[\alpha_n,1]}.$$

If $U_2 \neq \varnothing$ and $|U_1 \cup U_3| = k$, then $U_1 \cup U_3$ separates U_2 into a union of $k+1$ disjoint subsets $U_{2,1}, \dots, U_{2,k+1}$ (some of which might be empty). Let $i_1, \dots, i_k$ be the indexes such that $\alpha_{i_j} \in U_1 \cup U_3$. Since $H_U = H_{U_2} \cap H_{U_1 \cup U_3}$, the above equation (for the case where $U_2 = \varnothing$) implies that

$$\begin{aligned} H_U = (F_{[0,\alpha_{i_1}]} \cap H_{U_{2,1}}) &\times (F_{[\alpha_{i_1},\alpha_{i_2}]} \cap H_{U_{2,2}}) \times \dots \\ &\times (F_{[\alpha_{i_{k-1}},\alpha_{i_k}]} \cap H_{U_{2,k}}) \times (F_{[\alpha_{i_k},1]} \cap H_{U_{2,k+1}}). \end{aligned}$$

For a set of numbers $V = \{\beta_1, \dots, \beta_m\}$ in $[0,1]$, if $\beta_1, \beta_m \notin U_2$, we let

$$B_V = F_{[\beta_1,\beta_m]} \cap H_{V \setminus \{\beta_1,\beta_m\}}.$$

Theorem 4.1 implies the following.

Lemma 4.9. *Let $U = \{\alpha_1, \dots, \alpha_n\}$ and $V = \{\beta_1, \dots, \beta_n\}$ be sets of numbers from $[0,1]$. Suppose that $\tau(U) \equiv \tau(V)$ and $\alpha_1, \alpha_n \notin U_2$. Then the groups B_U and B_V are isomorphic.*

Proof. It follows from the proof of Theorem 4.1. Indeed, using conjugation by an orientation preserving element of $\mathrm{PLF}_2(\mathbb{R})$, one can assume that U and V are sets of numbers in $(0,1)$. By Theorem 4.1, we see that

$$H_U = F_{[0,\alpha_1]} \times B_U \times F_{[\alpha_n,1]} \text{ and } H_V = F_{[0,\beta_1]} \times B_V \times F_{[\beta_n,1]}.$$

are isomorphic. The isomorphism constructed in the proof of Theorem 4.1 maps $F_{[0,\alpha_1]}$ onto $F_{[0,\beta_1]}$, $F_{[\alpha_n,1]}$ onto $F_{[\beta_n,1]}$, and B_U onto B_V. □

Let $U = \{\alpha_1, \dots, \alpha_n\} \subseteq [0,1]$ be such that $\alpha_1, \alpha_n \notin U_2$. By Lemma 4.9, the isomorphism class of B_U depends only on $\tau(U)$. Thus, if $w \equiv \tau(U)$, we will use the notation B_w for the group B_U. If $\alpha_1, \alpha_n \in U_1$, then Lemma 4.9 implies that $B_w \cong B_{U \setminus \{\alpha_1,\alpha_n\} \cup \{0,1\}} = H_{U \setminus \{\alpha_1,\alpha_n\}}$.

Lemma 4.10. *Let $U = \{\alpha_1, \dots, \alpha_n\}$ and $V = \{\beta_1, \dots, \beta_n\}$ be finite sets of numbers in $[0,1]$ such that $\tau(U)$ does not start or end with 2. Assume that the word $\tau(U)$ is equal to $\tau(V)$ read backwards. Then B_U is isomorphic to B_V.*

Proof. If one conjugates B_V, where $V = \{\beta_1, \dots, \beta_n\}$, by the function $\zeta(t) = 1 - t$ in $\mathrm{Homeo}([0,1])$, one gets B_W, where $W = \{1-\beta_n, \dots, 1-\beta_1\} \subseteq [0,1]$. Since $\tau(U) \equiv \tau(W)$, the result follows from Lemma 4.9. □

Now let $U = \{\alpha_1, \dots, \alpha_n\}$ be a finite set of numbers in $(0,1)$. Let $\alpha_0 = 0$, $\alpha_{n+1} = 1$ and $U' = U \cup \{\alpha_0, \alpha_{n+1}\}$. Let $w \equiv \tau(U')$. Let us represent w as the product $1u_1i_1u_2i_2\dots u_{k+1}1$, where $i_j \in \{1,3\}, j = 1, \dots, k$, and each word u_i contains only letter 2. The discussion above and Lemma 4.9 clearly imply that

$$H_U \cong B_{1u_1i_1} \times B_{i_1u_2i_2} \times \dots \times B_{i_ku_{k+1}1}.$$

Note that by Lemma 4.10, to completely describe the structure of H_U, we only need to describe the groups B_w, where w is a word of one of the following 6 kinds: 11, 33, 13, 12^m1, 32^m3 and 12^m3 for $m \in \mathbb{N}$. We note that B_{11} is isomorphic to F, B_{33} is isomorphic to the derived subgroup of F (see Lemma 3.1) and that B_{13} is isomorphic to the normal subgroup $\mathcal{L}$ of F of all functions with slope 1 at 1 (indeed, $B_{13} \cong F_{[0,\frac{\pi}{4})} \cong F_{[0,1)} = \mathcal{L}$ by Lemma 4.3). The group B_w is more difficult to describe in the remaining 3 cases. We leave the cases $w \equiv 32^m3$ and $w \equiv 12^m3$ to the reader. For $w \equiv 12^m1, m \in \mathbb{N}$, the proof of Theorem 4.1 shows the following.

Lemma 4.11. *Let U be a set of m rational numbers in $(0,1)\backslash\mathbb{Z}[\frac{1}{2}]$. Then $H_U \cong B_{12^m1}$ is isomorphic to an iterated ascending HNN-extension of $K = F^{m+1}$ with m stable letters $t_1,\dots,t_m$ such that t_j commutes with t_i and $K^{t_j} < K$, so each t_j corresponds to an endomorphism of K. That endomorphism ϕ_j is defined as follows. We denote by ι_i the injection of F into the ith direct summand of F^{m+1}. Then*

$$\begin{aligned}
\phi_j\colon\quad \iota_i(x_0) &\to \iota_i(x_0) && \text{for all } i \notin \{j,j+1\},\\
\iota_i(x_1) &\to \iota_i(x_1) && \text{for all } i \notin \{j,j+1\},\\
\iota_j(x_0) &\to \iota_j(x_0x_1^{-1}),\\
\iota_j(x_1) &\to \iota_j(x_1^2x_2^{-1}x_1^{-1}),\\
\iota_{j+1}(x_0) &\to \iota_{j+1}(x_0x_1x_0^{-1}),\\
\iota_{j+1}(x_1) &\to \iota_{j+1}(x_1).
\end{aligned}$$

Proof. This follows from an analysis of the proof of Theorem 4.1 in the case where $U = U_2$. In the notations of the proof, the intervals $[a_i,b_i]$ for $i = 1,\dots,m+1$ all have endpoints in $\mathbb{Z}[\frac{1}{2}]$. Thus, the group $K_U = \prod_{i=1}^{m+1} F_{[a_i,b_i]}$ is isomorphic to the direct sum of $m+1$ copies of F. Each of the generators g_{β_j} fixes the intervals $[a_i,b_i]$ for all $i \neq \{j,j+1\}$ and thus acts trivially on $F_{[a_i,b_i]}$. In addition, g_{β_j} fixes the left half of $[a_j,b_j]$ and maps the right half of $[a_j,b_j]$ to its own left half. Similarly, g_{β_j} fixes the right half of $[a_{j+1},b_{j+1}]$ and maps the left half of $[a_{j+1},b_{j+1}]$ to its right half. Using these facts and the isomorphism of $F_{[a_j,b_j]}$ with F and $F_{[a_{j+1},b_{j+1}]}$ with F one gets the above description of the endomorphism ϕ_j. □

Notice that Lemma 4.11 implies that if $U = U_2$, then H_U has a generating set with $3m+2$ generators. In the following section we will improve this result and find the minimal number of generators.

§5. Stabilizers of finite sets of rational numbers

Let $U \subset (0,1)$ be a finite set of rational numbers. In this section, we are going to show that the rank of the first homology group of H_U with integral coefficients (given by Theorem 3.2) coincides with the smallest number of generators of H_U.

We first consider the case where $U = U_2$. Recall (see Subsection 2.5), that if $f \in F$ and u is a finite binary word, then $f_{[u]}$ denotes the copy of f in $F_{[u]}$. Recall also that if f consists of pairs of branches $v_i \to w_i$, then $f_{[u]}$ consists of pairs of branches $uv_i \to uw_i$ and some pairs of branches $p \to p$. We will need the following three lemmas.

Lemma 5.1. *Let u be a finite binary word, and let $g \in F$ be an element with a pair of branches $u \to u0$. Let $f \in F$ and consider the copy $f_{[u]}$ of f in $F_{[u]}$. Then ${f_{[u]}}^g = (f \oplus \mathbf{1})_{[u]}$. That is, conjugating the copy of f in $F_{[u]}$ by g gives the copy of $f \oplus \mathbf{1}$ in $F_{[u]}$.*

Proof. First, notice that $(f \oplus \mathbf{1})_{[u]} = f_{[u0]}$. To prove that ${f_{[u]}}^g$ is equal to the copy of f in $F_{[u0]}$ we will show that ${f_{[u]}}^g$ has support in the interval $[u0]$ and that for any pair of branches $v \to w$ of f, the element ${f_{[u]}}^g$ takes the branch $u0v$ to the branch $u0w$.

Since $f_{[u]}$ has support in $[u]$ and g maps the interval $[u]$ onto $[u0]$, the conjugate ${f_{[u]}}^g$ has support in the interval $[u0]$. For any pair of branches $v \to w$ of f, the copy $f_{[u]}$ has a pair of branches $uv \to uw$. The pair of branches $u \to u0$ of g implies that ${f_{[u]}}^g$ has a pair of branches $u0v \to u0w$ for any such v and w. Indeed, g^{-1} takes $u0v$ to uv, $f_{[u]}$ takes uv to uw, and g takes uw to $u0w$. □

Similarly, we have the following.

Lemma 5.2. *Let u be a finite binary word, and let $g \in F$ be an element with a pair of branches $u \to u1$. Let $f \in F$ and consider the copy $f_{[u]}$ of f in $F_{[u]}$. Then ${f_{[u]}}^g = (\mathbf{1} \oplus f)_{[u]}$. That is, conjugating the copy of f in $F_{[u]}$ by g gives the copy of $\mathbf{1} \oplus f$ in $F_{[u]}$.*

Lemma 5.3. *Let u be a finite binary word. Let h_ℓ and h_r be functions in F with supports disjoint from $[u]$. Let ρ_ℓ be a function in F that takes the branch u to $u0$ and fixes all points in the support of h_ℓ. Similarly, let $\rho_r \in F$ be a function that takes the branch u to $u1$ and fixes all points in the support of h_r. Let $f_\ell = h_\ell (x_0)_{[u]}$ and $f_r = h_r (x_0)_{[u]}$. Then $(x_0)_{[u]}$ belongs to the subgroup $\langle f_\ell, f_r, \rho_\ell, \rho_r \rangle$.*

Proof. Let $G = \langle f_\ell, f_r, \rho_\ell, \rho_r \rangle$. We consider the conjugate of f_ℓ by ρ_ℓ. Since ρ_ℓ fixes all points in the support of h_ℓ, the functions ρ_ℓ and h_ℓ commute. By Lemma 5.1, ${(x_0)_{[u]}}^{\rho_\ell} = (x_0 \oplus \mathbf{1})_{[u]}$. Thus, ${f_\ell}^{\rho_\ell} = h_\ell (x_0 \oplus \mathbf{1})_{[u]} \in G$. Similarly, using Lemma 5.2 we get ${f_r}^{\rho_r} = h_r (\mathbf{1} \oplus x_0)_{[u]} = h_r (x_1)_{[u]}$. Note that $x_0 \oplus \mathbf{1} = x_0^2 x_1^{-1} x_0^{-1}$. Since $[u]$ is disjoint from the support of h_r, $(x_0)_{[u]}$ and $(x_1)_{[u]}$ commute with h_r. Thus, we have ${f_r}^2 ({f_r}^{\rho_r})^{-1} f_r^{-1} = (x_0^2 x_1^{-1} x_0^{-1})_{[u]} = (x_0 \oplus \mathbf{1})_{[u]} \in G$. That implies that $h_\ell \in G$ and, in turn, that $(x_0)_{[u]} \in G$. □

Theorem 5.4. *Let $U = U_2$. Then H_U has a generating set with $|U| + 2$ elements (in fact, a generating set of this size can be described explicitly).*

Proof. Let $U = U_2 = \{\alpha_1, \dots, \alpha_m\}$. We start as in the proof of Theorem 4.1 by constructing a subgroup K_U and elements $g_1, \dots, g_m$ such that K_U and $g_1, \dots, g_m$ generate H_U. As above, K_U will be the direct product of groups of the form $F_{[a_i, b_i]}$. It will be convenient to assume that $[a_i, b_i]$ are dyadic intervals.

For $j = 1, \dots, m$, let $\alpha_j = .p_j s_j^{\mathbb{N}}$, where s_j is a minimal period. We assume that the prefixes p_j are long enough so that the intervals $[p_1], \dots, [p_m]$ are pairwise disjoint and such that $0 \notin [p_1]$ and $1 \notin [p_m]$.

Let T be an arbitrary finite binary tree with $2m+1$ leaves represented by binary words $v_1 \dots, v_{2m+1}$. The m intervals associated with the even numbered branches, that is, $[v_2], [v_4], \dots, [v_{2m}]$, are pairwise disjoint, $0 \notin [v_2]$, and $1 \notin [v_{2m}]$. Thus, by Lemma 2.2 there exists an element $f \in F$ with pairs of branches $p_j \to v_{2j}$ for $j = 1, \dots, m$. Conjugating H_U by f results in a group H_V, where $V = \{\beta_1, \dots, \beta_m\}$ and for each $j = 1, \dots, m$, $\beta_j = v_{2j} s_j^{\mathbb{N}}$. Clearly, it suffices to prove that H_V is $(m+2)$-generated. For the rest of the proof, we rename the set V and its elements by U and $\alpha_1, \dots, \alpha_m$, respectively.

Notice that if one removes from $I = [0,1]$ the endpoints $0, 1$ and the intervals $[v_2], \dots, [v_{2m}]$, one remains with a union of $m+1$ open intervals (a_i, b_i) for $i = 1, \dots, m+1$. It is obvious that $[a_i, b_i] = [v_{2i-1}]$. Thus, we define the subgroup

$$K_U = \prod_{i=1}^{m+1} F_{[v_{2i-1}]}.$$

Next, we choose elements $g_1, \dots, g_m \in H_U$. Unlike in the proof of Theorem 4.1, we do not require that the elements commute. For each $j = 1, \dots, m$ we let g_j be an element with the following pairs of branches:

$$\begin{cases} v_{2j-1} & \to \ v_{2j-1}0, \\ v_{2j}s_j & \to \ v_{2j}, \\ v_{2j+1} & \to \ v_{2j+1}1, \\ v_k & \to \ v_k, \text{ for all } k \in \{1,\ldots,2m+1\} \setminus \{2j-1, 2j, 2j+1\}. \end{cases}$$

Note that the existence of an element g_j with the required pairs of branches follows from Lemma 2.2.

Lemma 5.5. *The group H_U is generated by the subgroup K_U and the elements $g_1, \ldots, g_m$.*

Proof. The proof is identical to the proof of Lemma 4.6. Indeed, the only conditions in the definition of the elements $g_1, \ldots, g_m$ necessary for the proof are that g_j has a pair of branches $v_{2j}s_j \to v_{2j}$ and that the support of g_j is disjoint from the interval $[v_{2k}]$ for all $k \neq j$. Then, given an element $h \in H_U$, one can multiply h from the left and from the right by powers of $g_1, \ldots, g_m$ to get an element $h' \in K_U$. □

Lemma 5.6. *The group H_U is generated by the set*

$$S = \{(x_0)_{[v_1]}, (x_0)_{[v_3]}, \ldots, (x_0)_{[v_{2m-1}]}, (x_0)_{[v_{2m+1}]}, g_1, \ldots, g_m\}.$$

Proof. By Lemma 5.5 and the definition of K_U, it suffices to prove that for all $j = 1, \ldots, m+1$, the subgroup $\langle S \rangle$ contains the subgroup $F_{[v_{2j-1}]}$. For each $j = 1, \ldots, m$, the element g_j has a pair of branches $v_{2j-1} \to v_{2j-1}0$. Thus, by Lemma 5.1,

$${(x_0)_{[v_{2j-1}]}}^{g_j} = (x_0 \oplus \mathbf{1})_{[v_{2j-1}]}.$$

Since $(x_0)_{[v_{2j-1}]}$ and $(x_0 \oplus \mathbf{1})_{[v_{2j-1}]}$ generate $F_{[v_{2j-1}]}$, we have the inclusion $F_{[v_{2j-1}]} \subseteq \langle S \rangle$. For $j = m+1$, we note that g_m has a pair of branches $v_{2m+1} \to v_{2m+1}1$. Thus, by Lemma 5.2,

$${(x_0)_{[v_{2m+1}]}}^{g_m} = (\mathbf{1} \oplus x_0)_{[v_{2m+1}]} = (x_1)_{[v_{2m+1}]}.$$

Since $(x_0)_{[v_{2m+1}]}$ and $(x_1)_{[v_{2m+1}]}$ generate $F_{[v_{2m+1}]}$, it is also contained in $\langle S \rangle$. □

To prove that H_U is $(m+2)$-generated, we choose two elements x and y in K_U as follows. If m is odd, we let

$$x = \prod_{i=1}^{\frac{m+1}{2}} (x_0)_{[v_{4i-3}]} \text{ and } y = \prod_{i=1}^{\frac{m+1}{2}} (x_0)_{[v_{4i-1}]}.$$

If m is even, we let

$$x = \left[\prod_{i=1}^{\frac{m}{2}} (x_0)_{[v_{4i-3}]}\right] (x_0)_{[v_{2m-1}]} \text{ and } y = \left[\prod_{i=1}^{\frac{m}{2}} (x_0)_{[v_{4i-1}]}\right] (x_0)_{[v_{2m+1}]}.$$

Notice that in both cases, all elements appearing in the product defining x have disjoint supports, thus all elements in the product commute. The same is true for the product defining y.

We claim that H_U is generated by $x, y, g_1, \ldots g_m$. By Lemma 5.6, it suffices to prove the following.

Lemma 5.7. *The group $H = \langle x, y, g_1, \ldots, g_m \rangle$ contains the elements $(x_0)_{[v_{2j-1}]}$ for $j = 1, \ldots, m+1$.*

Proof. We first consider the case where m is odd. Recall that in that case,

$$x = \prod_{i=1}^{\frac{m+1}{2}} (x_0)_{[v_{4i-3}]}.$$

For each $k \in \{2, \dots, \frac{m+1}{2}\}$ we apply Lemma 5.3 with $u \equiv v_{4k-3}$,

$$h_\ell = h_r = \prod_{\substack{i=1 \\ i \neq k}}^{\frac{m+1}{2}} (x_0)_{[v_{4i-3}]},$$

$\rho_\ell = g_{2k-1}$ and $\rho_r = g_{2k-2}$. Note, that all the conditions of Lemma 5.3 are satisfied. Indeed, the support of $h_\ell = h_r$ is disjoint from $[u] = [v_{4k-3}]$, $\rho_\ell = g_{2k-1}$ fixes all intervals $[v_i]$ for $i \notin \{4k-3, 4k-2, 4k-1\}$ and in particular fixes the support of h_ℓ. Similarly, ρ_r fixes the support of h_r. By construction, $\rho_\ell = g_{2k-1}$ takes $u \equiv v_{4k-3}$ to $u0$ and ρ_r takes u to $u1$. Note that f_ℓ and f_r from the lemma are both equal to x. Thus, by Lemma 5.3,

$$(x_0)_{[v_{4k-3}]} \in \langle f_\ell, f_r, \rho_\ell, \rho_r \rangle = \langle x, g_{2k-1}, g_{2k-2} \rangle \subseteq \langle x, g_1, \dots, g_m \rangle.$$

If one multiplies x by the inverses of $(x_0)_{[v_{4k-3}]}$ for $k = 2, \dots, \frac{m+1}{2}$, one remains with $(x_0)_{[v_1]}$. Thus, $(x_0)_{[v_1]} \in \langle x, g_1, \dots, g_m \rangle$ as well.

Next, one should consider the element

$$y = \prod_{i=1}^{\frac{m+1}{2}} (x_0)_{[v_{4i-1}]}.$$

If one applies the same arguments as above, one sees that for all $k \in \{1, \dots, \frac{m+1}{2}\}$, the element $(x_0)_{[v_{4k-1}]}$ is in $\langle y, g_1, \dots, g_m \rangle$. Combining, we conclude that for all $j \in \{1, \dots, m+1\}$, the element $(x_0)_{[v_{2j-1}]}$ belongs to H, as necessary.

The proof for m even is very similar. Recall that in that case, we have

$$x = \left[\prod_{i=1}^{\frac{m}{2}} (x_0)_{[v_{4i-3}]} \right] (x_0)_{[v_{2m-1}]} \text{ and } y = \left[\prod_{i=1}^{\frac{m}{2}} (x_0)_{[v_{4i-1}]} \right] (x_0)_{[v_{2m+1}]}.$$

We apply Lemma 5.3 with $u \equiv v_{2m-1}$,

$$h_\ell = \prod_{i=1}^{\frac{m}{2}} (x_0)_{[v_{4i-3}]}, \quad h_r = \left[\prod_{i=1}^{\frac{m-2}{2}} (x_0)_{[v_{4i-1}]} \right] (x_0)_{[v_{2m+1}]},$$

$\rho_\ell = g_m$ and $\rho_r = g_{m-1}$. One can check that all the conditions in Lemma 5.3 are satisfied. Notice, in addition, that f_ℓ from the lemma is equal to x and $f_r = y$. Thus, Lemma 5.3 implies that $(x_0)_{[v_{2m-1}]} \in \langle f_\ell, f_r, \rho_\ell, \rho_r \rangle \subseteq H$. Multiplying x and y by $((x_0)_{[v_{2m-1}]})^{-1}$ results in elements

$$x' = \prod_{i=1}^{\frac{m}{2}} (x_0)_{[v_{4i-3}]} \text{ and } y' = \left[\prod_{i=1}^{\frac{m-2}{2}} (x_0)_{[v_{4i-1}]} \right] (x_0)_{[v_{2m+1}]}.$$

Proceeding as in the case where m is odd, for each $k = 2, \dots, \frac{m}{2}$ one can apply Lemma 5.3 with $u \equiv v_{4k-3}$, $h_\ell = h_r = x'((x_0)_{[v_{4k-3}]})^{-1}$, $\rho_\ell = g_{2k-1}$ and $\rho_r = g_{2k-2}$. As a result, one sees that $(x_0)_{[v_{4k-3}]} \in \langle x', g_1, \dots, g_m \rangle$ for $k = 2, \dots, \frac{m}{2}$. Multiplying x' by the inverses of $(x_0)_{[v_{4k-3}]}$ for $k = 2, \dots, \frac{m}{2}$ shows that $(x_0)_{[v_1]} \in \langle x', g_1, \dots, g_m \rangle$. In a similar way, one shows that $(x_0)_{[v_{4k-1}]} \in \langle y', g_1, \dots, g_m \rangle$ for $k = 1, \dots, \frac{m-2}{2}$, and as such so does $(x_0)_{[v_{2m+1}]}$. All together, we have that for all $j \in \{1, \dots, m+1\}$, the element $(x_0)_{[v_{2j-1}]}$ belongs to H, which completes the proof of the lemma. $\square$

The proof of Theorem 5.4 is complete. □

Remark 5.8. Suppose now that U_1 is not empty (but $U_3 = \varnothing$). If $|U_1| = k$, then the numbers from U_1 separate U_2 into a disjoint union $U_{2,1} \cup \cdots \cup U_{2,k+1}$ of subsets (some of which might be empty). By the results of Subsection 4.2, H_U is isomorphic to the direct product of subgroups $H_{U_{2,i}}$. Note that if $U_{2,i} = \varnothing$, then $H_{U_{2,i}} \cong F$.

This allows us to compute the presentation of H_U and, in particular, the minimal number of generators of that subgroup.

Theorem 5.9. *Let U be a finite set of rational numbers in $(0, 1)$, and let $U = U_1 \cup U_2$ be its natural partition. Then the smallest number of generators of H_U is $2|U_1| + |U_2| + 2$.*

Proof. The fact that the smallest number of generators of H_U cannot be smaller than $2|U_1| + |U_2| + 2$ follows from Theorem 3.2. Let us prove the upper bound. The proof is by induction on the number $n = |U_1|$. If $n = 0$, the result is Theorem 5.4. Assume that $n > 0$ and $m = |U_2|$. Let α be the smallest number in U_1 and assume that there are $c \geq 0$ numbers in U_2 smaller than α. From Remark 5.8 it follows that H_U is isomorphic to the direct product of H_V and H_W, where $V = V_2$, $|V| = c$, $|W_1| = n-1$, $|W_2| = m-c$. Thus, the induction hypothesis shows that H_V is generated by $c+2$ elements and H_W is generated by $2(n-1)+m-c+2$ elements. Hence, the direct product $H_U \cong H_V \times H_W$ is generated by $2n+m+2$ elements. □

§6. Finitely generated subgroups H_U are undistorted

Recall that if G is a group generated by a finite set S and H is a subgroup of G generated by a finite set T, then the distortion function $\delta_{S,T}$ is the smallest function $\mathbb{N} \to \mathbb{N}$ such that if an element $h \in H$ is a product of n elements of S, then it is a product of at most $\delta_{S,T}(n)$ elements of T. For fixed G, H but different (finite) S, T, the functions $\delta_{S,T}$ are equivalent[2]. The subgroup H is said to be *undistorted* in G if the distortion function is linear. Although many subgroups of the Thompson group F are known to be undistorted (see, for example, [19, 9, 20, 28]), F has distorted subgroups [20, 13].

Theorem 6.1. *Let U be a finite set of rational numbers in $(0, 1)$. Then the subgroup H_U is undistorted in F.*

Proof. Let $U = \{\alpha_1, \ldots, \alpha_n\}$. If $|U_1| = k$, then U_1 separates U_2 into a union of $k+1$ disjoint subsets $U_{2,1}, \ldots, U_{2,k+1}$. Let $i_1, \ldots, i_k$ be the indexes such that $\alpha_{i_j} \in U_1$. Let $\alpha_0 = 0$, $\alpha_{n+1} = 1$, $i_0 = 0$ and $i_{k+1} = n+1$. By the results of Subsection 4.2,

$$H_U = B_{\{\alpha_{i_0},\ldots,\alpha_{i_1}\}} \times B_{\{\alpha_{i_1},\ldots,\alpha_{i_2}\}} \times \cdots \times B_{\{\alpha_{i_{k-1}},\ldots,\alpha_{i_k}\}} \times B_{\{\alpha_{i_k},\ldots,\alpha_{i_{k+1}}\}},$$

where $B_{\{\alpha_{i_{j-1}},\ldots,\alpha_{i_j}\}} = F_{[\alpha_{i_{j-1}},\alpha_{i_j}]} \cap H_{U_{2,j}}$ for $j = 1, \ldots, k+1$.

Since a direct product is undistorted if and only if each factor is undistorted, it suffices to prove that if $a < b$ belong to $[0, 1] \cap \mathbb{Z}[\frac{1}{2}]$ and $U' = U'_2$ is a set of rational numbers in (a, b), then $F_{[a,b]} \cap H_{U'}$ is undistorted in F. We claim that $F_{[a,b]}$ is undistorted in F and that $F_{[a,b]} \cap H_{U'}$ is undistorted in $F_{[a,b]}$. That will imply that $F_{[a,b]} \cap H_{U'}$ is undistorted in F.

Proposition 9 in [9] implies that $F_{[0,\frac{1}{2}]}$ and $F_{[\frac{1}{2},1]}$ are undistorted in F. Clearly, one can replace $\frac{1}{2}$ by any number $\alpha \in (0, 1) \cap \mathbb{Z}[\frac{1}{2}]$ by conjugting F by a function $f \in F$ such that $f(\frac{1}{2}) = \alpha$. Hence, if $a = 0$ or $b = 1$, then $F_{[a,b]}$ is undistorted in F. Otherwise, $a, b \in (0, 1)$. Let $g \in \mathrm{PLF}_2(\mathbb{R})$ be a function such that $g(0) = 0$, $g(\frac{1}{2}) = a$, and $g(1) = b$,

[2]Two functions $f, g \colon \mathbb{N} \to \mathbb{N}$ are called *equivalent* if for some $c > 1$, $\frac{1}{c} f(\frac{n}{c}) - c \leq g(n) \leq cf(cn) + c$ for every $n \in \mathbb{N}$.

then $F^g = F_{[0,b]}$ and $F_{[\frac{1}{2},1]}{}^g = F_{[a,b]}$. It follows that $F_{[a,b]}$ is undistorted in $F_{[0,b]}$. Since $F_{[0,b]}$ is undistorted in F, it follows that $F_{[a,b]}$ is undistorted in F.

To prove that $F_{[a,b]} \cap H_{U'}$ is undistorted in $F_{[a,b]}$ we note that there is an isomorphism from $F_{[a,b]}$ to F that maps $F_{[a,b]} \cap H_{U'}$ onto a subgroup $H_{U''} \leq F$, where $U'' = U''_2$ and $|U''| = |U'|$. Clearly, it suffices to prove that $H_{U''}$ is undistorted in F. In other words, it suffices to prove the theorem for subsets U such that $U = U_2$.

Let $U = U_2$. We shall use the proof of Theorem 5.4. Let $h \in H_U$ be a product of at most n elements from $\{x_0, x_1\}$. Then, viewed as a reduced tree-diagram, h has at most cn vertices for some constant c. By the proof of Lemma 5.5, H_U is equal to GK_UG, where K_U is a direct product of $m+1$ copies $F_{[v_{2i-1}]}$ of F, $i = 1, \ldots, m+1$ (where $m = |U|$), and G is the subgroup generated by the elements $g_1, \ldots, g_m$ defined before Lemma 5.5.

Let us take the generators $[x_0]_{v_{2i-1}}, [x_1]_{v_{2i-1}}$ in each $F_{[v_{2i-1}]}$ and all the elements $g_1, \ldots, g_m$ as the elements of a generating set T of H_U.

The proof of Theorem 5.4 shows that by multiplying h from the left and from the right by powers of $g_1, \ldots, g_m$ one can get an element h' in K_U. The number of elements $g_1, \ldots, g_m$ required for the product is bounded from above by the number of vertices of h. Hence, the element h' has at most $cn + cn(c_1 + \cdots + c_m)$ vertices, where $c_1, \ldots, c_m$ are constants depending on $g_1, \ldots, g_m$. Then $h' = h_1 \cdots h_{m+1}$, where $h_i \in F_{[v_{2i-1}]}$, hence all h_i have pairwise disjoint supports. Then each h_i is represented by a diagram with at most $d_i n$ vertices, where d_i is a constant. By Property B of Burillo (see [9, 2]), h_i is a product of at most $d'_i n$ generators from $\{[x_0]_{v_{2i-1}}, [x_1]_{v_{2i-1}}\}$. Hence h is a product of at most $c'n$ generators of H_U for some constant c'. □

§7. Isomorphism vs conjugacy

In this section we show that the isomorphism between H_U and H_V (provided $\tau(U) \equiv \tau(V)$) is induced by conjugacy in some bigger group. In fact, we construct a chain $F < \mathcal{F} < \overline{F} < \mathrm{Homeo}([0,1])$ such that $\mathcal{F}$ is similar to F and consists of possibly infinite tree-diagrams, $\overline{F}$ is the completion of F with respect to a certain natural metric, and H_U, H_V are conjugate inside $\mathcal{F}$. This strengthens Theorem 4.1.

7.1. The completion of F with respect to the Hamming metric. Let μ be the standard Lebesgue measure on $[0,1]$. Consider the following metric on the group F:

$$\mathrm{dist}_H(f,g) = \mu(\mathrm{Supp}(fg^{-1})) + \mu(\mathrm{Supp}(f^{-1}g))$$

or, equivalently,

$$\mathrm{dist}_H(f,g) = \mu(\{x \mid f(x) \neq g(x)\}) + \mu(\{x \mid f^{-1}(x) \neq g^{-1}(x)\}).$$

Clearly, dist_H is a distance function on F.

The metric dist_H is similar to the standard Hamming metric on the symmetric group S_n although, unlike the Hamming metric on S_n, dist_H is not invariant with respect to left or right multiplication by elements of F. Thus, we shall call dist_H the *Hamming* metric on F.

Remark 7.1. It is easy to show that the group operations of F (the multiplication and the inverse) are continuous with respect to dist_H. This follows from the fact that for every $f \in F$ and every $\epsilon > 0$ there exists $\delta > 0$ such that if $\mu(S) < \delta$, then $\mu(f(S)) < \epsilon$ (one can take $\delta = \frac{\epsilon}{2^n}$, where 2^n is the maximal slope of f). Note that this fact is not true for arbitrary $f \in \mathrm{Homeo}([0,1])$. Hence, although dist_H can be obviously extended to the whole $\mathrm{Homeo}([0,1])$, the multiplication in $\mathrm{Homeo}([0,1])$ is not continuous with respect to dist_H.

The Hamming metric is of course related in a standard way to the norm $|f| = \mu(\mathrm{Supp}(f))$. A similar norm on the group of diffeomorphisms of arbitrary manifolds was considered in [8, Example 1.19]. For the group of measure preserving maps of a measure space, this is sometimes called the *uniform metric* [15].

Definition 7.2. We denote by $\overline{F}$ the (Raǐkov) completion of F with respect to dist_H (see [1, Section 3.6]). It consists of Cauchy sequences $(f_n), n \geq 1$, of elements of F with two sequences $(f_n), (g_n)$ being equivalent if

$$\lim_{n\to\infty}(\mathrm{dist}_H(f_n, g_n)) = 0.$$

Theorem 7.3. *The standard embedding* $F \to \mathrm{Homeo}([0,1])$ *extends to an embedding* $\overline{F} \to \mathrm{Homeo}([0,1])$.

Proof. Let a sequence of functions (g_m) in F be Cauchy with respect to the Hamming metric. We claim that the sequence (g_m) converges uniformly to a function g. Indeed, for each ϵ there is some $n \in \mathbb{N}$ such that for all $m_1, m_2 > n$ we have $\mathrm{dist}_H(g_{m_1}, g_{m_2}) < \epsilon$. It suffices to prove that for the same n, for every $x \in [0,1]$, the diameter of the set $B_n(x) = \{g_m(x) : m > n\}$ is at most ϵ. Assume that for some x the diameter of $B_n(x) = \{g_m(x) : m > n\}$ is greater than ϵ, and let a, b be elements of $B_n(x)$ such that $a < b$ and $b - a > \epsilon$. Let $m_1, m_2 > n$ be such that $g_{m_1}(x) = a$ and $g_{m_2}(x) = b$. Clearly, $g_{m_1}^{-1}([a,b])$ is contained in $[x,1]$. Similarly, $g_{m_2}^{-1}([a,b])$ is contained in $[0,x]$. As such, for all y in the open interval (a,b) we have $g_{m_1}^{-1}(y) \neq g_{m_2}^{-1}(y)$. Therefore, $\mathrm{dist}_H(g_{m_1}, g_{m_2}) \geq b - a > \epsilon$, in contradiction to the assumption.

Thus, (g_m) converges uniformly in $[0,1]$. Let $g(x)$ be the limit of $(g_m(x))$ for each $x \in [0,1]$. Then $g(0) = 0, g(1) = 1$. Moreover $g(x)$ is a nondecreasing function. Indeed, if $x < y$, then for each m we have $g_m(x) < g_m(y)$ and in particular the limits satisfy $g(x) \leq g(y)$.

Note that the sequence (g_m^{-1}), $m \geq 1$, is also a Cauchy sequence (by the definition of dist_H). Hence, we can define a function $g'(x)$ as $\lim_{m\to\infty} g_m^{-1}(x)$. We claim that $gg'(x) = x$ for all $x \in [0,1]$. Indeed, assume that for some x, $gg'(x) = y \neq x$. We can assume that $y > x$, the argument for $y < x$ being similar. Let $\epsilon = \frac{y-x}{4}$, and let $n \in \mathbb{N}$ be such that for all $m_1, m_2 > n$, $\mathrm{dist}_H(g_{m_1}, g_{m_2}) < \epsilon$. Since $g_m^{-1}(g(x))$ has limit $g'(g(x)) = y$, there is $m_1 > n$ such that $|g_{m_1}^{-1}(g(x)) - y| < \epsilon$. Let $y_1 = g_{m_1}^{-1}(g(x))$. By assumption, $|y_1 - y| < \epsilon$. Therefore, for $z = \frac{x+y}{2}$ we have $z \in (x, y_1)$. Let $c = g_{m_1}(z) < g_{m_1}(y_1) = g(x)$. Clearly, $g_{m_1}([x,z]) \subseteq [0,c]$. Since the sequence $g_m(x)$ converges to $g(x)$ and $c < g(x)$, for some $m_2 > m_1$ we have $g_{m_2}(x) > c$. In particular, $g_{m_2}([x,z]) \subseteq [c,1]$. It follows that $[x,z] \subseteq \{t : g_{m_1}(t) \neq g_{m_2}(t)\}$. Thus, $\mathrm{dist}_H(g_{m_1}, g_{m_2}) \geq z - x = 2\epsilon$, in contradiction to m_1, m_2 being greater than n. Thus, $gg'(x) = x$ for all $x \in [0,1]$. A similar argument shows that $g'g(x) = x$. Thus, g is a one-to-one increasing function $[0,1] \to [0,1]$, hence g is an element of $\mathrm{Homeo}([0,1])$.

Now, let (h_m) be a Cauchy sequence in F equivalent to (g_m), and let h be the pointwise limit of (h_m), that is, for every $x \in [0,1]$, $h(x) = \lim_{m\to\infty} h_m(x)$. We claim that $g = h$. Indeed, assume by contradiction that $g(x) \neq h(x)$ for some $x \in (0,1)$. Without loss of generality we can assume that $g(x) < h(x)$. Let a, b be such that $g(x) < a < b < h(x)$. By the definition of $g(x)$ and $h(x)$, for every sufficiently large m, we have $g_m(x) < a$ and $h_m(x) > b$. Thus, $g_m^{-1}([a,b]) \subseteq [x,1]$ and $h_m^{-1}([a,b]) \subseteq [0,x]$. As before, this implies that for all sufficiently large m, $\mathrm{dist}_H(g_m, h_m) \geq b - a$, in contradiction to (g_m) and (h_m) being equivalent Cauchy sequences.

Thus, mapping a Cauchy sequence (g_m) in F to its pointwise-limit as defined above gives a well-defined mapping ψ from $\overline{F}$ to $\mathrm{Homeo}([0,1])$.

Showing that the map $\psi\colon \overline{F} \to \mathrm{Homeo}([0,1])$ is a homomorphism is a straightforward exercise. It follows from the uniform convergence of Cauchy sequences in F and the uniform continuity of the limit functions. Clearly, the restriction of ψ to F coincides with the standard embedding of F into $\mathrm{Homeo}([0,1])$.

We claim that ψ is injective. To prove the claim, assume that (f_m) is a Cauchy sequence of elements in F with $\lim_{m\to\infty} f_m(x) = x$ for every $x \in [0,1]$. We need to prove that then

$$\lim_{m\to\infty} \mathrm{dist}_H(f_m, \mathbf{1}) = 0.$$

Suppose that this is not true. Then we can assume that for infinitely many

$$m : \mu(\mathrm{Supp}(f_m)) > d \ \text{ for some } \ d > 0.$$

Restricting to a subsequence, we can assume that for all m, $\mu(\mathrm{Supp}(f_m)) > d$. Since (f_m) is Cauchy, there is $n \in \mathbb{N}$ such that $\mathrm{dist}_H(f_m, f_n) < d$ for all $m > n$.

For each $k \in \mathbb{N}$, let

$$C_k = \bigcap_{m>k} \mathrm{Supp}(f_n f_m^{-1}).$$

Then the sequence C_k, $k \in \mathbb{N}$ is an increasing sequence of closed subsets of $[0,1]$. We claim that $\bigcup_{k\in\mathbb{N}} C_k$ contains the interior of $\mathrm{Supp}(f_n)$, denoted by $\mathrm{Int}(\mathrm{Supp}(f_n))$. Indeed, let $x \in \mathrm{Int}(\mathrm{Supp}(f_n))$ and assume by contradiction that $x \notin C_k$ for all k. Then for all $k \in \mathbb{N}$, there is some $m > k$ such that $x \notin \mathrm{Supp}(f_n f_m^{-1})$, so that $f_m(x) = f_n(x)$. It follows that the value $f_n(x)$ appears infinitely many times in the sequence $(f_m(x))$. Since the sequence $(f_m(x))$ is convergent, we must have $\lim_{m\to\infty} f_m(x) = f_n(x) \neq x$, in contradiction to the assumption.

Thus, the union of C_k, $k \in \mathbb{N}$, contains the interior of the support of f_n. Since C_k is increasing, we have

$$\lim_{k\to\infty} \mu(C_k) = \mu\Big(\bigcup_{k\in\mathbb{N}} C_k\Big) \geq \mu(\mathrm{Supp}(f_n)) > d.$$

It follows that there is some $k > n$ such that $\mu(C_k) > d$. This clearly implies that $\mathrm{dist}_H(f_{k+1}, f_n) > d$ in contradiction to the choice of n. □

7.2. A subgroup of $\overline{F}$. Let T be an infinite binary tree. Then there is a natural left-to-right (lexicographic) order on the branches of T, and a natural subdivision of the unit interval into possibly infinite number of intervals corresponding to the branches of the tree. Infinite branches of T correspond to intervals with empty interior of that subdivision. Other intervals have finite dyadic endpoints.

Definition 7.4. Consider the set $\mathcal{F}$ of triples (T_+, T_-, ϕ) where T_+, T_- are binary trees and ϕ is a bijection from the set of branches of T_+ to the set of branches of T_- satisfying the following properties:

(1) T_+ and T_- have the same finite number of infinite branches;
(2) if a branch p in T_+ is to the left of branch q, then $\phi(p)$ is to the left of $\phi(q)$ in T_-;
(3) ϕ takes infinite branches to infinite branches.

If T_+ and T_- are finite trees, then the function ϕ taking leaves of T_+ to leaves of T_- is defined uniquely. So F is naturally a subset of $\mathcal{F}$.

One can extend the equivalence relation (inserting and reducing pairs of common carets as in Subsection 2.2) and the group operations of F, multiplication and taking inverse (see Remark 2.1), to the set $\mathcal{F}$, making $\mathcal{F}$ a group containing F as a subgroup.

Every triple (T_+, T_-, ϕ) in $\mathcal{F}$ corresponds to a homeomorphism $\gamma = \gamma(T_+, T_-, \phi)$ of $[0,1]$ that takes linearly each interval $[p]$ of the partition corresponding to a branch p of T_+ to the interval $[\phi(p)]$ corresponding to the branch $\phi(p)$ of T_-. Properties 1, 2, 3 of Definition 7.4 imply that γ is indeed a homeomorphism of $[0,1]$. Note that for every point x of $[0,1]$, except for finitely many points corresponding to infinite branches of T_+, the homeomorphism γ has left and right derivatives at x that are powers of 2, and all break points of the derivative of γ, except for finitely many points, are from $\mathbb{Z}[\frac{1}{2}]$. As for the R. Thompson group F, this gives an embedding of $\mathcal{F}$ into Homeo$([0,1])$ that extends the natural embedding of F. We shall identify $\mathcal{F}$ with its image in Homeo$([0,1])$.

Remark 7.5. Let $g = (T_+, T_-, \phi)$ be a triple in $\mathcal{F}$, and let $\omega \to \omega'$ be a pair of infinite branches of g (i.e., ω is a branch of T_+, ω' is a branch of T_-, and $\phi(\omega) = \omega'$). Then ω ends with an infinite tail of zeros (respectively, ones) if and only if ω' ends with an infinite tail of zeros (resp. ones). Indeed, the infinite branch ω has a predecessor in the set of branches of T_+ if and only if it terminates with a tail of zeros. It has a successor in the set of branches of T_+ if and only if it terminates with a tail of ones. Similarly, for the branch ω' of T_-. The result follows since ϕ is order preserving.

Theorem 7.6. *The image of $\mathcal{F}$ in* Homeo$([0,1])$ *is inside $\overline{F}$, viewed as a subgroup of* Homeo$([0,1])$.

Proof. Let $g = (T_+, T_-, \phi)$ be a triple in $\mathcal{F}$. It suffices to show that there is a sequence $(g_m), m \geq 1$, of elements in F that converges to g in the Hamming metric on Homeo$([0,1])$. Let $\omega_i \to \omega_i'$, $i = 1, \ldots, n$, be the pairs of infinite branches of (T_+, T_-, ϕ). For every $m \in \mathbb{N}$ and every $i = 1, \ldots, n$, let $u_{i,m}$ (respectively, $v_{i,m}$) be the prefix of ω_i (respectively, ω_i') of length m. For each i, the sequence $[u_{i,m}]$, $m \in \mathbb{N}$ (respectively, $[v_{i,m}]$, $m \in \mathbb{N}$) is a nested sequence of intervals with intersection $\{.\omega_i\}$ (respectively, $\{.\omega_i'\}$). If ω_i has an infinite tail of zeros (respectively, ones), then for every sufficiently large m, $[u_{i,m}]$ is a right (respectively, left) neighborhood of $.\omega_i$. By Remark 7.5, in that case, for all sufficiently large m, $[v_{i,m}]$ is a left (respectively, right) neighborhood of $.\omega_i'$. If ω_i is not eventually constant, then $.\omega_i$ belongs to the interior of $[u_{i,m}]$ for all $m \in \mathbb{N}$. In that case, by Remark 7.5, for each $m \in \mathbb{N}$, $.\omega_i'$ belongs to the interior of $[v_{i,m}]$.

Let $m_0 \in \mathbb{N}$ be such that for each $i = 1, \ldots, n$, if ω_i is eventually constant, then $\omega_i \equiv u_{i,m_0} a_i^{\mathbb{N}}$ and $\omega_i' \equiv v_{i,m_0} a_i^{\mathbb{N}}$, where $a_i \in \{0,1\}$. Then for each $m \geq m_0$, $[u_{i,m}]$ is a left (respectively, right, two-sided) neighborhood of $.\omega_i$ if and only if $[v_{i,m}]$ is a left (respectively, right, two-sided) neighborhood of $.\omega_i'$.

Let $m \geq m_0$. Since g is continuous and $g(.\omega_i) = .\omega_i'$, $i = 1, \ldots, n$, there exists some $m' \geq m$ such that for all $i = 1, \ldots, n$, $g([u_{i,m'}]) \subseteq [v_{i,m}]$. Consider the subtree $S_{m'}$ of T_+ of all vertices x such that none of the $u_{i,m'}$'s is a strict prefix of the path from the root to x. Then $S_{m'}$ is a rooted binary subtree of T_+. Clearly, $S_{m'}$ does not have any infinite branch. Since $S_{m'}$ is a binary tree, that implies that it has only finitely many branches. Clearly, n of these branches are $u_{1,m'}, \ldots, u_{n,m'}$.

The rest of the branches of $S_{m'}$ are some branches of T_+. For any branch u of $S_{m'}$ that is not one of the branches $u_{1,m'}, \ldots, u_{n,m'}$, there is a branch v of T_- such that $v = \phi(u)$. There exists a function $g_m \in F$ whose tree-diagram contains all these branches $u \to \phi(u)$. In particular, g_m coincides with g on $[0,1] \setminus [u_{1,m'}] \cup \cdots \cup [u_{n,m'}] \supseteq [0,1] \setminus [u_{1,m}] \cup \cdots \cup [u_{n,m}]$. The choice of m' implies that g_m^{-1} coincides with g^{-1} on $[0,1] \setminus [v_{1,m}] \cup \cdots \cup [v_{n,m}]$. It follows that the sequence $(g_m), m \geq m_0$, converges to g in the Hamming metric on Homeo$([0,1])$. □

Theorem 7.7. *If U and V are two finite sets of numbers in $(0,1)$ and $\tau(U) \equiv \tau(V)$, then the subgroups H_U and H_V are conjugate in $\mathcal{F}$.*

To prove Theorem 7.7, we need the following two lemmas.

Lemma 7.8. *Let $\alpha = .pu^{\mathbb{N}}$ and $\beta = .pv^{\mathbb{N}}$ be two rational numbers in $(0,1)$ that are not in $\mathbb{Z}[\frac{1}{2}]$. Then the stabilizers $H_{\{\alpha\}}$ and $H_{\{\beta\}}$ are conjugate in $\mathcal{F}$. Moreover, the conjugator f can be chosen to have support in $[p]$.*

Proof. We can assume that u and v are minimal periods of α and β.

Let (R_+, R_-) be a reduced tree-diagram with pairs of branches $u_i \to v_i$, $i \in \mathbb{N}$, such that for some $k \in \{2, \ldots, n-1\}$, $u_k = u$ and $v_k = v$. We let $f_1 = (T_+^1, T_-^1)$ be the copy of (R_+, R_-) in $F_{[p]}$. The pairs of branches of (T_+^1, T_-^1) are of the form $pu_i \to pv_i$ for $i = 1, \ldots, n$, along with pairs of branches $b \to b$ for some words b such that p is not a prefix of b.

We construct tree-diagrams (T_+^j, T_-^j) by induction on $j \in \mathbb{N}$, so that for each j, the tree-diagram (T_+^j, T_-^j) has a pair of branches $pu^j \to pv^j$. For $j = 1$, we are done. If (T_+^j, T_-^j) is already constructed, we attach a copy of the tree R_+ to the end of the branch pu^j of T_+^j and a copy of the tree R_- to the end of the branch pv^j of the tree T_-^j. We let (T_+^{j+1}, T_-^{j+1}) be the resulting tree-diagram.

Let $f_j = (T_+^j, T_-^j)$. We let f be the "limit" of the tree-diagrams f_j for $j \in \mathbb{N}$, in the following sense. The triple f can be defined by listing its pairs of branches $p \to q$, where p and q are either both finite or both infinite, and $\phi(p) = q$.

The pairs of branches of (T_+, T_-, ϕ) are $pu^j u_i \to pv^j v_i$ for $j \geq 0$, $i \in \{1, \ldots, n\} \setminus \{k\}$; pairs of branches $b \to b$ for some words b such that p is not a prefix of b (namely, the pairs of branches of (T_+^1, T_-^1) of this form) and the pair of infinite branches $pu^{\mathbb{N}} \to pv^{\mathbb{N}}$. It is obvious that $f \in \mathcal{F}$ and that, as an element of Homeo($[0,1]$), f has support in $[p]$.

We claim that $H_{\{\alpha\}}{}^f = H_{\{\beta\}}$. It suffices to show the inclusion $\subseteq$ (to prove the other inclusion one would just replace f by f^{-1}). Let $g \in H_{\{\alpha\}}$. We view f as a function from Homeo($[0,1]$). By definition, $f(\alpha) = \beta$. It is obvious that g^f fixes β because g fixes α and $f(\alpha) = \beta$.

It remains to show that $g^f \in F$, that is, as a pair of trees, g^f has only finitely many pairs of branches. To prove that, it suffices to find a partition of $[0,1]$ into a finite number of intervals $I_1, \ldots, I_s$ with finite dyadic endpoints and elements $g_1, \ldots, g_s$ from F such that for every $k = 1, \ldots, s$, f coincides with g_k on I_k.

Since $g(\alpha) = \alpha$, there are natural numbers $m_1, m_2 \geq 1$ such that g has a pair of branches $pu^{m_1} \to pu^{m_2}$ (see Lemma 2.6). It is easily seen that by the definition of f, on $[0,1] \setminus [pv^{m_1}]$ (which is a union of two intervals), the function g^f coincides with $f_{m_1}^{-1} g f_{m_2}$. Consider a number x from $[pv^{m_1}]$. Then $x = .pv^{m_1}\omega$ for some infinite binary word ω. It is easy to check that g^f maps x to $y = .pv^{m_2}\omega$. Thus, if g_1 is any element of F that has the pair of branches $pv^{m_1} \to pv^{m_2}$, then g^f coincides with g_1 on $[pv^{m_1}]$. Therefore, $g^f \in F$. □

Lemma 7.9. *Let α and β be two irrational numbers in $(0,1)$. Then $H_{\{\alpha\}}$ and $H_{\{\beta\}}$ are conjugate in $\mathcal{F}$. If α and β belong to a dyadic interval $[p]$, then the conjugator of $H_{\{\alpha\}}$ and $H_{\{\beta\}}$ in $\mathcal{F}$ can be taken to have support in $[p]$.*

Proof. The proof is similar to the proof of Lemma 3.1. Let $\alpha = .\omega$ and $\beta = .\omega'$. For $i \in \mathbb{N}$, we define prefixes u_i of ω inductively as follows. For $i = 1$, we let u_1 be an arbitrary finite prefix of ω. If the prefix u_i is defined, we let u_{i+1} be the minimal prefix of ω such that u_i is a prefix of u_{i+1} and $u_{i+1} \equiv u_i s$, where the suffix s contains both digits 0 and 1. We define prefixes v_i of ω' in a similar way. We note that $[u_i]$ (respectively, $[v_i]$) is a nested sequence of dyadic intervals whose intersection is $\{\alpha\}$ (respectively, $\{\beta\}$). We note also that for each i, $[u_{i+1}]$ (respectively, $[v_{i+1}]$) is contained in the interior of $[u_i]$ (respectively, $[v_i]$).

We construct a sequence of elements $f_i = (T^i_+, T^i_-)$ in F such that for any i, (T^i_+, T^i_-) has the pair of branches $u_i \to v_i$. For $i = 1$ we let (T^1_+, T^1_-) be a tree-diagram with a pair of branches $u_1 \to v_1$. For any $i > 1$, we let (T^i_+, T^i_-) be a tree-diagram which has a pair of branches $u_i \to v_i$ and has all the pairs of branches of (T^{i-1}_+, T^{i-1}_-), other than the pair of branches $u_{i-1} \to v_{i-1}$. This is possible by Remark 2.3.

We let the triple $f = (T_+, T_-, \phi)$ be defined (as a collection of pairs of branches) as follows. The only pair of infinite branches in f is $\omega \to \omega'$. The pairs of finite branches are all the pairs of branches $u \to v$ of (T^i_+, T^i_-) other than the pair $u_i \to v_i$, for each $i \in \mathbb{N}$. Note that every finite pair of branches of f occurs in all but finitely many tree-diagrams (T^i_+, T^i_-). Clearly, for $i \in \mathbb{N}$, f coincides with f_i on $[0,1] \setminus [u_i]$.

We claim that $H^f_{\{\alpha\}} = H_{\{\beta\}}$. It suffices to show the inclusion $\subseteq$. Let $g \in H_{\{\alpha\}}$. By definition, $f(\alpha) = \beta$. Thus, g^f fixes β since g fixes α. To show that $g^f \in F$, we note that by Corollary 2.5, since g fixes α, it fixes a neighborhood $[u_i]$ of α for a sufficiently large $i \in \mathbb{N}$. Since f and f_i coincide on $[0,1] \setminus [u_i]$, in particular they coincide on the support of g. It follows that $g^f = g^{f_i} \in F$.

If α and β belong to $[p]$, then we let the prefix u_1 of ω and the prefix v_1 of ω' be $u_1 \equiv v_1 \equiv p$. Then one can take the function f_1 in the sequence above to be the identity (with a pair of branches $p \to p$). Continuing the construction of the sequence $\{f_i\}$ as above, we see that the limit function f has support in $[p]$. □

Proof of Theorem 7.7. Let $U = U_1 \cup U_2 \cup U_3$ be the natural partition of U. For each $\beta \in U_2 \cup U_3$ we choose a small dyadic interval $[p_\beta]$ such that $\beta \in [p_\beta]$ (i.e., $\beta = .p_\beta\omega$ for some infinite binary word ω). We can assume that the intervals $[p_\beta]$, $\beta \in U_2 \cup U_3$ are pairwise disjoint and that each of these intervals contains exactly one element of $U \cup \{0,1\}$, the number β.

Since $\tau(U) \equiv \tau(V)$, using conjugation in F if necessary (as in the proof of Theorem 4.1), we can assume that the set $V = V_1 \cup V_2 \cup V_3$ satisfies the following conditions.

(1) $V_1 = U_1$.
(2) For each $\delta \in V_2 \cup V_3$, $\delta \in [p_\beta]$, where β occupies the same position in the ordered set U as δ does in V.

By Lemmas 7.8 and 7.9, for each $\beta \in U_2 \cup U_3$ there is a homeomorphism $f_\beta \in \mathcal{F}$ with support in $[p_\beta]$ such that ${H_{\{\beta\}}}^{f_\beta} = H_{\{\delta\}}$. It is easy to check that $f = \prod_{\beta \in U_2 \cup U_3} f_\beta \in \mathcal{F}$ conjuagtes H_U to H_V. □

7.3. $\mathcal{F}$ and $\overline{F}$ are not amenable.

Theorem 7.10. *The group $\mathcal{F}$ contains a non-Abelian free subgroup.*

Proof. Let O be the set of all finite tuples of elements of F. Since O is countable, there is a bijection $\phi\colon \mathbb{N} \to O$. One can list the elements of F using the function ϕ, by listing the elements of the tuple $\phi(1)$, followed by the elements of the tuple $\phi(2)$ and so on. Clearly, every element of F is listed infinitely many times. Also, if we associate a function $\psi\colon \mathbb{N} \to F$ with this listing, then for any tuple t in O of length k there is some $n \in \mathbb{N}$ such that $t = (\psi(n), \dots, \psi(n+k-1))$. For each $i \in \mathbb{N}$, we let (R^i_+, R^i_-) be the reduced tree-diagram of $\psi(i)$.

Let T be the minimal infinite binary tree with $0^{\mathbb{N}}$ as a branch (i.e., the branches of T are $0^{\mathbb{N}}$, 0^k1 for all $k \geq 0$). We construct an element of $\mathcal{F}$ as follows. Let T_+, T_- be two copies of T. For each $k > 0$ we attach to the tree T_+ the tree R^k_+ at the end of the branch 0^k1. Similarly, to the end of the branch 0^k1 of T_- we attach the tree R^k_-. We denote the resulting tree-diagram with the natural mapping ϕ by (T_+, T_-, ϕ) and let g be the function in $\mathcal{F}$ represented by it. We note that for all $k \geq 0$, if $\alpha \in [0^k1]$, then $g(\alpha) \in [0^k 1]$. In addition, for any $k \in \mathbb{N}$ and any number $\alpha = .0^k 1\omega$, we have $g(\alpha) = g(.0^k 1\omega) = .0^k 1\omega'$, where $(\psi(k))(.\omega) = .\omega'$. In other words, when restricted to the interval $[0^k 1]$, g coincides with the copy of $\psi(k)$ in $F_{[0^k 1]}$.

Let f be an element of the Thompson group F with a pair of branches $00 \to 0$. We claim that the group generated by g and f is free. Assume by contradiction that $w(x,y)$ is a word in $\{x, y, x^{-1}, y^{-1}\}$ such that $w(x,y)$ is not trivial in the free group over $\{x,y\}$ and such that $w(g,f) = 1$. Since for every $k > 1$, g maps the interval $[0^k 1]$ onto itself and f maps $[0^k 1]$ onto $[0^{k-1} 1]$, it is easy to check that the sum of powers of y in the word $w(x,y)$ must be 0. Thus, $w(x,y)$ is equivalent in the free group to a word in conjugates of x by powers of y. Clearly, we can assume that all conjuagtors are positive powers of y.

We note that for all $k, j \in \mathbb{N}$, the function g^{f^j} restricted to $[0^k 1]$ coincides with the copy of $\psi(k+j)$ in $F_{[0^k 1]}$. Indeed, for any $\alpha = .0^k 1\omega$,

$$\begin{aligned} g^{f^j}(\alpha) &= f^j(g(f^{-j}(\alpha))) = f^j(g(f^{-j}(.0^k 1\omega))) \\ &= f^j(g(.0^{k+j} 1\omega)) = f^j(.0^{k+j} 1\omega') = .0^k 1\omega', \end{aligned}$$

where ω' here is such that $(\psi(k+j))(.\omega) = .\omega'$.

One can think of the situation as follows. If one fixes $k \in \mathbb{N}$ and considers only the dyadic interval $[0^k 1]$, then for every $j > 0$, the function g^{f^j} restricted to the chosen interval behaves exactly like (the relevant copy of) the function $\psi(k+j)$ of F. As noted above, the word $w(f,g)$ can be viewed as a word w' in conjugates of g by positive powers of f. If ℓ different conjugates $g^{f^{r_1}}, \ldots, g^{f^{r_\ell}}$ participate in the word, we can replace each of them by a letter in an alphabet $\{z_1, \ldots, z_\ell\}$ and consider the resulting word $w'(z_1, \ldots, z_\ell)$. Since F does not satisfy any law (see [23, Theorem 5.6.37]), there is a sequence of elements $f_1, \ldots, f_\ell$ in F such that $w'(f_1, \ldots, f_\ell) \neq 1$. It remains to notice that since the function ψ enumerates all tuples of elements of F, one after the other, it is possible to choose k such that $\psi(k + r_i) = f_i$ for all $i = 1, \ldots, \ell$. Then the word w' in the conjugates $g^{f^{r_1}}, \ldots, g^{f^{r_\ell}}$ behaves exactly as the (copy of the) function $w'(f_1, \ldots, f_\ell)$ in the interval $[0^k 1]$. As noted, it is not the identity. □

§8. Open problems

8.1. The number of isomorphism classes of maximal subgroups of F. Note that in every noncyclic countable free group every maximal subgroup of infinite index is free of countable rank. Thus, even though the set of maximal subgroups of infinite index of a noncyclic free group is of cardinality continuum, there is only one isomorphism class of these subgroups. From the results of [24] it follows that the set of maximal subgroups of infinite index of F also has cardinality continuum (it contains all subgroups $H_{\{\alpha\}}$, $\alpha \in (0,1)$). Still, the results of this paper and [18] show that, up to isomorphism, only 4 maximal subgroups of F of infinite index are known. The representatives of isomorphism classes are $H_{\{\frac{1}{2}\}}$, $H_{\{\frac{1}{3}\}}$, $H_{\{\frac{\sqrt{2}}{2}\}}$ and a subgroup that is isomorphic to the Thompson group F_3 (which consists of piecewise linear functions $[0,1] \to [0,1]$ where slopes are powers of 3 and break points are 3-adic). In [18] we also showed how to implicitly construct many other maximal subgroups of F, but we do not know whether these subgroups are isomorphic to each other or to some of the 4 maximal subgroups listed above. It is quite possible that one can construct copies of F_n as maximal subgroups of infinite index in F for every $n > 2$. Thus up to isomorphism, there should be at least countably many maximal subgroups of F of infinite index.

Problem 8.1. Is the set of isomorphism classes of maximal subgroups of F countable?

Note that a similar problem is interesting for many other groups (say, $\mathrm{SL}_n(\mathbb{Z})$, whose maximal subgroups have been extensively studied by Margulis and Soifer). Note also that for some finitely generated groups, say, direct products of two noncyclic free groups, the set of isomorphism classes of maximal subgroups is uncountable [12].

8.2. Distortion and closed subgroups. Closed subgroups of F were defined in [18]. The first author in [16] showed that one can alternatively define closed subgroups as follows.

Definition 8.2. Let $h \in F$. Then the *components* of h are all elements of F that coincide with h on a closed interval $[a, b]$ with finite dyadic a, b and are identity outside $[a, b]$ (in that case h necessarily fixes a and b). A subgroup $H \leq F$ is *closed* if for any $h \in H$, the subgroup H contains all components of h.

It is clear from this definition that for every (not necessarily finite) set $U \subseteq (0, 1)$ the stabilizer H_U of U is closed. It is quite possible that Theorem 6.1 can be generalized to arbitrary closed subgroups of F.

Problem 8.3. Is it true that every finitely generated closed subgroup of F is undistorted?

Note that it is an open problem (see [18]) whether all maximal subgroups of infinite index in F are closed. If the answer is "yes" and the answer to Problem 8.3 is affirmative, then all finitely generated maximal subgroups of F would be undistorted (because the subgroups of finite index in any finitely generated group are obviously undistored).

Note also that by [18] the distortion function of every finitely generated closed subgroup of F is recursive, because the membership problem for every such subgroup is decidable.

8.3. Subgroups of quasifinite index. The examples of Jones' subgroup $\vec{F}$ and Savchuk's subgroups H_U show that subgroups of F are of quasifinite index surprisingly often. Using the 2-core of subgroups of F as defined in [18], one can construct more examples of subgroups of F of infinite index that have quasifinite index in F.

Problem 8.4. Is there an algorithm to check if a finite set of elements of F generates a maximal subgroup or a subgroup of quasifinite index?

Note that the first author has found an algorithm checking whether a finite set of elements of F generates the whole F [16].

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