Алгебра и анализ
Том 29 (2017), № 1

St. Petersburg Math. J.
Vol. 29 (2018), No. 1, Pages 177–207
http://dx.doi.org/10.1090/spmj/1488
Article electronically published on December 27, 2017

SHARP CORRESPONDENCE PRINCIPLE AND QUANTUM MEASUREMENTS

L. CHARLES AND L. POLTEROVICH

To Yuriĭ Dmitrievich Burago
on the occasion of his 80th birthday

Abstract. We prove sharp remainder bounds for the Berezin–Toeplitz quantization and present applications to semiclassical quantum measurements.

§1. Introduction and main results

1.1. An outlook. The subject of this paper is quantization, a formalism behind the quantum-classical correspondence, a fundamental principle stating that quantum mechanics contains the classical one as a limiting case when the Planck constant $\hbar$ tends to 0. Mathematically, the correspondence is given by a linear map between smooth functions on a symplectic manifold and Hermitian operators on an $\hbar$-depending complex Hilbert space. It is assumed that, up to error terms (a.k.a. *remainders*) that are small with $\hbar$, some basic operations on functions correspond to their counterparts on operators. For instance, the Poisson bracket of functions corresponds to (a properly rescaled) commutator of the operators. In the present paper we study the size of the remainders focusing on the following facets of this problem.

First, *given a quantization, find explicit upper bounds for the remainders.* In this direction, we obtain such bounds for several basic quantization schemes, including the standard Berezin–Toeplitz quantization of closed Kähler manifolds, in terms of the $\mathcal{C}^k$-norms on functions with $k \leq 3$ (see Subsections 1.2 and 1.4 below). The motivation comes from semiclassical quantum mechanics: having a good control on the remainders, one can zoom into small regions of the phase space up to the quantum scale $\sim \sqrt{\hbar}$, the smallest scale allowed by the uncertainty principle. As an illustration, in Subsection 1.5 we present applications to noise production in semiclassical quantum measurements.

Second, according to the classical no-go theorems, no ideal quantizations, i.e., the ones with vanishing remainders, exist. This naturally leads to the following quantitative question: *can one find a quantization with arbitrarily small remainders*? It turns out that for a meaningful class of quantizations, the answer is negative. The remainders are subject to constraints that depend only on geometry of the phase space, which is the essence of the rigidity of remainders phenomenon discussed in Subsection 1.3.

1.2. Sharp remainder estimates. Let (M^{2n}, ω) be a closed symplectic manifold. We assume that (M, ω) is quantizable, i.e., the cohomology class $[\omega]/(2\pi)$ is integral. We write $\{f, g\}$ for the Poisson bracket of smooth functions f and g.

2010 *Mathematics Subject Classification.* Primary 53D20.

Key words and phrases. Berezin–Toeplitz quantization, symplectic manifold, quantum measurement.

The second author was partially supported by the Israel Science Foundation grant 178/13 and the European Research Council Advanced grant 338809.

Fix an auxiliary Riemannian metric ρ on M. For a function $f \in \mathcal{C}^\infty(M)$ its $\mathcal{C}^k$-norm with respect to ρ is denoted by $|f|_k$. For a pair of smooth functions f, g put $|f,g|_N = \sum_{j=0}^N |f|_j \cdot |g|_{N-j}$. We write $\|f\| = |f|_0 = \max |f|$ for the uniform norm, and $\|f\|_{L_1}$ for the L_1-norm of f with respect to the symplectic volume $\omega^n/n!$.

We also introduce a *reduced* version of $|f,g|_4$,

$$|f,g|_{1,3} := |f|_1 \cdot |g|_3 + |f|_2 \cdot |g|_2 + |f|_3 \cdot |g|_1 \tag{1}$$

which does not include fourth derivatives, and which plays an important role below.

For a finite-dimensional complex Hilbert space $\mathcal{H}$ write $\mathcal{L}(\mathcal{H})$ for the space of Hermitian operators on $\mathcal{H}$. The operator norm is denoted by $\|\cdot\|_{\mathrm{op}}$ and $[A, B]$ stands for the commutator $AB - BA$.

A *Berezin* quantization of M is given by the following data:

- a subset $\Lambda \subset \mathbb{R}_{>0}$ having 0 as a limit point;
- a family $\mathcal{H}_\hbar$ of finite-dimensional complex Hilbert spaces, $\hbar \in \Lambda$;
- a family $T_\hbar \colon \mathcal{C}(M, \mathbb{R}) \to \mathcal{L}(\mathcal{H}_\hbar)$ of positive surjective linear maps such that $T_\hbar(1) = \mathrm{id}$, $\hbar \in \Lambda$.

That $T_\hbar$ is positive means that for any $f \in \mathcal{C}(M, \mathbb{R})$, $f \geq 0$ implies that $T_\hbar(f) \geq 0$. We will also assume that there exist α, β, γ and $\delta > 0$ such that for any $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$, we have

(P1) **(norm correspondence)** $\|f\| - \alpha|f|_2\hbar \leq \|T_\hbar(f)\|_{\mathrm{op}} \leq \|f\|$;

(P2) **(the correspondence principle)**

$$\left\| -\frac{i}{\hbar} \cdot [T_\hbar(f), T_\hbar(g)] - T_\hbar(\{f,g\}) \right\|_{\mathrm{op}} \leq \beta |f,g|_{1,3}\hbar;$$

(P3) **(quasi-multiplicativity)** $\|T_\hbar(fg) - T_\hbar(f)T_\hbar(g)\|_{\mathrm{op}} \leq \gamma |f,g|_2\hbar$;

(P4) **(trace correspondence)**

$$\left| \mathrm{trace}(T_\hbar(f)) - (2\pi\hbar)^{-n} \int_M f \frac{\omega^n}{n!} \right| \leq \delta \|f\|_{L_1} \hbar^{-(n-1)},$$

for all $f, g \in \mathcal{C}^\infty(M)$ and all $\hbar \in \Lambda$.

A few elementary remarks are in order. The upper bound in (P1) is an immediate consequence of the positivity of $T_\hbar$. Furthermore, since $fg = gf$, from (P3) it follows that $[T_\hbar(f), T_\hbar(g)] = \mathcal{O}(\hbar)$. Property (P2) can be viewed as a refinement of this formula. Note also that substituting $f = 1$ into (P4), we see that the dimension of the space $\mathcal{H}_\hbar$ tends to ∞ as $\hbar \to 0$.

Theorem 1.1. *Every quantizable symplectic manifold admits a Berezin quantization satisfying* (P1)–(P4).

The novelty here is a fine structure of the remainders in (P1)-(P3). In particular, for quantizable Kähler manifolds, the standard Berezin–Toeplitz quantization satisfies (P1)–(P4).

Interestingly enough, for fixed ω and ρ, the coefficients α, β and γ are subject to constraints which manifest the optimality of inequalities (P1)–(P3). We discuss them in the next section. As a counterpoint, for certain quantization the constant δ in (P4) can be made arbitrarily small, see Remark 1.4 below. Furthermore, we present applications of (P1)–(P3) to semiclassical quantum measurements.

The seminal reference on Berezin quantization is the book [BdMG81] by Boutet de Monvel and Guillemin. In [BMS94, Gui95] or [BU96], it was deduced from [BdMG81] the existence of a quantization satisfying the following version of (P1)–(P3): for any smooth functions f, g we have

$$\|T_\hbar(f)\|_{\mathrm{op}} = \|f\| + \mathcal{O}(\hbar), \quad T_\hbar(f)T_\hbar(g) = T_\hbar(fg) + \mathcal{O}(\hbar), \tag{2}$$

$$[T_\hbar(f), T_\hbar(g)] = \tfrac{\hbar}{i} T_\hbar(\{f,g\}) + \mathcal{O}(\hbar^2), \tag{3}$$

where the $\mathcal{O}$'s are in the uniform norm and depend on f and g. More recently, Barron et al [BMMP14] have extended (2) to functions of class $\mathcal{C}^2$ and (3) to functions of class $\mathcal{C}^4$. We will prove in this paper that actually (3) holds for functions of class $\mathcal{C}^3$ (as we shall see in Remark 1.12 below, the absence of fourth derivatives in the correspondence principle (P2) has meaningful applications). Additionally, in Proposition 3.8 below we prove a slightly stronger version of quasi-multiplicativity (P3). More importantly, we make explicit the dependence in f and g of the remainders in the sense of (P1)–(P3).

We refer the reader to the lecture notes [LF16] by Y. Le Floch for a skillfully written exposition of our Theorem 1.1 in the Kähler case and useful preliminaries.

1.3. Rigidity of remainders. Confronting matrix analysis with geometry of the phase space, we get the following constraints on the remainders in (P1), (P2) and (P3) for any Berezin quantization.

Theorem 1.2. *Let (M, ω) be a closed quantizable symplectic manifold equipped with a Riemannian metric ρ. There exist positive constants C_1, C_2, C_3 depending on (M, ω, ρ) such that for every Berezin quantization we have*

(i) $\alpha \geq C_1$;
(ii) $\beta \geq C_2 \alpha^{-2}$;
(iii) $\gamma \geq C_3$.

The proof is given in §2 below.

The lower bound (ii) on the β-remainder in the correspondence principle (P2) deserves a special discussion. According to the classical no-go theorem (see, e.g., [GM00] and references therein), there is no linear map $T_\hbar \colon C^\infty(M) \to \mathcal{L}(\mathcal{H}_\hbar)$ that sends (up to a multiplicative constant) Poisson brackets of functions to commutators of operators. In other words, $\beta \neq 0$. As we shall see below, the proof of (ii), in addition to (P2), involves only the norm correspondence (P1) with fixed α. Therefore, in the presence of (P1), inequality (ii) can be regarded as a quantitative version of the no-go theorem.

Furthermore, the proof of (ii) involves both the lower and the upper bounds in the norm correspondence (P1). Recall that the latter uses that the Berezin quantization is positive. Interestingly enough, without the preservation of positivity (ii) is not necessarily valid. For instance, for the *geometric quantization* of a compact quantizable Kähler manifold in the presence of the metaplectic correction, the remainder in (P2) is of the order $\mathcal{O}(\hbar^2)$, see formula (19) in [Cha07]. Let us mention that the preservation of positivity is crucial for our applications to quantum measurements.

Another mysterious feature of the lower bound (ii) is that it involves the α-remainder appearing in the norm correspondence (P1). In particular, it does not rule out the existence of a quantization with the large error coefficient α in the norm correspondence (P1) and a small error coefficient β in the correspondence principle (P2). At the moment, we do not know whether such a trade-off between α and β can actually happen. This leads to the following question.

Question 1.3. Does there exist a constant $C_4 > 0$ such that for every Berezin quantization $\beta \geq C_4$?

Remark 1.4. There is no rigidity for the δ-remainder in (P4). Indeed, for the Berezin–Toeplitz quantization with the half-form correction [Cha06, formula (11)] the trace correspondence (P4) upgrades to

$$\left| \text{trace } (T_\hbar(f)) - (2\pi\hbar)^{-n} \int_M f \frac{\omega^n}{n!} \right| \leq \delta' \cdot \|f\|_{L_1} \hbar^{-(n-2)},$$

mind the power $-(n-2)$ on the right-hand side. Thus, decreasing $\hbar$, we can make $\delta = \delta' \hbar$ arbitrarily small.

1.4. Bargmann space. Even though Berezin–Toeplitz quantization of certain noncompact manifolds has been studied since the foundational paper [Ber75] (see [MM08] for more recent developments), no general statement in the spirit of Theorem 1.1 is currently available in the absence of compactness. Nevertheless, we perform a case study and explain how (P1)–(P4) extend to a noteworthy quantization of the symplectic vector space $\mathbb{R}^{2n} = \mathbb{C}^n$, namely to Toeplitz operators in Bargmann space. We also give a general estimate of the remainder in the composition of Toeplitz operators. Surprisingly, we did not find these results in the literature. The Bargmann space serves as a source of intuition for several aspects of our exploration of compact manifolds. First, for the Bargmann space, the quasimultiplicativity (P3) follows from an elementary (albeit tricky) algebraic consideration and thus enables one to guess the structure of the remainders in the compact case. Second, for the Bargmann space, quantization commutes with the phase-space rescaling (see formula (63) below). This highlights the significance of the rescaling, which serves as a useful tool for proving the rigidity of remainders in the compact case.

Recall that for any $\hbar > 0$, the Bargmann space $\mathcal{B}_\hbar$ is the space of holomorphic functions of $\mathbb{C}^n$ that are square integrable against the weight $e^{-\hbar^{-1}|z|^2}\mu$. Here μ is the measure $|dz_1 \dots dz_n \, d\bar{z}_1 \dots d\bar{z}_n|$ and $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$. For any $f \in L^\infty(\mathbb{C}^n)$, define the Toeplitz operator with multiplicator f:

$$T_\hbar(f) = \Pi_\hbar f \colon \mathcal{B}_\hbar \to \mathcal{B}_\hbar,$$

where $\Pi_\hbar$ is the orthogonal projector of $L^2(\mathbb{C}^n, e^{-\hbar^{-1}|z|^2}\mu)$ onto $\mathcal{B}_\hbar$. $T_\hbar(f)$ is a bounded operator with uniform norm $\|T_\hbar(f)\|_{\mathrm{op}} \leq \sup |f|$. If $f \in L^1(\mathbb{C}^n, \mu)$, one readily checks that the operator $T_\hbar(f)$ is of the trace class, and (P4) holds with the vanishing error term (i.e., $\delta = 0$).

For any integer k and function $f \colon \mathbb{C}^n \to \mathbb{C}$ of class $\mathcal{C}^k$, introduce the seminorm

$$|f|'_k = \sup_{\substack{|\alpha|=k,\\ x\in\mathbb{C}^n}} |\partial^\alpha f(x)|. \tag{4}$$

Let $\mathcal{C}^k_{\mathrm{b}}(\mathbb{C}^n)$ be the space of functions f of class $\mathcal{C}^k$ such that $|f|'_0, |f|'_1, \dots, |f|'_k$ are bounded.

Theorem 1.5. *For any $N \in \mathbb{N}$, there exists $C_N > 0$ such that for any $f \in \mathcal{C}^{2N}_{\mathrm{b}}(\mathbb{C}^n)$ and $g \in \mathcal{C}^N_{\mathrm{b}}(\mathbb{C}^n)$, for any $\hbar$ we have*

$$T_\hbar(f)T_\hbar(g) = \sum_{\ell=0}^{N-1} (-1)^\ell \hbar^\ell \sum_{\substack{\alpha\in\mathbb{N}^n,\\ |\alpha|=\ell}} \frac{1}{\alpha!} T_\hbar\big((\partial_z^\alpha f)(\partial_{\bar{z}}^\alpha g)\big) + \hbar^N R_N(f,g),$$

where $\|R_N(f,g)\|_{\mathrm{op}} \leq C_N \sum_{m=0}^N |f|'_{N+m} |g|'_{N-m}$.

To our knowledge, the estimate of the remainder is better than what can be found in the literature. For instance, in [Co92] or [Chi09], the number of derivatives involved in the estimates depends on the dimension n. However, in [Le10, Section 2.4.3], Lerner obtained estimates similar to Theorem 1.5 with $N = 2$ for the quantity $T_1(f)T_1(g) + T_1(g)T_1(f)$.

Interestingly, it seems that Berezin–Toeplitz quantization behaves better than Weyl quantization. Indeed, in all the results we know, the number of derivatives needed to estimate the remainder when we truncate the Moyal product, depends on the dimension. Actually, the Weyl symbol of the Toeplitz operator with multiplicator f is obtained by smoothing out f, cf. for instance [BS91, Section 5.2] or [Fo89, Section 2.7]. So the higher derivatives of the Weyl symbol are controlled in a sense by the lower derivatives of the multiplicator. Another observation is that the norm estimates of pseudodifferential operators, as for instance in the Calderon–Vaillancourt theorem, are generally obtained by the Cotlar–Stein lemma, whereas the Schur lemma is generally sufficient to estimate the norm of Toeplitz operators. However, many authors use Cotlar–Stein lemma for Toeplitz operators even if it is not necessary. For general results on Weyl quantization with symbols of limited regularity, we refer the reader to [Sj08].

By Theorem 1.5 with $N = 1$ and $N = 2$, we obtain the following version of (P3) and (P2), respectively: for any $f \in \mathcal{C}^2_{\mathrm{b}}(\mathbb{C}^n)$ and $g \in \mathcal{C}^1_{\mathrm{b}}(\mathbb{C}^n)$, we have

$$\big\|T_\hbar(fg) - T_\hbar(f)T_\hbar(g)\big\|_{\mathrm{op}} \leq \gamma'(|f|'_1|g|'_1 + |f|'_2|g|'_0)\hbar,$$

and for any $f \in \mathcal{C}^4_{\mathrm{b}}(\mathbb{C}^n)$ and $g \in \mathcal{C}^2_{\mathrm{b}}(\mathbb{C}^n)$,

$$\Big\|-\frac{i}{\hbar}\big[T_\hbar(f), T_\hbar(g)\big] - T_\hbar(\{f, g\})\Big\|_{\mathrm{op}} \leq \beta'\big(|f|'_2|g|'_2 + |f|'_3|g|'_1 + |f|'_4|g|'_0\big)\hbar,$$

where the constants γ', β' do not depend on f, g. Adapting the proof of (P2) in the closed symplectic case, we will also show that for any $f, g \in \mathcal{C}^3_{\mathrm{b}}(\mathbb{C}^n)$,

$$\Big\|-\frac{i}{\hbar}\big[T_\hbar(f), T_\hbar(g)\big] - T_\hbar(\{f, g\})\Big\|_{\mathrm{op}} \leq \beta''\big(|f|'_1|g|'_3 + |f|'_2|g|'_2 + |f|'_3|g|'_1\big)\hbar. \tag{5}$$

It would be interesting to calculate the constants α', β', β'' and γ'.

Let us mention finally that by using the explicit form (64) of the Schwartz kernel of $\Pi_\hbar$ and arguing as in the compact case, one gets a lower bound for $\|T_\hbar(f)\|_{\mathrm{op}}$ as in (P1), provided f is C^2-smooth and its uniform norm is attained at some point of $\mathbb{C}^n$. It is unclear whether the latter condition can be relaxed.

1.5. Joint approximate measurements. In this section we present an application of our results to joint approximate measurements of semiclassical observables. A slightly different measurement scheme was discussed in a similar context in [Po14] (see also [PR14, Chapter 9]). The main novelty is that the sharp remainder bounds enable us to work on smaller scales including the quantum length scale.

1.5.1. Preliminaries on quantum measurements. We start with some preliminaries on positive operator-valued measures and quantum measurements (we refer the reader to [BLW14], [PR14, Chapter 9] and references therein). Recall that $\mathcal{L}(\mathcal{H})$ denotes the space of all Hermitian operators on a finite-dimensional complex Hilbert space $\mathcal{H}$.

Consider a set Θ equipped with a σ-algebra $\mathcal{C}$ of its subsets. An $\mathcal{L}(\mathcal{H})$-valued *positive operator-valued measure* (POVM) W on $(\Theta, \mathcal{C})$ is a countably additive map $W : \mathcal{C} \to \mathcal{L}(\mathcal{H})$ that takes each subset $X \in \mathcal{C}$ to a positive operator $W(X) \in \mathcal{L}(\mathcal{H})$ and is normalized by $W(\Theta) = \mathbb{1}$.

In quantum measurement theory, W represents a measuring device coupled with the system, while Θ is interpreted as the space of device readings. When the system is in a pure state $\xi \in \mathcal{H}$, $|\xi| = 1$, the probability of finding the device in a subset $X \in \mathcal{C}$ equals $\langle W(X)\xi, \xi\rangle$. Given a bounded measurable function $f : \Theta \to \mathbb{R}$, one can define the integral $\mathbb{E}_W(f) := \int_\Theta f\, dW \in \mathcal{L}(\mathcal{H})$ as follows. Introduce a measure $\mu_{W,\xi}(X) = \langle W(X)\xi, \xi\rangle$ on Θ and put $\langle \mathbb{E}_W(f)\xi, \xi\rangle = \int_\Theta f\, d\mu_{W,\xi}$, for every state $\xi \in \mathcal{H}$. In a state ξ, the function f

becomes a random variable on Θ with respect to the measure $\mu_{W,\xi}$ with the expectation $\langle \mathbb{E}_W(f)\xi, \xi\rangle$.

Example 1.6. An important class of POVMs is formed by the *projector valued measures* P, for which all the operators $P(X)$, $X \in \mathcal{C}$ are orthogonal projectors. For instance, every quantum observable $A \in \mathcal{L}(\mathcal{H})$ with N pairwise distinct eigenvalues gives rise to the projector-valued measure $P := \{P_i\}$ on the set $\Theta_N := \{1, \dots, N\}$ and a function $\lambda \colon \Theta_N \to \mathbb{R}$ such that $A = \sum_{i=1}^N \lambda_i P_i$ is the spectral decomposition of A. At a state ξ, the probability of the outcome $\lambda = \lambda_i$ equals $p_i = \langle P\xi, \xi\rangle$, which agrees with the standard statistical postulate of quantum mechanics: the observable A takes value λ_i with probability p_i. Let us mention that at a pure state ξ the expectation of A equals $\langle A\xi, \xi\rangle$ and the variance $\mathbb{V}ar(A, \xi)$ equals $\langle A^2\xi, \xi\rangle - \langle A\xi, \xi\rangle^2$.

Example 1.7. POVMs naturally appear in the context of quantization. This will be fundamental for our discussion on quantum measurements of semiclassical observables. Let $\Theta = M$ be a quantizable symplectic manifold equipped with the Borel σ-algebra. Consider a Berezin quantization $(T_\hbar \colon \mathcal{C}(M, \mathbb{R}) \to \mathcal{L}(\mathcal{H}_\hbar), \hbar \in \Lambda)$. The maps $T_\hbar$ being positive, by Riesz theorem, we have $\mathbb{E}_{G_\hbar}(f) = T_\hbar(f)$ for a POVM $G_\hbar$ of M. Fix a sequence of quantum states $\xi_\hbar \in \mathcal{H}_\hbar$, $|\xi_\hbar| = 1$. In this case the measure $\mu_{G_\hbar, \xi_\hbar}$ governs the distribution of the quantum state $\xi_\hbar$ in the phase space. Limits of such measures as $\hbar \to 0$, which are called semiclassical defect measures (or Husimi measures), has been studied in the literature, see, e.g., [Z12, Chapter 5].

A somewhat simplistic description of quantum measurement is as follows: an experimentalist, after setting a quantum measuring device (i.e., an $\mathcal{L}(\mathcal{H})$-valued POVM W on Θ), performs a measurement whose outcome, at every state ξ, is the measure $\mu_{W,\xi}$ on Θ. Given a function f on Θ (experimentalist's choice), this procedure yields *an unbiased approximate measurement* of the quantum observable $A := \mathbb{E}_W(f)$. The expectation of A in every state ξ coincides with the one of the measurement procedure (hence *unbiased*), in spite of the fact that actual probability distributions determined by the observable A (see Example 1.6 above) and the pair $(f, \mu_{W,\xi})$ could be quite different (hence *approximate*). In particular, in general, the variance increases under an unbiased approximate measurement: $\mathbb{V}ar(f, \mu_{W,\xi}) = \mathbb{V}ar(A, \xi) + \langle \Delta_W(f)\xi, \xi\rangle$, where

$$\Delta_W(f) := \mathbb{E}_W(f^2) - \mathbb{E}_W(f)^2$$

is *the noise operator*. This operator, which is known to be positive, measures the increment of the variance. Furthermore, $\Delta_W(f) = 0$ provided W is a projector-valued measure, and hence every quantum observable admits a noiseless measurement in the light of Example 1.6.

Example 1.8. In the setting of Example 1.7, the noise operator $\Delta_{G_\hbar}(f)$ is given by the expression $T_\hbar(f^2) - T_\hbar(f)^2$ appearing on the left-hand side of the quasimultiplicativity property (P3). Look now at the case when ξ is an eigenvector of $T_\hbar(f)$ with an eigenvalue λ for a smooth classical observable f on M. The expectation of f with respect to $\mu_{G_\hbar, \xi}$ equals λ, while the variance coincides with the noise $\langle \Delta_{G_\hbar}(f)\xi, \xi\rangle$. By (P3), the latter does not exceed $\gamma |f, f|_2 \hbar$. It follows from the Chebyshev inequality that for every $r > 0$ (perhaps depending on $\hbar$)

$$\mu_{G_\hbar, \xi}(\{|f - \lambda| \geq r\}) \leq \frac{\gamma |f, f|_2 \hbar}{r^2}. \tag{6}$$

This inequality manifests the fact that in the semiclassical limit the eigenfunctions are concentrated near the energy level $\{f = \lambda\}$, see, e.g., [Z12, 6.2.1]. Note also that (6) provides a meaningful estimate for the concentration at the quantum length scale $r \sim \sqrt{\hbar}$.

To compare with, the usual method to estimate the left-hand side of (6) is to build a local inverse of $T(f) - \lambda$ on the region $\{|f - \lambda| \geq r\}$. In this way, we prove that $\mu_{G_\hbar,\xi}(\{|f - \lambda| \geq r\})$ is in $\mathcal{O}(\hbar^\infty)$ for smooth f and fixed r. More precisely, assuming that f is smooth, for any N and $r > 0$ one has

$$\mu_{G_\hbar,\xi}(\{|f - \lambda| \geq r\}) \leq C_N(f, r)\hbar^N,$$

where $C_N(f, r)$ is a positive constant independent of $\hbar$.

Let $A, B \in \mathcal{L}(\mathcal{H})$ be a pair of quantum observables. A *joint* unbiased approximate measurement of A and B consists of an $\mathcal{L}(\mathcal{H})$-valued POVM W on some space Θ and a pair of random variables f and g on Θ such that $\mathbb{E}_W(f) = A$ and $\mathbb{E}_W(g) = B$.

Definition 1.9. *The minimal noise* associated with the pair (A, B) is given by

$$\nu(A, B) := \inf_{W,f,g} \|\Delta_W(f)\|_{\mathrm{op}}^{1/2} \cdot \|\Delta_W(g)\|_{\mathrm{op}}^{1/2},$$

where the infimum is taken over all W, f, g as above. (Note: the space Θ is not fixed, it is varying together with the POVM W.)

The following *unsharpness principle* (see [PR14, Theorem 9.4.16]) provides a lower bound on the minimal noise:

$$\nu(A, B) \geq \frac{1}{2} \cdot \|[A, B]\|_{\mathrm{op}}. \tag{7}$$

It reflects impossibility of a noiseless joint unbiased approximate measurement of a pair of noncommuting observables.

Fix now any scheme $T_\hbar$ of Berezin quantization of a closed quantizable symplectic manifold (M, ω). Let $G_\hbar$ be the corresponding $\mathcal{L}(\mathcal{H}_\hbar)$-valued POVM on M, i.e., $T_\hbar(f) = \int f\, dG_\hbar$ for every smooth function f on M. In the light of Examples 1.7 and 1.8 above, the unsharpness principle yields

$$\begin{aligned} &\|T_\hbar(f^2) - T_\hbar(f)^2\|_{\mathrm{op}}^{1/2} \cdot \|T_\hbar(g^2) - T_\hbar(g)^2\|_{\mathrm{op}}^{1/2} \\ &\quad \geq \nu(T_\hbar(f), T_\hbar(g)) \geq \frac{1}{2} \cdot \|[T_\hbar(f), T_\hbar(g)]\|_{\mathrm{op}} \text{ for all } f, g \in \mathcal{C}^\infty(M). \end{aligned} \tag{8}$$

Combining this with (P3) and (P2), we get the following estimate for the minimal noise of a pair of semiclassical observables.

Proposition 1.10.

$$\gamma |f, f|_2^{1/2} |g, g|_2^{1/2} \hbar \geq \nu(T_\hbar(f), T_\hbar(g)) \geq \frac{1}{2} \cdot (\hbar \|\{f, g\}\| - \beta\hbar^2 |f, g|_{1,3}). \tag{9}$$

1.5.2. Joint measurements of the sign. Fix a monotone increasing function $u\colon \mathbb{R} \to [-1, 1]$ that equals -1 on $(-\infty, -1]$, satisfies $u(z) = z$ when z is near 0, and equals 1 on $[1, +\infty)$. For a function f on M and a positive $s \ll 1$, the classical observable $f_s := u(f(x)/s)$ can be viewed as a smooth approximation to the sign of $f(x)$. We refer to s as *the fuzziness* parameter.

Suppose that that the set $\{f = 0, g = 0\}$ is nonempty and

$$\sup_{\{f=0,g=0\}} |\{f, g\}| > 0. \tag{10}$$

Assume also that $0 < s \leq t \ll 1$. In what follows we are focusing on simultaneous approximate measurements of $T_\hbar(f_s)$ and $T_\hbar(g_t)$, the semiclassical observables that correspond to the signs of f and g with the fuzziness parameters s and t, respectively. Note that the operators $(1 + T_\hbar(f_s))/2$ and $(1 + T_\hbar(g_t))/2$ can be interpreted as "quasiprojectors" to the phase space regions $\{f > 0\}$ and $\{g > 0\}$, respectively. Since f and g do not Poisson commute, such a measurement is in general noisy due to the unsharpness principle.

Suppose now that $s = r\hbar^p$ and $t = R\hbar^q$ with $R, r > 0$, $p \geq 0$, $q \geq 0$, and $p + q \leq 1$. The standing assumption $s \leq t$ yields $p \geq q$, and $r \leq R$ if $p = q$.

Theorem 1.11. *Let $0 \leq q \leq p \leq 1/2$. There exists constants $c_+ > c_- > 0$ depending only on f, g, u and the metric such that*

$$c_- s^{-1} t^{-1} \hbar \leq \nu(T_\hbar(f_s), T_\hbar(g_t)) \leq c_+ s^{-1} t^{-1} \hbar \tag{11}$$

for all sufficiently small $\hbar$ and, in case $p = 1/2$, for all sufficiently large r.

In particular, if $p < 1/2$, the minimal noise ν is positive and $\sim \hbar^{1-p-q}$, and at the quantum length scale $p = q = 1/2$, it is bounded from below by $c_+ R^{-1} r^{-1} > 0$.

Proof. The result immediately follows from Proposition 1.10. Indeed the upper bound in (9) is not greater than $c_1 s^{-1} t^{-1} \hbar$, and the lower bound is not less than

$$c_2 s^{-1} t^{-1} \hbar - c_3 s^{-3} t^{-1} \hbar^2 = c_2 \hbar s^{-1} t^{-1} (1 - c_4 \hbar s^{-2}), \tag{12}$$

where the c_i are positive constants depending only on f, g, u and the metric. Note that the positivity of c_2 follows from assumption (10) on the Poisson non-commutativity of f and g. Since $s^{-2}\hbar = r^{-2}\hbar^{1-2p}$, we see that (11) holds for all sufficiently small $\hbar$ if $p < 1/2$ and for all sufficiently small $\hbar$ and all sufficiently large r if $p = 1/2$. □

This result deserves a discussion.

Remark 1.12. Now we are ready to explain the advantages of the reduced expression $|f, g|_{1,3}$ (see formula (1)) appearing in the remainder term of the correspondence principle (P2) as compared to $|f, g|_4$, which includes fourth derivatives. To this end, replace for a moment $|f, g|_{1,3}$ in the remainder of (P2) by $|f, g|_4$. Then, accordingly, the lower bound (12) will be modified as

$$c_2 \hbar s^{-1} t^{-1} - c_3 \hbar^2 s^{-4}.$$

The first term on the right-hand side is $\sim \hbar^{1-p-q}$ and the second term is $\sim \hbar^{2-4p}$. Thus, for the positivity of the right-hand side it is necessary that $3p - q \leq 1$. This inequality is violated, for instance, in the case where $p = 1/2$, $q = 0$, i.e., $s \sim r\hbar^{1/2}$ and $t \sim 1$. However, this case can be handled by using the reduced remainder. Indeed, inequality (11) above yields $\nu \sim R^{-1} r^{-1} \hbar^{1/2}$ for $p = 1/2, q = 0$.

Remark 1.13. Consider the following example. Let

$$M = S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$$

be the standard sphere equipped with the symplectic form of the total area 2π. Put $f = x$ and $g = y$. According to the prediction of [Po14] (which was made for a slightly different measurement scheme), the noise ν of such a measurement satisfies

$$\nu \gtrapprox \frac{\hbar}{\text{Area}(\Pi)}, \tag{13}$$

where Π is a "rectangle"

$$\Pi = \{|x| \lessapprox s,\ |y| \lessapprox t,\ z > 0\} \subset S^2.$$

Inequality (13) was proved in [Po14] for $s, t \sim 1$. Our methods confirm this prediction for smaller fuzziness parameters, including the quantum length scales $s \sim r\hbar^{1/2}, t \sim R\hbar^{1/2}$ as well as $s \sim r\hbar^{1/2}, t \sim 1$. Indeed, observe that $\text{Area}(\Pi) \approx st$ and hence inequality (13) follows from (11). Note that for $s \sim r\hbar^{1/2}, t \sim R\hbar^{1/2}$ the rectangle Π is a "quantum box": its area is $\sim rR\hbar$, the minimal possible (in terms of the power of $\hbar$) area occupied by a quantum state.

Remark 1.14. The minimal noise ν is well defined for arbitrary small fuzziness parameters s, t. When s and t are smaller than the quantum length $\sim \sqrt{\hbar}$, it is unclear how to calculate/estimate the minimal noise ν by standard methods of semiclassical analysis. Indeed, the derivatives of f_s and g_t blow up and the remainders in (P2), (P3) dominate the leading terms. In particular, the lower bound in (9) could become negative, and hence useless. We refer to a forthcoming paper [LFPS16] for a progress in this direction.

1.6. Phase space localization on quantum length scale. Our next application of the fine structure of the remainders in the Berezin quantization deals with phase space localization of a quantum particle at small scales. We use a model proposed in [Po14].

Let $\mathcal{U} = \{U_1, \dots, U_N\}$ be a finite open cover of a closed quantizable symplectic manifold M. Given a partition of unity $\{f_1, \dots, f_N\}$ subordinate to $\mathcal{U}$, consider the following *registration procedure*: if the system is prepared in a quantum state $\xi \in \mathcal{H}_\hbar$, $|\xi| = 1$, it is registered in the set U_i with probability $\langle T_\hbar(f_i)\xi, \xi\rangle$. Here the cover and the partition of unity may depend on $\hbar$. The registration procedure enables one to localize a semiclassical system in the phase space.

For $x \in Q := [-1,1]^N$ put $f_x := \sum x_i f_i$. Define

$$\mathcal{N}_+ := \max_{x \in Q} \|T_\hbar(f_x^2) - T_\hbar(f_x)^2\|_{\mathrm{op}}$$

and

$$\mathcal{N}_- := \frac{1}{2} \cdot \max_{x,y \in Q} \|[T_\hbar(f_x), T_\hbar(f_y)]\|_{\mathrm{op}}.$$

Observe that $\mathcal{N}_+ \geq \mathcal{N}_-$ by the unsharpness principle (8) above.

The above registration procedure is known to exhibit *inherent noise* which measures the unsharpness of the registration procedure. We refer to [Po14] and Chapter 9 of [PR14] for the precise definition. It is important for us that this noise lies in the interval $I_{\text{noise}} := [\mathcal{N}_-, \mathcal{N}_+]$, which we shall call *the noise interval*. The fine remainder estimates obtained in this paper yield meaningful bounds on the noise interval of the phase space localization procedure on small scales, up to the quantum length scale.

(♣) **The choice of the partition of unity.** To start with, let us choose a cover of M together with a subordinate partition of unity in a special way. Fix $r_0 > 0$ sufficiently small and for $0 < r \leq r_0$ consider a maximal $r/2$-net $\{z_i\}$ of points in M (with respect to the Riemannian distance d associated with the metric ρ). This means that $d(z_i, z_j) \geq r/2$ for $i \neq j$ and $\{z_i\}$ is a maximal collection with this property. Let $\mathcal{U}$ be the cover of M by metric balls $U_i := B(z_i, r)$. Let $u\colon [0, +\infty) \to [0,1]$ be a smooth cut-off function that equals 1 on $[0, 0.6]$ and vanishes on $[0.7, +\infty)$. Define functions g_i on M by $g_i(x) = u(d(x, z_i)/r)$. It was shown in [Po14] that for all sufficiently small $r < r_0(M, \omega, \rho)$ there exists p (depending only on the dimension of M) such that every $x \in M$ is covered by at most p balls U_i. Moreover, the balls $B(z_i, 0.6r)$ cover M. Thus, the functions

$$f_i := \frac{g_i}{\sum_{i=1}^N g_i}, \quad i = 1, \dots, N,$$

form a partition of unity subordinate to $\mathcal{U}$, and moreover, there exists $C > 0$ such that for every $r \in (0, r_0)$ and every $i = 1, \dots, N$

$$|f_i|_k \leq C r^{-k}, \quad k = 1, 2, 3. \tag{14}$$

In what follows we focus on the registration procedure associated with the cover $\mathcal{U}$ and the partition of unity $\{f_i\}$ described in (♣), where the radius $r \in (0, r_0]$ plays the role of a parameter. The next result provides bounds for the corresponding noise interval.

Theorem 1.15. *There exist constants* $0 < c_- < c_+$ *and* $\kappa > 0$ *depending only on* (M, ρ, ω) *such that*

$$I_{\text{noise}} \subset [c_- \hbar r^{-2}, c_+ \hbar r^{-2}] \tag{15}$$

for any sufficiently small $\hbar > 0$ *and* $r \in [\kappa \hbar^{1/2}, r_0]$.

Few remarks are in order. Choose $R > 0$, $\epsilon \in [0, 1/2]$ and apply Theorem 1.15 to $r = R\hbar^{1/2-\epsilon}$. If $\epsilon = 1/2$ and $R \in (0, r_0)$ is independent of $\hbar$, then the noise is strictly positive and of order $\sim \hbar$ as $\hbar \to 0$. This result, which was proved in [Po14], does not require fine remainder estimates. The latter enter the play when $\epsilon < 1/2$. Let us emphasize also that *for* $\epsilon = 0$ *and a fixed* $R \geq \kappa$, *i.e., on the quantum length scale, the noise is strictly positive and of order* ~ 1 *as* $\hbar \to 0$.

We also mention that the registration procedure above satisfies *noise-localization uncertainty relation*:

$$\text{Noise} \times \max_i \ \text{Size}\ (U_i) \geq c\hbar, \tag{16}$$

where Size is a properly defined symplectic invariant of U_i, and $c > 0$ is independent of $\hbar$. Indeed, Noise $\sim \hbar r^{-2}$ and since the U_i are Riemannian balls of a sufficiently small radius r, the size of U_i is $\sim r^2$. Relation (16) was established in [Po14] for the case $\epsilon = 1/2$ (i.e., for $r \sim 1$) for any partition of unity subordinate to the cover $\{U_i\}$. Here we work on smaller scales up to the quantum length scale. As a price for that, we have to assume that the derivatives of the functions forming the partition of unity are controlled by (14).

Proof of Theorem 1.15. Throughout the proof we denote by $c_1, c_2, \ldots$ positive constants independent of r and $\hbar$. We assume that $r \leq r_0$.

Observe that, by (14),

$$|f_x|_k \leq pCr^{-k}, \quad k = 1, 2, 3. \tag{17}$$

Thus by (P3),

$$\mathcal{N}_+ \leq c_1 \hbar r^{-2}.$$

In [Po14, Example 4.5 and formula (28)] it was shown that

$$\mu := \max_{x,y \in Q} \|\{f_x, f_y\}\| \geq c_2 r^{-2}.$$

By (P1), (P2), and (17), we have

$$\mathcal{N}_- \geq \frac{1}{2} \cdot \mu\hbar - c_3 \hbar^2 \max_{x,y \in Q} (|\{f_x, f_y\}|_2 + |f_x, f_y|_{1,3}) \geq c_4 \hbar r^{-2} - c_5 \hbar^2 r^{-4} \geq \hbar r^{-2} \Big(c_4 - \frac{c_5}{\kappa^2} \Big)$$

provided $r \geq \kappa \hbar^{1/2}$. If κ is sufficiently large, then $c_4 - c_5/\kappa^2 = c_6 > 0$. Combining the upper bound on $\mathcal{N}_+$ with the lower bound on $\mathcal{N}_-$, we get the desired result. □

Remark 1.16. A reader with a semiclassical background has certainly recognized some symbol in exotic classes in Theorems 1.11 and 1.15. Recall that these symbols are smooth functions depending on $\hbar$ and satisfying an estimate of the type $|\partial^\alpha f| \leq C_\alpha \hbar^{-\delta|\alpha|}$ for some fixed $\delta \in [0, 1/2]$. The theory of pseudodifferential operators can be extended to these symbols, providing an important tool in semiclassical analysis. Here these symbols appear in the functions f_s, g_t of Theorem 1.11 and in the functions f_x of Theorem 1.15. Observe that (P1)–(P4) are perfectly suited to handle these exotic symbols.

§2. Constraints on the remainders

In order to illustrate properties (P1)–(P4), we start with proving Theorem 1.2. Our strategy is to apply these properties to specially chosen symbols pushed to the limits of pseudo-differential calculus, that is symbols supported in a ball of radius $\sim \sqrt{\hbar}$. Items (i), (ii), (iii) of the theorem are proved in Subsections 2.2, 2.3 and 2.4, respectively.

2.1. Test balls and scaling relations. Certain constructions below are local, i.e., the action takes place in a neighbourhood of a point in M. To facilitate the discussion, we shall fix a *test ball* $B(r) \subset M$, that is an open ball whose closure lies in a Darboux chart equipped with coordinates $(x_1, \dots, x_{2n})$. The ball B is given by $\left\{\sum x_i^2 < r^2\right\}$, where $r \leq 1$. In the chart the symplectic form ω is given by $dx_1 \wedge dx_2 + \dots + dx_{2n-1} \wedge dx_{2n}$. It would be convenient to assume, without loss of generality, that the metric ρ in the chart is Euclidean. This assumption will change various bounds on the norms of derivatives $|f|_N$ as well as the bounds on the quantities α, β, γ entering (P1)–(P3) by multiplicative constants whose precise values are irrelevant for our discussion.

In what follows every compactly supported smooth function $f \in \mathcal{C}_c^\infty(B)$ is viewed as a smooth function on M: we extend it by 0.

For a function $f \in \mathcal{C}_c^\infty(B(1))$ and a number $s \in (0,1]$, we define a rescaled function $f_s \in \mathcal{C}^\infty(B)$ as follows: $f_s(x) = f(x/s)$ for $x \in B(s)$ and $f_s(x) = 0$ otherwise. The following obvious *scaling relations* turn out to be very useful below:

$$\begin{aligned} &|f_s|_k = s^{-k}|f|_k, \quad |f_s, g_s|_k = s^{-k}|f, g|_k, \\ &|f_s, g_s|_{1,3} = s^{-4}|f, g|_{1,3}, \quad |\{f_s, g_s\}|_k = s^{-(k+2)}|\{f, g\}_s|_k. \end{aligned} \tag{18}$$

2.2. α-remainder. We shall show that, for a test ball $B = B^{2n}(1)$,

$$\alpha \geq c \cdot \sup_f \frac{\|f\|^{1+1/n}}{\|f\|_{L_1}^{1/n} \cdot |f|_2}, \tag{19}$$

where the supremum is taken over all smooth nonconstant nonnegative compactly supported functions f on B, and $c > 0$ is a numerical constant. Incidentally, the finiteness of the supremum on the right-hand side of (19) follows from a generalized interpolation inequality in [CZ98]. Additionally, our proof shows that the constant c is independent of the dimension $2n$.

Indeed, fix any function f as above. Put $f_s(x) = f(x/s)$ with $s = \sqrt{t\hbar}$. Combining the scaling relations (18) with (P1) and (P4), we see that

$$\|T_\hbar(f_s)\|_{\mathrm{op}} \geq \|f\| - \alpha t^{-1}|f|_2$$

and

$$\operatorname{trace}(T_\hbar(f_s)) \leq (t/(2\pi))^n \cdot \|f\|_{L_1}(1 + \delta\hbar).$$

Noticing that $\|T_\hbar(f_s)\|_{\mathrm{op}} \leq \operatorname{trace}(T_\hbar(f_s))$ because $f_s \geq 0$, we get

$$\|f\| - \alpha t^{-1}|f|_2 \leq (t/(2\pi))^n \cdot \|f\|_{L_1}(1 + \delta\hbar).$$

Here t is fixed, and this inequality holds for all $\hbar$. Sending $\hbar \to 0$, we conclude that

$$\alpha \geq u(t) := \frac{\|f\| \cdot t - \frac{|f|_{L_1}}{(2\pi)^n} \cdot t^{n+1}}{|f|_2}.$$

One readily calculates that the maximal value of u equals

$$c(n) \cdot \frac{\|f\|^{1+1/n}}{\|f\|_{L_1}^{1/n} \cdot |f|_2},$$

where $c(n) = 2\pi n/(n+1)^{1+1/n} \to 2\pi$ as $n \to \infty$. This proves (19) with $c > 0$ independent on n. □

Remark 2.1. By (P4), $d_\hbar := \dim \mathcal{H}_\hbar = (2\pi\hbar)^{-n} \cdot \mathrm{Vol}(M) + \mathcal{O}(\hbar^{-(n-1)})$. It turns out that a weaker dimension bound, still capturing the correct order of $d_\hbar$ in $\hbar$, follows from the norm correspondence (P1):

$$d_\hbar \geq c\alpha^{-n}\hbar^{-n}, \quad c > 0. \tag{20}$$

The standard quantum mechanical intuition behind this formula is as follows: consider the partition of M into $\sim \mathrm{Vol}(M)\hbar^{-n}$ "quantum boxes", i.e., cubes of side $\sim \sqrt{\hbar}$. Since each box carries ~ 1 quantum state, and the states corresponding to different boxes are approximately orthogonal, $\mathcal{H}$ contains a subspace of dimension $\sim \hbar^{-n}$, which yields (20). The actual proof follows this idea, with the following amendment. Instead of partitioning M into quantum boxes, we cover M by $\sim \hbar^{-n}$ balls of radii $\sim \sqrt{\hbar}$ as in (♣) of Subsection 1.6 above, and apply (P1) to a specially chosen subordinate partition of unity. Let us present the formal argument. Denote by $c_0, c_1, \ldots$ positive constants depending on the manifold M and the metric ρ. From (♣) it readily follows that for every sufficiently small $r > 0$, the manifold M admits a partition of unity $f_1, \ldots, f_N$ with $N \geq c_0 r^{-2n}$, $\|f_i\| \geq c_1$ and $|f_i|_2 \leq c_2 r^{-2}$ for all i. By (P1), for all i we have

$$\|A_i\|_{\mathrm{op}} \geq c_1 - \alpha \cdot c_2 r^{-2}\hbar. \tag{21}$$

Since $A_i \geq 0$, it follows that $\mathrm{trace}(A_i) \geq \|A_i\|_{\mathrm{op}}$. Therefore,

$$d_\hbar = \mathrm{trace}(\mathbb{1}) = \sum_{i=1}^{N} \mathrm{trace}(A_i) \geq \sum_{i=1}^{N} \|A_i\|_{\mathrm{op}}.$$

Combining (21) with $N \geq c_0 r^{-2n}$, we get

$$d_\hbar \geq c_0 r^{-2n}(c_1 - \alpha \cdot c_2 r^{-2}\hbar).$$

Choosing $r = c_3\alpha^{1/2}\hbar^{1/2}$ with $c_3 > 0$ sufficiently large, we arrive at (20). □

2.3. β-remainder. Again, we work in a test ball $B = B^{2n}(1)$.

Step 1. Write $B = B^{2n}(1)$. Fix a pair of noncommuting functions $f, g \in \mathcal{C}_c^\infty(B)$. Observe that

$$\|[T_\hbar(f), T_\hbar(g)]\|_{\mathrm{op}} \leq 2\|T_\hbar(f)\|_{\mathrm{op}} \cdot \|T_\hbar(g)\|_{\mathrm{op}} \leq 2\|f\| \cdot \|g\|.$$

We emphasize that the inequality on the right uses the positivity of the Berezin quantization and in general fails for Weyl-like quantizations (cf. discussion after Theorem 1.2 above). Furthermore,

$$\|T_\hbar(\{f,g\})\|_{\mathrm{op}} \geq \|\{f,g\}\| - \alpha\hbar|\{f,g\}|_2.$$

Combining these inequalities with the correspondence principle (P2), we see that

$$\hbar^2(\alpha|\{f,g\}|_2 + \beta|f,g|_{1,3}) \geq \hbar\|\{f,g\}\| - 2\|f\| \cdot \|g\|. \tag{22}$$

The rest of the proof proceeds by two successive optimizations: first, on the "size" of f, g through a rescaling to the quantum scale $\sim \sqrt{\hbar}$, and second, on the "shapes" of f and g.

Step 2. Applying the scaling relations to equation (22) above we get

$$\hbar^2(\alpha s^{-4}|\{f,g\}|_2 + \beta s^{-4}|f,g|_{1,3}) \geq \hbar s^{-2}\|\{f,g\}\| - 2\|f\| \cdot \|g\|.$$

Put $t = s^2\hbar^{-1}$, $a = \|\{f,g\}\|$, $b = 2\|f\| \cdot \|g\|$ and rewrite this inequality as

$$\alpha|\{f,g\}|_2 + \beta|f,g|_{1,3} \geq ta - t^2 b.$$

The right-hand side attains the maximum $a^2/(4b)$ for $t = a/(2b)$. Note that $t = a/(2b)$ means that $s = \sqrt{a/(2b)} \cdot \hbar$, and so $s \in (0,1)$ for $\hbar$ sufficiently small. Therefore, for all $f, g \in \mathcal{C}_c^\infty(B)$ with $\{f, g\} \neq 0$ we have

$$\alpha|\{f,g\}|_2 + \beta|f,g|_{1,3} \geq \frac{\|\{f,g\}\|^2}{8\|f\| \cdot \|g\|}. \tag{23}$$

Step 3. Next, we take f, g in the following special form. Choose non-commuting functions $F, G \in \mathcal{C}_c^\infty(B)$, and put

$$f = z^{1/2} F \sin(z^{-1}G), \quad g = z^{1/2} F \cos(z^{-1}G),$$

where $z > 0$ plays the role of a parameter. A direct calculation shows that the Poisson bracket

$$u := \{f, g\} = \{-F^2/2, G\}$$

is independent of z. At the same time, if $\|F\| \leq 1$, we have $\|f\| \cdot \|g\| \leq z$.

Recall that by Theorem 1.2 we have a bound $\alpha \geq C_1 > 0$ with C_1 depending only on (M, ω, ρ). Put $Z = \|u\|^2/(12C_1|u|_2)$ and observe that for all $z \in (0, Z]$ we have $|f,g|_{1,3} \leq K \cdot z^{-3}$ with some $K > 0$. Combining this with (23) we conclude that

$$\alpha|u|_2 + K\beta z^{-3} \geq z^{-1}\|u\|^2/8 \text{ for all } z \in (0, Z],$$

which yields

$$K\beta \geq v(z) := z^2\|u\|^2/8 - z^3\alpha|u|_2 \text{ for all } z \in (0, Z]. \tag{24}$$

Observe now that the function $v(z)$ attains its maximal value $c\alpha^{-2}$ with $c = \|u\|^6/(2 \cdot 12^3|u|_2^2)$ at

$$z_0 = \|u\|^2/(12\alpha|u|_2) \in (0, Z].$$

Substituting z_0 into (24), we see that $\beta \geq K^{-1}c\alpha^{-2}$, as required.

2.4. γ-remainder. Applying (P3) to $T_\hbar(fg)$ and $T_\hbar(gf)$ and subtracting we get

$$\|[T_\hbar(f), T_\hbar(g)]\|_{\text{op}} \leq 2\gamma|f,g|_2\hbar.$$

On the other hand, by (P1) and (P2),

$$\|[T_\hbar(f), T_\hbar(g)]\|_{\text{op}} \geq \hbar\|\{f,g\}\| + O(\hbar^2).$$

Combining these inequalities and letting $\hbar \to 0$, we obtain

$$\gamma \geq \sup_{f,g} \frac{\|\{f,g\}\|}{2|f,g|_2} > 0,$$

where the supremum is taken over all pairs of smooth noncommuting functions f and g on M. □

§3. Quantization of symplectic manifolds

3.1. Preliminaries. Consider a compact manifold M endowed with a volume form μ and a Hermitian line bundle $A \to M$. The space $\mathcal{C}^0(M, A)$ of continuous sections of A has a natural scalar product $\langle \cdot, \cdot \rangle$ given by integrating the pointwise scalar product against μ. We denote by $\|\psi\| = \langle \psi, \psi \rangle^{\frac{1}{2}}$ the corresponding norm. A bounded operator P of $\mathcal{C}^0(M, A)$ is by definition a continuous endomorphism of the normed vector space $(\mathcal{C}^0(M, A), \| \cdot \|)$. Its norm is defined by

$$\|P\|_{\text{op}} = \sup \frac{\|Ps\|}{\|s\|},$$

where s runs over the nonvanishing continuous sections of A. Equivalently, we could introduce the completion $L^2(M, A)$ of the pre-Hilbert space $\mathcal{C}^0(M, A)$ with $\langle \cdot, \cdot \rangle$, extend

P to a bounded operator of $L^2(M,A)$, and define $\|P\|_{\mathrm{op}}$ as the norm of the extension. Actually, the space $\mathcal{C}^0(M,A)$ is sufficient for our needs, and we will not use its completion in the sequel.

With any continuous section K of $A \boxtimes \overline{A} \to M^2$, [1] we associate the endomorphism P of $\mathcal{C}^0(M,A)$ given by

$$(P\Psi)(x) = \int_M K(x,y)\cdot \Psi(y)\,\mu(y).$$

Here the dot stands for the contraction $A_y \times \overline{A}_y \to \mathbb{C}$ induced by the metric of A. K is uniquely determined by P and is called the Schwartz kernel of P. M being compact, P is bounded. The basic estimate we need is the Schur test, see, e.g., [HS78, Theorem 5.2].

Proposition 3.1. *Let P be the endomorphism of $\mathcal{C}^0(M,A)$ with Schwartz kernel $K \in \mathcal{C}^0(M^2, A\boxtimes \overline{A})$. Then $\|P\|^2_{\mathrm{op}} \leq C_1C_2$ where C_1, C_2 are the nonnegative real numbers given by*

$$C_1 = \sup_{x\in M}\int_M |K(x,\,\cdot\,)|\mu, \quad C_2 = \sup_{y\in M}\int_M |K(\,\cdot\,,y)|\mu.$$

We shall also need the following easy properties. Let $K \in \mathcal{C}^0(M^2, A\boxtimes\overline{A})$ be the Schwartz kernel of P.

- For any $f \in \mathcal{C}^0(M,\mathbb{C})$, $(1\boxtimes f)K$ and $(f\boxtimes 1)K$ are the Schwartz kernels of Pf and fP, respectively.
- Let ∇ be a Hermitian connection of A and assume that K is of class $\mathcal{C}^1$. Then for any continuous vector field X of M, $(\nabla_X \boxtimes \mathrm{id})K$ is the Schwartz kernel of $\nabla_X \circ P$. Furthermore, if X is of class $\mathcal{C}^1$, the operator $P\circ\nabla_X\colon \mathcal{C}^1(M,A)\to\mathcal{C}^0(M,A)$ extends to the bounded operator of $\mathcal{C}^0(M,A)$ with the kernel $-(\mathrm{id}\boxtimes(\nabla_X + \operatorname{div} X))K$. Here the divergence is defined by the formula $\mathcal{L}_X\mu = \operatorname{div}(X)\mu$.

In the sequel, we often denote an operator and its Schwartz kernel by the same letter.

3.2. Toeplitz operators. As in Subsection 3.1, consider a compact manifold M endowed with a volume form and a Hermitian line bundle $A\to M$. Let $\mathcal{H}$ be a finite-dimensional subspace of $\mathcal{C}^\infty(M,A)$. Let B be the section of $A\boxtimes\overline{A}$ defined by

$$B(x,y) = \sum_{i=1}^N e_i(x)\otimes \overline{e}_i(y), \quad x,y\in M, \tag{25}$$

where $(e_i,\ i=1,\dots,N)$ is any orthonormal basis of $\mathcal{H}$. The operator Π with Schwartz kernel B is the projector from $\mathcal{C}^0(M,A)$ onto $\mathcal{H}$ whose kernel is the orthogonal complement of $\mathcal{H}$ in $\mathcal{C}^0(M,A)$. Even if we are not in a genuine Hilbert space, we call Π an orthogonal projector. For any $f\in\mathcal{C}^0(M)$, define the Toeplitz operator

$$T(f) := \Pi f\colon \mathcal{H}\to\mathcal{H}.$$

Here f stands for the multiplication operator by f. The map sending f to $T(f)$ is clearly linear and positive. Furthermore, $T(1)=\mathrm{id}$.

3.3. Bergman kernels and generalizations. Consider a quantizable symplectic compact manifold (M,ω). Our aim is to produce a Berezin quantization $(T_\hbar\colon \mathcal{C}^0(M)\to\mathcal{L}(\mathcal{H}_\hbar), \hbar\in\Lambda)$. We will use the integral parameter k instead of $\hbar\in\mathbb{N}$ [2] having in mind that $\hbar = 1/k$. The Hilbert space $\mathcal{H}_k$ will be defined as a finite-dimensional subspace of $\mathcal{C}^\infty(M,A_k)$ with A_k a conveniently defined Hermitian line bundle. The linear map $T_k\colon\mathcal{C}^0(M)\to\mathcal{L}(\mathcal{H}_k)$ will be the corresponding Toeplitz quantization as in Subsection 3.2.

[1] If $E\to M$ and $F\to N$ are two vector bundles, $E\boxtimes F\to M\times N$ is the vector bundle $(\pi_M^*E)\otimes(\pi_N^*F)$, where π_M, π_N are the projections from $M\times N$ onto M and N, respectively.

[2] $\mathbb{N}$ is the set $\mathbb{Z}_{\geq 0}$ of nonnegative integers.

M being quantizable, it admits a prequantum bundle L, that is a Hermitian line bundle endowed with a connection ∇ of curvature $\frac{1}{i}\omega$. Consider a complex structure j, which is not necessarily integrable, but is compatible with ω, meaning that $\omega(jX, jY) = \omega(X, Y)$ for any tangent vectors $X, Y \in T_pM$, and if X does not vanish, $\omega(X, jX) > 0$. We denote by $T^{1,0}M$ the subbundle $\ker(j - i)$ of $TM \otimes \mathbb{C}$. Consider also an auxiliary Hermitian line bundle A.

For any $k \in \mathbb{N}$, let $A_k = L^k \otimes A$, and endow the space $\mathcal{C}^0(M, A_k)$ with the scalar product defined by integrating the pointwise scalar product against the Liouville volume $\mu = \omega^n/n!$. With any finite-dimensional subspace $\mathcal{H}_k$ of $\mathcal{C}^\infty(M, A_k)$ we associate a smooth kernel $B_k \in \mathcal{C}^\infty(M^2, A_k \boxtimes \overline{A}_k)$ defined as in (25). The metrics of L and A induce the identifications $L_x \otimes \overline{L}_x \simeq \mathbb{C}$ and $A_x \otimes \overline{A}_x \simeq \mathbb{C}$. In the sequel, we shall often view $B_k(x, x)$ as a complex number through these identifications.

Theorem 3.2. *There exists a family* $(\mathcal{H}_k \subset \mathcal{C}^\infty(M, L^k \otimes A),\ k \in \mathbb{N})$ *of finite-dimensional subspaces such that for the corresponding family* (B_k) *and any* $m \in \mathbb{N}$ *we have*

$$B_k(x, y) = \Big(\frac{k}{2\pi}\Big)^n E^k(x, y) \sum_{\ell \in \mathbb{Z} \cap [-m, m/2]} k^{-\ell} \sigma_\ell(x, y) + \mathcal{O}_\infty(k^{n-(m+1)/2}), \tag{26}$$

where $2n$ *is the dimension of* M, *and*

- E *is a section of* $L \boxtimes \overline{L}$ *satisfying* $E(x, x) = 1$, $|E(x, y)| < 1$ *if* $x \neq y$ *and for any vector field* $Z \in \mathcal{C}^\infty(M, T^{1,0}M)$, $(\nabla_{\overline{Z}} \boxtimes \mathrm{id})E$ *and* $(\mathrm{id} \boxtimes \nabla_Z)E$ *vanish to second order along the diagonal of* $M^2 = M \times M$;
- *for any* $\ell \in \mathbb{Z}$, σ_ℓ *is a section of* $A \boxtimes \overline{A}$. *If* ℓ *is negative,* σ_ℓ *vanishes to order* -3ℓ *along the diagonal.*

Furthermore, $\sigma_0(x, x) = 1$ *for any* $x \in M$.

The notation $\mathcal{O}_\infty(k^N)$ has been introduced by the first author in previous papers and refers to a uniform control of the section and its successive derivatives. The precise meaning is as follows. A family $(\Psi(\cdot, k) \in \mathcal{C}^\infty(M^2, A_k \boxtimes \overline{A}_k),\ k \in \mathbb{N})$ is in $\mathcal{O}_\infty(k^N)$ if for any open set U of M^2, for any compact subset K of U, for any unitary frames $\tau_A \colon U \to A \boxtimes \overline{A}$ and $\tau_L \colon U \to L \boxtimes \overline{L}$, for any $m \in \mathbb{N}$, and for any vector fields $X_1, \ldots, X_m$ of M^2, there exists $C > 0$ such that for any k,

$$\Psi(\cdot, k) = f_k \tau_L^k \otimes \tau_A \text{ on } U \ \Rightarrow \sup_K |X_1 \ldots X_m f_k| \leq C k^{N+m}. \tag{27}$$

Observe that one loses a factor k at each derivative, so that condition (27) does not depend on the choice of the frame τ_L.

It is not difficult to show that for any $\sigma \in \mathcal{C}^\infty(M^2, A \boxtimes \overline{A})$ vanishing to order p along the diagonal, the family $(E^k \otimes \sigma, k \in \mathbb{N})$ is in $\mathcal{O}_\infty(k^{-p/2})$. So, in Theorem 3.2, the family $(k^{-\ell} E^k \otimes \sigma_\ell)$ is in $\mathcal{O}_\infty(k^{-\ell})$ if ℓ is nonnegative and in $\mathcal{O}_\infty(k^{\ell/2})$ if ℓ is negative. We refer the reader to Subsections 2.2 and 2.3 of [Cha14] for more details and other basic properties of $\mathcal{O}_\infty(k^N)$.

In the Kähler case, that is when j is integrable and L, A are holomorphic line bundles, we can define $\mathcal{H}_k$ as the space of holomorphic sections of A_k. The corresponding kernel B_k is called the Bergman kernel. The asymptotics of B_k given in Theorem 3.2 were deduced in [Cha03] (Corollary 1) from the seminal paper [BdMS76]. A direct proof was given in [BBS08], cf. also [MM07] and [SZ02] for similar results. In this case, we can even choose E so that the σ_ℓ's with negative ℓ are identically null.

In the general symplectic case, the spaces $\mathcal{H}_k$ are defined in such a way that B_k admits an asymptotic expansion of this form. The existence of such a quantization was proved in [Cha14] using the ideas of [BdMG81], cf. also [MM07] and [SZ02] for similar results. In the construction proposed in [Cha14], we start with any sections E and

σ_0 satisfying the assumptions of Theorem 3.2. We assume also that $\overline{E}(x,y) = E(y,x)$ and $\overline{\sigma}_0(x,y) = \sigma_0(y,x)$, so that the operator P_k with Schwartz kernel $\left(\frac{k}{2\pi}\right)^n E^k \sigma_0$ is selfadjoint. One proves that the spectrum P_k concentrates onto 0 and 1, in the sense that

$$\operatorname{spec}(P_k) \subset [-Ck^{-1/2}, Ck^{-1/2}] \cup [1 - Ck^{-1/2}, 1 + Ck^{-1/2}],$$

where C is a positive constant independent of k. Furthermore, for any k, $\operatorname{spec}(P_k) \cap [1 - Ck^{-1/2}, 1 + Ck^{-1/2}]$ consists of a finite number of eigenvalues, each having a finite multiplicity, and the corresponding eigenvectors are smooth. We define $\mathcal{H}_k$ as the sum of the corresponding eigenspaces

$$\mathcal{H}_k := \bigoplus_{\lambda \in \operatorname{spec}(P_k) \cap [1 - Ck^{-1/2}, 1 + Ck^{-1/2}]} \ker(P_k - \lambda).$$

Then one proves that the corresponding kernel has the expected behaviour.

3.4. Berezin–Toeplitz operators, (P1) and (P4). Consider a family

$$(\mathcal{H}_k \subset \mathcal{C}^\infty(M, L^k \otimes A),\ k \in \mathbb{N})$$

satisfying the conditions of Theorem 3.2. For any $f \in \mathcal{C}^0(M)$, define the Toeplitz operator

$$T_k(f) := \Pi_k f \colon \mathcal{H}_k \to \mathcal{H}_k,$$

where Π_k is the orthogonal projector of $\mathcal{C}^0(M, A_k)$ onto $\mathcal{H}_k$, as in Subsection 3.2. We shall show that this construction satisfies properties (P1)–(P4) of Theorem 1.1. We start with the norm and trace correspondences, because their proofs are very short.

For the estimation of the norm, we will use special vectors of $\mathcal{H}_k$ called coherent states. Let B_k be the Schwartz kernel of Π_k. Let $x \in M$ and u, v be unitary vectors of L_x and A_x respectively. Let Ψ_k be the section of $L^k \otimes A$ defined by

$$\Psi_k(y) = B_k(y,x) \cdot (u^k \otimes v) \quad \text{for all} \quad y \in M, \tag{28}$$

where the dot stands for the contractions $A_{k,y} \otimes \overline{A}_{k,x} \otimes A_{k,x} \to A_{k,y}$ induced by the metrics of L and A. Expanding B_k in an orthonormal basis $(e_{k,i}, i = 1, \dots, N_k)$ as in (25), we see that Ψ_k belongs to $\mathcal{H}_k$. Furthermore,

$$\|\Psi_k\|^2 = \sum_{i=1}^{N_k} |e_{k,i}(x)|^2 = B_k(x,x), \tag{29}$$

where we view $B_k(x,x)$ as a number, as explained before Theorem 3.2. We deduce from Theorem 3.2 that $\|\Psi_k\|^2 \sim (k/2\pi)^n$. When k is sufficiently large, we set $\Psi_k^{\mathrm{n}} = \Psi_k / \|\Psi_k\|$.

Proposition 3.3. *There exists $\alpha > 0$ such that for any $f \in \mathcal{C}^2(M)$ whose x is a critical point, and for any k, we have*

$$\left\| T_k(f)\Psi_k^{\mathrm{n}} - f(x)\Psi_k^{\mathrm{n}} \right\| \le \alpha k^{-1} |f|_2.$$

Furthermore, α does not depend on x, u, and v.

Applying this to a point x where $|f|$ attains its maximum, we deduce that the spectrum of $T_k(f)$ intersects $\|f\| + \alpha k^{-1}|f|_2[-1,1]$. This implies property $(P1)$.

Proof. Let $\lambda = f(x)$. Let (U, y_i) be a coordinate system centered at x. Let V be a relatively compact open neighborhood of x contained in U. Write $\delta = \sum y_i^2$. Then if x is a critical point of f, we have

$$|f(y) - \lambda| \le C_1 |f|_2 \delta(y) \quad \text{for all} \quad y \in V, \tag{30}$$

where C_1 does not depend on y and f. By Theorem 3.2, we have

$$\int_V |\Psi_k^{\mathrm{n}}|^2\delta^2\mu = \mathcal{O}(k^{-2}), \quad \int_{M\setminus V} |\Psi_k^{\mathrm{n}}|^2\mu = \mathcal{O}(k^{-\infty}). \tag{31}$$

Indeed, we can adapt the standard proof of the Kähler case as follows. Recall that $\|\Psi_k\|^2 \sim (k/2\pi)^n$. Furthermore, there exists $0 < r < 1$ such that for any $y \in M \setminus V$ we have $|E(y,x)| \leq r$. The second estimate of (31) follows easily from Theorem 3.2. For the first estimate, we use the fact that there exists $C_2 > 0$ such that $|E(y,x)| \leq e^{-\delta(y)/C_2}$ for any $y \in V$. So, by Theorem 3.2, there exists $C_3 > 0$ such that

$$|\Psi_k^{\mathrm{n}}(y)| \leq k^{n/2}C_3 e^{-k\delta(y)/C_2} \quad \text{for all} \quad y \in V,$$

on U. Write $\mu = g dy_1 \wedge \cdots \wedge dy_{2n}$ and let $C_4 > 0$ be such that $|g| \leq C_4$ on V. We have

$$\begin{aligned}\int_U |\Psi_k^{\mathrm{op}n}|^2\delta^2\mu &\leq k^n C_3^2 C_4 \int_{\mathbb{R}^{2n}} e^{-2k|u|^2/C_2}|u|^4\,du \\ &= k^{-2}C_3^2C_4\int_{\mathbb{R}^{2n}} e^{-2|u|^2/C_2}|u|^4\,du.\end{aligned}$$

This proves the first equation in (31).

Now, using equations (31), (30) and the fact that $|f(y)-\lambda| \leq 2|f|_2$ on M, we see that

$$\|(f-\lambda)\Psi_k^{\mathrm{op}n}\|^2 = \int_M |f(y)-\lambda|^2|\Psi_k^{\mathrm{op}n}(y)|^2\mu(y) \leq Ck^{-2}|f|_2^2$$

for some $C > 0$ independent of f. Since $\|\Pi_k\|_{\mathrm{op}} \leq 1$, it follows that

$$\|\Pi_k f\Psi_k^{\mathrm{n}} - \lambda\Psi_k^{\mathrm{n}}\| \leq \alpha k^{-1}|f|_2,$$

where $\alpha = C^{1/2}$. The fact that α may be chosen independently of x, u, and v, follows from the compactness of M. □

We prove property (P4).

Proposition 3.4. *For any k, there exists a sequence $(\rho(\,\cdot\,,k))$ in $\mathcal{C}^\infty(M)$ such that for any $f \in \mathcal{C}^0(M)$ we have*

$$\operatorname{tr}(T_k(f)) = \Big(\frac{k}{2\pi}\Big)^n \int_M f\rho(\,\cdot\,,k)\mu,$$

where μ is the Liouville volume. Furthermore, $\rho(\,\cdot\,,k) = 1 + \mathcal{O}(k^{-1})$ uniformly on M.

Proof. Denote by h_k the metric of A_k. Then

$$\begin{aligned}\operatorname{tr}(T_k(f)) = \sum_i \langle fe_{k,i}, e_{k,i}\rangle &= \sum_i \int_M f(x)h_k(e_{k,i}(x), e_{k,i}(x))\mu(x) \\ &= \int_M f(x)B_k(x,x)\mu(x),\end{aligned}$$

where we identify $B_k(x,x)$ with a number as previously. By Theorem 3.2, we know that $B_k(x,x) = \big(k/2\pi\big)^n\rho(x,k)$, where $\rho(\,\cdot\,,k)$ has the asymptotic expansion $1+k^{-1}\sigma_1(x,x)+k^{-2}\sigma_2(x,x)+\ldots$ □

3.5. Proof of sharp remainder estimates, (P2) and (P3). Our strategy is to make a detour through the Kostant–Souriau operators and the corresponding Toeplitz operators, which are well-behaved in terms of commutator estimates. In particular, these modified Toeplitz operators satisfy a correspondence principle with a remainder better than (P2), involving only second derivatives. We will then analyse the Toeplitz operators as perturbation of the former.

3.5.1. Kostant–Souriau operators. Let us introduce a covariant derivative ∇^A of A. We denote by ∇^k the covariant derivative of $L^k \otimes A$ induced by ∇^A and the covariant derivative ∇ of L. Let $f \in \mathcal{C}^1(M)$ and denote by X its Hamiltonian vector field.[3] The Kostant–Souriau operator associated with f acting on sections of $L^k \otimes A$ is given by

$$H_k(f) = f + \tfrac{1}{ik}\nabla^k_X. \tag{32}$$

It was discovered independently by Kostant [Ko70] and Souriau [Sou70] that when A is the trivial line bundle and ∇^A the de Rham derivative, H_k satisfies an exact correspondence principle. For a general pair (A, ∇^A), we have

$$[H_k(f), H_k(g)] = \tfrac{i}{k} H_k(\{f, g\}) - \tfrac{1}{k^2}\Omega_A(X, Y) \tag{33}$$

for any $f, g \in \mathcal{C}^2(M)$, where Ω_A is the curvature of ∇^A.

When f is of class $\mathcal{C}^2$, $H_k(f)$ sends $\mathcal{C}^1(M, A_k)$ into $\mathcal{C}^0(M, A_k)$, so the same holds for the commutator $[H_k(f), \Pi_k]$. By the properties recalled after Proposition 3.1, this commutator extends to a bounded operator of $(\mathcal{C}^0(M, A_k), \| \cdot \|)$. When f is smooth, in [Cha14] it was proved that the norm of $[H_k(f), \Pi_k]$ is in $\mathcal{O}(k^{-1})$. We will extend this to functions of class $\mathcal{C}^2$ and prove that the $\mathcal{O}(k^{-1})$ only depends on the $\mathcal{C}^2$-norm of f.

Theorem 3.5. *There exists $C > 0$ such that for any $f \in \mathcal{C}^2(M)$ and any $k \in \mathbb{N}$ we have*

$$\big\| [H_k(f), \Pi_k] \big\|_{\mathrm{op}} \leq C k^{-1} |f|_2.$$

The proof will be given in §4. It is a consequence of Theorem 3.2. Denote by $T^c_k(f)$ the operator

$$T^c_k(f) = \Pi_k H_k(f) : \mathcal{H}_k \to \mathcal{H}_k. \tag{34}$$

The superscript c stands for correction. Surprisingly, we only need to assume f and g of class $\mathcal{C}^2$ to get the sharp correspondence principle for T^c_k.

Proposition 3.6. *For any f and g in $\mathcal{C}^2(M)$, we have*

$$[T^c_k(f), T^c_k(g)] = \tfrac{i}{k} T^c_k(\{f, g\}) + \mathcal{O}(k^{-2})|f|_2|g|_2. \tag{35}$$

Here it is implicitly meant that the $\mathcal{O}(k^{-2})$'s do not depend on f or g. More precisely, the $\mathcal{O}(k^{-2})$ is a term whose uniform norm does not exceed Ck^{-2}, where C depends only on the family $(\mathcal{H}_k)$, but **not** on f or g. We use the same convention in the sequel.

Proof. A straightforward computation shows that

$$\Pi_k[\Pi_k, H_k(f)][\Pi_k, H_k(g)]\Pi_k = T^c_k(f)T^c_k(g) - \Pi_k H_k(f) H_k(g) \Pi_k.$$

By Theorem 3.5, the left-hand side is $\mathcal{O}(k^{-2})|f|_2|g|_2$. Therefore,

$$[T^c_k(f), T^c_k(g)] = \Pi_k[H_k(f), H_k(g)]\Pi_k + \mathcal{O}(k^{-2})|f|_2|g|_2.$$

Using the Kostant–Souriau formula (33) and the fact that $\Pi_k \Omega_A(X, Y)\Pi_k = \mathcal{O}(1)|f|_1|g|_1$, we get (35). $\square$

[3] In this paper the Hamiltonian vector field X_f of a function f is defined by $i_{X_f}\omega + df = 0$, and the Poisson bracket is given by $\{f, g\} = -\omega(X_f, X_g)$.

3.5.2. Kähler case. We assume in this section that (M, ω, j) is a Kähler manifold, L, A are holomorphic Hermitian line bundles over M, and the connections ∇ and ∇^A are the Chern connections. Furthermore, $\mathcal{H}_k$ is the space of holomorphic sections of A_k.

Lemma 3.7. *For any vector field X of M of class $\mathcal{C}^1$, we have*

$$\Pi_k \nabla^k_X \Pi_k = -\Pi_k \operatorname{div}(Z) \Pi_k,$$

where $Z = \frac{1}{2}(X - ijX)$ and $\operatorname{div}(Z)$ is the divergence of Z with respect to the Liouville form.

Proof. Since $\mathcal{H}_k$ consists of holomorphic sections and $\overline{Z}$ is a section of $T^{0,1}M$, $\Pi_k \nabla^k_{\overline{Z}} \Pi_k = 0$. Since Z is of class $\mathcal{C}^1$, the integral $\int \mathcal{L}_Z(f\mu)$ vanishes for any smooth function f. We see that for any s, $t \in \mathcal{C}^\infty(M, A_k)$,

$$\langle \nabla^k_Z s, t\rangle + \langle s, \nabla^k_{\overline{Z}} t\rangle + \langle \operatorname{div}(Z) s, t\rangle = 0.$$

Applying this to s, $t \in \mathcal{H}_k$, we deduce that $\Pi_k(\nabla_Z + \operatorname{div}(Z))\Pi_k = 0$. Consequently,

$$\Pi_k \nabla^k_X \Pi_k = \Pi_k \nabla^k_Z \Pi_k + \Pi_k \nabla^k_{\overline{Z}} \Pi_k = -\Pi_k \operatorname{div}(Z) \Pi_k,$$

which was to be proved. □

When X is the Hamiltonian vector field of $f \in \mathcal{C}^2(M)$, we have $\operatorname{div} X = 0$, so that $\operatorname{div}(Z) = -i/2 \operatorname{div}(jX) = i\Delta f$, where Δ is the holomorphic Laplacian. We deduce Tuynman's formula [Tuy87]:

$$\Pi_k i \nabla_X \Pi_k = \Pi_k(\Delta f)\Pi_k. \tag{36}$$

Recall that $T_k(f)$ is the Toeplitz operator $\Pi_k f \colon \mathcal{H}_k \to \mathcal{H}_k$. By (36), we have

$$T^c_k(f) = T_k(f) - \tfrac{1}{k} T_k(\Delta f) = T_k(f) + \mathcal{O}(k^{-1})|f|_2. \tag{37}$$

Let us prove that T_k satisfies the quasi-multiplicativity (P3).

Proposition 3.8. *For any functions $f \in \mathcal{C}^1(M)$ and $g \in \mathcal{C}^2(M)$, we have*

$$T_k(f)T_k(g) = T_k(fg) + \mathcal{O}(k^{-1})(|f|_0|g|_2 + |f|_1|g|_1),$$
$$T_k(g)T_k(f) = T_k(fg) + \mathcal{O}(k^{-1})(|f|_0|g|_2 + |f|_1|g|_1).$$

Proof. Let Y be the Hamiltonian vector field of g. We have

$$\Pi_k f[\Pi_k, H_k(g)]\Pi_k = T_k(f)T_k(g) - T_k(fg) + \Pi_k f \Pi_k \tfrac{1}{ik} \nabla^k_Y \Pi_k - \Pi_k \tfrac{1}{ik} \nabla^k_{fY} \Pi_k.$$

By Theorem 3.5, the left-hand side is $\mathcal{O}(k^{-1})|f|_0|g|_2$. By Lemma 3.7, $\Pi_k \frac{1}{ik} \nabla^k_Y \Pi_k = \mathcal{O}(k^{-1})|Y|_1$, so that

$$\begin{aligned} \Pi_k f \Pi_k \tfrac{1}{ik} \nabla^k_Y \Pi_k &= \mathcal{O}(k^{-1})|f|_0|g|_2, \\ \Pi_k \tfrac{1}{ik} \nabla^k_{fY} \Pi_k &= \mathcal{O}(k^{-1})|fY|_1 = \mathcal{O}(k^{-1})(|f|_0|g|_2 + |f|_1|g|_1), \end{aligned} \tag{38}$$

which concludes the proof of the first equation. To get the second, we take the adjoint. □

Finally, we show the sharp correspondence principle (P2).

Proposition 3.9. *For any $f, g \in \mathcal{C}^3(M)$, we have*

$$[T_k(f), T_k(g)] = \tfrac{i}{k} T_k(\{f, g\}) + \mathcal{O}(k^{-2})(|f|_1|g|_3 + |f|_2|g|_2 + |f|_3|g|_1).$$

Proof. By Proposition 3.8, $[T_k(u), T_k(v)] = \mathcal{O}(k^{-1})(|u|_0|v|_2 + |u|_1|v|_1)$ for any $u \in \mathcal{C}^1(M)$ and $v \in \mathcal{C}^2(M)$. Consequently,

$$[T_k(\Delta f), T_k(g)] = \mathcal{O}(k^{-1})(|f|_2|g|_2 + |f|_3|g|_1).$$

Similarly,

$$[T_k(f), T_k(\Delta g)] = \mathcal{O}(k^{-1})(|f|_1|g|_3 + |f|_2|g|_2).$$

Since $T_k(u) = \mathcal{O}(1)|u|_0$, we have

$$[T_k(\Delta f), T_k(\Delta g)] = \mathcal{O}(1)|f|_2|g|_2, \quad T_k(\Delta\{f,g\}) = \mathcal{O}(1)|f,g|_3.$$

We conclude with Proposition 3.6 by using $T_k^c(f) = T_k(f) - \frac{1}{k}T_k(\Delta f)$. □

3.5.3. Symplectic case. Let us return to the general symplectic case. We do not know how to generalize Lemma 3.7. Instead, we will use the following result.

Theorem 3.10. *There exists $C > 0$ such that for any $Z \in \mathcal{C}^0(M, T^{1,0}M)$,*

$$\left\| \tfrac{i}{k} \nabla^k_{\overline{Z}} \Pi_k \right\|_{\mathrm{op}} \leq C k^{-1} |Z|_0 \quad \textit{for all} \quad k \in \mathbb{N}, \tag{39}$$

and if Z is of class $\mathcal{C}^1$, then for any

$$f \in \mathcal{C}^2(M), \quad [H_k(f), \tfrac{i}{k}\nabla^k_{\overline{Z}}\Pi_k] \colon \mathcal{C}^\infty(M, A_k) \to \mathcal{C}^0(M, A_k)$$

extends continuously to a bounded operator of $(\mathcal{C}^0(M, A_k), \|\cdot\|_k)$ satisfying

$$\left\| [H_k(f), \tfrac{i}{k}\nabla^k_{\overline{Z}}\Pi_k] \right\|_{\mathrm{op}} \leq C k^{-2}(|f|_2 |Z|_0 + |f|_1|Z|_1) \quad \textit{for all} \quad k \in \mathbb{N}. \tag{40}$$

The proof will be given in §4. A consequence of the first inequality is the following lemma.

Lemma 3.11. *For any $X \in \mathcal{C}^1(M, TM)$, we have*

$$\Pi_k \tfrac{i}{k} \nabla^k_X \Pi_k = \mathcal{O}(k^{-1})|X|_1.$$

Proof. Write $X = Z + \overline{Z}$ with Z a section of $T^{1,0}M$. By (39), we have

$$\Pi_k \tfrac{i}{k} \nabla^k_{\overline{Z}} \Pi_k = \mathcal{O}(k^{-1})|Z|_0. \tag{41}$$

Taking the adjoint, we get $\Pi_k \frac{i}{k}(\nabla^k_Z + \operatorname{div}(Z))\Pi_k = \mathcal{O}(k^{-1})|Z|_0$. Since $\operatorname{div}(Z)$ is in $\mathcal{O}(|Z|_1)$ in the uniform norm, we obtain

$$\Pi_k \tfrac{i}{k} \nabla^k_Z \Pi_k = \mathcal{O}(k^{-1})|Z|_1. \tag{42}$$

Adding (41) and (42), we get the result. □

As a consequence, we have

$$T_k(f) = T_k^c(f) + \mathcal{O}(k^{-1})|f|_2. \tag{43}$$

We also deduce the quasi-multiplicativity (P3).

Proposition 3.12. *For any functions $f \in \mathcal{C}^1(M)$ and $g \in \mathcal{C}^2(M)$, we have*

$$T_k(f)T_k(g) = T_k(fg) + \mathcal{O}(k^{-1})(|f|_0|g|_2 + |f|_1|g|_1),$$
$$T_k(g)T_k(f) = T_k(fg) + \mathcal{O}(k^{-1})(|f|_0|g|_2 + |f|_1|g|_1).$$

Proof. The proof is exactly the same as that of Proposition 3.8 except that we deduce relations (38) from Lemma 3.11 instead of Lemma 3.7. □

Finally we show the sharp correspondence principle (P2).

Proposition 3.13. *For any $f, g \in \mathcal{C}^3(M)$,*

$$[T_k(f), T_k(g)] = \tfrac{i}{k} T_k(\{f,g\}) + \mathcal{O}(k^{-2})(|f|_1|g|_3 + |f|_2|g|_2 + |f|_3|g|_1).$$

Proof. Denote by X and Y the Hamiltonian vector fields of f and g. By (43), we have $T_k^c(\{f,g\}) = T_k(\{f,g\}) + \mathcal{O}(k^{-1})|f,g|_3$. By Lemma 3.11,

$$[\Pi_k \tfrac{1}{ik}\nabla^k_X \Pi_k, \Pi_k \tfrac{1}{ik}\nabla^k_Y \Pi_k] = \mathcal{O}(k^{-2})|f|_2|g|_2.$$

So by Proposition 3.6, it suffices to show that

$$[\Pi_k f \Pi_k, \Pi_k \tfrac{1}{ik} \nabla^k_Y \Pi_k] = \mathcal{O}(k^{-2})(|f|_1|g|_3 + |f|_2|g|_2). \tag{44}$$

Write $Y = Z + \overline{Z}$ with Z a section of $T^{1,0}M$. Doing a straightforward computation, we obtain

$$[\Pi_k H_k(f)\Pi_k, \Pi_k \tfrac{1}{ik}\nabla^k_{\overline{Z}}\Pi_k] = \Pi_k[H_k(f), \Pi_k]\tfrac{1}{ik}\nabla^k_{\overline{Z}}\Pi_k + \Pi_k[H_k(f), \tfrac{1}{ik}\nabla^k_{\overline{Z}}\Pi_k]\Pi_k.$$

By Theorem 3.5 and equation (39), the first term of the left-hand side is $\mathcal{O}(k^{-2})|f|_2|Z|_0$. By (40), the second term is $\mathcal{O}(k^{-2})(|f|_1|Z|_1 + |f|_2|Z|_0)$. Consequently,

$$[\Pi_k H_k(f)\Pi_k, \Pi_k \tfrac{1}{ik}\nabla^k_{\overline{Z}}\Pi_k] = \mathcal{O}(k^{-2})(|f|_1|Z|_1 + |f|_2|Z|_0). \tag{45}$$

Using Lemma 3.11 and equation (39), we deduce from (45) that

$$[\Pi_k f\Pi_k, \Pi_k \tfrac{1}{ik}\nabla^k_{\overline{Z}}\Pi_k] = \mathcal{O}(k^{-2})(|f|_1|Z|_1 + |f|_2|Z|_0). \tag{46}$$

Taking the adjoint, we get

$$[\Pi_k f\Pi_k, \Pi_k \tfrac{1}{ik}(\nabla^k_Z + \operatorname{div}(Z))\Pi_k] = \mathcal{O}(k^{-2})(|f|_1|Z|_1 + |f|_2|Z|_0). \tag{47}$$

By Proposition 3.12, $[T_k(f), T_k(v)] = \mathcal{O}(k^{-1})(|f|_1|v|_1 + |f|_2|v|_0)$ for any function $v \in \mathcal{C}^1(M)$. Applying this to $v = \operatorname{div}(Z)$ and using (47), we obtain

$$[\Pi_k f\Pi_k, \Pi_k \tfrac{1}{ik}\nabla^k_Z\Pi_k] = \mathcal{O}(k^{-2})(|f|_1|Z|_1 + |f|_2|Z|_0). \tag{48}$$

Finally, equation (44) follows from (46) and (48). □

§4. Proofs of Theorems 3.5 and 3.10

Now use explicitly the structure of the kernel B_k given in Theorem 3.2.

4.1. The fundamental estimates. Consider a section E satisfying the same assumptions as in Theorem 3.2. The existence of such a section is proved in Lemma 3.2 of [Cha14]. Let U be the open set where E does not vanish. Let $\varphi \in \mathcal{C}^\infty(U)$ and $\alpha_E \in \Omega^1(U)$ be defined by

$$\varphi = -2\ln|E|, \quad \nabla^{L\boxtimes\overline{L}}E = \tfrac{1}{i}\alpha_E \otimes E. \tag{49}$$

Here $\nabla^{L\boxtimes\overline{L}}$ is the connection of $L\boxtimes\overline{L}$ induced by ∇. So for any vector fields X and Y of M we have $\nabla^{L\boxtimes\overline{L}}_{(X,Y)} = \nabla_X \boxtimes \mathrm{id} + \mathrm{id}\boxtimes\overline{\nabla}_Y$, where (X, Y) is the vector field of M^2 sending (p, q) into $X(p) \oplus Y(q)$.

By Theorem 3.2, φ vanishes along the diagonal Δ of M^2 and is positive outside Δ. Furthermore, φ and α_E possess the following properties:

(i) α_E vanishes on $T_\Delta(M^2)$;

(ii) φ vanishes to second order along Δ. For any $x \in M$, the kernel of the Hessian of φ at (x, x) is the tangent space to the diagonal;

(iii) for any $f \in \mathcal{C}^\infty(M)$ with Hamiltonian vector field X, $g - \alpha_E(X, X)$ vanishes to second order along Δ, where $g(x, y) = f(x) - f(y)$.

For the proof of these properties, see Proposition 2.15, Remark 2.16, and Proposition 2.18 in [Cha14].

For any continuous section σ of $A\boxtimes\overline{A}$ and $k \in \mathbb{N}$, we let $P_k(\sigma)$ be the operator acting on $\mathcal{C}^0(M, L^k \otimes A)$ with Schwartz kernel $k^nE^k \otimes \sigma$. Here and in the sequel, μ is the Liouville form $\omega^n/n!$.

Lemma 4.1. *For any compact subset K of U and any $p \in \mathbb{N}$, there exists $C_{K,p}$ such that for any $\sigma \in \mathcal{C}^0(M^2, A\boxtimes\overline{A})$ whose support is contained in K, we have*

$$\text{for all}\quad k \in \mathbb{N}, \quad \|P_k(\sigma)\|_{\mathrm{op}} \le C_{K,p}|\sigma|_{K,p}k^{-p/2},$$

where $|\sigma|_{K,p} \in \mathbb{R}_+ \cup \{\infty\}$ is the supremum of $|\sigma(z)|(\varphi(z))^{-p/2}$ over $K \setminus \Delta$.

Proof. Assume first that K does not intersect the diagonal of M. Then φ takes positive values on K, so there exists $C > 0$ such that $1/C \leq \varphi \leq C$ on K. Consequently

$$|E^k \otimes \sigma| \leq |\sigma|_{K,p} C^{p/2} e^{-k/(2C)}$$

on K and we conclude easily.

Assume now that $K \subset V^2$ where (V, x_i) is a coordinate system of M such that $V^2 \subset U$. By Property (ii) and the fact that φ is positive outside the diagonal, there exists $C > 0$ such that

$$|x-y|^2/C \leq \varphi(x,y) \leq C|x-y|^2 \tag{50}$$

on K. If the support of σ is contained in K, then $|\sigma(x,y)| \leq C^{p/2}|\sigma|_{K,p}|x-y|^p$ on V^2. Identify V with an open set of $\mathbb{R}^{2n}$. Then we have

$$\begin{aligned}\int_M \big|P_k(\sigma)(x,y)\big|\mu(y) &\leq k^n C^{p/2}|\sigma|_{K,p}\int_V e^{-k|x-y|^2/C}|x-y|^p\,dy\\ &\leq k^n C^{p/2}|\sigma|_{K,p}\int_{\mathbb{R}^{2n}} e^{-k|x-y|^2/C}|x-y|^p\,dy\\ &= k^{-p/2} C^{p/2}|\sigma|_{K,p}\int_{\mathbb{R}^{2n}} e^{-|x-y|^2/C}|x-y|^p\,dy\end{aligned}$$

by doing a convenient change of variable

$$= k^{-p/2} C_1 |\sigma|_{K,p}.$$

In the same way we show that

$$\int_M \big|P_k(\sigma)(x,y)\big|\mu(x) \leq k^{-p/2} C_2 |\sigma|_{K,p}$$

for some $C_2 > 0$ independent of σ and k. Applying Proposition 3.1, we conclude that

$$\|P_k(\sigma)\|_{\mathrm{op}} \leq C|\sigma|_{K,p} k^{-p/2}$$

with $C = \max(C_1, C_2)$.

Consider now any compact subset K of U. The diagonal Δ being compact, there exists a finite family $(V_i)_{i=1,\dots,N}$ of open sets of M such that each V_i is the domain of a coordinate system, $V_i^2 \subset U$, and $\Delta \subset \bigcup V_i^2$. Then U is covered by the $N+1$ open sets $U_0 = U \setminus \Delta$, $U_i = V_i^2$, $i = 1, \dots, N$. Choose a subordinate partition of unity $f_i \in \mathcal{C}^\infty(U)$, $i = 0, \dots, N$. If σ is supported in K, $f_i\sigma$ is supported in $K \cap \operatorname{supp} f_i$, and by the first part of the proof for $i = 0$ and the second part for $i = 1, \dots, \ell$, we have

$$\|P_k(f_i\sigma)\|_{\mathrm{op}} \leq C_i |f_i\sigma|_{K\cap \operatorname{supp} f_i, p} k^{-p/2} \leq C_i |\sigma|_{K,p} k^{-p/2}$$

for some constants $C_i > 0$. □

Recall that we denote by $H_k(f)$ the Kostant–Souriau operator (32).

Lemma 4.2. *For any $p \in \mathbb{N}$ and any $\sigma \in \mathcal{C}^\infty(M^2, A \boxtimes \overline{A})$ supported in U and vanishing to order p along the diagonal, there exists $C > 0$ such that for any $f \in \mathcal{C}^2(M)$, we have*

$$\|P_k(\sigma)\|_{\mathrm{op}} \leq C k^{-p/2}, \quad \big\|[H_k(f), P_k(\sigma)]\big\|_{\mathrm{op}} \leq C k^{-p/2-1}|f|_2.$$

Proof. This is a consequence of Lemma 4.1. Set $K = \operatorname{supp}\sigma$. Using Property (ii) as in (50) and the Taylor formula, we see that $|\sigma|_{K,p}$ is finite, which proves the first estimate.

To prove the second estimate, we introduce $g(x,y) = f(x) - f(y)$ and the vector field $Y = (X, X)$ of M^2, where X is the Hamiltonian vector field of f. Then on U we have

$$\big[(f - i\nabla_X) \boxtimes \mathrm{id} - \mathrm{id} \boxtimes (f + i\nabla_X)\big]E = (g - \alpha_E(Y))E.$$

Thus,

$$\big[(f+\tfrac{1}{ik}\nabla^k_X)\boxtimes \mathrm{id}-\mathrm{id}\boxtimes(f-\tfrac{1}{ik}\nabla^k_X)\big](E^k\otimes\sigma)=E^k\big((g-\alpha_E(Y)+\tfrac{1}{ik}\nabla^{A\boxtimes\overline{A}}_Y)\sigma\big).$$

Consequently, using the basic facts on Schwartz kernels recalled in Subsection 3.1, we obtain

$$[H_k(f),P_k(\sigma)]=P_k\big((g-\alpha_E(Y))\sigma\big)+\tfrac{1}{ik}P_k\big(\nabla^{A\boxtimes\overline{A}}_Y\sigma\big). \tag{51}$$

We claim that there exists $C>0$ such that for any $f\in\mathcal{C}^2(M)$, we have

$$|g-\alpha_E(Y)|\le C\varphi|f|_2 \tag{52}$$

on K. This has the consequence that $|(g+\alpha_E(Y))\sigma|_{K,p+2}\le C|f|_2|\sigma|_{K,p}$. So by Lemma 4.1, the first term of the right-hand side of (51) is $\mathcal{O}(k^{-p/2-1})|f|_2$. To prove (52), introduce a coordinate system (V,x_i) such that the closure of V is a compact subset of U. On V^2 we have

$$g(x,y)=\sum_{i=1}^{2n}g_i(x)(y_i-x_i)+\mathcal{O}(\varphi)|f|_2$$

for some functions $g_i\in\mathcal{C}^1(V)$. Similarly, by Property (i), α_E vanishes along the diagonal Δ, whence

$$\alpha_E(Y)=\sum_{i=1}^{2n}h_i(x)(y_i-x_i)+\mathcal{O}(\varphi)|X|_0.$$

By Property (iii), $(g-\alpha_E(Y))$ vanishes to second order along Δ, so $g_i(x)=h_i(x)$ for any i. Inequality (52) follows.

We shall show that there exists $C>0$ such that for any vector field Z of M^2 tangent to Δ, we have

$$|\nabla^{A\boxtimes\overline{A}}_Z\sigma|\le C\varphi^{p/2}|Z|_1. \tag{53}$$

By Lemma 4.1, this has the consequence that the second term on the right in (51) is $\mathcal{O}(k^{-p/2-1})|f|_2$, which concludes the proof.

Let us prove (53). We denote by $\mathcal{O}(N)$ any section vanishing to order N along the diagonal. Observe that for any vector field Z of M^2, $\nabla^{A\boxtimes\overline{A}}_Z\sigma$ is $\mathcal{O}(p-1)$. Whenever Z is tangent to Δ, $\nabla^{A\boxtimes\overline{A}}_Z\sigma$ is $\mathcal{O}(p)$. So if (V,x_i) is a coordinate system as above,

$$\nabla^{A\boxtimes\overline{A}}\sigma=\sum_{i=1}^{2n}(dy_i-dx_i)\otimes a_i+dx_i\otimes b_i,$$

where $a_i=\nabla^{A\boxtimes\overline{A}}_{\partial_{y_i}}\sigma$ is $\mathcal{O}(p-1)$ and $b_i=\nabla^{A\boxtimes\overline{A}}_{\partial_{x_i}+\partial_{y_i}}\sigma$ is $\mathcal{O}(p)$.

Now, there exists $C'>0$ such that for any vector field Z of M^2 tangent to Δ of class $\mathcal{C}^1$ and supported in V^2, we have

$$|(dy_i-dx_i)(Z)|\le C'\varphi^{1/2}|Z|_1,\quad |dx_i(Z)|\le C'|Z|_0.$$

This proves (53) for the vector fields supported in V^2. We prove the general case with a partition of unity argument. ∎

4.2. The proof. Recall that we denote by B_k the Schwartz kernel of Π_k. Let $\psi \in \mathcal{C}^\infty(M \times M, \mathbb{R})$ be supported in U and equal to 1 on a neighborhood of the diagonal. Let R_k be the operator with Schwartz kernel $(1-\psi)B_k$. Let $(\sigma_\ell, \ell \in \mathbb{Z})$ be the same family as in Theorem 3.2. For any $m \in \mathbb{N}$, introduce the operator $R_{m,k}$ so that

$$\Pi_k = (2\pi)^{-n} \sum_{\ell \in \mathbb{Z} \cap [-m, m/2]} k^{-\ell} P_k(\psi \sigma_\ell) + R_k + R_{m,k}. \tag{54}$$

Each term of the right-hand side of (54) will be denoted generically by Q_k. We will prove that, if m is sufficiently large, then for any vector field $Z \in \mathcal{C}^\infty(M, T^{1,0}M)$ we have

$$\big[H_k(f), Q_k\big] = \mathcal{O}(k^{-1})|f|_2, \tag{55}$$

$$\tfrac{i}{k}\nabla^k_{\overline{Z}} Q_k = \mathcal{O}(k^{-1}), \tag{56}$$

$$\big[H_k(f), \tfrac{i}{k}\nabla^k_{\overline{Z}} Q_k\big] = \mathcal{O}(k^{-2})|f|_2. \tag{57}$$

After that, we will prove that (56) holds whenever Z is continuous, and (57) holds for any Z of class $\mathcal{C}^1$. Finally we will make explicit the dependence in Z of the $\mathcal{O}$.

The principal terms. For any $\ell \in \mathbb{Z}$, let $Q_{\ell,k} = k^{-\ell} P_k(\psi\sigma_\ell)$. By Lemma 4.2, we have

$$[H_k(f), Q_{\ell,k}] = \begin{cases} \mathcal{O}(k^{-\ell-1})|f|_2 & \text{if } \ell \geq 0, \\ \mathcal{O}(k^{\ell/2-1})|f|_2 & \text{if } \ell \leq 0. \end{cases} \tag{58}$$

This proves that $Q_{\ell,k}$ satisfies (55). To prove the remaining estimates, we use the fact that

$$\tfrac{i}{k}\nabla^k_{\overline{Z}} Q_{\ell,k} = P_k(\alpha_E(\overline{Z}, 0)\psi\sigma_\ell) + \tfrac{i}{k} P_k\big((\nabla^A_{\overline{Z}} \boxtimes \mathrm{id})\psi\sigma_\ell\big), \tag{59}$$

where α_E is as in (49). By Theorem 3.2, $\alpha_E(\overline{Z}, 0)$ vanishes to second order along the diagonal. By Lemma 4.2, we have

$$\tfrac{i}{k}\nabla^k_{\overline{Z}} Q_{\ell,k} = \begin{cases} \mathcal{O}(k^{-\ell-1}) & \text{if } \ell \geq 0, \\ \mathcal{O}(k^{\ell/2-1}) & \text{if } \ell \leq 0, \end{cases} \tag{60}$$

and

$$\big[H_k(f), \tfrac{i}{k}\nabla^k_{\overline{Z}} Q_{\ell,k}\big] = \begin{cases} \mathcal{O}(k^{-\ell-2})|f|_2 & \text{if } \ell \geq 0, \\ \mathcal{O}(k^{\ell/2-2})|f|_2 & \text{if } \ell \leq 0, \end{cases} \tag{61}$$

which proves (56) and (57) for $Q_k = Q_{k,\ell}$.

The remainders. Denote by $B'_k = (1-\psi)B_k$ and $B_{m,k}$ the Schwartz kernels of R_k and $R_{m,k}$, respectively. Let ∇^k be the connection of $A_k \boxtimes \overline{A}_k$ induced by the connections of A and L.

Recall the class $\mathcal{O}_\infty(k^N)$ introduced after Theorem 3.2. Set

$$\mathcal{O}_\infty(k^{-\infty}) := \bigcap_{N>0} \mathcal{O}_\infty(k^{-N}).$$

Lemma 4.3. *(B_k) belongs to $\mathcal{O}_\infty(k^{-\infty})$. $(B_{m,k})$ belongs to $\mathcal{O}_\infty(k^{n-(m+1)/2})$. In particular, for any smooth vector fields X_1, X_2 of M^2 and any N we have*

$$\nabla^k_{X_1} B'_k = \mathcal{O}(k^{-N}), \qquad \nabla^k_{X_1}\nabla^k_{X_2} B'_k = \mathcal{O}(k^{-N}),$$

$$\nabla^k_{X_1} B_{m,k} = \mathcal{O}(k^{n+1/2-m/2}), \qquad \nabla^k_{X_1}\nabla^k_{X_2} B_{m,k} = \mathcal{O}(k^{n+3/2-m/2})$$

uniformly on M^2.

Proof. Observe first that if $(\Psi(\,\cdot\,,k))$ belongs to $\mathcal{O}_\infty(k^N)$ and $\xi \in \mathcal{C}^\infty(M)$, then $(\xi\Psi(\,\cdot\,,k))$ belongs to $\mathcal{O}_\infty(k^N)$. Since the remainder in (26) is in $\mathcal{O}_\infty(k^{n-(m+1)/2})$, the same holds for $(B_{m,k})$. For the same reason, (B_k) being in $\mathcal{O}_\infty(k^n)$, the same holds for (B'_k). Since the pointwise norm of the section E appearing in (26) satisfies $|E| < 1$ outside the diagonal, (B_k) is in $\mathcal{O}(k^{-\infty})$ on any compact set not intersecting the diagonal. So (B'_k) is in $\mathcal{O}(k^{-\infty})$. This actually implies that (B'_k) is in $\mathcal{O}_\infty(k^{-\infty})$. Indeed, $\mathcal{O}_\infty(k^n) \cap \mathcal{O}(k^{-\infty}) = \mathcal{O}_\infty(k^{-\infty})$, which follows from the basic interpolation formula: for any open set V of $\mathbb{R}^m$ and compact subset K of V, there exists $C > 0$ such that for any smooth function f on V, we have

$$\sum_{|\alpha|=1} \sup_K |\partial^\alpha f| \leq C\big(\sup_V |f|\big)^{1/2}\bigg(\sup_V |f| + \sum_{|\alpha|=2} \sup_V |\partial^\alpha f|\bigg)^{1/2}.$$

A proof may be found [Shu78, Lemma 3.2]. □

By writing the Schwartz kernels of $[H_k(f), Q_k]$, $\nabla^k_{\overline{Z}} Q_k$, and $[H_k(f), \nabla^k_{\overline{Z}} Q_k]$ in terms of the Schwartz kernel of Q_k, we deduce from Lemma 4.3 that if m is sufficiently large, then R_k and $R_{m,k}$ satisfy (55), (56), (57) for smooth f; so far, without specifying the dependence of the $\mathcal{O}$'s on f.

To make explicit this dependence, we use the following fact. Consider any family $(\tau_k \in \mathcal{C}^\infty(M^2, A_k \boxtimes \overline{A}_k),\ k \in \mathbb{N})$ and assume that there exists $N \in \mathbb{R}$ such that for any smooth vector field X of M^2, we have

$$\nabla^k_X \tau_k = \mathcal{O}(k^{-N})$$

uniformly on M^2. Then there exists $C > 0$ such that, for any continuous vector field X,

$$|\nabla^k_X \tau_k| \leq C k^{-N} |X|_0 \tag{62}$$

on M^2. To prove this, we write X in local smooth frames and use the identities $\nabla^k_{X_1+X_2} = \nabla^k_{X_1} + \nabla^k_{X_2}$ and $\nabla^k_{gX} = g\nabla^k_X$. This proves (55) and (57) for $Q_k = R_k$ or $R_{m,k}$ with actually $|f|_1$ instead of $|f|_2$, because $|X_0|$ only depends on the first derivatives of f.

Dependence in Z. We claim that if $(Q_k, k \in \mathbb{N})$ is such that (56) is true for any $Z \in \mathcal{C}^\infty(M, T^{1,0}M)$, then for any $Z \in \mathcal{C}^0(M, T^{1,0}M)$ we have

$$\tfrac{i}{k} \nabla^k_{\overline{Z}} Q_k = \mathcal{O}(k^{-1})|Z|_0.$$

The proof is the same as that of (62): write Z in local smooth frames of $T^{1,0}M$.

Similarly, assume that (56) and (57) hold for any smooth section of $T^{1,0}M$; then for any $Z \in \mathcal{C}^1(M, T^{1,0}M)$, we have

$$\big[H_k(f), \tfrac{i}{k}\nabla^k_{\overline{Z}} Q_k\big] = \mathcal{O}(k^{-2})(|f|_2 |Z|_0 + |f|_1 |Z|_1).$$

The proof is the same, now with the use of the formula

$$\big[H_k(f), \tfrac{i}{k}\nabla^k_{g\overline{Z}} Q_k\big] = g\big[H_k(f), \tfrac{i}{k}\nabla^k_{\overline{Z}} Q_k\big] + \tfrac{i}{k}(X.g)\tfrac{i}{k}\nabla^k_{\overline{Z}} Q_k,$$

where X is the Hamiltonian vector field of f.

§5. Bargmann space

In this section, we prove Theorem 1.5 and the version (5) of (P2). It suffices to prove these estimates for $\hbar = 1$. Indeed, recall that for any $\hbar > 0$ and $f \in L^\infty(\mathbb{C}^n)$, we denote by $\mathcal{B}_\hbar$ the Bargmann space and by $T_\hbar(f)$ the Toeplitz operator $\Pi_\hbar f \colon \mathcal{B}_\hbar \to \mathcal{B}_\hbar$. Then we have a Hilbert space isomorphism

$$U_\hbar \colon \mathcal{B}_\hbar \to \mathcal{B}_1, \quad \xi \to \hbar^{n/2}\xi(\hbar^{1/2}\,\cdot).$$

We easily check that

$$T_\hbar(f) = U_\hbar^* T_1(f_\hbar) U_\hbar, \tag{63}$$

where $f_\hbar(x) = f(\hbar^{1/2}x)$. Recall the semi-norm $|\cdot|'_k$ introduced in (4). Since $|f_\hbar|'_k = \hbar^{k/2}|f|'_k$, we see that relation (5) and Theorem 1.5 with $\hbar = 1$ imply the same results for any $\hbar$.

Instead of $\mathcal{B}_1$, it will be more convenient to work with the closed subspace $\mathcal{B}$ of $L^2(\mathbb{C}^n, \mu)$ consisting of all functions ξ satisfying $\partial\xi/\partial\overline{z}_i = -\frac{1}{2}z_i\xi$ for $i = 1, \dots, n$. $\mathcal{B}_1$ and $\mathcal{B}$ are isomorphic Hilbert spaces due to the unitary map $\xi \to \xi e^{-|z|^2/2}$. Furthermore, for any $f \in L^\infty(\mathbb{C}^n)$, this unitary map conjugates $T_1(f)$ with $T(f) := \Pi f \colon \mathcal{B} \to \mathcal{B}$, where Π is the orthogonal projector of $L^2(\mathbb{C}^n, \mu)$ onto $\mathcal{B}$. So our goal is to prove the following.

Theorem 5.1. *For any $N \in \mathbb{N}$, there exists $C_N > 0$ such that for any $f \in \mathcal{C}_{\mathrm{b}}^{2N}(\mathbb{C}^n)$ and $g \in \mathcal{C}_{\mathrm{b}}^{N}(\mathbb{C}^n)$, we have*

$$T(f)T(g) = \sum_{\ell=0}^{N-1} (-1)^\ell \hbar^\ell \sum_{\alpha\in\mathbb{N}^n,\ |\alpha|=\ell} \frac{1}{\alpha!} T\big((\partial_z^\alpha f)(\partial_{\overline{z}}^\alpha g)\big) + R_N(f,g),$$

where $\|R_N(f,g)\|_{\mathrm{op}} \le C_N \sum_{m=0}^N |f|'_{N+m}|g|'_{N-m}$.

It is well known that the Schwartz kernel of Π is given by

$$\Pi(u,v) = (2\pi)^{-n} e^{-\frac{1}{2}(|u|^2+|v|^2)+u\cdot\overline{v}}, \quad u, v \in \mathbb{C}^n. \tag{64}$$

It satisfies the identities

$$|\Pi(u,v)| = (2\pi)^{-n} e^{-\frac{1}{2}|u-v|^2}, \tag{65}$$

$$\Pi(u,v)\Pi(v,w) = (2\pi)^{-n} e^{-(v-u)\cdot(\overline{v}-\overline{w})}\Pi(u,w). \tag{66}$$

Let $W \colon \mathbb{C}^{4n} \to \mathbb{R}$ be the weight given by

$$W = 1 + |z_1 - z_2| + |z_2 - z_3| + |z_3 - z_4|.$$

Let $N \in \mathbb{N}$. For any measurable function $g \colon \mathbb{C}^{4n} \to \mathbb{C}$ such that $|g|W^{-N}$ is bounded, we introduce the following function on $\mathbb{C}^{2n}$:

$$K(g)(x_1, x_4) = \int_{\mathbb{C}^{2n}} \Pi(x_1,x_2)\Pi(x_2,x_3)\Pi(x_3,x_4) g(x_1,x_2,x_3,x_4)\, \mu(x_2)\mu(x_3).$$

Lemma 5.2. *$K(g)$ is the Schwartz kernel of a bounded operator of $L^2(\mathbb{C}^n, \mu)$. Its uniform norm satisfies*

$$\|K(g)\| \le C_N \sup_{x\in\mathbb{C}^{4n}} \big(|g(x)|W^{-N}(x)\big)$$

for some constant C_N independent of g.

Proof. This follows from the Schur test. Indeed, by (65), for any $x_1 \in \mathbb{C}^n$ we have

$$\begin{aligned}
&\int_{\mathbb{C}^n} |K(g)(x_1,x_4)|\mu(x_4) \\
&\qquad \le \int_{\mathbb{C}^{3n}} |\Pi(x_1,x_2)\Pi(x_2,x_3)\Pi(x_3,x_4)g(x_1,x_2,x_3,x_4)|\, \mu(x_2)\mu(x_3)\mu(x_4) \\
&\qquad \le C_N \sup_{x\in\mathbb{C}^{4n}} \big(|g(x)|W^{-N}(x)\big),
\end{aligned}$$

where C_N is the constant

$$C_N = \int_{\mathbb{C}^{3n}} e^{-\frac{1}{2}(|x_1-x_2|^2+|x_2-x_3|^2+|x_3-x_4|^2)} W(x_1,x_2,x_3,x_4)^N \ \mu(x_2)\mu(x_3)\mu(x_4)$$
$$= \int_{\mathbb{C}^{3n}} e^{-\frac{1}{2}(|y_1|^2+|y_2|^2+|y_3|^2)}(1+|y_1|+|y_2|+|y_3|)^N \ \mu(y_1)\mu(y_2)\mu(y_3).$$

Similarly, $\int_{\mathbb{C}^n} |K(g)(x_1,x_4)|\mu(x_1) \le C_N \sup_{x\in\mathbb{C}^{4n}} \big(|g(x)|W^{-N}(x)\big)$. □

Observe that for f_1, $f_2 \in \mathcal{C}^0_{\mathrm{b}}(\mathbb{C}^n)$, $K(1\boxtimes f_1 \boxtimes f_2 \boxtimes 1)$ is the product of Toeplitz operators $T(f_1)T(f_2)$. In particular $K(1\boxtimes 1 \boxtimes f_2 \boxtimes 1) = T(f_2)$. Here, we denote by $1\boxtimes f_1 \boxtimes f_2 \boxtimes 1$ the functions sending (x_1,x_2,x_3,x_4) to $f_1(x_2)f_2(x_3)$. In the sequel, abusing notations, we sometimes write $K(f_1(z_2)f_2(z_3))$ instead of $K(1\boxtimes f_1 \boxtimes f_2 \boxtimes 1)$.

Lemma 5.3. *Let $g \in \mathcal{C}^1(\mathbb{C}^{4n})$ be such that $|g|W^{-N}$ and $|f|W^{-N}$ are bounded, where f is any partial derivative of g. Then*

$$K\big((\overline{z}_{2,i}-\overline{z}_{3,i})g\big) = K\big(\partial g/\partial z_{2,i}\big), \quad K\big((z_{3,i}-z_{2,i})g\big) = K\big(\partial g/\partial \overline{z}_{3,i}\big).$$

Proof. By (66), we have

$$\Big(\frac{\partial}{\partial z_{2,i}} + (\overline{z}_{2,i}-\overline{z}_{3,i})\Big)\big(\Pi(z_1,z_2)\Pi(z_2,z_3)\big) = 0,$$
$$\Big(\frac{\partial}{\partial \overline{z}_{3,i}} + (z_{3,i}-z_{2,i})\Big)\big(\Pi(z_2,z_3)\Pi(z_3,z_4)\big) = 0,$$

and the result follows by integrating by part. □

Consider now $N\in\mathbb{N}$ and $f,g\in\mathcal{C}^{2N}_{\mathrm{b}}(\mathbb{C}^n)$. We compute $K(f(z_2)g(z_3))$ by replacing $f(z_2)$ with its Taylor expansion around z_3,

$$f(z_2) = \sum_{\substack{\alpha,\beta\in\mathbb{N}^n,\\ |\alpha|+|\beta|<2N}} \frac{1}{\alpha!\beta!} f_{\alpha,\beta}(z_3)(z_2-z_3)^\alpha(\overline{z}_2-\overline{z}_3)^\beta + r_N(z_2,z_3).$$

Here for any α, β, we denote by $f_{\alpha,\beta}$ the derivative $\partial_z^\alpha \partial_{\overline{z}}^\beta f$. Furthermore, for any $z_2, z_3 \in \mathbb{C}^n$, we have

$$|r_N(z_2,z_3)| \le C'_N |f|'_{2N}\big(1+|z_2-z_3|\big)^{2N}$$

with some constant C'_N independent of f. By Lemma 5.2,

$$\|K(r_N(z_2,z_3)g(z_3))\| \le C_{2N}C'_N|f|'_{2N}|g|'_0.$$

Denote by $P_{\alpha,\beta}$ the operator $K\big(f_{\alpha,\beta}(z_3)(z_2-z_3)^\alpha(\overline{z}_2-\overline{z}_3)^\beta g(z_3)\big)$. We have

$$T(f)T(g) = \sum_{\substack{\alpha,\beta\in\mathbb{N}^n,\\ |\alpha|+|\beta|<2N}} \frac{P_{\alpha,\beta}}{\alpha!\beta!} + K((r_N(z_2,z_3)g(z_3)).$$

In the sequel we say that $\beta\le\alpha$ if $\beta(i)\le\alpha(i)$ for any $i=1,\dots,n$.

Lemma 5.4. *Let α, $\beta\in\mathbb{N}^n$ be such that $|\alpha|+|\beta|<2N$. If $|\alpha|<N$ and $\beta\le\alpha$, then*

$$P_{\alpha,\beta} = \frac{\alpha!(-1)^{|\alpha-\beta|}}{(\alpha-\beta)!}T(\partial_{\overline{z}}^{\alpha-\beta}(f_{\alpha,\beta}g)), \tag{67}$$

and otherwise

$$\|P_{\alpha,\beta}\| \le C_N \sum_{m=0}^{N} |f|'_{N+m}|g|'_{N-m} \tag{68}$$

with some constant C_N independent of f and g.

Proof. First, if for some i we have $\beta(i) > \alpha(i)$, then by the first identity in Lemma 5.3, we have $P_{\alpha,\beta} = 0$ and (68) is satisfied. Assume now that $\beta \leq \alpha$. By the first identity in Lemma 5.3,

$$P_{\alpha,\beta} = \frac{\alpha!}{(\alpha-\beta)!} K\big(f_{\alpha,\beta}(z_3)(z_2 - z_3)^{\alpha-\beta} g(z_3)\big).$$

If $|\alpha| \geq N$, then $|\alpha| + |\beta| + |\alpha - \beta| = 2|\alpha| \geq 2N$, so we can find a multiindex $\gamma \in \mathbb{N}^n$ such that $|\alpha| + |\beta| + |\gamma| = 2N$ and $\gamma \leq \alpha - \beta$. By the second identity in Lemma 5.3, we have

$$P_{\alpha,\beta} = \frac{\alpha!(-1)^{|\gamma|}}{(\alpha-\beta)!} K\big((z_2 - z_3)^{\alpha-\beta-\gamma}\big(\partial_{\overline{z}}^{\gamma}(f_{\alpha,\beta}g)\big)(z_3)\big).$$

Then expanding $\partial_{\overline{z}}^{\gamma}(f_{\alpha,\beta}g)$ and applying Lemma 5.2 with the weight W^{-2N}, we deduce that (68) is satisfied. Finally, assume that $|\alpha| \leq N$ and $\beta \leq \alpha$, so that $f_{\alpha,\beta}g$ is of class $\mathcal{C}^{|\alpha-\beta|}$. Then by second identity in Lemma 5.3, we have

$$P_{\alpha,\beta} = \frac{\alpha!(-1)^{|\alpha-\beta|}}{(\alpha-\beta)!} K\big(\big(\partial_{\overline{z}}^{\alpha-\beta}(f_{\alpha,\beta}g)\big)(z_3)\big),$$

and we arrive at (67). $\square$

To finish the proof of Theorem 5.1, we use the following algebraic identity based on the fact that $\partial_{\overline{z}}^{\gamma} f_{\alpha,\beta} = f_{\alpha,\beta+\gamma}$.

Lemma 5.5. *For any $\alpha \in \mathbb{N}^n$, we have*

$$\sum_{\substack{\beta\in\mathbb{N}^n,\\ \beta\leq\alpha}} \frac{(-1)^{|\alpha-\beta|}}{\beta!(\alpha-\beta)!} \partial_{\overline{z}}^{\alpha-\beta}(f_{\alpha,\beta}g) = \frac{1}{\alpha!} f_{\alpha,0}\, g_{0,\alpha}.$$

Proof. Setting $\gamma = \alpha - \beta$, we have

$$\begin{aligned}\sum_{\substack{\beta\in\mathbb{N}^n,\\ \beta\leq\alpha}} \frac{(-1)^{|\alpha-\beta|}}{\beta!(\alpha-\beta)!} \partial_{\overline{z}}^{\alpha-\beta}(f_{\alpha,\beta}g) &= \sum_{\substack{\gamma\in\mathbb{N}^n,\\ \gamma\leq\alpha}} \frac{(-1)^{|\gamma|}}{(\alpha-\gamma)!\gamma!} \partial_{\overline{z}}^{\gamma}(f_{\alpha,\alpha-\gamma}g)\\ &= \sum_{\substack{\delta,\gamma\in\mathbb{N}^n,\\ \delta\leq\gamma\leq\alpha}} \frac{(-1)^{|\gamma|}}{(\alpha-\gamma)!\delta!(\gamma-\delta)!} f_{\alpha,\alpha-\delta}\, g_{0,\delta} = \sum_{\substack{\delta\in\mathbb{N}^n,\\ \delta\leq\alpha}} \frac{1}{\delta!} C(\delta,\alpha) f_{\alpha,\alpha-\delta}\, g_{0,\delta},\end{aligned}$$

where

$$C(\delta,\alpha) = \sum_{\substack{\gamma\in\mathbb{N}^n,\\ \delta\leq\gamma\leq\alpha}} \frac{(-1)^{|\gamma|}}{(\alpha-\gamma)!(\gamma-\delta)!} = \sum_{\substack{\lambda\in\mathbb{N}^n,\\ \lambda\leq\alpha-\delta}} \frac{(-1)^{|\lambda-\delta|}}{(\alpha-\delta-\lambda)!\lambda!} = \begin{cases} (-1)^{|\alpha|} & \text{if } \alpha = \delta,\\ 0 & \text{otherwise,}\end{cases}$$

which concludes the proof. $\square$

Now we explain the proof of estimate (5). As for the proof of Theorem 1.5, we may assume that $\hbar = 1$ and work in $\mathcal{B}$ instead of $\mathcal{B}_1$. So we need to prove the following.

Theorem 5.6. *There exists β'' such that for any $f, g \in \mathcal{C}^3_{\mathrm{b}}(\mathbb{C}^n)$, we have*

$$\big\|\big[T(f), T(g)\big] - iT(\{f,g\})\big\|_{\mathrm{op}} \leq \beta''\big(|f|'_1|g|'_3 + |f|'_2|g|'_2 + |f|'_3|g|'_1\big).$$

Consider the trivial holomorphic line bundle L over $\mathbb{C}^n$ with canonical frame s. Define the metric of L so that $|s|^2(z) = e^{-|z|^2}$. Then the space $\mathcal{H}$ of holomorphic sections of L with finite L^2-norm is isomorphic to $\mathcal{B}_1$ by the map sending a function $\xi \in \mathcal{B}_1$ to the section $\xi s \in \mathcal{H}$. If we work with the unitary frame $t = e^{-|z|^2/2}s$ instead of s, we get an isomorphism between $\mathcal{B}$ and $\mathcal{H}$ by sending ξ to ξt. In the sequel, we identify $\mathcal{H}$ with $\mathcal{B}$ in this way and more generally, identify the space of continuous sections of L with $\mathcal{C}^0(\mathbb{C}^n)$.

A straightforward computation shows that the Chern connection ∇ of L is given in terms of t by

$$\nabla t = \alpha \otimes t \text{ with } \alpha = \frac{i}{2} \sum (z_i d\overline{z}_i - \overline{z}_i dz_i).$$

The curvature is $\frac{1}{i}\omega$, where ω is the symplectic form $\omega = i \sum dz_i \wedge d\overline{z}_i$. For any function $f \in \mathcal{C}^1(\mathbb{C}^n)$, introduce the Kostant–Souriau operator

$$H(f) = f - i\nabla_X,$$

where X is the Hamiltonian vector field of f. It acts on functions of class $\mathcal{C}^\infty(\mathbb{C}^n)$ by $H(f) = f - iX - i\alpha(X)$. One easily checks the Kostant–Souriau and Tuynman formulas

$$[H(f), H(g)] = iH(\{f, g\}), \quad f, g \in \mathcal{C}^1_{\mathrm{b}}(\mathbb{C}^n),$$
$$\Pi H(f)\Pi = T(f - \Delta f), \quad f \in \mathcal{C}^2_{\mathrm{b}}(\mathbb{C}^n),$$

where $\Delta = -\sum \partial^2/\partial z_i \partial \overline{z}_i$. Furthermore, we have the following result similar to Theorem 3.5.

Lemma 5.7. *There exists $C > 0$ such that for any $f \in \mathcal{C}^2_{\mathrm{b}}(\mathbb{C})$, we have*

$$\big\| [H(f), \Pi] \big\|_{\mathrm{op}} \leq C|f_2|'.$$

Proof. By a straightforward computation, we check first that the Hamiltonian vector field X of f is given by $X = i\sum\big((\partial f/\partial \overline{z}_i)\partial_{z_i} - (\partial f/\partial z_i)\partial_{\overline{z}_i}\big)$. Then the Schwartz kernel of $\nabla_X \circ \Pi$ is $i\sum(\overline{v}_i - \overline{u}_i)(\partial f/\partial \overline{z}_i)(u)\Pi(u, v)$. So, the Schwartz kernel of the commutator $[H(f), \Pi]$ is $m(u, v)\Pi(u, v)$ with

$$m(u, v) = f(u) - f(v) - \sum (u_i - v_i)\frac{\partial f}{\partial z_i}(v) - \sum (\overline{u}_i - \overline{v}_i)\frac{\partial f}{\partial \overline{z}_i}(u).$$

Replacing $f(u)$ by its Taylor expansion at v, we see that

$$|m(u, v)| \leq C|f|'_2(1 + |u - v|^2)$$

for some constant C independent of f. Applying the Schur test as in the proof of Lemma 5.2, we conclude the proof. ☐

Now the proof of Theorem 5.6 is completely similar to that of Proposition 3.9, where instead of Proposition 3.8 one uses directly Theorem 5.1 with $N = 1$.

Acknowledgments

We thank Yohann Le Floch, Nicolas Lerner, Stéphane Nonnenmacher, Johannes Sjöstrand, San Vũ Ngọc, and Steve Zelditch for useful discussions.

References

[BMMP14] T. Barron, X. Ma, G. Marinescu, and M. Pinsonnault, *Semi-classical properties of Berezin–Toeplitz operators with C^k-symbol*, J. Math. Phys. **55** (2014), no. 4, 042108. MR3390584

[Ber75] F. Berezin, *General concept of quantization*, Comm. Math. Phys. **40** (1975), 153–174. MR0411452

[BS91] F. Berezin and M. Shubin, *The Schrödinger equation*, Moskov. Gos. Univ., Moscow, 1983; English transl., Kluwer Acad. Publ., Dordreht, 1991. MR1186643

[BBS08] R. Berman, B. Berndtsson, and J. Sjöstrand, *A direct approach to Bergman kernel asymptotics for positive line bundles*, Ark. Mat. **46** (2008), no. 2, 197–217. MR2430724

[BMS94] M. Bordemann, E. Meinrenken, and M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and* $\mathrm{gl}(N)$*, $N \to \infty$ limits*, Comm. Math. Phys. **165** (1994), no. 2, 281–296. MR1301849

[BU96] D. Borthwick and A. Uribe, *Almost complex structure and geometric quantization*, Math. Res. Lett. **3** (1996), no. 6, 845–861. MR1426541

[BdMG81] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, Ann. of Math. Stud., vol. 99, Princeton Univ. Press, Princeton, NJ, 1981. MR620794

[BdMS76] L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegő*, Journées: Équations aux Dérivées Partielles de Rennes (1975), Astérisque **34–35** (1976), 123–164. MR0590106

[BLW14] P. Busch, P. Lahti, and R. F. Werner, *Quantum root-mean-square error and measurement uncertainty relations*, Rev. Modern Phys. **86** (2014), 1261–1281. MR3390587

[Cha03] L. Charles, *Berezin–Toeplitz operators, a semi-classical approach*, Comm. Math. Phys. **239** (2003), no. 1–2, 1–28. MR1997113

[Cha06] ———, *Symbolic calculus for Toeplitz operators with half-form*, J. Symplectic Geom. **4** (2006), no. 2, 171–198. MR2275003

[Cha07] ———, *Semi-classical properties of geometric quantization with metaplectic correction*, Comm. Math. Phys. **270** (2007), no. 2, 445–480. MR2276452

[Cha14] ———, *Quantization of compact symplectic manifold*, J. Geom. Anal. **26** (2016), no. 4, 2664–2710. MR3544936

[CZ98] Q. Chen and X. Zhou, *Generalization of interpolation inequalities*, J. Math. Anal. Appl. **226** (1998), no. 1, 130–142. MR1646469

[Chi09] H. Chihara, *Bounded Berezin–Toeplitz operators on the Segal–Bargmann space*, Integral Equations Operator Theory **63** (2009), no. 3, 321–335. MR2491034

[Co92] L. A. Coburn, *Deformation estimates for the Berezin–Toeplitz quantization*, Comm. Math. Phys. **149** (1992), no. 2, 415–424. MR1186036

[GM00] V. L. Ginzburg and R. Montgomery, *Geometric quantization and no-go theorems*, Poisson Geometry (Warsaw, 1998), Banach Center Publ., vol. 51, Polish Acad. Sci. Inst. Math., Warsaw, 2000, pp. 69–77. MR1764435

[Fo89] G. B. Folland, *Harmonic analysis in phase space*, Ann. of Math. Stud., vol. 122, Princeton Univ. Press, Princeton, NJ, 1989. MR983366

[HS78] P. R. Halmos and V. S. Sunder, *Bounded integral operators on L^2 spaces*, Ergeb. Math. Grenzgeb. (3), Springer-Verlag, Berlin–New York, 1978. MR517709

[Gui95] V. Guillemin, *Star products on compact pre-quantizable symplectic manifolds*, Lett. Math. Phys. **35** (1995), no. 1, 85–89. MR1346047

[Ko70] B. Kostant, *Quantization and unitary representations.* I. *Prequantization*, Lectures in Modern Analysis and Applications. III, Lecture Notes in Math., vol. 170, Springer, Berlin, 1970, pp. 87–208. MR0294568

[La98] N. P. Landsman, *Mathematical topics between classical and quantum mechanics*, Springer Monogr. Math., Springer-Verlag, New York, 1998. MR1662141

[Le10] N. Lerner, *Metrics on the phase space and non-selfadjoint pseudo-differential operators*, Pseudo-Differential Operators. Theory and Applications, vol. 3, Birkhäuser Verlag, Basel, 2010. MR2599384

[LF16] Y. Le Floch, *Berezin–Toeplitz operators on compact Kähler manifolds: an introduction*, Preprint, 2016; `https://sites.google.com/site/yohannleflochhomepage/english-version/teaching`

[LFPS16] Y. Le Floch, L. Polterovich, and O. Shabtai, (in preparation)

[MM07] X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, Progr. Math., vol. 254, Birkhäuser Verlag, Basel, 2007. MR2339952

[MM08] ———, *Toeplitz operators on symplectic manifolds*, J. Geom. Anal. **18** (2008), no. 2, 565–611. MR2393271

[Po14] L. Polterovich, *Symplectic geometry of quantum noise*, Comm. Math. Phys. **327** (2014), no. 4, 481–519. MR3183407

[PR14] L. Polterovich and D. Rosen, *Function theory on symplectic manifolds*, CRM Monogr. Ser., vol. 34. Amer. Math. Soc., Providence, RI, 2014. MR3241729

[SZ02] B. Shiffman and S. Zelditch, *Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds*, J. Reine Angew. Math. **544** (2002), 181–222. MR1887895

[Shu78] M. A. Shubin, *Pseudodifferential operators and spectral theory*, Nauka, Moscow, 1978; English transl., Springer-Verlag, Berlin, 2001. MR0509034 (2002d:47073)

[Sj08] J. Sjöstrand, *Pseudodifferential operators and weighted normed symbol spaces*, Serdica Math. J. **34** (2008), no. 1, 1–38. MR2414412

[Sou70] J.-M. Souriau, *Structure des systèmes dynamiques*, Maîtrises de mathématiques, Dunod, Paris, 1970. MR0260238

[Tuy87] G. M. Tuynman, *Quantization: towards a comparison between methods*, J. Math. Phys. **28** (1987), no. 12, 2829–2840. MR917637

[Z12] M. Zworski, *Semiclassical analysis*, Grad. Stud. Math., vol. 138, Amer. Math. Soc., Providence, RI, 2012. MR2952218

UMR 7586, Institut de Mathématiques, de Jussieu-Paris Rive Gauche, Sorbonne Universités, UPMC Univ Paris 06, F-75005, Paris, France

Email address: laurent.charles@imj-prg.fr

Faculty of Exact Sciences, School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel

Email address: polterov@post.tau.ac.il

Received 13/OCT/2016

Originally published in English