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ON THE TOTAL CURVATURE OF MINIMIZING GEODESICS ON CONVEX SURFACES

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Dedicated to Yu. D. Burago
on the occasion of his 80th birthday

Abstract. A universal upper bound is given for the total curvature of a minimizing geodesic on a convex surface in the Euclidean space.

§1. Introduction

In this paper we give an affirmative answer to the question asked by Dmitry Burago; the same question was also stated in [1, 2] and [3]. Namely, we prove the following.

1.1. Main theorem. *The total curvature of a minimizing geodesic on a convex surface in* $\mathbb{R}^3$ *cannot exceed* 1000^{1000}.

The value $2 \cdot \pi$ is the optimal bound for the analogous problem in the plane. The total curvature of a minimizing geodesic on a convex surface in $\mathbb{R}^3$ can exceed $2 \cdot \pi$ and the optimal bound is expected to be slightly bigger than $2 \cdot \pi$. The former example was constructed by Bárány, Kuperberg, and Zamfirescu in [3].

Let us list other related results.

- In [4], Liberman gave a bound on the total curvature of a short geodesic in terms of the ratio diameter and inradius of K. In the proof he used an analog of Lemma 3.1 discussed below.
- In [5], Usov gave an optimal bound for the total curvature of geodesics on the graphs of ℓ-Lipschitz convex function. Namely, he proved that if $f\colon \mathbb{R}^2 \to \mathbb{R}$ is ℓ-Lipschitz and convex, then any geodesic in its graph

$$\Gamma_f = \big\{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\big\}$$

  has total curvature of at most $2 \cdot \ell$. An amusing generalization of Usov's result was given by Berg in [6].
- In [7], Pogorelov conjectured that the spherical image of a geodesic on a convex surface must be (locally) rectifiable. It is easy to check that the length of the spherical image of a geodesic cannot be smaller than its total curvature, so this conjecture (if true) would be stronger than Liberman's theorem. Counterexamples to various forms of this conjecture were found by Zalgaller in [8], Milka in [9], and Usov in [10]; these results were partly rediscovered later by Pach in [2].

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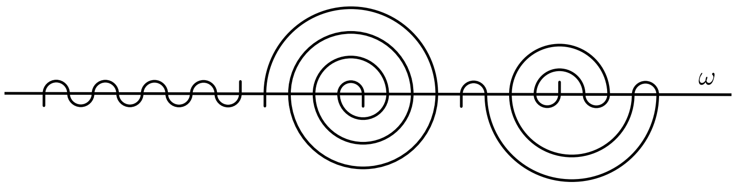
- In [3], Bárány, Kuperberg, and Zamfirescu constructed a corkscrew minimizing geodesic on a closed convex surface; that is, a minimizing geodesic that twists around a given line arbitrarily many times. They also rediscovered the results of Liberman and Usov mentioned above.

Idea of the proof. First we show that it suffices to estimate total curvature for the minimizing geodesics with almost constant velocity vector, say $\dot\gamma(t) \approx \boldsymbol{i}$.

To estimate total curvature in this case it suffices to estimate the integral

$$\int \langle \ddot\gamma(t), \boldsymbol{j}\rangle \cdot dt$$

for a vector $\boldsymbol{j} \perp \boldsymbol{i}$. To understand the idea of this estimate, imagine that the surface is lighten in the direction of $\boldsymbol{j}$, so that it is divided into the dark and light sides by a curve ω. On the diagram you see different combinatorics in which γ meets ω.



In the first case the total curvature is estimated by the integral of the Gauss curvature of the regions squeezed between γ and ω. This follows from the Tongue Lemma 4.2, which is a heart of our proof.

The second case might look impossible, but the corkscrew geodesic constructed in [3] can meet ω in this order. Here we show that the total curvature of the twists grows geometrically from a middle twist to the ends and at the ends the integral of the full twists cannot be larger than $2 \cdot \pi$. This suffices to estimate the total curvature of the whole geodesic.

The last case is a mixture of the first two and it is done by mixing both techniques.

§2. Preliminaries

Total curvature. Recall that the *total curvature* of a curve $\gamma\colon [0,\ell] \to \mathbb{R}^3$ (briefly $\operatorname{TotCurv}\gamma$) is defined as the supremum of the sums of exterior angles for the polygonal lines inscribed in γ.

Note that for a polygonal line σ, its total curvature coincides with the sum of its exterior angles.

If γ is a smooth curve with unit-speed parametrization, then

$$\operatorname{TotCurv}\gamma = \int_0^\ell \kappa(t)\cdot dt,$$

where $\kappa(t) = |\ddot\gamma(t)|$ is the curvature of γ at t.

2.1. Proposition. *Assume $\gamma_n\colon \mathbb{I} \to \mathbb{R}^3$ is a sequence of curves converging pointwise to a curve $\gamma_\infty\colon \mathbb{I} \to \mathbb{R}^3$. Then*

$$\liminf_{n\to\infty} \operatorname{TotCurv}\gamma_n \geq \operatorname{TotCurv}\gamma_\infty.$$

Proof. Fix a polygonal line σ_∞ inscribed in γ_∞. Let $\gamma_\infty(t_0), \dots, \gamma_\infty(t_k)$ be its vertices as they appear on γ_∞. Consider the polygonal line σ_n inscribed in γ_n with the vertices $\gamma_n(t_0), \dots, \gamma_n(t_k)$. Note that

$$\operatorname{TotCurv}\sigma_n \to \operatorname{TotCurv}\sigma_\infty \quad\text{as}\quad n \to \infty.$$

By the definition of the total curvature,

$$\operatorname{TotCurv} \sigma_n \le \operatorname{TotCurv} \gamma_n.$$

The statement follows because the broken line σ_∞ can be chosen in such a way that $\operatorname{TotCurv} \sigma_\infty$ is arbitrarily close to $\operatorname{TotCurv} \gamma_\infty$. $\square$

Convergence of sets. Given a closed set $\Sigma \subset \mathbb{R}^3$, denote by $\operatorname{dist}_\Sigma$ the distance function from Σ, i.e.,

$$\operatorname{dist}_\Sigma(x) = \inf\{|x-y| \mid y \in \Sigma\}.$$

We say that a sequence of closed sets $\Sigma_n \subset \mathbb{R}^3$ converges to a closed set $\Sigma_\infty \subset \mathbb{R}^3$ if for any $x \in \mathbb{R}^3$ we have $\operatorname{dist}_{\Sigma_n}(x) \to \operatorname{dist}_{\Sigma_\infty}(x)$ as $n \to \infty$.

Convex surfaces. By a *convex surface* Σ in the Euclidean 3-space $\mathbb{R}^3$ we understand the boundary of a closed convex set K with nonempty interior. If K is compact, we say that the Σ is *closed.*

Assume Σ is smooth. If at every point of Σ the principle curvatures are positive, we say that Σ is *strongly convex.*

2.2. Proposition. *Assume Σ_n is a sequence of convex surfaces that converge to a convex surface Σ_∞. Then for any minimizing geodesic γ_∞ in Σ_∞ there is a sequence of minimizing geodesics γ_n in Σ_n that converge pointwise to γ_∞ as $n \to \infty$.*

Proof. Assume that $\gamma_\infty\colon [0,\ell] \to \Sigma_\infty$ is parametrized by its arc-length.

Fix a subinterval $[a,b] \subset (0,\ell)$. Set $p_\infty = \gamma_\infty(a)$ and $q_\infty = \gamma_\infty(b)$. Let $p_n, q_n \in \Sigma_n$ be two sequences of points that converge to p_∞ and q_∞, respectively.

Denote by γ_n a minimizing geodesic from p_n to q_n in Σ_n. Note that γ_n converges to $\gamma_\infty|_{[a,b]}$ as $n \to \infty$.

Taking the subinterval $[a,b]$ closer and closer to $[0,\ell]$ and applying diagonal procedure, we get the result. $\square$

§3. Liberman's lemma

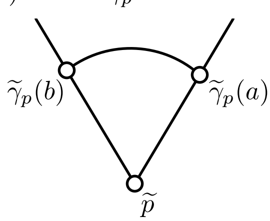
In this section we give a slight generalization of the construction given by Liberman in [4]; see also [11].

Development. Let $\gamma\colon [0,\ell] \to \mathbb{R}^3$ be a curve parametrized by arc-length, and suppose a point p does not lie on γ.

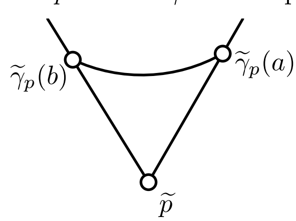
Assume that $\widetilde{\gamma}_p\colon [0,\ell] \to \mathbb{R}^2$ is a plane curve parametrized by arc-length and $\widetilde{p}$ is a point in the plane such that

$$|\widetilde{p} - \widetilde{\gamma}(t)| = |p - \gamma(t)|$$

for any $t \in [0,\ell]$; moreover, the direction from $\widetilde{p}$ to $\widetilde{\gamma}(t)$ changes monotonically (clockwise or counterclockwise). Then $\widetilde{\gamma}_p$ is called the *development* of γ with respect to p.



Convex development.



Concave development.

We say that the development $\widetilde{\gamma}_p$ is *convex* (*concave*) in the interval $[a,b]$ if the arc $\widetilde{\gamma}_p|_{[a,b]}$ cuts a convex bounded (respectively, unbounded) domain from the solid angle $\angle\widetilde{p}^{\widetilde{\gamma}_p(a)}_{\widetilde{\gamma}_p(b)}$.

We say that $\widetilde{\gamma}_p$ is locally convex (concave) in the interval $[a,b]$ if any point $x\in[a,b]$ admits a closed neighborhood $[a',b']$ in $[a,b]$ such that $\widetilde{\gamma}_p$ is convex (respectively, concave) in the interval $[a',b']$.

If we pass to the limit of this construction as p moves to infinity along a half-line in the direction of a unit vector $\boldsymbol{u}$, then the limit curve is called the development of γ in the direction $\boldsymbol{u}$ and is denoted by $\widetilde{\gamma}_{\boldsymbol{u}}$.

We can define the development $\widetilde{\gamma}_{\boldsymbol{u}}$ directly: (1) the development $\widetilde{\gamma}_{\boldsymbol{u}}\colon[0,\ell]\to\mathbb{R}^2$ is parametrized by arc-length, (2) for a fixed unit vector $\widetilde{\boldsymbol{u}}\in\mathbb{R}^2$, we have

$$\langle\widetilde{\boldsymbol{u}},\widetilde{\gamma}_{\boldsymbol{u}}(t)\rangle=\langle\boldsymbol{u},\gamma(t)\rangle$$

for any $t\in[0,\ell]$, and (3) the projection of $\widetilde{\gamma}_{\boldsymbol{u}}(t)$ to the line normal to $\widetilde{\boldsymbol{u}}$ is monotone in t.

We may assume that $\widetilde{\boldsymbol{u}}$ is the vertical vector in the coordinate plane. In this case we say that $\widetilde{\gamma}_{\boldsymbol{u}}$ is concave (convex) in the interval $[a,b]$ if the lune bounded by arc $\widetilde{\gamma}_{\boldsymbol{u}}|_{[a,b]}$ and the segment $[\widetilde{\gamma}_{\boldsymbol{u}}(a)\widetilde{\gamma}_{\boldsymbol{u}}(b)]$ is convex and lies above (respectively, below) the line segment $[\widetilde{\gamma}_{\boldsymbol{u}}(a)\widetilde{\gamma}_{\boldsymbol{u}}(b)]$.

Dark and light sides. Suppose $\Sigma\subset\mathbb{R}^3$ is a convex surface, $p\in\Sigma$, and $z\neq p$.

We say that p lies on the dark (light) side of Σ from z if none of the points $p+t\cdot(p-z)$ lie inside of Σ for $t>0$ (respectively, for $t<0$). The intersection of the dark and the light side is called the horizon of z; it is denoted by ω_z.

Note that if z lies inside Σ, then all the points of Σ lie on the dark side from z and its horizon ω_z is empty.

If Σ is smooth, we can define the outer normal vector ν_p to Σ at p. In this case p lies on the dark (light) side of Σ from z if and only if $\langle p-z,\nu_p\rangle\geq 0$ (respectively, $\langle p-z,\nu_p\rangle\leq 0$). If in addition Σ is closed and strongly convex, then the horizon is empty for z inside Σ, and it is formed by a closed smooth curve for z outside Σ.

We could also define the light/dark sides and the horizon in the limit case, as p escapes to infinity along a half-line in direction $\boldsymbol{u}$.

The last notions can be defined directly. We say that a point $p\in\Sigma$ lies on the dark (light) side for the unit vector $\boldsymbol{u}$ if none of the points $p+\boldsymbol{u}\cdot t$ lie inside Σ for $t>0$, (respectively, $t<0$). As before, the intersection of the light and the dark side is called the horizon of $\boldsymbol{u}$ and it is denoted by $\omega_{\boldsymbol{u}}$.

In the smooth case, this definition means that $\langle\nu_p,\boldsymbol{u}\rangle\geq 0$ (respectively, $\langle\nu_p,\boldsymbol{u}\rangle\leq 0$). If in addition Σ is closed and strongly convex, then $\omega_{\boldsymbol{u}}$ is a closed smooth curve.

3.1. Liberman's lemma. *Assume γ is a geodesic on a convex surface $\Sigma\subset\mathbb{R}^3$. Then for any point $z\notin\Sigma$ the development $\widetilde{\gamma}_z$ is locally convex (locally concave) if γ lies on the dark (respectively, light) side of Σ from z.*

Similarly, for any unit vector $\boldsymbol{u}$, the development $\widetilde{\gamma}_{\boldsymbol{u}}$ is locally convex (locally concave) if γ lies on the dark (respectively, light) side of Σ for $\boldsymbol{u}$.

Note that for any space curve γ and any unit vector $\boldsymbol{u}$ we have

$$\operatorname{TotCurv}\widetilde{\gamma}_{\boldsymbol{u}}\leq\operatorname{TotCurv}\gamma.$$

On the other hand, the total curvature of few developments gives an estimate for the total curvature of the original curve. For example, if $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ is the orthonormal basis, then

$$\operatorname{TotCurv}\gamma\leq\operatorname{TotCurv}\widetilde{\gamma}_{\boldsymbol{i}}+\operatorname{TotCurv}\widetilde{\gamma}_{\boldsymbol{j}}+\operatorname{TotCurv}\widetilde{\gamma}_{\boldsymbol{k}}.$$

If γ lies completely on the dark (or light) side for the direction $\boldsymbol{u}$, then by Liberman's lemma we get

$$\operatorname{TotCurv}\widetilde{\gamma}_{\boldsymbol{u}} \le \pi.$$

It follows that if γ crosses the horizons $\omega_{\boldsymbol{i}}$, $\omega_{\boldsymbol{j}}$, and $\omega_{\boldsymbol{k}}$ at most N times, then

$$\operatorname{TotCurv}\gamma \le \operatorname{TotCurv}\widetilde{\gamma}_{\boldsymbol{i}} + \operatorname{TotCurv}\widetilde{\gamma}_{\boldsymbol{j}} + \operatorname{TotCurv}\widetilde{\gamma}_{\boldsymbol{k}} \le (N+1)\cdot\pi.$$

Therefore, to violate Main Theorem, γ must cross the horizons $\omega_{\boldsymbol{i}}$, $\omega_{\boldsymbol{j}}$, and $\omega_{\boldsymbol{k}}$ a huge number of times.

§4. Curvature of development

Let $\Sigma \subset \mathbb{R}^3$ be a closed smooth strongly convex surface and $\gamma\colon [0,\ell] \to \Sigma$ a unit-speed geodesic. Assume that for some unit vector $\boldsymbol{u}$, the geodesic γ crosses the horizon $\omega_{\boldsymbol{u}}$ transversely at $t_0 < \dots < t_k$. Set $\alpha_i = \measuredangle(\dot\gamma(t_i), \boldsymbol{u}) - \frac{\pi}{2}$ for each i. Note that $|\alpha_i| \le \frac{\pi}{2}$.

The values t_i and α_i will be called, respectively, the *meeting moments* and the *meeting angles* of the geodesic γ with the horizon $\omega_{\boldsymbol{u}}$.

We introduce the new notation

$$\operatorname{TotCurv}_{\boldsymbol{u}}\gamma \stackrel{\text{def}}{=} \operatorname{TotCurv}\widetilde{\gamma}_{\boldsymbol{u}}.$$

From Liberman's Lemma 3.1, we get the following.

4.1. Corollary. *Let $\Sigma \subset \mathbb{R}^3$ be a strongly convex smooth surface, $\gamma\colon [0,\ell] \to \Sigma$ a unit-speed geodesic, and $\boldsymbol{u}$ is a unit vector. Assume that γ crosses the horizon $\omega_{\boldsymbol{u}}$ transversely and $t_0 < \dots < t_k$ are its meeting moments and $\alpha_0, \dots, \alpha_k$ its meeting angles with the horizon $\omega_{\boldsymbol{u}}$. Then*

$$\operatorname{TotCurv}_{\boldsymbol{u}}\gamma \le 3\cdot\pi + 2\cdot\left|\alpha_0 - \alpha_1 + \dots + (-1)^k\cdot\alpha_k\right|.$$

As you will see in what follows, in order to find the required estimate for the total curvature of a geodesic, we will get an upper bound for the sum

$$\left|\alpha_0 - \alpha_1 + \dots + (-1)^k\cdot\alpha_k\right|.$$

Finding such an upper bound is the most important ingredient in the proof of the Main Theorem.

Proof. By Liberman's lemma,

$$\operatorname{TotCurv}_{\boldsymbol{u}}(\gamma|_{[t_{i-1},t_i]}) = \pm(\alpha_{i-1} - \alpha_i),$$

where the sign is "+" if $\gamma_{[t_i,t_{i+1}]}$ lies on the dark side and "−" if it lies on the light side from $\boldsymbol{u}$. Summing all this up, we get

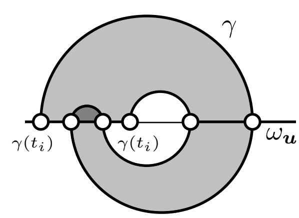
$$\operatorname{TotCurv}_{\boldsymbol{u}}\left(\gamma|_{[t_0,t_k]}\right) = \left|\alpha_0 - 2\cdot\alpha_1 + \dots + (-1)^{k-1}\cdot 2\cdot\alpha_{k-1} + (-1)^k\cdot\alpha_k\right|.$$

By Liberman's lemma, we also have

$$\operatorname{TotCurv}_{\boldsymbol{u}}\left(\gamma|_{[0,t_0]}\right), \operatorname{TotCurv}_{\boldsymbol{u}}\left(\gamma|_{[t_k,\ell]}\right) \le \pi.$$

Since $\alpha_0, \alpha_k \le \frac{\pi}{2}$, the statement follows. □

If Σ is a surface in $\mathbb{R}^3$ and $p \in \Sigma$, then we denote by K_p the Gauss curvature of Σ at p.



Assume a, b are the meeting moments of the minimizing geodesic γ with $\omega_{\boldsymbol{u}}$. The arc $\gamma|_{[a,b]}$ will be called an $\omega_{\boldsymbol{u}}$-*tongue* if there is an immersed disk $\iota\colon \mathbb{D} \looparrowright \Sigma$ such that the closed curve $\iota|_{\partial D}$ is formed by the joint of the arc $\gamma|_{[a,b]}$ and an arc of $\omega_{\boldsymbol{u}}$. In this case the immersion ι is called the *disk of the tongue*.

4.2. Tongue lemma. *Let $\boldsymbol{u}$ be a unit vector, let $\gamma\colon [a,b] \to \Sigma$ be a minimizing geodesic on the strongly convex surface $\Sigma \subset \mathbb{R}^3$ which is an $\omega_{\boldsymbol{u}}$-tongue, and $\iota\colon \mathbb{D} \looparrowright \Sigma$ its disk.*

Then

$$\int_{\mathbb{D}} K_{\iota(x)} \cdot d_{\iota(x)} \operatorname{area}_\Sigma$$

takes one of the values

$$\alpha - \beta,\ -\alpha + \beta,\ \pi - \alpha - \beta,\ \pi + \alpha + \beta \pmod{2 \cdot \pi}.$$

In particular,

$$\big|\alpha - \beta\big| \le \int_{\mathbb{D}} K_{\iota(x)} \cdot d_{\iota(x)} \operatorname{area}_\Sigma . \tag{1}$$

If in addition the image $\iota(\mathbb{D})$ lies completely in the dark or the light side for $\boldsymbol{u}$, then

$$\operatorname{TotCurv} \gamma \le \int_{\iota(\mathbb{D})} K_p \cdot d_p \operatorname{area}_\Sigma . \tag{2}$$

Proof. Since γ is a geodesic, the parallel translation along γ sends $\dot\gamma(a)$ to $\dot\gamma(b)$.

Note also that $\boldsymbol{u}$ belongs to the tangent plane to Σ at any point on the horizon $\omega_{\boldsymbol{u}}$; in particular, $\boldsymbol{u}$ extends to a parallel tangent vector field on $\omega_{\boldsymbol{u}}$.

It follows that parallel translation along $\iota|_{\partial\mathbb{D}}$ rotates the tangent plane by the angle

$$\pm(\tfrac{\pi}{2} + \alpha) \pm (\tfrac{\pi}{2} + \beta).$$

To prove the main statement of the lemma, it remains to apply the Gauss–Bonnet formula.

Denote by R the right-hand side in (1). Note that $R \ge 0$ and $|\alpha|, |\beta| \le \frac{\pi}{2}$. From the main statement of the lemma it then follows that the minimal possible value for R is $\big|\alpha - \beta\big|$.

To prove (2), note that in this case ι is an embedding. Further, note that the spherical image of the dark side of Σ is a hemisphere. Therefore, $2 \cdot \pi$ is the integral of the Gauss curvature along the dark side. It follows that

$$\int_{\mathbb{D}} K_{\iota(x)} \cdot d_{\iota(x)} \operatorname{area}_\Sigma = \int_{\iota(\mathbb{D})} K_p \cdot d_p \operatorname{area}_\Sigma < 2 \cdot \pi.$$

By Liberman's lemma, the statement follows. ☐

§5. Almost straight arcs

Let $\varepsilon > 0$. We say that a curve $\gamma : [a,b] \to \mathbb{R}^3$ is ε-straight if

$$(1 - \varepsilon) \cdot \operatorname{length} \gamma \le |\gamma(b) - \gamma(a)|.$$

5.1. Lemma. *Assume $\varepsilon > 0$ and n is a positive integer such that $n \cdot \varepsilon > 2$. Then any minimizing geodesic on a convex surface Σ in $\mathbb{R}^3$ can be subdivided into ε-straight arcs $\gamma_1, \dots, \gamma_n$.*

Proof. Let $\theta \in (0, \pi)$ be such that

$$1 - \cos\theta = \varepsilon.$$

Assume two points p and q lie on the convex surface Σ. Denote by ν_p and ν_q the outer normal vectors at p and q, respectively. Note that if

$$\measuredangle(\nu_p, \nu_q) \le 2 \cdot \theta,$$

then any minimizing geodesic from p to q on Σ is ε-straight.

Let $\gamma\colon [0, \ell] \to \Sigma$ be a minimizing geodesic parametrized by its arc-length.

Assume $\gamma_{[t,\ell]}$ is not ε-straight. Set t' to be the maximal value in $[t, \ell)$ such that the interval $[t, t']$ is ε-straight. Consider a sequence $0 = t_0 < t_1 < \cdots < t_n < \ell$ such that $t_{i+1} = t_i'$ for each i. Denote by ν_i the outer unit normal vector to Σ at $\gamma(t_i)$. From above we get

$$\measuredangle(\nu_i, \nu_j) \ge 2 \cdot \theta$$

for all i and j. In other words, the open balls $\mathrm{B}_\theta(\nu_i)$ do not overlap in $\mathbb{S}^2$.

It remains to note that

$$\operatorname{area}[\mathrm{B}_\theta(\nu_i)] = 2 \cdot \pi \cdot \varepsilon \text{ and } \operatorname{area} \mathbb{S}^2 = 4 \cdot \pi.$$

Hence, the result follows. □

5.2. Corollary. *Assume $\gamma\colon [0, \ell] \to \Sigma$ is a unit-speed minimizing geodesic on the convex surface Σ in $\mathbb{R}^3$. Then $\operatorname{diam} \gamma \ge \frac{\ell}{10}$.*

Proof. Apply Lemma 5.1 for $\varepsilon = \frac{1}{2}$. □

§6. An arc in an almost constant direction

6.1. Proposition. *For any $\varepsilon > 0$ there is $\delta > 0$ such that the following holds true.*

If $\gamma\colon [a, b] \to \Sigma$ is a minimizing geodesic on a smooth strongly convex surface Σ in $\mathbb{R}^3$, then there is an interval $[a', b'] \subset [a, b]$ such that

$$\operatorname{TotCurv}(\gamma|_{[a',b']}) > \delta \cdot \operatorname{TotCurv} \gamma$$

and

$$\measuredangle(\dot\gamma(t), \boldsymbol{u}) < \varepsilon$$

for a fixed unit vector $\boldsymbol{u}$ and any $t \in [a', b']$.

Moreover, if $\varepsilon = \frac{1}{10}$, then we can take $\delta = \frac{1}{100^{100}}$.

In the proof we will need the following two lemmas.

6.2. Lemma. *For any ε there is $\delta > 0$ such that the following holds true.*

Assume γ is a curve, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are two vectors in $\mathbb{R}^3$, and $0 \le \alpha_1, \alpha_2 \le \pi$ are such that

$$\varepsilon < \measuredangle(\boldsymbol{v}_1, \boldsymbol{v}_2) < \pi - \varepsilon,$$
$$\alpha_i - \delta < \measuredangle(\boldsymbol{v}_i, \dot\gamma(t)) < \alpha_i + \delta.$$

Then there is a vector $\boldsymbol{u}$ such that $\measuredangle(\boldsymbol{u}, \dot\gamma(t)) < \varepsilon$.

Moreover, if $\varepsilon < \frac{1}{10}$, then one can take $\delta = \varepsilon^{10}$.

The proof of the above lemma is straightforward computation; we omit it.

6.3. Lemma. *For any $\varepsilon > 0$ there is $\delta > 0$ such that the following holds true.*

Let $\gamma\colon [a, b] \to \Sigma$ be a δ-straight minimizing geodesic on a smooth strongly convex surface Σ in $\mathbb{R}^3$. Set $\boldsymbol{v}_\gamma = \gamma(b) - \gamma(a)$. Then there in a subinterval $[a', b']$ in $[a, b]$ such that

$$\operatorname{TotCurv}(\gamma|_{[a',b']}) \ge \delta \cdot \operatorname{TotCurv} \gamma$$

and

$$\alpha - \varepsilon \le \measuredangle(\dot\gamma(t), \boldsymbol{v}_\gamma) \le \alpha + \varepsilon$$

for some fixed α and any $t \in [a', b']$.

Moreover, if $\varepsilon < \frac{1}{10}$, then one can take $\delta = \varepsilon^{10}$.

Proof. Without loss of generality we may assume that $a = 0$, $b = 2$, and

$$\operatorname{TotCurv}(\gamma|_{[1,2]}) \geq \tfrac{1}{2} \cdot \operatorname{TotCurv} \gamma.$$

Set $p = \gamma(0)$. Let $\theta \in (0, \pi)$ be such that $1 - \cos\theta = \delta$. Note that

$$\measuredangle(\boldsymbol{v}_\gamma, \gamma(t) - p) \leq \measuredangle(\widetilde{\gamma}_p(1) - \widetilde{p}, \widetilde{\gamma}_p(2) - \widetilde{p}) < \theta \tag{1}$$

for any $t \geq 1$.

By Liberman's lemma,

$$\operatorname{TotCurv}_p(\gamma|_{[1,2]}) < \pi + \theta.$$

Assume $N = \lceil \frac{\pi}{\theta} + 1 \rceil$. Then we can subdivide $\gamma|_{[1,2]}$ into N arcs $\gamma_1, \gamma_2, \dots, \gamma_N$ such that

$$\operatorname{TotCurv}_p(\gamma_n) \leq \theta \tag{2}$$

for each n.

From (1) and (2), it follows that for each n, there is α_n with

$$\alpha_n - \theta \leq \measuredangle(\dot\gamma_n(t), \boldsymbol{v}_\gamma) \leq \alpha_n + \theta.$$

The arc γ_n with the maximal total curvature will solve the proposition.

It remains to choose δ so that $\theta(\delta) < \frac{\varepsilon}{100}$. □

Proof of Proposition 6.1. Set $\gamma_0 = \gamma$.

Fix $\delta > 0$, set $n = \lceil \frac{2}{\delta} \rceil$. By Lemma 5.1, the geodesic γ_0 can be subdivided into n arcs that are δ-straight. We choose the arc γ_0' with maximal total curvature. Assuming $\delta < \frac{1}{10}$, we get

$$\operatorname{TotCurv} \gamma_0' \geq \tfrac{\delta}{10} \cdot \operatorname{TotCurv} \gamma_0.$$

Let α_1 be the angle and γ_1 the arc in γ_0' provided by Lemma 6.3. In particular,

$$\operatorname{TotCurv} \gamma_1 \geq \delta \cdot \operatorname{TotCurv} \gamma_0' \geq \tfrac{\delta^2}{10} \cdot \operatorname{TotCurv} \gamma_0.$$

If $\alpha_1 \leq \frac{\varepsilon}{2}$ or $\alpha_1 \geq \pi - \frac{\varepsilon}{2}$ and δ is sufficiently small, then the statement holds true for the arc γ_1 and the vector $\boldsymbol{u} = \pm \boldsymbol{v}_{\gamma_0'}$.

Otherwise we repeat the above construction for γ_1. Namely, apply Lemma 5.1 to the geodesic γ_1 and denote by γ_1' the δ-straight arc with maximal total curvature. If δ is small, we get

$$\tfrac{\varepsilon}{3} < \measuredangle(\boldsymbol{v}_{\gamma_1'}, \boldsymbol{v}_{\gamma_0'}) < \pi - \tfrac{\varepsilon}{3}. \tag{3}$$

Again, we get

$$\operatorname{TotCurv} \gamma_1' \geq \tfrac{\delta}{10} \cdot \operatorname{TotCurv} \gamma_1 \geq \tfrac{\delta^3}{100} \cdot \operatorname{TotCurv} \gamma_0.$$

Next, we apply Lemma 6.3 to γ_1'. Denote by γ_2 and α_2 the angle and the sub-arc of γ_1'. Again

$$\operatorname{TotCurv} \gamma_2 \geq \tfrac{\delta^4}{100} \cdot \operatorname{TotCurv} \gamma_0.$$

The curve γ_2 runs under nearly constant angle to $\boldsymbol{v}_{\gamma_0'}$ and $\boldsymbol{v}_{\gamma_1'}$. The inequality (3) makes it possible to apply Lemma 6.2. Hence, the main statement in the proposition follows.

Straightforward computations prove the last statement. □

§7. Drifting geodesics

In this section we fix the notation to be used further without additional explanation.

Fix a system of (x, y, z)-coordinates on the Euclidean space; denote by $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ the standard basis.

A plane parallel to, say, the (y, z)-coordinate plane will be called a (y, z)-plane.

7.1. Definition. A smooth curve $\gamma\colon [0, \ell] \to \mathbb{R}^3$ is said to be $\boldsymbol{i}$-*drifting* if both ends $\gamma(0)$ and $\gamma(\ell)$ lie on the x-axis and $\measuredangle(\dot\gamma(t), \boldsymbol{i}) < \frac{1}{10}$ for all t.

$(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$-frame. Let Σ be a convex surface and $\gamma\colon [0, \ell] \to \Sigma$ an $\boldsymbol{i}$-drifting minimizing geodesic with unit-speed parametrization.

Given $t \in [0, \ell]$, consider the oriented orthonormal frame $\lambda(t), \mu(t), \nu(t)$ such that $\nu(t)$ is the outer normal to Σ at $\gamma(t)$, the vector $\mu(t)$ lies in a (y, z)-plane and therefore the vector $\lambda(t)$ lies in the plane spanned by $\nu(t)$ and the x-axis. We assume in addition that $\langle \lambda, \boldsymbol{i} \rangle \geq 0$.

Since $\langle \dot\gamma(t), \boldsymbol{i} \rangle > 0$, we have $\nu(t) \neq \boldsymbol{i}$, so that the frame (λ, μ, ν) is uniquely determined for any $t \in [0, \ell]$.

Angle functions. Set

$$\phi(t) = \measuredangle(\boldsymbol{i}, \dot\gamma(t)), \quad \psi(t) = \tfrac{\pi}{2} - \measuredangle(\boldsymbol{i}, \nu(t)), \quad \theta(t) = \tfrac{\pi}{2} - \measuredangle(\mu(t), \dot\gamma(t)).$$

From the above definitions it follows that $|\theta(t)|, |\psi(t)| \leq \frac{\pi}{2}$ and for each t there is a right spherical triangle with legs $|\theta(t)|, |\psi(t)|$ and hypotenuse $\phi(t)$. In particular, $\cos\theta \cdot \cos\psi = \cos\phi$. We get the following.

7.2. *Claim.* For any t we have

$$\phi(t) \geq |\psi(t)| \quad \text{and} \quad \phi(t) \geq |\theta(t)|.$$

Applying Liberman's lemma in the direction $\boldsymbol{i}$, we also get the following.

7.3. *Claim.* If an arc $\gamma|_{[a,b]}$ lies in the dark (light) side for $\boldsymbol{i}$, then the angle function ϕ is monotone nondecreasing (respectively, nonincreasing) in $[a, b]$.

§8. Plane sections

Assume γ is a curve on a smooth strongly convex surface Σ in $\mathbb{R}^3$. Consider a plane L passing through two points of γ, say $p = \gamma(a)$ and $q = \gamma(b)$ with $a < b$. Let $L_\pm$ be half-planes in L bounded by the line trough p and q. Set $\sigma_\pm = \Sigma \cap L_\pm$.

8.1. *Observation.* If γ is a minimizing geodesic in the smooth strongly convex surface $\Sigma \subset \mathbb{R}^3$ and a, b, and $\sigma_\pm$ are as above, then

$$\operatorname{length}\sigma_\pm \geq \operatorname{length}(\gamma|_{[a,b]}).$$

To prove the observation, it suffices to note that the $\sigma_\pm$ are smooth convex plane curves connecting p to q in Σ.

Based on this observation, we give a couple of estimates on drifting minimizing geodesics.

Let $\gamma\colon [a, b] \to \mathbb{R}^3$ be a curve and ℓ a line that does not pass through points of γ. Assume $\phi\colon [a, b] \to \mathbb{R}$ is a continuous azimuth angle of γ in the cylindrical coordinates with the axis ℓ. If

$$|\phi(b) - \phi(a)| \geq 2 \cdot n \cdot \pi,$$

we will say that γ *goes around* the line ℓ at least n times.

8.2. Proposition. *Assume $\gamma\colon[0,\ell]\to\Sigma$ is an $\boldsymbol{i}$-drifting minimizing geodesic in the convex surface $\Sigma\subset\mathbb{R}^3$, a subsegment $[a,b]$ is included in $[0,\ell]$, and the following conditions are fulfilled.*

(i) *The points $\gamma(a)$ and $\gamma(b)$ lie in a half-plane with boundary line formed by the x-axis and the arc $\gamma|_{[a,b]}$ goes around the x-axis, at least once.*
(ii) *The x-coordinate of $\gamma(a)$ is larger than the x-coordinate of $\frac12\cdot(\gamma(0)+\gamma(\ell))$.*

Then $\gamma(b)$ lies on the dark side for $\boldsymbol{i}$.

Proof. Let us apply Observation 8.1 to the plane containing the x-axis, $\gamma(a)$ and $\gamma(b)$.

We may assume that $\gamma(0)$ is the origin of the (x,y,z)-coordinate system, and that the two points $p=\gamma(a)$ and $q=\gamma(b)$ lie in the (x,z)-coordinate half-plane with $x\ge0$, denoted by Π. We may assume that $\sigma_+\subset\Pi$. Let $(x_p,0,z_p)$ and $(x_q,0,z_q)$ be the coordinates of p and q.

From the assumptions we get $x_p<x_q<2\cdot x_p$.

Suppose the contrary, then $\gamma(b)$ lies on the light side for $\boldsymbol{i}$. Then from the convexity of the curve $\Pi\cap\Sigma$ we get

$$\operatorname{length}\sigma_+\le\sqrt{(x_q-x_p)^2+z_p^2}.$$

On the other hand, since $\gamma|_{[a,b]}$ goes around the x-axis at least once, we get

$$\operatorname{length}\gamma|_{[a,b]}\ge\sqrt{(x_q-x_p)^2+(z_p+z_q)^2}.$$

These two estimates contradict Observation 8.1. □

8.3. Corollary. *If Σ, γ, ℓ, a, and b are as in the Proposition 8.2 and the arc $\gamma|_{[a,b]}$ goes around the x-axis at least twice, then the arc $\gamma|_{[b,\ell]}$ lies on the dark side with respect to $\boldsymbol{i}$.*

Proof. Fix $b'\in[b,\ell]$. Note that one can find $a'\in[a,b]$ such that the assumptions of Proposition 8.2 are fulfilled for the interval $[a',b']$. Applying the proposition, we get the result. □

8.4. Proposition. *Assume $\gamma\colon[0,\ell]\to\Sigma$ is an $\boldsymbol{i}$-drifting minimizing geodesic in the convex surface $\Sigma\subset\mathbb{R}^3$. Assume that the arc $\gamma|_{[b,\ell]}$ lies on the dark side of Σ with respect to $\boldsymbol{i}$. If $b\le s<t\le\ell$ and the point $\gamma(s)$ lies in the plane Π through $\gamma(t)$ spanned by $\nu(t)$ and $\lambda(t)$, then*

$$\phi(s)\le\psi(t).$$

Proof. We apply Observation 8.1 to the plane Π and $p=\gamma(s)$ and $q=\gamma(t)$.

Let x_p and x_q be the x-coordinates of p and q.

Since $\gamma|_{[s,t]}$ lies on the dark side, its Liberman development $\widetilde\gamma|_{[s,t]}$ with respect to $\boldsymbol{i}$ is concave. In particular,

$$\operatorname{length}(\gamma|_{[s,t]})=\operatorname{length}(\widetilde\gamma|_{[s,t]})\ge\tfrac{x_q-x_p}{\cos\phi(s)}.$$

On the other hand, the convexity of σ_+ implies that

$$\operatorname{length}\sigma_+\le\tfrac{x_q-x_p}{\cos\psi(t)}.$$

It remains to apply Observation 8.1. □

§9. s-pairs

Let $\Sigma \subset \mathbb{R}^3$ be a strongly convex surface and $\gamma\colon [0,\ell] \to \Sigma$ an $\boldsymbol{i}$-drifting minimizing geodesic.

We may assume that the horizon $\omega_{\boldsymbol{j}}$ is a smooth curve and γ intersects the horizons transversely.

Let $t_0 < t_1 < \dots < t_k$ be the meeting moments of γ with $\omega_{\boldsymbol{j}}$. Set

$$\phi_n = \phi(t_n), \quad \psi_n = \psi(t_n), \quad \theta_n = \theta(t_n).$$

Note that $\theta_n = \pm\alpha_n$ so we can say $s_n \cdot \theta_n = (-1)^m \cdot \alpha_n$ for some sequence of signs $s_i = \pm 1$. In particular,

$$\alpha_0 - \alpha_1 + \dots + (-1)^k \cdot \alpha_k = s_0 \cdot \theta_0 + s_1 \cdot \theta_1 + \dots + s_k \cdot \theta_k.$$

Note that for the right choice of orientation, if $s_n = +1$, then $\nu_{\gamma(t)}$ moves clockwise in $\mathbb{S}^2$ at t_n and if $s_n = -1$, then it moves counterclockwise.

We say that a pair of indices $i < j$ forms an *s-pair* if

$$\sum_{n=i}^{j} s_n = 0 \text{ and } \sum_{n=i}^{j'} s_n > 0$$

for $i < j' < j$.

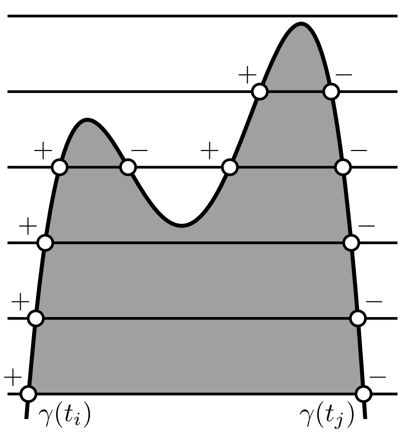
If you exchange "+1" and "−1" in s by "(" and ")" respectively, then (i,j) is an s-pair if and only if the ith bracket forms a pair with the j-bracket.

Note that any index i appears in at most one s-pair and for any s-pair (i,j) we have

- $s_i = 1$; that is, the ith bracket must be opening.
- $s_j = -1$; that is, the jth bracket must be closing.

In particular,

$$s_i \cdot \theta_i + s_j \cdot \theta_j = \theta_i - \theta_j == (-1)^i \cdot \alpha_i + (-1)^j \cdot \alpha_j.$$



Tongue interpretation. Assume (i,j) is an s-pair. Note that in this case there is an arc of $\omega_{\boldsymbol{j}}$ from $\gamma(t_i)$ to $\gamma(t_j)$ with monotonic x-coordinate. Moreover, a disk of the tongue has this arc in the boundary.

The proof can be guessed from the diagram. It shows a lift of γ in the universal cover of the strip of Σ between (y,z)-planes containing $\gamma|_{[t_i,t_j]}$; the solid horizontal lines are the lifts of $\omega_{\boldsymbol{j}}$.

We say that q is the depth of an s-pair (i,j) (briefly, $q=\operatorname{depth}_s(i,j)$) if q is the maximal number such that there is a q-long nested sequence of s-pairs starting with (i,j). For example, the s-pair on the diagram has depth 5.

More precisely, the depth of (i,j) is the maximal number q for which there is a sequence of s-pairs $(i,j)=(i_1,j_1),(i_2,j_2),\dots,(i_q,j_q)$ such that

$$i=i_1<\dots<i_q<j_q<\dots<j_1=j.$$

Note that the s-pairs of the same depth do not overlap, i.e., if $\operatorname{depth}(i,j)=\operatorname{depth}(i',j')$ for two distinct s-pairs (i,j) and (i',j'), then either $i<j<i'<j'$ or $i'<j'<i<j$.

The following proposition follows directly from the discussion above.

9.1. Proposition. *Let (i,j) be an s-pair. Then the arcs $\gamma|_{[t_i,t_j]}$ and an arc of ω_j bound an immersed disk in Σ that lies between (y,z)-planes through $\gamma(t_i)$ and $\gamma(t_j)$. Moreover, the maximal multiplicity of the disk is at most* $\operatorname{depth}_s(i,j)$.

9.2. Corollary. *Denote by S_q the subset of indices $\{1,\dots,k\}$ that are the parts of s-pairs with depth q. Then*

$$\Big|\sum_{n\in S_q}(-1)^n\cdot\alpha_n\Big|=\Big|\sum_{n\in S_q}s_n\cdot\theta_n\Big|\le 4\cdot\pi\cdot q.$$

Proof. For each n denote by K_n the integral of the Gauss curvature of the part of the surface Σ with the x-coordinate less than the x-coordinate of $\gamma(t_n)$. Note that

$$0\le K_1\le\dots\le K_k\le 4\cdot\pi.$$

By Proposition 9.1 and the Tongue Lemma, we get

$$s_i\cdot\theta_i+s_j\cdot\theta_j=\theta_i-\theta_j\le q\cdot(K_j-K_i).$$

The statement follows, because the s-pairs with the same depth do not overlap. □

9.3. Corollary. *Assume*

$$q=\max_{1\le i<j\le k}\left\{\Big|\sum_{n=i}^{j}s_n\Big|\right\}.$$

Then

$$\Big|\sum_{n=1}^{k}s_n\cdot\theta_n\Big|\le 2\cdot q\cdot\left(q+\frac{3}{2}\right)\cdot\pi.$$

Proof. Denote by S the set of all indices that appear in some s-pair.

Note that the depth of any s-pair is at most q. That is,

$$S=S_1\cup\dots\cup S_q.$$

By Corollary 9.2,

$$\Big|\sum_{n\in S}s_n\cdot\theta_n\Big|\le 2\cdot q\cdot(q+1)\cdot\pi. \tag{1}$$

Set $R=\{1,\dots,k\}\setminus S$; this is the set of indices that do not appear in an s-pair.

Given r, set $i\in Q_r$ if

$$\sum_{n=1}^{i}s_n=r.$$

Note that $Q_r\ne\varnothing$ for at most q values of r, and in each set Q_r there are at most 2 indices that do not appear in an s-pair; that is, $Q_r\cap R$ has at most two indices for each r.

Since $|a_n| < \frac{\pi}{2}$, we get

$$\left|\sum_{n\in R} s_n \cdot \theta_n\right| \le q \cdot \pi.$$

This inequality together with (1) implies the claim. $\square$

§10. Geometric growth

10.1. *Claim.* Assume $\psi(t) > \varepsilon$ for $t \in [t_i, t_{i+1}]$ and $s_i = s_{i+1}$. Then

$$|\theta_{i+1} - \theta_i| > \pi \cdot \sin\varepsilon.$$

Proof. Note that the arc $\gamma|_{[t_i,t_{i+1}]}$ is a tongue with embedded disk $\iota\colon \mathbb{D}^2 \to \Sigma$. Since $\psi(t) > \varepsilon$, the spherical image $\nu \circ \iota(\mathbb{D}^2)$ of $\iota(\mathbb{D}^2)$ lies in a half-disk of radius $\frac{\pi}{2} - \varepsilon$ in $\mathbb{S}^2$. Note that

$$K(\iota(\mathbb{D}^2)) = \mathrm{area}(\nu \circ \iota(\mathbb{D}^2)) < \pi \cdot (1 - \sin\varepsilon).$$

It remains to apply Tongue Lemma 4.2. $\square$

10.2. *Claim.* Assume γ lies on the dark side for $\boldsymbol{i}$. Then for any pair of indices $j > i$ such that

$$\left|\sum_{n=i}^{j} s_n\right| > 5,$$

we have

$$\phi_j > \tfrac{3}{2} \cdot \phi_i.$$

Proof. By Claim 7.3, we may assume that

$$\sum_{n=i}^{j} s_n = 6.$$

Let j' be the smallest index such that

$$\left|\sum_{n=i}^{j'} s_n\right| = 5.$$

Note that for any $b > t_j$ there is $a \in [t_i, t_j]$ such that interval $[a, b]$ satisfies the assumptions of Proposition 8.4. In particular, $\psi(b) > \phi_i$ for any $b > t_j$. Applying Claim 10.1, we get $|\theta_j| > \frac{\pi}{2} \cdot \phi_i$ or $|\theta_{j'}| > \frac{\pi}{2} \cdot \phi_i$. By Claim 7.3, ϕ_n is monotone nondecreasing, and $\phi_n \ge |\theta_n|$ for any n, in both cases we get

$$\phi_j > \tfrac{\pi}{2} \cdot \phi_i,$$

and the result follows. $\square$

10.3. Proposition. *If γ is an $\boldsymbol{i}$-drifting minimizing geodesic on the dark side for $\boldsymbol{i}$, then*

$$\mathrm{TotCurv}_{\boldsymbol{j}}\, \gamma \le 100 \cdot \pi.$$

Proof. We may assume that γ crosses the $\boldsymbol{j}$-horizon $\omega_{\boldsymbol{j}}$ transversely. Let $t_0 < \dots < t_k$ be the meeting moments of γ with $\omega_{\boldsymbol{j}}$ and $s_0, \dots, s_k$ the signs.

Recall that S_q denotes the subset of indices $\{1, \dots, k\}$ that appear in an s-pair with depth q. By Corollary 9.2,

$$\left|\sum_{n\in S_q} s_n \cdot \theta_n\right| \le 4 \cdot q \cdot \pi.$$

In particular,

$$\left|\sum_{n\in S_1\cup\dots\cup S_5} s_n \cdot \theta_n\right| \le 40 \cdot \pi.$$

Set $R = \{1, \dots, k\} \setminus (S_1 \cup \dots \cup S_5)$; this is the set of indices that appear in s-pairs with depth at least 6 as well as those that do not appear in any s-pair.

By Claim 7.2,
$$\left|\sum_{n\in R} s_n \cdot \theta_n\right| \le \sum_{n\in R} |\theta_n| \le \sum_{n\in R} \phi_n.$$
To estimate the last sum, we shall use the results in §10. First, we subdivide R into 5 subsets $R_1, \dots, R_5$ by setting $n \in R_m$ if $m \equiv n \pmod 5$.

Given $n \in R_m$, denote by n' the smallest index in R_m that is larger than n; n' is defined for any $n \in R_m$ except the largest one. According to Claim 10.2, $\phi_{n'} > \frac{3}{2} \cdot \phi_n$; that is, the sequence $(\phi_n)_{n\in R_m}$ grows faster than the geometric progression with coefficient $\frac{3}{2}$. Since ϕ_n is monotone nondecreasing in n, we get
$$\sum_{n\in R_m} \phi_n < 3 \cdot \phi_k.$$
It follows that
$$\sum_{n\in R} \phi_n < 15 \cdot \phi_k \le \tfrac{15}{2} \cdot \pi.$$
By Corollary 4.1,
$$\operatorname{TotCurv}_{\boldsymbol{j}} \gamma \le 2 \cdot \pi + 2 \cdot [\alpha_0 - \alpha_1 + \dots + (-1)^k \cdot \alpha_k] < 100 \cdot \pi. \qquad \square$$

§11. Assembling the proof

Assume $\gamma\colon [0, \ell] \to \Sigma$ is a minimizing geodesic in a convex surface $\Sigma \subset \mathbb{R}^3$.

By Propositions 2.1 and 2.2, we may assume that Σ is closed, strongly convex, and smooth and the geodesic γ has finite length.

According to Proposition 6.1, we can pass to an $\boldsymbol{i}$-drifting arc γ' of γ for some (x, y, z)-coordinate system such that
$$\operatorname{TotCurv} \gamma' > \tfrac{1}{100^{100}} \cdot \operatorname{TotCurv} \gamma. \tag{1}$$
We shall use the notation of §7 for γ'.

Rotating the (y, z)-coordinate plane, we can ensure that
$$\operatorname{TotCurv} \gamma' \le 10 \cdot \operatorname{TotCurv}_{\boldsymbol{j}} \gamma'$$
and that γ' crosses the horizon $\omega_{\boldsymbol{j}}$ transversally.

By Corollary 8.3, we can subdivide γ' into at most three arcs:

- *Left arc* γ'_- that lies on the light side for $\boldsymbol{i}$.
- *Middle arc* γ'_0 that rotates around the x-axis at most 4 times.
- *Right arc* γ'_+ that lies on the dark side for $\boldsymbol{i}$.

Indeed, choose an arc $\gamma'|_{[a,b]}$ on the right from the (y, z)-plane through $\frac{1}{2} \cdot (\gamma'(0) + \gamma'(\ell))$ that rotates around the x-axis 2 times and assume that b takes the minimal possible value. Note that if $\gamma'(s)$ lies on the (y, z)-plane through $\frac{1}{2} \cdot (\gamma'(0) + \gamma'(\ell))$, then $[s, b] \supset [a, b]$ and any subarc of $[s, b]$ rotates around the x-axis at most 2 times.

By Corollary 8.3, we can take $\gamma'_+ = \gamma'|_{[b,\ell]}$; in case there is no such arc $[a, b]$, we assume that γ'_+ is not present. Repeat the construction reverting the direction of the x-axis; we get the lower arc γ'_-. The remaining arc is assumed to be γ'_0; note that any subarc of γ'_0 is divided by the (y, z)-plane through $\frac{1}{2} \cdot (\gamma'(0) + \gamma'(\ell))$ into two each of which rotates around the x-axis at most 2 times. Therefore, the number of rotations of any arc in γ'_0 is at most 4.

Let us estimate the total curvature of γ'_-, γ'_0, and γ'_+ separately.

By Proposition 10.3, we get

$$\operatorname{TotCurv}_{\boldsymbol{j}} \gamma'_+ \le 100 \cdot \pi. \tag{2}$$

Similarly,

$$\operatorname{TotCurv}_{\boldsymbol{j}} \gamma'_- \le 100 \cdot \pi. \tag{3}$$

By Corollary 9.3,

$$\operatorname{TotCurv}_{\boldsymbol{j}} \gamma'_0 \le 100 \cdot \pi. \tag{4}$$

Together with (2), (3), and (4) the last inequality implies that

$$\operatorname{TotCurv}_{\boldsymbol{j}} \gamma' \le 300 \cdot \pi.$$

Now, the result follows from (1). □

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