Алгебра и анализ
Том 29 (2017), № 1

St. Petersburg Math. J.
Vol. 29 (2018), No. 1, Pages 223–225
http://dx.doi.org/10.1090/spmj/1490
Article electronically published on December 27, 2017

A QUEST FOR 5-POINT CONDITION OF ALEXANDROV'S TYPE

A. PETRUNIN

Easy reading for professionals

Abstract. A description of Alexandrov's 4-point comparison via quadratic forms is given and a natural 5-point condition which might have future applications is proposed.

Associated form

We construct a quadratic form $W_{\boldsymbol{x}}$ on $\mathbb{R}^{n-1}$ for a given n-point array $\boldsymbol{x} = (x_1, \dots, x_n)$ in a metric space X.

For this, we fix a nondegenerate simplex $\triangle$ in $\mathbb{R}^{n-1}$ and denote by $v_1, \dots, v_n$ its vertices. If $(e_1, \dots, e_{n-1})$ is the standard basis on $\mathbb{R}^{n-1}$, we may assume that $v_i = e_i$ for $i < n$ and $v_n = 0$.

Let $|a - b|_X$ denote the distance between points a and b in the metric space X. Note that the formula

$$W_{\boldsymbol{x}}(v_i - v_j) = |x_i - x_j|_X^2$$

for all i and j determines uniquely a quadratic form W_x.

This quadratic form $W_{\boldsymbol{x}}$ will be called the *form of the point array $\boldsymbol{x}$ with respect to the simplex $\triangle$.*

Note that an array $\boldsymbol{x} = (x_1, \dots, x_n)$ in a metric space X is isometric to an array in Euclidean space if and only if $W_{\boldsymbol{x}}(v) \geq 0$ for any $v \in \mathbb{R}^{n-1}$.

In particular, the condition $W_{\boldsymbol{x}} \geq 0$ for triples of points means that all three triangle inequalities hold true.

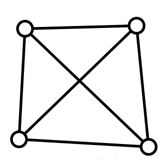
Alexandrov's 4-point comparison

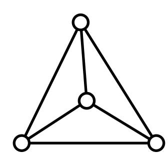
Now we discuss the relationship between the form of quadruples and geometry of the space. In this case $\triangle$ is a tetrahedron on $\mathbb{R}^3$.

From the 3-point case, it follows that $W_{\boldsymbol{x}}$ is nonnegative on every plane parallel to a face of the tetrahedron $\triangle$. In particular, $W_{\boldsymbol{x}}$ can have at most one negative eigenvalue.

Assume that $W_{\boldsymbol{x}}(w) < 0$ for some $w \in \mathbb{R}^3$. From the above it follows that w is transversal to each of 4 planes parallel to a faces of $\triangle$.

Consider the projection of $\triangle$ along w to a transversal plane. Note that in the projection the 4 vertices of $\triangle$ lie in general position, that is, no three of them lie on one line. Therefore, we can see one of the two combinatorial pictures shown on the diagram. It is easily seen that the combinatorics of the picture does not depend on the choice of w. If we see the diagram on the left, we say that $\boldsymbol{x}$ is of type Quad(4) and otherwise we say that it is of type Quad(3).


Quad(4)


Quad(3)

2010 *Mathematics Subject Classification.* Primary 51E12.

Key words and phrases. Alexandrov's 4-point comparison, quadratic forms.

The following statements give a relationship between the forms $W_{\boldsymbol{x}}$ of the quadruple $\boldsymbol{x}$ and the curvature bounds in the sense of Alexandrov. The proofs are left to the reader.

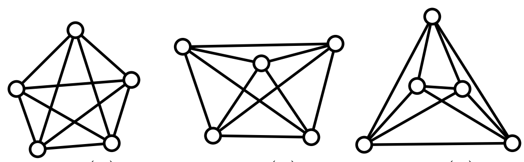
Assume X is a complete space with intrinsic metric. The following statements hold true.

- If $W_{\boldsymbol{x}} \geq 0$ for any quadrilateral $\boldsymbol{x} = (x_1, \dots, x_4)$, then X is isometric to a closed convex set in a Hilbert space.
- X has no quadruples of type $\mathrm{Quad}(3)$ if and only if X has nonnegative curvature in the sense of Alexandrov; in this case we say that X is an Alex[0] space.
- X has no quadruples of type $\mathrm{Quad}(4)$ if and only if X is a CAT[0] space, which is also called a Hadamard space

5-point conditions

Let us try to do the same for 5-points arrays $\boldsymbol{x} = (x^1, \dots, x^5)$ in a metric space. Its form $W_{\boldsymbol{x}}$ is defined on $\mathbb{R}^4$ and it must be nonnegative on any plane parallel to any of 10 two-dimensional faces of the 4-simplex $\triangle$. In particular, $W_{\boldsymbol{x}}$ has at most two negative eigenvalues.

In the case where $W_{\boldsymbol{x}}$ has exactly two negative eigenvalues, one can choose a plane Π such that the restriction of $W_{\boldsymbol{x}}$ to Π is negative. Let us project $\triangle$ along Π to a transversal plane. The same argument as in the case of $n = 4$ shows that after projection the vertices of $\triangle$ lie in general position. Therefore we may get one of the following three combinatorial pictures.



Pent(5) Pent(4) Pent(3)

That is, for any 5-point array $\boldsymbol{x}$, either $W_{\boldsymbol{x}}$ has at most one negative eigenvalue or it has exactly two negative eigenvalues and belongs to one of the three types $\mathrm{Pent}(5)$, $\mathrm{Pent}(4)$ or $\mathrm{Pent}(3)$.

We may consider metric spaces that do not admit 5-point arrays of some of these types. For example, a $\mathrm{Pent}(\widehat{3}, \widehat{4})$ space is a complete length-metric space without 5-point arrays of type $\mathrm{Pent}(3)$ and $\mathrm{Pent}(4)$.

Here are some easy observations about these new classes of metric spaces.

(i) Any CAT[0] space is a $\mathrm{Pent}(\widehat{3}, \widehat{4}, \widehat{5})$ space. In other words, the form $W_{\boldsymbol{x}}$ for any 5-point array $\boldsymbol{x}$ in any CAT[0] space has at most one negative eigenvalue.

(ii) Any Alex[0] space has no 5-point arrays of type $\mathrm{Pent}(3)$ or $\mathrm{Pent}(4)$, that is, it is a $\mathrm{Pent}(\widehat{3}, \widehat{4})$ space.

(iii) If a complete Riemannian manifold M has no 5-point arrays of type $\mathrm{Pent}(5)$, then it is simply connected and it has nonpositive sectional curvature.

Question. Do $\mathrm{Pent}(\widehat{3}, \widehat{4})$ spaces have meaningful geometry?

I find this question interesting because the $\mathrm{Pent}(\widehat{3}, \widehat{4})$ spaces include all CAT[0] as well as Alex[0] spaces, i.e., all Alexandrov spaces with nonnegative and nonpositive curvature. Therefore, a positive answer to the above question might lead to a uniform treatment of these two types of spaces.

We present a couple of examples of properties shared by CAT[0] and Alex[0] spaces.

- Two minimizing geodesics with common ends and yet one common point must coinside.
- A plane with metric induced by norm is CAT[0] or Alex[0] only if the norm is Euclidean.

At the moment I do not know if the same holds true for $\mathrm{Pent}(\widehat{3}, \widehat{4})$ spaces.

Here is another question related to observations (i) and (iii).

Question. Is it true that any $\mathrm{Pent}(\widehat{3},\widehat{4},\widehat{5})$ space with intrinsic metric is $\mathrm{CAT}[0]$?

Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA
Email address: petrunin@math.psu.edu

Received 10/OCT/2016

Translated by THE AUTHOR