Алгебра и анализ
Том 29 (2017), № 2

St. Petersburg Math. J.
Vol. 29 (2018), No. 2, Pages 325–362
http://dx.doi.org/10.1090/spmj/1496
Article electronically published on March 12, 2018

HOMOGENIZATION OF THE DIRICHLET PROBLEM FOR HIGHER-ORDER ELLIPTIC EQUATIONS WITH PERIODIC COEFFICIENTS

T. A. SUSLINA

To the memory of Vladimir Savel'evich Buslaev

ABSTRACT. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class C^{2p}. The object under study is a selfadjoint strongly elliptic operator $A_{D,\varepsilon}$ of order $2p$, $p \geq 2$, in $L_2(\mathcal{O};\mathbb{C}^n)$, given by the expression $b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$, $\varepsilon > 0$, with the Dirichlet boundary conditions. Here $g(\mathbf{x})$ is a bounded and positive definite $(m \times m)$-matrix-valued function in $\mathbb{R}^d$, periodic with respect to some lattice; $b(\mathbb{D}) = \sum_{|\alpha|=p} b_\alpha \mathbf{D}^\alpha$ is a differential operator of order p with constant coefficients; and the b_α are constant $(m \times n)$-matrices. It is assumed that $m \geq n$ and the symbol $b(\boldsymbol{\xi})$ has maximal rank. Approximations are found for the resolvent $(A_{D,\varepsilon} - \zeta I)^{-1}$ in the $L_2(\mathcal{O};\mathbb{C}^n)$-operator norm and in the norm of operators acting from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^p(\mathcal{O};\mathbb{C}^n)$, with error estimates depending on ε and ζ.

INTRODUCTION

An extensive literature is devoted to homogenization problems for differential operators (DOs) with periodic rapidly oscillating coefficients. To start with, we mention the books [BeLPa, BaPan, ZhKO].

0.1. Operator error estimates for homogenization problems in $\mathbb{R}^d$. In a series of papers [BSu1, BSu2, BSu3, BSu4] by Birman and Suslina, an operator-theoretic approach to homogenization problems was suggested and developed. This approach was applied to the study of a wide class of matrix selfadjoint strongly elliptic second order DOs acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ and admitting a factorization of the form

$$\mathcal{A}_\varepsilon = b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D}), \quad \varepsilon > 0. \tag{0.1}$$

Here an $(m \times m)$-matrix-valued function $g(\mathbf{x})$ is bounded, uniformly positive definite, and periodic with respect to some lattice $\Gamma \subset \mathbb{R}^d$. Next, $b(\mathbf{D})$ is a first order DO of the form $b(\mathbf{D}) = \sum_{j=1}^d b_j D_j$, where the b_j are constant $(m \times n)$-matrices. It is assumed that $m \geq n$ and that the symbol $b(\boldsymbol{\xi})$ has rank n for any $0 \neq \boldsymbol{\xi} \in \mathbb{R}^d$. The simplest example of an operator like (0.1) is the acoustics operator $-\operatorname{div} g(\mathbf{x}/\varepsilon)\nabla$; the operator of elasticity theory can also be written in the required form. These and other examples were considered in [BSu2] in detail.

In [BSu1, BSu2], it was shown that, as $\varepsilon \to 0$, the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ converges in the $L_2(\mathbb{R}^d;\mathbb{C}^n)$-operator norm to the resolvent of the *effective operator* $\mathcal{A}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$.

2010 *Mathematics Subject Classification.* Primary 35B27.
Key words and phrases. Periodic differential operators, higher-order elliptic equations, Dirichlet problem, homogenization, effective operator, corrector, operator error estimates.
Supported by RFBR (project no. 16-01-00087).

Here g^0 is a constant *effective matrix*. The following estimate was proved:

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C\varepsilon. \tag{0.2}$$

In [BSu3], a sharper approximation for the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ in the $L_2(\mathbb{R}^d;\mathbb{C}^n)$-operator norm with error $O(\varepsilon^2)$ was found. In [BSu4], an approximation for the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the Sobolev space $H^1(\mathbb{R}^d;\mathbb{C}^n)$ was obtained. It was proved that

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon\mathcal{K}(\varepsilon)\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \le C\varepsilon. \tag{0.3}$$

Here $\mathcal{K}(\varepsilon)$ is the so-called *corrector*. The operator $\mathcal{K}(\varepsilon)$ contains rapidly oscillating factors and so depends on ε; herewith, $\|\mathcal{K}(\varepsilon)\|_{L_2\to H^1} = O(\varepsilon^{-1})$.

Estimates of the form (0.2) and (0.3) are called *operator error estimates*. They are order-sharp. The method of [BSu1, BSu2, BSu3, BSu4] is based on the scaling transformation, the Floquet–Bloch theory, and analytic perturbation theory.

We also mention the recent papers [Su4, Su5], where two-parametric analogs of estimates (0.2) and (0.3) (depending on ε and ζ) for the resolvent $(\mathcal{A}_\varepsilon - \zeta I)^{-1}$ at an arbitrary point $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$ were obtained.

A different approach to operator error estimates (*the modified method of first-order approximation* or *the shift method*) was suggested by Zhikov; in [Zh] and [ZhPas1], this method was employed to get estimates of the form (0.2) and (0.3) for the acoustics operator and the elasticity operator. Concerning further results, see the recent survey [ZhPas2] by Zhikov and Pastukhova and the references therein.

A homogenization problem for periodic elliptic DOs of *high even order* is of separate interest. The operator-theoretic approach of Birman and Suslina was developed for such operators in the paper [V] by Veniaminov and in the recent paper [KuSu] by Kukushkin and Suslina.

In [V], operators of the form $\mathcal{B}_\varepsilon = (\mathbf{D}^p)^* g(\mathbf{x}/\varepsilon)\mathbf{D}^p$ were studied. Here $g(\mathbf{x})$ is a symmetric positive definite and bounded tensor of order $2p$, periodic with respect to a lattice Γ. Such an operator with $p = 2$ arises in the theory of elastic plates (see [ZhKO]). The effective operator is given by $\mathcal{B}^0 = (\mathbf{D}^p)^* g^0\mathbf{D}^p$, where g^0 is the effective tensor. In [V], the following analog of estimate (0.2) was proved:

$$\|(\mathcal{B}_\varepsilon + I)^{-1} - (\mathcal{B}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C\varepsilon.$$

In [KuSu], a more general class of higher-order elliptic DOs acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ and admitting a factorization of the form

$$A_\varepsilon = b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D}) \tag{0.4}$$

was studied. Here $g(\mathbf{x})$ is a bounded and uniformly positive definite $(m\times m)$-matrix-valued function, periodic with respect to Γ. The operator $b(\mathbf{D})$ of order $p \ge 2$ is of the form $b(\mathbf{D}) = \sum_{|\alpha|=p} b_\alpha \mathbf{D}^\alpha$, where the b_α are constant $(m\times n)$-matrices. It is assumed that $m \ge n$ and that the symbol $b(\boldsymbol{\xi})$ has rank n for any $0 \ne \boldsymbol{\xi} \in \mathbb{R}^d$. The main results of [KuSu] are approximations of the resolvent $(A_\varepsilon - \zeta I)^{-1}$, where $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$, in various operator norms with two-parametric error estimates (depending on ε and ζ). It was shown that the resolvent $(A_\varepsilon - \zeta I)^{-1}$ converges in the $L_2(\mathbb{R}^d;\mathbb{C}^n)$-operator norm to the resolvent of the effective operator $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ (where g^0 is the constant effective matrix), and

$$\|(A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_1(\zeta)\varepsilon. \tag{0.5}$$

Approximation was obtained for the resolvent in the "energy" norm (i. e., the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^p(\mathbb{R}^d;\mathbb{C}^n)$), with the corrector taken into account:

$$\|(A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1} - \varepsilon^p K(\zeta;\varepsilon)\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \le C_2(\zeta)\varepsilon. \tag{0.6}$$

The corrector $K(\zeta;\varepsilon)$ contains rapidly oscillating factors; we have $\|K(\zeta;\varepsilon)\|_{L_2\to H^p} = O(\varepsilon^{-p})$. The dependence of $C_1(\zeta)$ and $C_2(\zeta)$ on ζ is searched out.

Similar results on homogenization of higher-order elliptic operators were obtained in the recent papers [Pas1, Pas2] by Pastukhova with the help of the shift method (in those papers, estimates are one-parametric, it was assumed that $\zeta = -1$).

0.2. Operator error estimates for homogenization problems in a bounded domain. Operator error estimates were also studied for second order elliptic operators with rapidly oscillating coefficients in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with sufficiently smooth boundary. In [Zh, ZhPas1], the acoustics operator and the operator of elasticity theory with the Dirichlet or Neumann conditions on the boundary $\partial\mathcal{O}$ were studied; analogs of estimates (0.2) and (0.3), but with error terms of order $O(\varepsilon^{1/2})$, were obtained. The error deteriorates because of the boundary influence. (In the case of the Dirichlet problem for the acoustics operator, the $(L_2 \to L_2)$-estimate was improved in [ZhPas1], but the order was not sharp.)

Similar results for the operator $-\operatorname{div} g(\mathbf{x}/\varepsilon)\nabla$ in a bounded domain with the Dirichlet or Neumann conditions were obtained in the papers [Gr1, Gr2] by Griso with the help of the "unfolding" method. In [Gr2], an analog of estimate (0.2) of sharp order $O(\varepsilon)$ for the same operator was obtained for the first time.

For the second order matrix operators $\mathcal{A}_{D,\varepsilon}$ and $\mathcal{A}_{N,\varepsilon}$ given by expression (0.1) with the Dirichlet or Neumann conditions, respectively, operator error estimates were obtained in the papers [PSu1, PSu2, Su1, Su2, Su3]. In [PSu1, PSu2], the Dirichlet problem was studied and the following estimate was obtained:

$$\big\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} - \varepsilon\mathcal{K}_D(\varepsilon)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \le C\varepsilon^{1/2}. \tag{0.7}$$

Here $\mathcal{A}_D^0$ is the effective operator with the Dirichlet condition, and $\mathcal{K}_D(\varepsilon)$ is the corresponding corrector. In [Su1, Su2], a sharp-order estimate in the $L_2(\mathcal{O};\mathbb{C}^n)$-operator norm was proved:

$$\big\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le C\varepsilon. \tag{0.8}$$

Similar results for the Neumann problem were obtained in [Su3]. The method of [PSu1, PSu2, Su1, Su2, Su3] was based on using the results for the problem in $\mathbb{R}^d$, introduction of the boundary layer correction term, and estimation of this term in $H^1(\mathcal{O};\mathbb{C}^n)$ and in $L_2(\mathcal{O};\mathbb{C}^n)$. Some technical tricks were borrowed from [ZhPas1].

In the recent papers [Su4, Su5], approximations for the resolvents $(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1}$ and $(\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}$ at an arbitrary point $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$ (two-parametric analogs of estimates (0.7) and (0.8)) were obtained.

Independently, an estimate of the form (0.8) for uniformly elliptic second order systems with the Dirichlet or Neumann conditions, under some regularity assumptions on the coefficients, was obtained by a different method in the paper [KeLiS] by Kenig, Lin, and Shen.

0.3. Main results. In the present paper, we study the operator $A_{D,\varepsilon}$ of order $2p$ in a bounded domain $\mathcal{O}$ of class C^{2p}. This operator is given in a factorized form (0.4) under the Dirichlet conditions on the boundary $\partial\mathcal{O}$. *Our goal* is to find approximations for the resolvent $(A_{D,\varepsilon} - \zeta I)^{-1}$ at a regular point ζ with error estimates depending on ε and ζ.

Now we describe the main results. Let $\zeta = |\zeta|e^{i\varphi} \in \mathbb{C}\setminus\mathbb{R}_+$ be such that $|\zeta| \ge 1$. It is proved that

$$\big\|(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le \mathcal{C}_1(\varphi)\big(\varepsilon|\zeta|^{-1+1/2p} + \varepsilon^{2p}\big), \tag{0.9}$$

(0.10)
$$\big\|(A_{D,\varepsilon}-\zeta I)^{-1}-(A_D^0-\zeta I)^{-1}-\varepsilon^p K_D(\zeta;\varepsilon)\big\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})}\le \mathcal{C}_2(\varphi)\big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p}+\varepsilon^p\big)$$

for $0<\varepsilon\le\varepsilon_1$ (where ε_1 is a sufficiently small number depending on the domain $\mathcal{O}$ and the lattice Γ). Here A_D^0 is the effective operator given by the expression $b(\mathbf{D})^*g^0b(\mathbf{D})$ with the Dirichlet conditions. The corrector $K_D(\zeta;\varepsilon)$ involves rapidly oscillating factors, and $\|K_D(\zeta;\varepsilon)\|_{L_2\to H^p}=O(\varepsilon^{-p})$. The dependence of the constants $\mathcal{C}_1(\varphi)$ and $\mathcal{C}_2(\varphi)$ on the angle φ is traced; estimates (0.9) and (0.10) are uniform with respect to φ in any sector $\varphi\in[\varphi_0,2\pi-\varphi_0]$ with arbitrarily small $\varphi_0>0$. For fixed ζ, estimate (0.9) is of sharp order $O(\varepsilon)$, while estimate (0.10) is of order $O(\varepsilon^{1/2})$ (the order deteriorates because of the boundary influence). Estimates (0.9) and (0.10) show that the error becomes smaller as $|\zeta|$ grows.

In the general case, the corrector $K_D(\zeta;\varepsilon)$ involves an auxiliary smoothing operator. We distinguish an additional condition under which the standard corrector (without smoothing) can be used.

Besides approximation for the resolvent, we find approximation for the operator $g(\mathbf{x}/\varepsilon)b(\mathbf{D})(A_{D,\varepsilon}-\zeta I)^{-1}$ (corresponding to the "flux") in the $(L_2\to L_2)$-operator norm.

For completeness, we also find approximation for the resolvent $(A_{D,\varepsilon}-\zeta I)^{-1}$ in a larger domain of the parameter ζ; the character of dependence of estimates on ζ in this case is different. Let us describe these results. The operators $A_{D,\varepsilon}$ and A_D^0 are positive definite. Let $c_*>0$ be their common lower bound. Suppose that $\zeta\in\mathbb{C}\setminus[c_*,\infty)$. We put $\zeta-c_*=|\zeta-c_*|e^{i\psi}$. For $0<\varepsilon\le\varepsilon_1$ we have

$$\big\|(A_{D,\varepsilon}-\zeta I)^{-1}-(A_D^0-\zeta I)^{-1}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})}\le\mathcal{C}(\zeta)\varepsilon, \tag{0.11}$$

$$\big\|(A_{D,\varepsilon}-\zeta I)^{-1}-(A_D^0-\zeta I)^{-1}-\varepsilon^pK_D(\zeta;\varepsilon)\big\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})}\le\mathcal{C}(\zeta)\varepsilon^{1/2}, \tag{0.12}$$

where $\mathcal{C}(\zeta)=C(\psi)|\zeta-c_*|^{-2}$ for $|\zeta-c_*|\le1$ and $\mathcal{C}(\zeta)=C(\psi)$ for $|\zeta-c_*|>1$. The dependence of $C(\psi)$ on the angle ψ is traced. Estimates (0.11) and (0.12) are uniform with respect to ψ in any sector of the form $\psi\in[\psi_0,2\pi-\psi_0]$ with arbitrarily small $\psi_0>0$.

0.4. Method. We rely on the results for operator (0.4) of order $2p$ in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ obtained in [KuSu] (estimates (0.5) and (0.6)). First, we deduce yet another result for the problem in $\mathbb{R}^d$ (which is similar to (0.6)), where the Steklov smoothing is involved; see Theorem 3.3 below.

The method of investigation of the operator $A_{D,\varepsilon}$ is similar to the case of the second order operators: it is based on consideration of the associated problem in $\mathbb{R}^d$, introduction of the boundary layer correction term, and a careful analysis of this term. An important technical role is played by the Steklov smoothing (like in the paper [ZhPas1]) and estimates in the ε-neighborhood of the boundary. First, estimate (0.10) is proved. Next, we prove estimate (0.9), using the already proved inequality (0.10) and duality arguments.

Estimates (0.11) and (0.12) are deduced (in a relatively simple way) from the already proved estimates at the point $\zeta=-1$ and suitable identities for the resolvents.

0.5. Plan of the paper. The paper consists of two chapters. Chapter 1 (§§1–3) is devoted to the problem in $\mathbb{R}^d$. In §1, we introduce the class of operators A_ε in $L_2(\mathbb{R}^d;\mathbb{C}^n)$, describe the effective operator A^0, and introduce smoothing operators of two types. In §2, we describe the properties of the matrix-valued function $\Lambda(\mathbf{x})$ that is a periodic solution of the auxiliary problem (1.10). In §3, the results of the paper [KuSu] on approximation of the resolvent $(A_\varepsilon-\zeta I)^{-1}$ (Theorems 3.1 and 3.2) are given, and another version of approximation for the resolvent in the "energy" norm involving the Steklov smoothing (Theorem 3.3) is obtained. Chapter 2 (§§4–8) is devoted to the Dirichlet problem. §4 contains the statement of the problem, description of the effective operator, and auxiliary statements. In §5, the main results for the Dirichlet problem, namely, estimates (0.9), (0.10) are formulated (see Theorems 5.1 and 5.2). The first two steps of the proofs are presented: the associated problem in $\mathbb{R}^d$ is considered, and the boundary layer correction term $\mathbf{w}_\varepsilon$ is introduced; the problem is reduced to estimation of the correction term in $H^p(\mathcal{O};\mathbb{C}^n)$ and in $L_2(\mathcal{O};\mathbb{C}^n)$. In §6, we prove the required estimates for the correction term and complete the proof of Theorems 5.1 and 5.2. In §7, we distinguish the case where the smoothing operator can be removed and the standard corrector can be used. Some special cases are considered. In §8, approximation for the resolvent $(A_{D,\varepsilon} - \zeta I)^{-1}$ for $\zeta \in \mathbb{C} \setminus [c_*, \infty)$ is obtained (estimates (0.11) and (0.12) are proved).

0.6. Notation. Let $\mathfrak{H}$ and $\mathfrak{G}$ be complex separable Hilbert spaces. The symbols $\|\cdot\|_{\mathfrak{H}}$ and $(\cdot,\cdot)_{\mathfrak{H}}$ stand for the norm and the inner product in $\mathfrak{H}$, respectively; the symbol $\|\cdot\|_{\mathfrak{H}\to\mathfrak{G}}$ denotes the norm of a continuous linear operator acting from $\mathfrak{H}$ to $\mathfrak{G}$.

The inner product and the norm in $\mathbb{C}^n$ are denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively. Next, $\mathbf{1} = \mathbf{1}_n$ stands for the unit $(n\times n)$-matrix. If a is a matrix of size $m\times n$, then $|a|$ denotes the norm of the matrix a viewed as an operator from $\mathbb{C}^n$ to $\mathbb{C}^m$. The classes L_q of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O}\subset\mathbb{R}^d$ are denoted by $L_q(\mathcal{O};\mathbb{C}^n)$, $1\le q\le\infty$. The Sobolev classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O}\subseteq\mathbb{R}^d$ are denoted by $H^s(\mathcal{O};\mathbb{C}^n)$, $s\in\mathbb{R}$. Next, $H^s_0(\mathcal{O};\mathbb{C}^n)$ is the closure of the class $C_0^\infty(\mathcal{O};\mathbb{C}^n)$ in the space $H^s(\mathcal{O};\mathbb{C}^n)$. If $n=1$, we write simply $L_q(\mathcal{O})$ and $H^s(\mathcal{O})$, but sometimes we use this simpler notation also for spaces of vector-valued or matrix-valued functions.

The vectors are denoted by the bold font. We denote $\mathbf{x} = (x_1,\dots,x_d)\in\mathbb{R}^d$, $iD_j = \partial_j = \partial/\partial x_j$, $j=1,\dots,d$, and $\mathbf{D} = -i\nabla = (D_1,\dots,D_d)$. If $\alpha = (\alpha_1,\dots,\alpha_d)\in\mathbb{Z}_+^d$ is a multiindex, then $|\alpha| = \sum_{j=1}^d \alpha_j$ and $\mathbf{D}^\alpha = D_1^{\alpha_1}\dots D_d^{\alpha_d}$. For two multiindices α and β, we write $\beta\le\alpha$ if $\beta_j\le\alpha_j$, $j=1,\dots,d$; the binomial coefficients are denoted by $C_\alpha^\beta = C_{\alpha_1}^{\beta_1}\dots C_{\alpha_d}^{\beta_d}$.

We use the notation $\mathbb{R}_+ = [0,\infty)$. By C, c, $\mathfrak{c}$, $\mathcal{C}$, $\mathfrak{C}$ (possibly, with indices and marks) we denote various constants in estimates.

Chapter 1. Homogenization of operators in $\mathbb{R}^d$

§1. Periodic elliptic operators in $L_2(\mathbb{R}^d;\mathbb{C}^n)$

1.1. Lattices in $\mathbb{R}^d$. Let Γ be the lattice in $\mathbb{R}^d$ generated by a basis $\mathbf{n}_1,\dots,\mathbf{n}_d$:

$$\Gamma = \Big\{\mathbf{n}\in\mathbb{R}^d\,:\,\mathbf{n} = \sum_{i=1}^d l_i\mathbf{n}_i,\ l_i\in\mathbb{Z}\Big\},$$

and let Ω be the elementary cell of the lattice Γ:

$$\Omega = \Big\{\mathbf{x}\in\mathbb{R}^d\,:\,\mathbf{x} = \sum_{i=1}^d t_i\mathbf{n}_i,\ -\frac{1}{2}<t_i<\frac{1}{2}\Big\}.$$

The basis $\mathbf{s}_1,\dots,\mathbf{s}_d$ in $\mathbb{R}^d$ dual to the basis $\mathbf{n}_1,\dots,\mathbf{n}_d$ is defined by the relations $\langle\mathbf{s}_i,\mathbf{n}_j\rangle_{\mathbb{R}^d} = 2\pi\delta_{ij}$. This basis gives rise to a lattice $\widetilde{\Gamma}$ *dual to the lattice* Γ:

$$\widetilde{\Gamma} = \Big\{\mathbf{s}\in\mathbb{R}^d\,:\,\mathbf{s} = \sum_{i=1}^d q_i\mathbf{s}_i,\ q_i\in\mathbb{Z}\Big\}.$$

Instead of the cell of the dual lattice, it is more convenient to consider the *central Brillouin zone*

$$\widetilde{\Omega} = \big\{\mathbf{k}\in\mathbb{R}^d\,:\,|\mathbf{k}|<|\mathbf{k}-\mathbf{s}|,\ 0\ne\mathbf{s}\in\widetilde{\Gamma}\big\},$$

which is a fundamental domain of $\widetilde{\Gamma}$. Below we use the notation $|\Omega| = \operatorname{meas}\Omega$,

$$r_0 = \frac{1}{2}\min_{0\neq\mathbf{s}\in\widetilde{\Gamma}}|\mathbf{s}|, \quad r_1 = \frac{1}{2}\operatorname{diam}\Omega.$$

By $\widetilde{H}^s(\Omega;\mathbb{C}^n)$ we denote the subspace of all functions in $H^s(\Omega;\mathbb{C}^n)$ whose Γ-periodic extension to $\mathbb{R}^d$ belongs to $H^s_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{C}^n)$. If $\varphi(\mathbf{x})$ is a Γ-periodic function on $\mathbb{R}^d$, we denote

$$\varphi^\varepsilon(\mathbf{x}) := \varphi(\varepsilon^{-1}\mathbf{x}), \quad \varepsilon > 0.$$

1.2. The class of operators. In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, consider the DO A_ε of order $2p$ formally given by the differential expression

$$A_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}), \quad \varepsilon > 0. \tag{1.1}$$

Here $g(\mathbf{x})$ is a uniformly positive definite and bounded $(m\times m)$-matrix-valued function (in general, $g(\mathbf{x})$ is a Hermitian matrix with complex entries):

$$g,\ g^{-1} \in L_\infty(\mathbb{R}^d); \quad g(\mathbf{x}) > 0. \tag{1.2}$$

The operator $b(\mathbf{D})$ is given by

$$b(\mathbf{D}) = \sum_{|\alpha|=p} b_\alpha \mathbf{D}^\alpha, \tag{1.3}$$

where the b_α are constant $(m\times n)$-matrices (in general, with complex entries). It is assumed that $m \ge n$ and that the symbol $b(\boldsymbol{\xi}) = \sum_{|\alpha|=p} b_\alpha \boldsymbol{\xi}^\alpha$ satisfies

$$\operatorname{rank} b(\boldsymbol{\xi}) = n, \quad 0 \neq \boldsymbol{\xi} \in \mathbb{R}^d.$$

This condition is equivalent to the inequalities

$$\alpha_0 \mathbf{1}_n \le b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \le \alpha_1 \mathbf{1}_n, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}, \quad 0 < \alpha_0 \le \alpha_1 < \infty, \tag{1.4}$$

with some positive constants α_0 and α_1. Without loss of generality we assume that the norms of the matrices b_α do not exceed the constant $\alpha_1^{1/2}$:

$$|b_\alpha| \le \alpha_1^{1/2}, \quad |\alpha| = p. \tag{1.5}$$

The precise definition of the operator A_ε is given in terms of the quadratic form

$$a_\varepsilon[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}^d} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle \, d\mathbf{x}, \quad \mathbf{u} \in H^p(\mathbb{R}^d;\mathbb{C}^n). \tag{1.6}$$

Note that the following elementary inequalities are valid:

$$\sum_{|\alpha|=p} |\boldsymbol{\xi}^\alpha|^2 \le |\boldsymbol{\xi}|^{2p} \le \mathfrak{c}_p \sum_{|\alpha|=p} |\boldsymbol{\xi}^\alpha|^2, \quad \boldsymbol{\xi} \in \mathbb{R}^d, \tag{1.7}$$

where $\mathfrak{c}_p$ depends only on d and p. Using the Fourier transformation and relations (1.2), (1.4), and (1.7), it is easy to check that

$$c_0 \int_{\mathbb{R}^d} |\mathbf{D}^p \mathbf{u}|^2 \, d\mathbf{x} \le a_\varepsilon[\mathbf{u},\mathbf{u}] \le c_1 \int_{\mathbb{R}^d} |\mathbf{D}^p \mathbf{u}|^2 \, d\mathbf{x}, \quad \mathbf{u} \in H^p(\mathbb{R}^d;\mathbb{C}^n), \tag{1.8}$$

where $|\mathbf{D}^p\mathbf{u}|^2 := \sum_{|\alpha|=p} |\mathbf{D}^\alpha \mathbf{u}|^2$. Here

$$c_0 = \alpha_0 \|g^{-1}\|^{-1}_{L_\infty}, \quad c_1 = \mathfrak{c}_p \alpha_1 \|g\|_{L_\infty}. \tag{1.9}$$

Hence, the form (1.6) is closed and nonnegative. The selfadjoint operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ corresponding to this form is denoted by A_ε.

1.3. The effective operator. In order to formulate our results, we need to introduce the effective operator A^0. Let an $(n \times m)$-matrix-valued function $\Lambda \in \widetilde{H}^p(\Omega)$ be the (weak) Γ-periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x}) \left(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m \right) = 0, \qquad \int_\Omega \Lambda(\mathbf{x})\, d\mathbf{x} = 0. \tag{1.10}$$

The so-called *effective matrix* g^0 of size $m \times m$ is defined as follows:

$$g^0 = |\Omega|^{-1} \int_\Omega \widetilde{g}(\mathbf{x})\, d\mathbf{x}, \tag{1.11}$$

where

$$\widetilde{g}(\mathbf{x}) := g(\mathbf{x}) \left(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m \right). \tag{1.12}$$

It turns out that the matrix g^0 is positive definite. The *effective operator* A^0 for the operator (1.1) is given by the differential expression

$$A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}) \tag{1.13}$$

on the domain $H^{2p}(\mathbb{R}^d; \mathbb{C}^n)$. Below we need the following estimates for the symbol $L(\boldsymbol{\xi}) = b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi})$ of the effective operator:

$$c_0 |\boldsymbol{\xi}|^{2p} \mathbf{1}_n \le L(\boldsymbol{\xi}) \le C_* |\boldsymbol{\xi}|^{2p} \mathbf{1}_n, \quad \boldsymbol{\xi} \in \mathbb{R}^d, \tag{1.14}$$

where c_0 is defined by (1.9) and $C_* = \alpha_1 \|g\|_{L_\infty}$. These estimates follow from (1.4) and the properties of the effective matrix (its positivity and estimates (1.16)).

1.4. Properties of the effective matrix. The following properties of the effective matrix were checked in [KuSu, Proposition 5.3].

Proposition 1.1. *Denote*

$$\overline{g} := |\Omega|^{-1} \int_\Omega g(\mathbf{x}) d\mathbf{x}, \quad \underline{g} := \left(|\Omega|^{-1} \int_\Omega g(\mathbf{x})^{-1} d\mathbf{x} \right)^{-1}.$$

The effective matrix g^0 satisfies the estimates

$$\underline{g} \le g^0 \le \overline{g}. \tag{1.15}$$

If $m = n$, then $g^0 = \underline{g}$.

In homogenization theory for specific DOs, estimates (1.15) are known as the Voight–Reuss bracketing. From (1.15) it follows that

$$|g^0| \le \|g\|_{L_\infty}, \quad |(g^0)^{-1}| \le \|g^{-1}\|_{L_\infty}. \tag{1.16}$$

Now we distinguish the cases where one of the inequalities in (1.15) becomes an identity. The following two statements were checked in [KuSu, Propositions 5.4 and 5.5].

Proposition 1.2. *Let $\mathbf{g}_k(\mathbf{x})$, $k = 1, \dots, m$, be the columns of the matrix $g(\mathbf{x})$. The identity $g^0 = \overline{g}$ is equivalent to the relations*

$$b(\mathbf{D})^* \mathbf{g}_k(\mathbf{x}) = 0, \quad k = 1, \dots, m. \tag{1.17}$$

Proposition 1.3. *Let $\mathbf{l}_k(\mathbf{x})$, $k = 1, \dots, m$, be the columns of the matrix $g(\mathbf{x})^{-1}$. The identity $g^0 = \underline{g}$ is equivalent to the relations*

$$\mathbf{l}_k(\mathbf{x}) = \mathbf{l}_k^0 + b(\mathbf{D})\mathbf{v}_k(\mathbf{x}), \quad \mathbf{l}_k^0 \in \mathbb{C}^m, \quad \mathbf{v}_k \in \widetilde{H}^p(\Omega; \mathbb{C}^n); \quad k = 1, \dots, m. \tag{1.18}$$

The following property was mentioned in [KuSu, Remark 5.6].

Remark 1.4. If $g^0 = \underline{g}$, then the matrix (1.12) is constant: $\widetilde{g}(\mathbf{x}) = g^0 = \underline{g}$.

1.5. Smoothing operators. In what follows, we need two smoothing operators of different types. The first of them acts in $L_2(\mathbb{R}^d;\mathbb{C}^m)$ as follows:

$$(\Pi_\varepsilon \mathbf{u})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\widetilde{\Omega}/\varepsilon} e^{i\langle \mathbf{x},\boldsymbol{\xi}\rangle} \widehat{\mathbf{u}}(\boldsymbol{\xi})\, d\boldsymbol{\xi}, \tag{1.19}$$

where $\widehat{\mathbf{u}}(\boldsymbol{\xi})$ is the Fourier image of the function $\mathbf{u}(\mathbf{x})$. In other words, Π_ε is the pseudodifferential operator whose symbol is the characteristic function $\chi_{\widetilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$ of the set $\widetilde{\Omega}/\varepsilon$. Obviously, Π_ε is the orthogonal projection in $H^s(\mathbb{R}^d;\mathbb{C}^m)$ for any $s \geq 0$. We have $\mathbf{D}^\alpha \Pi_\varepsilon \mathbf{u} = \Pi_\varepsilon \mathbf{D}^\alpha \mathbf{u}$ for $\mathbf{u} \in H^s(\mathbb{R}^d;\mathbb{C}^m)$ and any multiindex α such that $|\alpha| \leq s$.

The following statement was checked in [PSu2, Proposition 1.4].

Proposition 1.5. *For any* $\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^m)$ *we have*

$$\|\Pi_\varepsilon \mathbf{u} - \mathbf{u}\|_{L_2(\mathbb{R}^d)} \leq \varepsilon r_0^{-1} \|\mathbf{D}\mathbf{u}\|_{L_2(\mathbb{R}^d)}.$$

The next statement was obtained in [BSu4, Subsection 10.2].

Proposition 1.6. *Suppose that* $f(\mathbf{x})$ *is a* Γ*-periodic function on* $\mathbb{R}^d$ *such that* $f \in L_2(\Omega)$*. Let* $[f^\varepsilon]$ *be the operator of multiplication by the function* $f(\varepsilon^{-1}\mathbf{x})$*. Then the operator* $[f^\varepsilon]\Pi_\varepsilon$ *is continuous in* $L_2(\mathbb{R}^d;\mathbb{C}^m)$ *and*

$$\|[f^\varepsilon]\Pi_\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq |\Omega|^{-1/2}\|f\|_{L_2(\Omega)}, \quad \varepsilon > 0.$$

The second operator in question is called the *Steklov smoothing operator* and is denoted by S_ε. It acts in $L_2(\mathbb{R}^d;\mathbb{C}^m)$ as follows:

$$(S_\varepsilon \mathbf{u})(\mathbf{x}) = |\Omega|^{-1} \int_\Omega \mathbf{u}(\mathbf{x} - \varepsilon \mathbf{z})\, d\mathbf{z}. \tag{1.20}$$

Note that $\|S_\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq 1$. Obviously, $\mathbf{D}^\alpha S_\varepsilon \mathbf{u} = S_\varepsilon \mathbf{D}^\alpha \mathbf{u}$ for $\mathbf{u} \in H^s(\mathbb{R}^d;\mathbb{C}^m)$ and any multiindex α such that $|\alpha| \leq s$.

We mention some properties of the operator (1.20); see [ZhPas1, Lemmas 1.1 and 1.2] or [PSu2, Propositions 3.1, 3.2].

Proposition 1.7. *For any* $\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^m)$ *we have*

$$\|S_\varepsilon \mathbf{u} - \mathbf{u}\|_{L_2(\mathbb{R}^d)} \leq \varepsilon r_1 \|\mathbf{D}\mathbf{u}\|_{L_2(\mathbb{R}^d)}.$$

Proposition 1.8. *Let* $f(\mathbf{x})$ *be a* Γ*-periodic function in* $\mathbb{R}^d$ *such that* $f \in L_2(\Omega)$*. Let* $[f^\varepsilon]$ *be the operator of multiplication by the function* $f(\varepsilon^{-1}\mathbf{x})$*. Then the operator* $[f^\varepsilon]S_\varepsilon$ *is continuous in* $L_2(\mathbb{R}^d;\mathbb{C}^m)$ *and*

$$\|[f^\varepsilon]S_\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq |\Omega|^{-1/2}\|f\|_{L_2(\Omega)}, \quad \varepsilon > 0.$$

§2. Properties of the matrix-valued function Λ

In what follows, we need estimates for the norms of the matrix-valued function Λ (see [KuSu, Corollary 5.8]):

$$\|\Lambda\|_{L_2(\Omega)} \leq |\Omega|^{1/2} C_\Lambda^{(1)}, \quad C_\Lambda^{(1)} = m^{1/2} \alpha_0^{-1/2} (2r_0)^{-p} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2}, \tag{2.1}$$

$$\|b(\mathbf{D})\Lambda\|_{L_2(\Omega)} \leq |\Omega|^{1/2} C_\Lambda^{(2)}, \quad C_\Lambda^{(2)} = m^{1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2}, \tag{2.2}$$

$$\|\Lambda\|_{H^p(\Omega)} \leq |\Omega|^{1/2} C_\Lambda, \quad C_\Lambda = C_\Lambda^{(2)} \alpha_0^{-1/2} \Bigl(\sum_{|\beta|\leq p} (2r_0)^{-2(p-|\beta|)} \Bigr)^{1/2}. \tag{2.3}$$

The following lemma is a generalization of Lemma 8.3 in [BSu4] to the case of higher-order operators.

Lemma 2.1. *Let $\Lambda(\mathbf{x})$ be a Γ-periodic solution of problem* (1.10) *and $\widetilde{g}(\mathbf{x})$ the matrix-valued function* (1.12). *Then for any $u \in C_0^\infty(\mathbb{R}^d)$ we have*

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathbf{D}^p(\Lambda(\mathbf{x})u(\mathbf{x}))|^2\,d\mathbf{x} &\\ \le \beta_1 \int_{\mathbb{R}^d} |u|^2\,d\mathbf{x} &+ \beta_2 \sum_{|\alpha|=p}\sum_{\beta\le\alpha\,:\,|\beta|\ge1}\int_{\mathbb{R}^d}\left(|\mathbf{D}^{\alpha-\beta}\Lambda|^2+|\widetilde{g}|^2\right)|\mathbf{D}^\beta u|^2\,d\mathbf{x}\\ &+\beta_3\sum_{|\alpha|=p}\sum_{\beta\le\alpha\,:\,|\beta|\ge1}\sum_{\gamma\le\alpha-\beta:|\gamma|\ge1}\int_{\mathbb{R}^d}|\mathbf{D}^{\alpha-\beta-\gamma}\Lambda|^2|\mathbf{D}^\gamma u|^2\,d\mathbf{x}.\end{aligned}\tag{2.4}$$

The constants β_l, $l=1,2,3$, depend only on d, p, m, α_0, α_1, $\|g\|_{L_\infty}$, and $\|g^{-1}\|_{L_\infty}$.

Proof. Let $\mathbf{e}_1,\dots,\mathbf{e}_m$ be the standard orthonormal basis in $\mathbb{C}^m$. Denote the columns of $\Lambda(\mathbf{x})$ by $\mathbf{v}_j(\mathbf{x})$, $j=1,\dots,m$. By (1.10), the Γ-periodic vector-valued function $\mathbf{v}_j(\mathbf{x})$ belongs to $H^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{C}^n)$ and satisfies the identity

$$\int_{\mathbb{R}^d}\langle g(\mathbf{x})(b(\mathbf{D})\mathbf{v}_j(\mathbf{x})+\mathbf{e}_j), b(\mathbf{D})\boldsymbol{\eta}(\mathbf{x})\rangle\,d\mathbf{x}=0\tag{2.5}$$

for any function $\boldsymbol{\eta}\in H^p(\mathbb{R}^d;\mathbb{C}^n)$ such that $\boldsymbol{\eta}(\mathbf{x})=0$ for $|\mathbf{x}|>R$ (with some $R>0$).

Clearly, it suffices to check (2.4) for a real-valued function u. So, let $u(\mathbf{x})$ be a real-valued function such that $u\in C_0^\infty(\mathbb{R}^d)$. We put $\boldsymbol{\eta}(\mathbf{x})=\mathbf{v}_j(\mathbf{x})u(\mathbf{x})^2$. By (1.3), we have

$$\begin{aligned}b(\mathbf{D})\boldsymbol{\eta} &= ub(\mathbf{D})(\mathbf{v}_ju)+\sum_{|\alpha|=p}b_\alpha\sum_{\beta\le\alpha\,:\,|\beta|\ge1}C_\alpha^\beta\mathbf{D}^{\alpha-\beta}(\mathbf{v}_ju)\mathbf{D}^\beta u\\ &= ub(\mathbf{D})(\mathbf{v}_ju)+u\sum_{|\alpha|=p}b_\alpha\sum_{\beta\le\alpha\,:\,|\beta|\ge1}C_\alpha^\beta(\mathbf{D}^{\alpha-\beta}\mathbf{v}_j)\mathbf{D}^\beta u\\ &\quad+\sum_{|\alpha|=p}b_\alpha\sum_{\beta\le\alpha:|\beta|\ge1}C_\alpha^\beta\mathbf{D}^\beta u\sum_{\gamma\le\alpha-\beta:|\gamma|\ge1}C^\gamma_{\alpha-\beta}(\mathbf{D}^{\alpha-\beta-\gamma}\mathbf{v}_j)\mathbf{D}^\gamma u.\end{aligned}$$

Substituting this expression in (2.5), we arrive at the identity

$$\begin{aligned}\int_{\mathbb{R}^d}\langle g(b(\mathbf{D})\mathbf{v}_j)u, b(\mathbf{D})(\mathbf{v}_ju)\rangle\,d\mathbf{x}+\sum_{|\alpha|=p}\sum_{\beta\le\alpha:|\beta|\ge1}C_\alpha^\beta\int_{\mathbb{R}^d}\langle g(b(\mathbf{D})\mathbf{v}_j)u, b_\alpha(\mathbf{D}^{\alpha-\beta}\mathbf{v}_j)\mathbf{D}^\beta u\rangle\,d\mathbf{x}&\\ +J_1+J_2+J_3=0,&\end{aligned}$$

where

$$\begin{aligned}J_1&:=\int_{\mathbb{R}^d}\langle g\mathbf{e}_ju, b(\mathbf{D})(\mathbf{v}_ju)\rangle\,d\mathbf{x},\\ J_2&:=\sum_{|\alpha|=p}\sum_{\beta\le\alpha:|\beta|\ge1}C_\alpha^\beta\int_{\mathbb{R}^d}\langle g\mathbf{e}_ju, b_\alpha(\mathbf{D}^{\alpha-\beta}\mathbf{v}_j)\mathbf{D}^\beta u\rangle\,d\mathbf{x},\\ J_3&:=\sum_{|\alpha|=p}\sum_{\beta\le\alpha:|\beta|\ge1}\sum_{\gamma\le\alpha-\beta:|\gamma|\ge1}C_\alpha^\beta C^\gamma_{\alpha-\beta}\int_{\mathbb{R}^d}\langle g(b(\mathbf{D})\mathbf{v}_j+\mathbf{e}_j), b_\alpha(\mathbf{D}^{\alpha-\beta-\gamma}\mathbf{v}_j)\mathbf{D}^\beta u\mathbf{D}^\gamma u\rangle\,d\mathbf{x}.\end{aligned}$$

Next, employing the formula

$$(b(\mathbf{D})\mathbf{v}_j)u=b(\mathbf{D})(\mathbf{v}_ju)-\sum_{|\alpha|=p}b_\alpha\sum_{\beta\le\alpha:|\beta|\ge1}C_\alpha^\beta\mathbf{D}^{\alpha-\beta}\mathbf{v}_j\mathbf{D}^\beta u$$

and denoting

$$J:=\int_{\mathbb{R}^d}\langle gb(\mathbf{D})(\mathbf{v}_ju), b(\mathbf{D})(\mathbf{v}_ju)\rangle\,d\mathbf{x},$$

we rewrite the above identity as

$$J=-J_1-J_2-J_3+J_4-J_5+J_6,$$

where

$$J_4 := \sum_{|\alpha|=p} \sum_{\beta\le\alpha:|\beta|\ge1} C_\alpha^\beta \int_{\mathbb{R}^d} \langle g b_\alpha (\mathbf{D}^{\alpha-\beta}\mathbf{v}_j)\mathbf{D}^\beta u, b(\mathbf{D})(\mathbf{v}_j u)\rangle\, d\mathbf{x},$$

$$J_5 := \sum_{|\alpha|=p} \sum_{\beta\le\alpha:|\beta|\ge1} C_\alpha^\beta \int_{\mathbb{R}^d} \langle g b(\mathbf{D})(\mathbf{v}_j u), b_\alpha (\mathbf{D}^{\alpha-\beta}\mathbf{v}_j)\mathbf{D}^\beta u\rangle\, d\mathbf{x},$$

$$J_6 := \sum_{|\alpha|=|\alpha'|=p} \sum_{\substack{\beta\le\alpha:|\beta|\ge1\\ \beta'\le\alpha':|\beta'|\ge1}} C_\alpha^\beta C_{\alpha'}^{\beta'} \int_{\mathbb{R}^d} \left\langle g b_{\alpha'} (\mathbf{D}^{\alpha'-\beta'}\mathbf{v}_j)\mathbf{D}^{\beta'} u, b_\alpha (\mathbf{D}^{\alpha-\beta}\mathbf{v}_j)\mathbf{D}^\beta u\right\rangle d\mathbf{x}.$$

The term J_1 is estimated with the help of the Cauchy inequality:

$$|J_1| \le \|g^{1/2}\mathbf{e}_j u\|_{L_2(\mathbb{R}^d)} \|g^{1/2} b(\mathbf{D})(\mathbf{v}_j u)\|_{L_2(\mathbb{R}^d)} \le \frac{1}{4} J + \|g\|_{L_\infty} \|u\|^2_{L_2(\mathbb{R}^d)}.$$

By (1.5), the term J_2 satisfies

$$|J_2| \le c_1^{(2)} \|u\|^2_{L_2(\mathbb{R}^d)} + c_2^{(2)} \sum_{|\alpha|=p} \sum_{\beta\le\alpha:|\beta|\ge1} \int_{\mathbb{R}^d} |\mathbf{D}^{\alpha-\beta}\mathbf{v}_j|^2 |\mathbf{D}^\beta u|^2\, d\mathbf{x},$$

where $c_l^{(2)} = \kappa_l(d,p)\|g\|_{L_\infty}\alpha_1^{1/2}$, $l=1,2$, and the constants $\kappa_l(d,p)$ depend only on d and p.

Next, the vectors $\widetilde{\mathbf{g}}_j(\mathbf{x}) := g(\mathbf{x})(b(\mathbf{D})\mathbf{v}_j(\mathbf{x}) + \mathbf{e}_j)$, $j=1,\dots,m$, are the columns of the matrix $\widetilde{g}(\mathbf{x})$ defined by (1.12). By (1.5), the term J_3 satisfies

$$\begin{aligned}|J_3| &\le c^{(3)} \sum_{|\alpha|=p} \sum_{\beta\le\alpha:|\beta|\ge1} \sum_{\gamma\le\alpha-\beta:|\gamma|\ge1} \int_{\mathbb{R}^d} |\widetilde{\mathbf{g}}_j||\mathbf{D}^{\alpha-\beta-\gamma}\mathbf{v}_j||\mathbf{D}^\beta u||\mathbf{D}^\gamma u|\, d\mathbf{x}\\ &\le c^{(3)} \sum_{|\alpha|=p} \sum_{\beta\le\alpha:|\beta|\ge1} \sum_{\gamma\le\alpha-\beta:|\gamma|\ge1} \int_{\mathbb{R}^d} |\mathbf{D}^{\alpha-\beta-\gamma}\mathbf{v}_j|^2|\mathbf{D}^\gamma u|^2\, d\mathbf{x}\\ &\qquad + c^{(4)} \sum_{|\alpha|=p} \sum_{\beta\le\alpha:|\beta|\ge1} \int_{\mathbb{R}^d} |\widetilde{\mathbf{g}}_j|^2 |\mathbf{D}^\beta u|^2\, d\mathbf{x},\end{aligned}$$

where $c^{(3)} = \kappa_3(d,p)\alpha_1^{1/2}$ and $c^{(4)} = \kappa_4(d,p)\alpha_1^{1/2}$.

The terms J_4 and J_5 are estimated in the same way. We have

$$|J_4| + |J_5| \le \frac{1}{4} J + c^{(5)} \sum_{|\alpha|=p} \sum_{\beta\le\alpha:|\beta|\ge1} \int_{\mathbb{R}^d} |\mathbf{D}^{\alpha-\beta}\mathbf{v}_j|^2 |\mathbf{D}^\beta u|^2\, d\mathbf{x},$$

where $c^{(5)} = \kappa_5(d,p)\alpha_1\|g\|_{L_\infty}$. Finally, the term J_6 admits the estimate

$$|J_6| \le c^{(6)} \sum_{|\alpha|=p} \sum_{\beta\le\alpha:|\beta|\ge1} \int_{\mathbb{R}^d} |\mathbf{D}^{\alpha-\beta}\mathbf{v}_j|^2 |\mathbf{D}^\beta u|^2\, d\mathbf{x},$$

where $c^{(6)} = \kappa_6(d,p)\alpha_1\|g\|_{L_\infty}$.

As a result, we arrive at the inequality

$$\begin{aligned}J &\le \breve{\beta}_1 \|u\|^2_{L_2(\mathbb{R}^d)} + \breve{\beta}_2 \sum_{|\alpha|=p} \sum_{\beta\le\alpha:|\beta|\ge1} \int_{\mathbb{R}^d} (|\mathbf{D}^{\alpha-\beta}\mathbf{v}_j|^2 + |\widetilde{\mathbf{g}}_j|^2)|\mathbf{D}^\beta u|^2\, d\mathbf{x}\\ &\qquad + \breve{\beta}_3 \sum_{|\alpha|=p} \sum_{\beta\le\alpha:|\beta|\ge1} \sum_{\gamma\le\alpha-\beta:|\gamma|\ge1} \int_{\mathbb{R}^d} |\mathbf{D}^{\alpha-\beta-\gamma}\mathbf{v}_j|^2|\mathbf{D}^\gamma u|^2\, d\mathbf{x},\end{aligned}$$

where $\breve{\beta}_1 = 2\|g\|_{L_\infty} + 2c_1^{(2)}$, $\breve{\beta}_2 = 2(c_2^{(2)} + c^{(4)} + c^{(5)} + c^{(6)})$, and $\breve{\beta}_3 = 2c^{(3)}$.

Taking the lower estimate (1.8) (with $\varepsilon = 1$) into account and summing over j, we arrive at the desired inequality (2.4). $\square$

Corollary 2.2. *For $u \in C_0^\infty(\mathbb{R}^d)$ and $\varepsilon > 0$ we have*

$$
\begin{aligned}
&\varepsilon^{2p} \int_{\mathbb{R}^d} |\mathbf{D}^p(\Lambda^\varepsilon u)|^2 \, d\mathbf{x} \\
&\quad \le \beta_1 \int_{\mathbb{R}^d} |u|^2 \, d\mathbf{x} + \beta_2 \sum_{|\alpha|=p} \sum_{\beta \le \alpha : |\beta| \ge 1} \varepsilon^{2|\beta|} \int_{\mathbb{R}^d} \left(|(\mathbf{D}^{\alpha-\beta}\Lambda)^\varepsilon|^2 + |\widetilde{g}^\varepsilon|^2 \right) |\mathbf{D}^\beta u|^2 \, d\mathbf{x} \\
&\qquad + \beta_3 \sum_{|\alpha|=p} \sum_{\beta \le \alpha : |\beta| \ge 1} \sum_{\gamma \le \alpha - \beta : |\gamma| \ge 1} \varepsilon^{2|\gamma|} \int_{\mathbb{R}^d} |(\mathbf{D}^{\alpha-\beta-\gamma}\Lambda)^\varepsilon|^2 |\mathbf{D}^\gamma u|^2 \, d\mathbf{x}.
\end{aligned} \tag{2.6}
$$

Proof. Let $u \in C_0^\infty(\mathbb{R}^d)$. We substitute $\mathbf{x} = \varepsilon \mathbf{y}$ and $u(\mathbf{x}) = v(\mathbf{y})$. Then

$$
\varepsilon^{2p} \int_{\mathbb{R}^d} |\mathbf{D}^p_{\mathbf{x}}(\Lambda^\varepsilon(\mathbf{x}) u(\mathbf{x}))|^2 \, d\mathbf{x} = \varepsilon^d \int_{\mathbb{R}^d} |\mathbf{D}^p_{\mathbf{y}}(\Lambda(\mathbf{y}) v(\mathbf{y}))|^2 \, d\mathbf{y}.
$$

Applying (2.4) to the integral on the right and using the inverse change, we arrive at estimate (2.6). ☐

§3. Results for the homogenization problem in $\mathbb{R}^d$

In this section, we formulate the results on homogenization for the operator A_ε in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ obtained in [KuSu], and obtain yet another result (involving the Steklov smoothing operator).

3.1. Approximation of the resolvent of A_ε in the $L_2(\mathbb{R}^d; \mathbb{C}^n)$-operator norm. A point $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$ is regular for both operators A_ε and A^0. We put $\zeta = |\zeta| e^{i\varphi}$, $\varphi \in (0, 2\pi)$, and denote

$$
c(\varphi) = \begin{cases} |\sin \varphi|^{-1} & \text{if } \varphi \in (0, \pi/2) \cup (3\pi/2, 2\pi), \\ 1 & \text{if } \varphi \in [\pi/2, 3\pi/2]. \end{cases} \tag{3.1}
$$

The following theorem was proved in [KuSu, Theorem 8.1].

Theorem 3.1. *Suppose that A_ε is the operator* (1.1) *and A^0 is the effective operator* (1.13). *Let $\zeta = |\zeta| e^{i\varphi} \in \mathbb{C} \setminus \mathbb{R}_+$, and let $c(\varphi)$ be given by* (3.1). *Then for $\varepsilon > 0$ we have*

$$
\left\| (A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \le C_1 c(\varphi)^2 \varepsilon |\zeta|^{-1+1/2p}.
$$

The constant C_1 depends only on d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

3.2. Approximation of the resolvent of A_ε in the $(L_2 \to H^p)$-operator norm. In order to approximate the resolvent $(A_\varepsilon - \zeta I)^{-1}$ in the norm of operators acting from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to the Sobolev space $H^p(\mathbb{R}^d; \mathbb{C}^n)$, we need to introduce the *corrector*

$$
K(\zeta; \varepsilon) := [\Lambda^\varepsilon] \Pi_\varepsilon b(\mathbf{D}) (A^0 - \zeta I)^{-1}. \tag{3.2}
$$

Recall that Λ is the periodic solution of problem (1.10) and Π_ε is the smoothing operator (1.19). The operator (3.2) is a continuous mapping of $L_2(\mathbb{R}^d; \mathbb{C}^n)$ into $H^p(\mathbb{R}^d; \mathbb{C}^n)$. This can easily be checked by using Proposition 1.6 and the relation $\Lambda \in \widetilde{H}^p(\Omega)$. Herewith, $\|K(\zeta; \varepsilon)\|_{L_2 \to H^p} = O(\varepsilon^{-p})$.

The following result was obtained in [KuSu, Theorem 8.2].

Theorem 3.2. *Under the assumptions of Theorem* 3.1, *let* $K(\zeta;\varepsilon)$ *be the operator* (3.2), *and let* $\widetilde{g}(\mathbf{x})$ *be the matrix-valued function* (1.12). *Then for* $\varepsilon > 0$ *we have*

$$\begin{aligned} &\big\|(A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1} - \varepsilon^p K(\zeta;\varepsilon)\big\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \\ &\qquad\qquad \le C_2 c(\varphi)^2 \varepsilon |\zeta|^{-1/2+1/2p}\big(1+|\zeta|^{-1/2}\big), \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\big\|g^\varepsilon b(\mathbf{D})(A_\varepsilon - \zeta I)^{-1} - \widetilde{g}^\varepsilon \Pi_\varepsilon b(\mathbf{D})(A^0 - \zeta I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\qquad\qquad \le C_3 c(\varphi)^2 \varepsilon |\zeta|^{-1/2+1/2p}. \end{aligned} \tag{3.4}$$

The constants C_2 *and* C_3 *depend only on* m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *and the parameters of the lattice* Γ.

3.3. Another approximation of the resolvent of A_ε in the $(L_2 \to H^p)$-operator norm. We put

$$\widetilde{K}(\zeta;\varepsilon) := [\Lambda^\varepsilon] S_\varepsilon b(\mathbf{D})(A^0 - \zeta I)^{-1}, \tag{3.5}$$

where S_ε is the Steklov smoothing operator defined by (1.20). The operator (3.5) is a continuous mapping of $L_2(\mathbb{R}^d;\mathbb{C}^n)$ into $H^p(\mathbb{R}^d;\mathbb{C}^n)$ (this can easily be checked by using Proposition 1.8 and the relation $\Lambda \in \widetilde{H}^p(\Omega)$). Herewith, $\|\widetilde{K}(\zeta;\varepsilon)\|_{L_2\to H^p} = O(\varepsilon^{-p})$.

Along with Theorem 3.2, the following result is true; this result turns out to be more convenient for further application to the study of problems in a bounded domain.

Theorem 3.3. *Under the assumptions of Theorem* 3.1, *let* $\widetilde{K}(\zeta;\varepsilon)$ *be the operator* (3.5), *and let* $\widetilde{g}(\mathbf{x})$ *be the matrix-valued function* (1.12). *Then for* $\varepsilon > 0$ *we have*

$$\begin{aligned} &\big\|(A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1} - \varepsilon^p \widetilde{K}(\zeta;\varepsilon)\big\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \\ &\qquad \le \big(C_4 c(\varphi)^2 \varepsilon |\zeta|^{-1/2+1/2p} + C_5 c(\varphi)\varepsilon^p\big)\big(1+|\zeta|^{-1/2}\big), \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\big\|g^\varepsilon b(\mathbf{D})(A_\varepsilon - \zeta I)^{-1} - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})(A^0 - \zeta I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\qquad\qquad \le C_6 c(\varphi)^2 \varepsilon |\zeta|^{-1/2+1/2p} + C_7 c(\varphi)\varepsilon^p. \end{aligned} \tag{3.7}$$

The constants C_4, C_5, C_6, *and* C_7 *depend only on* m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *and the parameters of the lattice* Γ.

Theorem 3.3 is deduced from Theorem 3.2 with the help of the following lemma.

Lemma 3.4. *For any* $\mathbf{u} \in H^{2p}(\mathbb{R}^d;\mathbb{C}^n)$ *and* $\varepsilon > 0$ *we have*

$$\varepsilon^{2p}\int_{\mathbb{R}^d} |\mathbf{D}^p(\Lambda^\varepsilon \mathbf{z}_\varepsilon)|^2\, d\mathbf{x} \le \beta_1 \int_{\mathbb{R}^d} |\mathbf{z}_\varepsilon|^2\, d\mathbf{x} + \beta_2 \mathcal{I}_2^\varepsilon[\mathbf{z}_\varepsilon] + \beta_3 \mathcal{I}_3^\varepsilon[\mathbf{z}_\varepsilon], \tag{3.8}$$

where $\mathbf{z}_\varepsilon := (\Pi_\varepsilon - S_\varepsilon) b(\mathbf{D})\mathbf{u}$,

$$\mathcal{I}_2^\varepsilon[\mathbf{z}_\varepsilon] := \sum_{|\alpha|=p}\ \sum_{\beta\le\alpha:|\beta|\ge 1} \varepsilon^{2|\beta|}\int_{\mathbb{R}^d} \big(|(\mathbf{D}^{\alpha-\beta}\Lambda)^\varepsilon|^2 + |\widetilde{g}^\varepsilon|^2\big)\, |\mathbf{D}^\beta \mathbf{z}_\varepsilon|^2\, d\mathbf{x},$$

$$\mathcal{I}_3^\varepsilon[\mathbf{z}_\varepsilon] := \sum_{|\alpha|=p}\ \sum_{\beta\le\alpha:|\beta|\ge 1}\ \sum_{\gamma\le\alpha-\beta:|\gamma|\ge 1} \varepsilon^{2|\gamma|}\int_{\mathbb{R}^d} |(\mathbf{D}^{\alpha-\beta-\gamma}\Lambda)^\varepsilon|^2 |\mathbf{D}^\gamma \mathbf{z}_\varepsilon|^2\, d\mathbf{x}.$$

The constants β_l, $l = 1,2,3$, *depend only on* d, p, m, α_0, α_1, $\|g\|_{L_\infty}$, *and* $\|g^{-1}\|_{L_\infty}$.

Proof. By Propositions 1.6, 1.8 and the relations $\Lambda \in \widetilde{H}^p(\Omega)$, $\widetilde{g} \in L_2(\Omega)$, all terms in inequality (3.8) are continuous functionals of $\mathbf{u}$ in the $H^{2p}(\mathbb{R}^d;\mathbb{C}^n)$-norm. Since $C_0^\infty(\mathbb{R}^d;\mathbb{C}^n)$ is dense in $H^{2p}(\mathbb{R}^d;\mathbb{C}^n)$, it suffices to prove (3.8) for $\mathbf{u} \in C_0^\infty(\mathbb{R}^d;\mathbb{C}^n)$.

Fixing a function $\chi \in C^\infty(\mathbb{R}_+)$ such that $0 \le \chi(t) \le 1$, $\chi(t) = 1$ for $0 \le t \le 1$, and $\chi(t) = 0$ for $t \ge 2$, we put $\chi_R(\mathbf{x}) := \chi(R^{-1}|\mathbf{x}|)$, $\mathbf{x} \in \mathbb{R}^d$, $R > 0$. Let $\mathbf{u} \in C_0^\infty(\mathbb{R}^d;\mathbb{C}^n)$, and let $\mathbf{z}_\varepsilon = (\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}$. Then $\chi_R\mathbf{z}_\varepsilon \in C_0^\infty(\mathbb{R}^d;\mathbb{C}^m)$ and, by Corollary 2.2,

$$\varepsilon^{2p}\int_{\mathbb{R}^d} |\mathbf{D}^p(\Lambda^\varepsilon\chi_R\mathbf{z}_\varepsilon)|^2\,d\mathbf{x} \le \beta_1\int_{\mathbb{R}^d}|\chi_R\mathbf{z}_\varepsilon|^2\,d\mathbf{x} + \beta_2\mathcal{I}_2^\varepsilon[\chi_R\mathbf{z}_\varepsilon] + \beta_3\mathcal{I}_3^\varepsilon[\chi_R\mathbf{z}_\varepsilon].$$

Combining this with the estimates $\max|\mathbf{D}^\alpha\chi_R| \le cR^{-|\alpha|}$ (for any α) and applying the Lebesgue theorem, we obtain inequality (3.8) by the limit procedure as $R \to \infty$. $\square$

Relation (3.8) and the Leibnitz formula

$$(\mathbf{D}^\alpha\Lambda^\varepsilon)\mathbf{z}_\varepsilon = \mathbf{D}^\alpha(\Lambda^\varepsilon\mathbf{z}_\varepsilon) - \sum_{\beta\le\alpha:|\beta|\ge1} C_\alpha^\beta(\mathbf{D}^{\alpha-\beta}\Lambda^\varepsilon)\mathbf{D}^\beta\mathbf{z}_\varepsilon$$

directly imply the following statement.

Corollary 3.5. *Under the assumptions of Lemma* 3.4, *we have*

$$\sum_{|\alpha|=p}\int_{\mathbb{R}^d}|(\mathbf{D}^\alpha\Lambda)^\varepsilon\mathbf{z}_\varepsilon|^2\,d\mathbf{x} \le \widetilde{\beta}_1\int_{\mathbb{R}^d}|\mathbf{z}_\varepsilon|^2\,d\mathbf{x} + \widetilde{\beta}_2\mathcal{I}_2^\varepsilon[\mathbf{z}_\varepsilon] + \widetilde{\beta}_3\mathcal{I}_3^\varepsilon[\mathbf{z}_\varepsilon].$$

The constants $\widetilde{\beta}_l$, $l = 1, 2, 3$, *depend only on* d, p, m, α_0, α_1, $\|g\|_{L_\infty}$, *and* $\|g^{-1}\|_{L_\infty}$.

Proof of Theorem 3.3. Note that

$$\|\mathbf{v}\|^2_{H^p(\mathbb{R}^d)} \le \check{\mathfrak{c}}_p\big(\|\mathbf{v}\|^2_{L_2(\mathbb{R}^d)} + \|\mathbf{D}^p\mathbf{v}\|^2_{L_2(\mathbb{R}^d)}\big), \quad \mathbf{v} \in H^p(\mathbb{R}^d;\mathbb{C}^n), \tag{3.9}$$

where $\check{\mathfrak{c}}_p$ depends only on d and p.

We estimate the difference of the operators (3.2) and (3.5). Let $\mathbf{F} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$, and let $\mathbf{u}_0 = (A^0 - \zeta I)^{-1}\mathbf{F}$. By (3.9), we have

$$\begin{aligned}\big\|\big(K(\zeta;\varepsilon) - \widetilde{K}(\zeta;\varepsilon)\big)\mathbf{F}\big\|^2_{H^p(\mathbb{R}^d)} &= \big\|\Lambda^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0\big\|^2_{H^p(\mathbb{R}^d)} \\ &\le \check{\mathfrak{c}}_p\big\|\mathbf{D}^p\left(\Lambda^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0\right)\big\|^2_{L_2(\mathbb{R}^d)} + \check{\mathfrak{c}}_p\big\|\Lambda^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0\big\|^2_{L_2(\mathbb{R}^d)}.\end{aligned} \tag{3.10}$$

The second term on the right-hand side of (3.10) is estimated with the help of Propositions 1.6 and 1.8. Taking (2.1) into account, we obtain

$$\begin{aligned}&\big\|\Lambda^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0\big\|_{L_2(\mathbb{R}^d)} \\ &\quad\le 2|\Omega|^{-1/2}\|\Lambda\|_{L_2(\Omega)}\|b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \le 2C_\Lambda^{(1)}\|b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathbb{R}^d)}.\end{aligned} \tag{3.11}$$

To estimate the first term on the right-hand side of (3.10), we apply Lemma 3.4. Estimate (3.8) with $\mathbf{z}_\varepsilon = (\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0$ is satisfied. The first summand on the right in (3.8) is estimated by using Propositions 1.5 and 1.7:

$$\int_{\mathbb{R}^d}|\mathbf{z}_\varepsilon|^2\,d\mathbf{x} \le C_1'\varepsilon^2\|\mathbf{D}b(\mathbf{D})\mathbf{u}_0\|^2_{L_2(\mathbb{R}^d)}, \tag{3.12}$$

where $C_1' = (r_0^{-1} + r_1)^2$. The terms $\mathcal{I}_2^\varepsilon[\mathbf{z}_\varepsilon]$ and $\mathcal{I}_3^\varepsilon[\mathbf{z}_\varepsilon]$ are estimated with the help of Propositions 1.6 and 1.8. We have

$$\begin{aligned}&\int_{\mathbb{R}^d}\big(|(\mathbf{D}^{\alpha-\beta}\Lambda)^\varepsilon|^2 + |\widetilde{g}^\varepsilon|^2\big)|\mathbf{D}^\beta\mathbf{z}_\varepsilon|^2\,d\mathbf{x} \\ &\qquad\le 4|\Omega|^{-1}\big(\|\mathbf{D}^{\alpha-\beta}\Lambda\|^2_{L_2(\Omega)} + \|\widetilde{g}\|^2_{L_2(\Omega)}\big)\|\mathbf{D}^\beta b(\mathbf{D})\mathbf{u}_0\|^2_{L_2(\mathbb{R}^d)}.\end{aligned}$$

Combining this with (2.2) and (2.3), we obtain

$$\mathcal{I}_2^\varepsilon[\mathbf{z}_\varepsilon] \le C_2'\sum_{l=1}^p\varepsilon^{2l}\|\mathbf{D}^l b(\mathbf{D})\mathbf{u}_0\|^2_{L_2(\mathbb{R}^d)}, \tag{3.13}$$

where $C_2' = \kappa_2'(d,p)\big(C_\Lambda^2 + \|g\|^2_{L_\infty}(1+C_\Lambda^{(2)})^2\big)$. Similarly,

$$\mathcal{I}_3^\varepsilon[\mathbf{z}_\varepsilon] \le C_3' \sum_{l=1}^p \varepsilon^{2l} \|\mathbf{D}^l b(\mathbf{D})\mathbf{u}_0\|^2_{L_2(\mathbb{R}^d)}, \tag{3.14}$$

where $C_3' = \kappa_3'(d,p)C_\Lambda^2$. As a result,

$$\varepsilon^{2p} \int_{\mathbb{R}^d} |\mathbf{D}^p \left(\Lambda^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0\right)|^2\, d\mathbf{x} \le C' \sum_{l=1}^p \varepsilon^{2l}\|\mathbf{D}^l b(\mathbf{D})\mathbf{u}_0\|^2_{L_2(\mathbb{R}^d)}, \tag{3.15}$$

where $C' = \beta_1 C_1' + \beta_2 C_2' + \beta_3 C_3'$. From (3.10), (3.11), and (3.15) it follows that

$$\begin{aligned} &\varepsilon^{2p}\big\|\big(K(\zeta;\varepsilon) - \widetilde{K}(\zeta;\varepsilon)\big)\mathbf{F}\big\|^2_{H^p(\mathbb{R}^d)} \\ &\quad\le 4\check{\mathfrak{c}}_p (C_\Lambda^{(1)})^2 \varepsilon^{2p}\|b(\mathbf{D})\mathbf{u}_0\|^2_{L_2(\mathbb{R}^d)} + \check{\mathfrak{c}}_p C' \sum_{l=1}^p \varepsilon^{2l}\|\mathbf{D}^l b(\mathbf{D})\mathbf{u}_0\|^2_{L_2(\mathbb{R}^d)}. \end{aligned} \tag{3.16}$$

Let us estimate the norms $\|\mathbf{D}^l b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathbb{R}^d)}$, $l = 0,1,\dots,p$. We have

$$\begin{aligned} &\big\|\mathbf{D}^l b(\mathbf{D})(A^0 - \zeta I)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\quad\le \big\|\mathbf{D}^l b(\mathbf{D})(A^0)^{-1/2-l/2p}\big\|_{L_2\to L_2}\big\|(A^0)^{1/2+l/2p}(A^0-\zeta I)^{-1}\big\|_{L_2\to L_2}. \end{aligned}$$

Taking (1.4) and (1.14) into account, we obtain

$$\big\|\mathbf{D}^l b(\mathbf{D})(A^0)^{-1/2-l/2p}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \alpha_1^{1/2} c_0^{-1/2-l/2p}.$$

Next,

$$\begin{aligned} \big\|(A^0)^{1/2+l/2p}(A^0-\zeta I)^{-1}\big\|_{L_2\to L_2} &\le \sup_{x\ge 0} x^{1/2+l/2p}|x-\zeta|^{-1} \\ &\le \big(\sup_{x\ge 0} x|x-\zeta|^{-1}\big)^{1/2+l/2p}\big(\sup_{x\ge 0}|x-\zeta|^{-1}\big)^{1/2-l/2p}. \end{aligned}$$

Calculating the suprema

$$\sup_{x\ge 0} x|x-\zeta|^{-1} \le c(\varphi), \quad \sup_{x\ge 0}|x-\zeta|^{-1} = c(\varphi)|\zeta|^{-1},$$

we arrive at the inequality

$$\big\|(A^0)^{1/2+l/2p}(A^0-\zeta I)^{-1}\big\|_{L_2\to L_2} \le c(\varphi)|\zeta|^{-1/2+l/2p}, \quad l = 0,1,\dots,p.$$

Thus,

$$\|\mathbf{D}^l b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \le \check{C}_l c(\varphi)|\zeta|^{-1/2+l/2p}\|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \quad l = 0,1,\dots,p, \tag{3.17}$$

where $\check{C}_l = \alpha_1^{1/2} c_0^{-1/2-l/2p}$. Together with (3.16), this implies

$$\begin{aligned} &\varepsilon^p\|K(\zeta;\varepsilon) - \widetilde{K}(\zeta;\varepsilon)\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \\ &\quad\le C_K^{(1)} c(\varphi)\varepsilon^p|\zeta|^{-1/2} + C_K^{(2)} c(\varphi)|\zeta|^{-1/2}\sum_{j=1}^p \big(\varepsilon|\zeta|^{1/2p}\big)^j, \end{aligned} \tag{3.18}$$

where $C_K^{(1)} = 2\check{\mathfrak{c}}_p^{1/2} C_\Lambda^{(1)}\check{C}_0$ and $C_K^{(2)} = \check{\mathfrak{c}}_p^{1/2}(C')^{1/2}\max_{1\le l\le p}\check{C}_l$. Obviously, we have $\sum_{j=1}^p \big(\varepsilon|\zeta|^{1/2p}\big)^j \le p(\varepsilon|\zeta|^{1/2p} + \varepsilon^p|\zeta|^{1/2})$. Combining this with (3.18), we obtain

$$\varepsilon^p\big\|K(\zeta;\varepsilon) - \widetilde{K}(\zeta;\varepsilon)\big\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \le C_K c(\varphi)\big(\varepsilon^p|\zeta|^{-1/2} + \varepsilon|\zeta|^{-1/2+1/2p} + \varepsilon^p\big), \tag{3.19}$$

where $C_K = \max\{C_K^{(1)}, pC_K^{(2)}\}$.

Finally, relations (3.3) and (3.19) imply the required inequality (3.6) with the constants $C_4 = C_2 + C_K$ and $C_5 = C_K$.

We proceed to the proof of inequality (3.7). By (1.3), (1.5), and (1.12), we have

$$\big\|\widetilde{g}^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0\big\|_{L_2(\mathbb{R}^d)} \le \|g\|_{L_\infty}\big\|(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0\big\|_{L_2(\mathbb{R}^d)} + \sum_{|\alpha|=p} \|g\|_{L_\infty}\alpha_1^{1/2}\big\|(\mathbf{D}^\alpha\Lambda)^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0\big\|_{L_2(\mathbb{R}^d)}.$$

The first summand on the right is estimated with the help of (3.12), and for the second we can use Corollary 3.5 (cf. (3.12)–(3.14)). We arrive at

$$\big\|\widetilde{g}^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0\big\|_{L_2(\mathbb{R}^d)} \le C'' \sum_{l=1}^{p} \varepsilon^l \|\mathbf{D}^l b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathbb{R}^d)},$$

where $C'' = \|g\|_{L_\infty}(r_0^{-1} + r_1) + \kappa''(d,p)\|g\|_{L_\infty}\alpha_1^{1/2}\big(\widetilde{\beta}_1 C_1' + \widetilde{\beta}_2 C_2' + \widetilde{\beta}_3 C_3'\big)^{1/2}$. Together with (3.17), this implies

$$\big\|\widetilde{g}^\varepsilon(\Pi_\varepsilon - S_\varepsilon)b(\mathbf{D})\mathbf{u}_0\big\|_{L_2(\mathbb{R}^d)} \le \widetilde{C}'' c(\varphi)\big(\varepsilon|\zeta|^{-1/2+1/2p} + \varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \tag{3.20}$$

where $\widetilde{C}'' = pC'' \max_{1\le l\le p} \check{C}_l$. Now, (3.4) and (3.20) yield the required inequality (3.7) with the constants $C_6 = C_3 + \widetilde{C}''$ and $C_7 = \widetilde{C}''$. □

3.4. Removal of the smoothing operator. It turns out that, under some additional assumptions about the properties of the matrix-valued function $\Lambda(\mathbf{x})$, it is possible to remove the smoothing operator from the corrector.

Condition 3.6. *We assume that the* Γ*-periodic solution* Λ *of problem* (1.10) *is bounded and is a multiplier from* $H^p(\mathbb{R}^d;\mathbb{C}^m)$ *to* $H^p(\mathbb{R}^d;\mathbb{C}^n)$:

$$\Lambda \in L_\infty(\mathbb{R}^d) \cap M(H^p(\mathbb{R}^d;\mathbb{C}^m) \to H^p(\mathbb{R}^d;\mathbb{C}^n)).$$

Due to the periodicity of the matrix-valued function Λ, Condition 3.6 is equivalent to the relation $\Lambda \in L_\infty(\Omega) \cap M(H^p(\Omega;\mathbb{C}^m) \to H^p(\Omega;\mathbb{C}^n))$. The norm of the operator $[\Lambda]$ of multiplication by the matrix-valued function $\Lambda(\mathbf{x})$ is denoted by

$$M_\Lambda := \|[\Lambda]\|_{H^p(\mathbb{R}^d)\to H^p(\mathbb{R}^d)}. \tag{3.21}$$

A description of the spaces of multipliers in the Sobolev classes can be found in the book [MSh]. Some sufficient conditions ensuring Condition 3.6 are known (see [KuSu, Proposition 7.10]).

Proposition 3.7. *Suppose that at least one of the following assumptions is fulfilled*:

1°. $2p > d$;

2°. $g^0 = \underline{g}$, *i.e., we have* (1.18).

Then Condition 3.6 *is satisfied, and* $\|\Lambda\|_{L_\infty}$ *and the multiplier norm* (3.21) *are controlled in terms of* m, n, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *and the parameters of the lattice* Γ.

Under Condition 3.6, instead of the corrector (3.2) (or the corrector (3.5)), one can use the operator

$$K^0(\zeta;\varepsilon) := \Lambda^\varepsilon b(\mathbf{D})(A^0 - \zeta I)^{-1}, \tag{3.22}$$

which in this case is continuous from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^p(\mathbb{R}^d;\mathbb{C}^n)$. Note that the operator (3.22) is the traditional corrector used in the homogenization theory.

The following statement is deduced from Proposition 7.12 of the paper [KuSu] by the scaling transformation.

Proposition 3.8. *Under the assumptions of Theorem* 3.1*, suppose that Condition* 3.6 *is satisfied. Let $K(\zeta;\varepsilon)$ and $K^0(\zeta;\varepsilon)$ be the operators defined by* (3.2) *and* (3.22)*, respectively. Then for $\varepsilon > 0$ we have*

$$\varepsilon^p \big\| K(\zeta;\varepsilon) - K^0(\zeta;\varepsilon) \big\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \le C_8 c(\varphi)(\varepsilon^p + \varepsilon^{2p}),$$

$$\big\| \widetilde{g}^\varepsilon (I - \Pi_\varepsilon) b(\mathbf{D})(A^0 - \zeta I)^{-1} \big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_9 c(\varphi)\varepsilon^p.$$

The constants C_8 and C_9 depend only on m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and also on $\|\Lambda\|_{L_\infty}$ and M_Λ.

Combining Theorem 3.2 and Proposition 3.8, we arrive at the following result.

Theorem 3.9. *Under Condition* 3.6 *and the assumptions of Theorem* 3.1*, let $K^0(\zeta;\varepsilon)$ be the operator* (3.22)*, and let $\widetilde{g}(\mathbf{x})$ be the matrix-valued function* (1.12)*. Then for $\varepsilon > 0$ we have*

$$\begin{aligned} &\big\| (A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1} - \varepsilon^p K^0(\zeta;\varepsilon) \big\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \\ &\qquad \le C_2 c(\varphi)^2 \varepsilon |\zeta|^{-1/2+1/2p}\big(1 + |\zeta|^{-1/2}\big) + C_8 c(\varphi)(\varepsilon^p + \varepsilon^{2p}), \\ &\big\| g^\varepsilon b(\mathbf{D})(A_\varepsilon - \zeta I)^{-1} - \widetilde{g}^\varepsilon b(\mathbf{D})(A^0 - \zeta I)^{-1} \big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\qquad \le C_3 c(\varphi)^2 \varepsilon |\zeta|^{-1/2+1/2p} + C_9 c(\varphi)\varepsilon^p. \end{aligned}$$

The constants C_2 and C_3 depend only on m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ. The constants C_8 and C_9 depend on the same parameters and also on $\|\Lambda\|_{L_\infty}$ and M_Λ.

Chapter 2. Homogenization of the Dirichlet problem

§4. The Dirichlet problem in a bounded domain

4.1. The statement of the problem. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class C^{2p}. In $L_2(\mathcal{O};\mathbb{C}^n)$, we consider the operator $A_{D,\varepsilon}$ given formally by the differential expression $b(\mathbf{D})^* g^\varepsilon b(\mathbf{D})$ with the Dirichlet conditions on $\partial\mathcal{O}$. The precise definition is as follows: $A_{D,\varepsilon}$ is the selfadjoint operator in $L_2(\mathcal{O};\mathbb{C}^n)$ generated by the quadratic form

$$a_{D,\varepsilon}[\mathbf{u},\mathbf{u}] = \int_{\mathcal{O}} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle \, d\mathbf{x}, \quad \mathbf{u} \in H^p_0(\mathcal{O};\mathbb{C}^n). \tag{4.1}$$

The form (4.1) is closed and positive definite. Indeed, let us extend $\mathbf{u} \in H^p_0(\mathcal{O};\mathbb{C}^n)$ by zero to $\mathbb{R}^d \setminus \mathcal{O}$. Then $\mathbf{u} \in H^p(\mathbb{R}^d;\mathbb{C}^n)$. By (1.8), we have

$$c_0 \int_{\mathcal{O}} |\mathbf{D}^p \mathbf{u}|^2 \, d\mathbf{x} \le a_{D,\varepsilon}[\mathbf{u},\mathbf{u}] \le c_1 \int_{\mathcal{O}} |\mathbf{D}^p \mathbf{u}|^2 \, d\mathbf{x}, \quad \mathbf{u} \in H^p_0(\mathcal{O};\mathbb{C}^n). \tag{4.2}$$

It remains to recall that the form $\|\mathbf{D}^p \mathbf{u}\|_{L_2(\mathcal{O})}$ determines a norm in $H^p_0(\mathcal{O};\mathbb{C}^n)$ equivalent to the standard one. By the Friedrichs inequality, (4.2) implies that

$$a_{D,\varepsilon}[\mathbf{u},\mathbf{u}] \ge c_2 \|\mathbf{u}\|^2_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H^p_0(\mathcal{O};\mathbb{C}^n), \quad c_2 = c_0 (\operatorname{diam}\mathcal{O})^{-2p}. \tag{4.3}$$

Our goal is to approximate the generalized solution $\mathbf{u}_\varepsilon \in H^p_0(\mathcal{O};\mathbb{C}^n)$ of the Dirichlet problem

$$\begin{aligned} &b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}_\varepsilon(\mathbf{x}) - \zeta \mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{O}; \\ &\mathbf{u}_\varepsilon(\mathbf{x}) = \partial_{\boldsymbol{\nu}} \mathbf{u}_\varepsilon(\mathbf{x}) = \dots = \partial^{p-1}_{\boldsymbol{\nu}} \mathbf{u}_\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\mathcal{O}, \end{aligned} \tag{4.4}$$

where $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$, for small ε. Here $\partial^l_{\boldsymbol{\nu}} \mathbf{u}(\mathbf{x})$ stands for the normal derivative of $\mathbf{u}$ of order l on $\partial\mathcal{O}$. As in §3, we assume that $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$. (The case of other admissible values of ζ is studied below in §8.) We have $\mathbf{u}_\varepsilon = (A_{D,\varepsilon} - \zeta I)^{-1}\mathbf{F}$. In operator terms, we study the behavior of the resolvent $(A_{D,\varepsilon} - \zeta I)^{-1}$.

Lemma 4.1. *Let $\zeta = |\zeta| e^{i\varphi} \in \mathbb{C} \setminus \mathbb{R}_+$, and let $c(\varphi)$ be defined by* (3.1). *Suppose that $\mathbf{u}_\varepsilon$ is the generalized solution of problem* (4.4). *Then for $\varepsilon > 0$ we have*

$$\|\mathbf{u}_\varepsilon\|_{L_2(\mathcal{O})} \le c(\varphi)|\zeta|^{-1}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{4.5}$$

$$\|\mathbf{D}^p \mathbf{u}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathcal{C}_0 c(\varphi)|\zeta|^{-1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{4.6}$$

where $\mathcal{C}_0 = 2^{1/2} c_0^{-1/2}$. In operator terms,

$$\begin{aligned} &\left\|(A_{D,\varepsilon} - \zeta I)^{-1}\right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le c(\varphi)|\zeta|^{-1}, \\ &\left\|\mathbf{D}^p (A_{D,\varepsilon} - \zeta I)^{-1}\right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le \mathcal{C}_0 c(\varphi)|\zeta|^{-1/2}. \end{aligned} \tag{4.7}$$

Proof. By (4.3), the spectrum of $A_{D,\varepsilon}$ is contained in $[c_2, \infty) \subset \mathbb{R}_+$. The norm of the resolvent $(A_{D,\varepsilon} - \zeta I)^{-1}$ does not exceed the inverse distance from the point ζ to $\mathbb{R}_+$. This implies (4.7).

To check (4.6), we write the integral identity for the solution $\mathbf{u}_\varepsilon \in H^p_0(\mathcal{O};\mathbb{C}^n)$ of problem (4.4):

$$(g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} - \zeta(\mathbf{u}_\varepsilon, \boldsymbol{\eta})_{L_2(\mathcal{O})} = (\mathbf{F}, \boldsymbol{\eta})_{L_2(\mathcal{O})}, \quad \boldsymbol{\eta} \in H^p_0(\mathcal{O};\mathbb{C}^n). \tag{4.8}$$

Substituting $\boldsymbol{\eta} = \mathbf{u}_\varepsilon$ in (4.8) and using (4.5), we obtain

$$(g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon, b(\mathbf{D})\mathbf{u}_\varepsilon)_{L_2(\mathcal{O})} \le 2c(\varphi)^2|\zeta|^{-1}\|\mathbf{F}\|^2_{L_2(\mathcal{O})}.$$

Combining this with (4.2), we arrive at (4.6). □

4.2. The effective operator A^0_D. In $L_2(\mathcal{O};\mathbb{C}^n)$, consider the selfadjoint operator A^0_D generated by the quadratic form

$$a^0_D[\mathbf{u},\mathbf{u}] = \int_{\mathcal{O}} \langle g^0 b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle \, d\mathbf{x}, \quad \mathbf{u} \in H^p_0(\mathcal{O};\mathbb{C}^n). \tag{4.9}$$

Here g^0 is the effective matrix given by (1.11). Taking (1.16) into account, we see that the form (4.9) satisfies estimates of the form (4.2) and (4.3) with the same constants.

Since $\partial\mathcal{O} \in C^{2p}$, the operator A^0_D is given by the expression $b(\mathbf{D})^* g^0 b(\mathbf{D})$ on the domain $H^{2p}(\mathcal{O};\mathbb{C}^n) \cap H^p_0(\mathcal{O};\mathbb{C}^n)$. We have

$$\|(A^0_D)^{-1}\|_{L_2(\mathcal{O})\to H^{2p}(\mathcal{O})} \le \widehat{c}, \tag{4.10}$$

where the constant $\widehat{c}$ depends only on α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the domain $\mathcal{O}$. To justify this fact, it suffices to refer to Theorems 2.2 and 2.3 of the paper [So].

Remark 4.2. Instead of the condition $\partial\mathcal{O} \in C^{2p}$, one could impose the following implicit condition on the domain: suppose that $\mathcal{O}$ is a bounded Lipschitz domain such that estimate (4.10) is true. The results of Chapter 2 remain valid for such domain. In the case of scalar elliptic operators, wide sufficient conditions on $\partial\mathcal{O}$ ensuring estimate (4.10) can be found in [KoE] and [MSh, Chapter 7] (in particular, it suffices that $\partial\mathcal{O} \in C^{2p-1,\nu}$, $\nu > 1/2$).

Let $\mathbf{u}_0 = (A^0_D - \zeta I)^{-1}\mathbf{F}$, where $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. Then $\mathbf{u}_0$ is the generalized solution of the problem

$$\begin{aligned} &b(\mathbf{D})^* g^0 b(\mathbf{D})\mathbf{u}_0(\mathbf{x}) - \zeta\mathbf{u}_0(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{O}; \\ &\mathbf{u}_0(\mathbf{x}) = \partial_{\boldsymbol{\nu}}\mathbf{u}_0(\mathbf{x}) = \cdots = \partial^{p-1}_{\boldsymbol{\nu}}\mathbf{u}_0(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\mathcal{O}. \end{aligned} \tag{4.11}$$

Lemma 4.3. *Let $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$. Let $\mathbf{u}_0$ be the generalized solution of problem* (4.11)*. Then for $\varepsilon > 0$ we have*

$$\|\mathbf{u}_0\|_{L_2(\mathcal{O})} \le c(\varphi)|\zeta|^{-1}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{4.12}$$

$$\|\mathbf{D}^p\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \mathcal{C}_0 c(\varphi)|\zeta|^{-1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{4.13}$$

$$\|\mathbf{u}_0\|_{H^p(\mathcal{O})} \le \widetilde{\mathcal{C}}_0 c(\varphi)(|\zeta|^{-1} + |\zeta|^{-1/2})\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{4.14}$$

$$\|\mathbf{u}_0\|_{H^{2p}(\mathcal{O})} \le \widehat{c}c(\varphi)\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

In operator terms,

$$\big\|(A_D^0 - \zeta I)^{-1}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le c(\varphi)|\zeta|^{-1},$$

$$\big\|\mathbf{D}^p(A_D^0 - \zeta I)^{-1}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le \mathcal{C}_0 c(\varphi)|\zeta|^{-1/2},$$

$$\big\|(A_D^0 - \zeta I)^{-1}\big\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})} \le \widetilde{\mathcal{C}}_0 c(\varphi)(|\zeta|^{-1} + |\zeta|^{-1/2}),$$

$$\big\|(A_D^0 - \zeta I)^{-1}\big\|_{L_2(\mathcal{O})\to H^{2p}(\mathcal{O})} \le \widehat{c}c(\varphi). \tag{4.15}$$

The constant $\widetilde{\mathcal{C}}_0$ depends only on d, p, α_0, and $\|g^{-1}\|_{L_\infty}$.

Proof. Estimates (4.12) and (4.13) are proved as in the proof of Lemma 4.1. Since (3.9) is extended to functions of class $H_0^p(\mathcal{O};\mathbb{C}^n)$, relations (4.12) and (4.13) imply (4.14) with the constant $\widetilde{\mathcal{C}}_0 = \check{\mathfrak{c}}_p^{1/2}\max\{1, \mathcal{C}_0\}$.

Estimate (4.15) follows from (4.10) and the inequality

$$\big\|A_D^0(A_D^0 - \zeta I)^{-1}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le \sup_{x\ge 0} x|x-\zeta|^{-1} \le c(\varphi). \qquad \square$$

4.3. Estimates near the boundary. In this subsection, we formulate two simple auxiliary statements valid for bounded Lipschitz domains $\mathcal{O}$. Precisely, we impose the following condition.

Condition 4.4. *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain. Denote $(\partial\mathcal{O})_\varepsilon = \{\mathbf{x} \in \mathbb{R}^d : \operatorname{dist}\{\mathbf{x}; \partial\mathcal{O}\} < \varepsilon\}$. Suppose that there exists a number $\varepsilon_0 \in (0,1]$ such that the strip $(\partial\mathcal{O})_{\varepsilon_0}$ can be covered by a finite number of open sets admitting diffeomorphisms of class $C^{0,1}$ that rectify the boundary $\partial\mathcal{O}$. Denote $\varepsilon_1 = \varepsilon_0(1+r_1)^{-1}$, where $2r_1 = \operatorname{diam}\Omega$.*

Obviously, Condition 4.4 is less restrictive than the above assumption $\partial\mathcal{O} \in C^{2p}$.

Lemma 4.5. *Suppose that Condition 4.4 is satisfied. Denote $B_\varepsilon = (\partial\mathcal{O})_\varepsilon \cap \mathcal{O}$. Then the following is true.*

1°. *For any $u \in H^1(\mathcal{O})$ we have*

$$\int_{B_\varepsilon} |u|^2\,d\mathbf{x} \le \beta_0\varepsilon\|u\|_{H^1(\mathcal{O})}\|u\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \le \varepsilon_0.$$

2°. *For any $u \in H^1(\mathbb{R}^d)$ we have*

$$\int_{(\partial\mathcal{O})_\varepsilon} |u|^2\,d\mathbf{x} \le \beta_0\varepsilon\|u\|_{H^1(\mathbb{R}^d)}\|u\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le \varepsilon_0.$$

The constant β_0 depends only on the domain $\mathcal{O}$.

Lemma 4.6. *Under Condition 4.4, suppose that $f(\mathbf{x})$ is a Γ-periodic function in $\mathbb{R}^d$ such that $f \in L_2(\Omega)$. Let S_ε be the operator given by* (1.20)*. Denote $\beta_* = \beta_0(1+r_1)$, $2r_1 = \operatorname{diam}\Omega$. Then for $0 < \varepsilon \le \varepsilon_1$ any function $\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^m)$ satisfies*

$$\int_{(\partial\mathcal{O})_\varepsilon} |f^\varepsilon(\mathbf{x})|^2|(S_\varepsilon\mathbf{u})(\mathbf{x})|^2\,d\mathbf{x} \le \beta_*\varepsilon|\Omega|^{-1}\|f\|^2_{L_2(\Omega)}\|\mathbf{u}\|_{H^1(\mathbb{R}^d)}\|\mathbf{u}\|_{L_2(\mathbb{R}^d)}.$$

Lemma 4.6 is an analog of Lemma 2.6 in [ZhPas1]. Lemmas 4.5 and 4.6 were checked in [PSu2, §5] under the condition $\partial\mathcal{O} \in C^1$, but the proofs work also under Condition 4.4.

§5. The results for the Dirichlet problem

5.1. Approximation of the resolvent for $|\zeta| \geq 1$. Now we formulate our main homogenization results for the operator $A_{D,\varepsilon}$.

Theorem 5.1. *Suppose that $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain of class C^{2p}. Let $\zeta = |\zeta|e^{i\varphi} \in \mathbb{C} \setminus \mathbb{R}_+$ and $|\zeta| \geq 1$. Suppose that $c(\varphi)$ is defined by* (3.1)*. Let $\mathbf{u}_\varepsilon$ be the solution of problem* (4.4)*, and let $\mathbf{u}_0$ be the solution of problem* (4.11) *for $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. Suppose that the number ε_1 is subject to Condition* 4.4*. Then for $0 < \varepsilon \leq \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \leq \mathcal{C}_1 c(\varphi)^5 \big(\varepsilon|\zeta|^{-1+1/2p} + \varepsilon^{2p}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{5.1}$$

In operator terms,

$$\big\|(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq \mathcal{C}_1 c(\varphi)^5 \big(\varepsilon|\zeta|^{-1+1/2p} + \varepsilon^{2p}\big). \tag{5.2}$$

The constant $\mathcal{C}_1$ depends only on d, p, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

To approximate the solution in $H^p(\mathcal{O};\mathbb{C}^n)$, we need to introduce a corrector. We fix a continuous linear extension operator

$$P_\mathcal{O} : H^s(\mathcal{O};\mathbb{C}^n) \to H^s(\mathbb{R}^d;\mathbb{C}^n), \quad s = 0, 1, \dots, 2p.$$

Such an operator exists (see, e.g., [St]). Denote

$$\|P_\mathcal{O}\|_{H^s(\mathcal{O})\to H^s(\mathbb{R}^d)} =: C_\mathcal{O}^{(s)}, \quad s = 0, 1, \dots, 2p. \tag{5.3}$$

The constants $C_\mathcal{O}^{(s)}$ depend only on the domain $\mathcal{O}$ and s. By $R_\mathcal{O}$ we denote the operator of restriction of functions on $\mathbb{R}^d$ to the domain $\mathcal{O}$. We introduce a corrector

$$K_D(\zeta;\varepsilon) = R_\mathcal{O}[\Lambda^\varepsilon] S_\varepsilon b(\mathbf{D}) P_\mathcal{O} (A_D^0 - \zeta I)^{-1}. \tag{5.4}$$

The operator $K_D(\zeta;\varepsilon)$ maps $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^p(\mathcal{O};\mathbb{C}^n)$ continuously. Indeed, the operator $b(\mathbf{D})P_\mathcal{O}(A_D^0 - \zeta I)^{-1}$ is a continuous mapping of $L_2(\mathcal{O};\mathbb{C}^n)$ into $H^p(\mathbb{R}^d;\mathbb{C}^m)$, and the operator $[\Lambda^\varepsilon]S_\varepsilon$ is continuous from $H^p(\mathbb{R}^d;\mathbb{C}^m)$ to $H^p(\mathbb{R}^d;\mathbb{C}^n)$ (this follows from Proposition 1.8 and the relation $\Lambda \in \widetilde{H}^p(\Omega)$).

Let $\mathbf{u}_0$ be the solution of problem (4.11). We denote $\widetilde{\mathbf{u}}_0 := P_\mathcal{O}\mathbf{u}_0$ and put

$$\widetilde{\mathbf{v}}_\varepsilon(\mathbf{x}) = \widetilde{\mathbf{u}}_0(\mathbf{x}) + \varepsilon^p \Lambda^\varepsilon(\mathbf{x})(S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \tag{5.5}$$

$$\mathbf{v}_\varepsilon := \widetilde{\mathbf{v}}_\varepsilon|_\mathcal{O}. \tag{5.6}$$

Then

$$\mathbf{v}_\varepsilon = (A_D^0 - \zeta I)^{-1}\mathbf{F} + \varepsilon^p K_D(\zeta;\varepsilon)\mathbf{F}. \tag{5.7}$$

Theorem 5.2. *Under the assumptions of Theorem* 5.1*, let $\mathbf{v}_\varepsilon$ be defined by* (5.4) *and* (5.7)*. Then for $0 < \varepsilon \leq \varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^p(\mathcal{O})} \leq \mathcal{C}_2 c(\varphi)^4 \big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{5.8}$$

In operator terms,

$$\begin{aligned}\big\|(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1} - \varepsilon^p K_D(\zeta;\varepsilon)\big\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})}& \\ \leq \mathcal{C}_2 c(\varphi)^4 \big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big)&.\end{aligned} \tag{5.9}$$

The flux $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ satisfies

$$\big\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\big\|_{L_2(\mathcal{O})} \leq \mathcal{C}_3 c(\varphi)^4 \big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} \tag{5.10}$$

for $0 < \varepsilon \le \varepsilon_1$. *The constants* $\mathcal{C}_2$ *and* $\mathcal{C}_3$ *depend only on* d, p, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

Remark 5.3.

1) For fixed $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$, $|\zeta| \ge 1$, the estimates of Theorem 5.1 are of sharp order $O(\varepsilon)$.

2) For fixed $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$, $|\zeta| \ge 1$, the estimates of Theorem 5.2 are of order $O(\varepsilon^{1/2})$. This is explained by the boundary influence.

3) The error of approximations in Theorems 5.1 and 5.2 becomes smaller as $|\zeta|$ grows.

4) The estimates of Theorems 5.1 and 5.2 are uniform with respect to the angle φ in a domain $\{\zeta = |\zeta|e^{i\varphi} : |\zeta| \ge 1, \varphi_0 \le \varphi \le 2\pi - \varphi_0\}$ with arbitrarily small $\varphi_0 > 0$.

5.2. The first step of the proof. The associated problem in $\mathbb{R}^d$. The proof of Theorems 5.1 and 5.2 is based on application of results for the problem in $\mathbb{R}^d$ and introduction of the boundary layer correction term.

By Lemma 4.3 and (5.3), we have

$$\|\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le C_{\mathcal{O}}^{(0)} c(\varphi)|\zeta|^{-1}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{5.11}$$

$$\|\widetilde{\mathbf{u}}_0\|_{H^p(\mathbb{R}^d)} \le C^{(p)} c(\varphi)|\zeta|^{-1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{5.12}$$

$$\|\widetilde{\mathbf{u}}_0\|_{H^{2p}(\mathbb{R}^d)} \le C^{(2p)} c(\varphi)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{5.13}$$

where $C^{(p)} = 2C_{\mathcal{O}}^{(p)}\widetilde{\mathcal{C}}_0$ and $C^{(2p)} = C_{\mathcal{O}}^{(2p)}\widehat{c}$. We have taken into account that $|\zeta| \ge 1$. Interpolating between (5.12) and (5.13), we obtain

$$\|\widetilde{\mathbf{u}}_0\|_{H^{p+k}(\mathbb{R}^d)} \le C^{(p+k)} c(\varphi)|\zeta|^{-1/2+k/2p}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad k = 0, 1, \dots, p. \tag{5.14}$$

Here $C^{(p+k)} = (C^{(p)})^{1-k/p}(C^{(2p)})^{k/p}$.

We put

$$\widetilde{\mathbf{F}} := A^0\widetilde{\mathbf{u}}_0 - \zeta\widetilde{\mathbf{u}}_0. \tag{5.15}$$

Then $\widetilde{\mathbf{F}} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$ and $\widetilde{\mathbf{F}}|_{\mathcal{O}} = \mathbf{F}$. From (1.14), (5.11), and (5.13) it follows that

$$\|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d)} \le C_*\|\widetilde{\mathbf{u}}_0\|_{H^{2p}(\mathbb{R}^d)} + |\zeta|\|\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le \mathcal{C}_4 c(\varphi)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{5.16}$$

where $\mathcal{C}_4 = C_* C^{(2p)} + C_{\mathcal{O}}^{(0)}$. Let $\widetilde{\mathbf{u}}_\varepsilon \in H^p(\mathbb{R}^d;\mathbb{C}^n)$ be the solution of the following equation in $\mathbb{R}^d$:

$$A_\varepsilon\widetilde{\mathbf{u}}_\varepsilon - \zeta\widetilde{\mathbf{u}}_\varepsilon = \widetilde{\mathbf{F}}, \tag{5.17}$$

i.e., $\widetilde{\mathbf{u}}_\varepsilon = (A_\varepsilon - \zeta I)^{-1}\widetilde{\mathbf{F}}$.

We can apply the results of §3. Combining Theorems 3.1 and 3.3 and relations (5.15)–(5.17), for $\varepsilon > 0$ we get

$$\begin{aligned} \big\|\widetilde{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{u}}_0\big\|_{L_2(\mathbb{R}^d)} &\le C_1 c(\varphi)^2\varepsilon|\zeta|^{-1+1/2p}\|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d)} \\ &\le C_1\mathcal{C}_4 c(\varphi)^3\varepsilon|\zeta|^{-1+1/2p}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \end{aligned} \tag{5.18}$$

$$\begin{aligned} \big\|\widetilde{\mathbf{u}}_\varepsilon - \widetilde{\mathbf{v}}_\varepsilon\big\|_{H^p(\mathbb{R}^d)} &\le \big(2C_4 c(\varphi)^2\varepsilon|\zeta|^{-1/2+1/2p} + 2C_5 c(\varphi)\varepsilon^p\big)\|\widetilde{\mathbf{F}}\|_{L_2(\mathbb{R}^d)} \\ &\le \big(2C_4\mathcal{C}_4 c(\varphi)^3\varepsilon|\zeta|^{-1/2+1/2p} + 2C_5\mathcal{C}_4 c(\varphi)^2\varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \end{aligned} \tag{5.19}$$

Next, by Proposition 1.8 and (2.1),

$$\|[\Lambda^\varepsilon]S_\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_\Lambda^{(1)}. \tag{5.20}$$

Together with (1.4) and (5.12), this yields

$$\|\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le C_\Lambda^{(1)}\alpha_1^{1/2}\|\widetilde{\mathbf{u}}_0\|_{H^p(\mathbb{R}^d)} \le \mathcal{C}_5 c(\varphi)|\zeta|^{-1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{5.21}$$

where $\mathcal{C}_5 = C_\Lambda^{(1)} \alpha_1^{1/2} C^{(p)}$. By (5.18) and (5.21),

$$\begin{aligned} \|\tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{v}}_\varepsilon\|_{L_2(\mathbb{R}^d)} &\le \|\tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} + \varepsilon^p \|\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \\ &\le \big(C_1 \mathcal{C}_4 c(\varphi)^3 \varepsilon |\zeta|^{-1+1/2p} + \mathcal{C}_5 c(\varphi)\varepsilon^p |\zeta|^{-1/2}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \end{aligned} \tag{5.22}$$

5.3. The second step of the proof. Introduction of the correction term $\mathbf{w}_\varepsilon$. The first-order approximation $\mathbf{v}_\varepsilon$ of the solution $\mathbf{u}_\varepsilon$ does not satisfy the Dirichlet conditions. We consider the "correction term" $\mathbf{w}_\varepsilon \in H^p(\mathcal{O};\mathbb{C}^n)$ satisfying the integral identity

$$(g^\varepsilon b(\mathbf{D})\mathbf{w}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta})_{L_2(\mathcal{O})} - \zeta(\mathbf{w}_\varepsilon, \boldsymbol{\eta})_{L_2(\mathcal{O})} = 0, \quad \boldsymbol{\eta} \in H_0^p(\mathcal{O};\mathbb{C}^n), \tag{5.23}$$

and the condition

$$\mathbf{w}_\varepsilon - \varepsilon^p \Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0 \in H_0^p(\mathcal{O};\mathbb{C}^n). \tag{5.24}$$

Lemma 5.4. *Let $\mathbf{u}_\varepsilon$ be the solution of problem* (4.4), *and let $\mathbf{v}_\varepsilon$ be given by* (5.7). *Suppose that $\mathbf{w}_\varepsilon \in H^p(\mathcal{O};\mathbb{C}^n)$ satisfies* (5.23) *and* (5.24). *Then for $\varepsilon > 0$ we have*

$$\|\mathbf{D}^p(\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon)\|_{L_2(\mathcal{O})} \le \mathcal{C}_6\big(c(\varphi)^4 \varepsilon |\zeta|^{-1/2+1/2p} + c(\varphi)^3 \varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{5.25}$$

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathcal{C}_7\big(c(\varphi)^4 \varepsilon |\zeta|^{-1+1/2p} + c(\varphi)^3 \varepsilon^p |\zeta|^{-1/2}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{5.26}$$

The constants $\mathcal{C}_6$ and $\mathcal{C}_7$ depend only on d, p, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

Proof. Denote $\mathbf{V}_\varepsilon := \mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon$. By (4.8), (5.23), and (5.24), the function $\mathbf{V}_\varepsilon$ belongs to $H_0^p(\mathcal{O};\mathbb{C}^n)$ and satisfies the identity

$$\big(g^\varepsilon b(\mathbf{D})\mathbf{V}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta}\big)_{L_2(\mathcal{O})} - \zeta(\mathbf{V}_\varepsilon, \boldsymbol{\eta})_{L_2(\mathcal{O})} = J_\varepsilon[\boldsymbol{\eta}], \quad \boldsymbol{\eta} \in H_0^p(\mathcal{O};\mathbb{C}^n), \tag{5.27}$$

where $J_\varepsilon[\boldsymbol{\eta}] := (\mathbf{F}, \boldsymbol{\eta})_{L_2(\mathcal{O})} - \big(g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta}\big)_{L_2(\mathcal{O})} + \zeta(\mathbf{v}_\varepsilon, \boldsymbol{\eta})_{L_2(\mathcal{O})}$. We extend $\boldsymbol{\eta}$ by zero to $\mathbb{R}^d \setminus \mathcal{O}$, keeping the same notation; then $\boldsymbol{\eta} \in H^p(\mathbb{R}^d;\mathbb{C}^n)$. Recall that $\widetilde{\mathbf{F}}$ is an extension of $\mathbf{F}$, and $\tilde{\mathbf{v}}_\varepsilon$ is an extension of $\mathbf{v}_\varepsilon$. Hence,

$$J_\varepsilon[\boldsymbol{\eta}] = (\widetilde{\mathbf{F}}, \boldsymbol{\eta})_{L_2(\mathbb{R}^d)} - \big(g^\varepsilon b(\mathbf{D})\tilde{\mathbf{v}}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta}\big)_{L_2(\mathbb{R}^d)} + \zeta(\tilde{\mathbf{v}}_\varepsilon, \boldsymbol{\eta})_{L_2(\mathbb{R}^d)}.$$

Combining this with (5.17), we arrive at

$$J_\varepsilon[\boldsymbol{\eta}] = \big(g^\varepsilon b(\mathbf{D})(\tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{v}}_\varepsilon), b(\mathbf{D})\boldsymbol{\eta}\big)_{L_2(\mathbb{R}^d)} - \zeta\big(\tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{v}}_\varepsilon, \boldsymbol{\eta}\big)_{L_2(\mathbb{R}^d)}.$$

Applying (1.4), (5.19), and (5.22), we obtain the inequality

$$|J_\varepsilon[\boldsymbol{\eta}]| \le k_1(\zeta,\varepsilon)\|\mathbf{F}\|_{L_2(\mathcal{O})}\big(\mathcal{C}_8\|(g^\varepsilon)^{1/2} b(\mathbf{D})\boldsymbol{\eta}\|_{L_2(\mathcal{O})} + \mathcal{C}_9|\zeta|^{1/2}\|\boldsymbol{\eta}\|_{L_2(\mathcal{O})}\big). \tag{5.28}$$

Here $k_1(\zeta,\varepsilon) := c(\varphi)^3 \varepsilon |\zeta|^{-1/2+1/2p} + c(\varphi)^2 \varepsilon^p$, $\mathcal{C}_8 = 2\alpha_1^{1/2}\|g\|_{L_\infty}^{1/2} \mathcal{C}_4 \max\{C_4, C_5\}$, and $\mathcal{C}_9 = \max\{C_1\mathcal{C}_4, \mathcal{C}_5\}$.

We substitute $\boldsymbol{\eta} = \mathbf{V}_\varepsilon$ in (5.27), take the imaginary part of the corresponding identity, and apply (5.28):

$$\begin{aligned} |\operatorname{Im}\zeta|\|\mathbf{V}_\varepsilon\|^2_{L_2(\mathcal{O})} \le k_1(\zeta,\varepsilon)\|\mathbf{F}\|_{L_2(\mathcal{O})} \\ \times \big(\mathcal{C}_8\|(g^\varepsilon)^{1/2} b(\mathbf{D})\mathbf{V}_\varepsilon\|_{L_2(\mathcal{O})} + \mathcal{C}_9|\zeta|^{1/2}\|\mathbf{V}_\varepsilon\|_{L_2(\mathcal{O})}\big). \end{aligned} \tag{5.29}$$

For $\operatorname{Re}\zeta \ge 0$ (in this case $\operatorname{Im}\zeta \ne 0$), we deduce the inequality

$$|\zeta|\|\mathbf{V}_\varepsilon\|^2_{L_2(\mathcal{O})} \le 2\mathcal{C}_8 k_2(\zeta,\varepsilon)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|(g^\varepsilon)^{1/2} b(\mathbf{D})\mathbf{V}_\varepsilon\|_{L_2(\mathcal{O})} + \mathcal{C}_9^2 k_2(\zeta,\varepsilon)^2\|\mathbf{F}\|^2_{L_2(\mathcal{O})}. \tag{5.30}$$

Here $k_2(\zeta,\varepsilon) := c(\varphi)^4 \varepsilon |\zeta|^{-1/2+1/2p} + c(\varphi)^3 \varepsilon^p$. If $\operatorname{Re}\zeta < 0$, we take the real part of the corresponding identity and apply (5.28):

$$\begin{aligned} |\operatorname{Re}\zeta|\|\mathbf{V}_\varepsilon\|^2_{L_2(\mathcal{O})} \le k_1(\zeta,\varepsilon)\|\mathbf{F}\|_{L_2(\mathcal{O})} \\ \times \big(\mathcal{C}_8\|(g^\varepsilon)^{1/2} b(\mathbf{D})\mathbf{V}_\varepsilon\|_{L_2(\mathcal{O})} + \mathcal{C}_9|\zeta|^{1/2}\|\mathbf{V}_\varepsilon\|_{L_2(\mathcal{O})}\big). \end{aligned} \tag{5.31}$$

Adding (5.29) and (5.31), we deduce an analog of (5.30). Finally, for all values of ζ under consideration we get

$$|\zeta| \|\mathbf{V}_\varepsilon\|^2_{L_2(\mathcal{O})} \le 4\mathcal{C}_8 k_2(\zeta,\varepsilon) \|\mathbf{F}\|_{L_2(\mathcal{O})} \|(g^\varepsilon)^{1/2} b(\mathbf{D})\mathbf{V}_\varepsilon\|_{L_2(\mathcal{O})} + 4\mathcal{C}_9^2 k_2(\zeta,\varepsilon)^2 \|\mathbf{F}\|^2_{L_2(\mathcal{O})}. \tag{5.32}$$

Now, relations (5.27) with $\boldsymbol{\eta} = \mathbf{V}_\varepsilon$, (5.28), and (5.32) imply that

$$a_{D,\varepsilon}[\mathbf{V}_\varepsilon, \mathbf{V}_\varepsilon] \le 9\mathcal{C}_8 k_2(\zeta,\varepsilon) \|\mathbf{F}\|_{L_2(\mathcal{O})} \|(g^\varepsilon)^{1/2} b(\mathbf{D})\mathbf{V}_\varepsilon\|_{L_2(\mathcal{O})} + 9\mathcal{C}_9^2 k_2(\zeta,\varepsilon)^2 \|\mathbf{F}\|^2_{L_2(\mathcal{O})}.$$

We see that

$$a_{D,\varepsilon}[\mathbf{V}_\varepsilon, \mathbf{V}_\varepsilon] \le \check{\mathcal{C}}_6^2 k_2(\zeta,\varepsilon)^2 \|\mathbf{F}\|^2_{L_2(\mathcal{O})}, \tag{5.33}$$

where $\check{\mathcal{C}}_6^2 = 18\mathcal{C}_9^2 + 81\mathcal{C}_8^2$. Combining (5.33) with (4.2), we obtain estimate (5.25) with the constant $\mathcal{C}_6 = \check{\mathcal{C}}_6 c_0^{-1/2}$. Finally, (5.32) and (5.33) imply (5.26) with the constant $\mathcal{C}_7 = 2(\mathcal{C}_8\check{\mathcal{C}}_6 + \mathcal{C}_9^2)^{1/2}$. ☐

Conclusions.

1) Relations (5.25) and (5.26) show that for $\varepsilon > 0$ we have

$$\|\mathbf{D}^p(\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon)\|_{L_2(\mathcal{O})} \le \mathcal{C}_6\big(c(\varphi)^4 \varepsilon |\zeta|^{-1/2+1/2p} + c(\varphi)^3 \varepsilon^p\big) \|\mathbf{F}\|_{L_2(\mathcal{O})} + \|\mathbf{D}^p \mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}, \tag{5.34}$$

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathcal{C}_7\big(c(\varphi)^4 \varepsilon |\zeta|^{-1+1/2p} + c(\varphi)^3 \varepsilon^p |\zeta|^{-1/2}\big) \|\mathbf{F}\|_{L_2(\mathcal{O})} + \|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}. \tag{5.35}$$

Clearly, for the proof of Theorem 5.2 it remains to estimate the norm $\|\mathbf{w}_\varepsilon\|_{H^p(\mathcal{O})}$.

2) From (5.21) we deduce an estimate for the difference $\mathbf{v}_\varepsilon - \mathbf{u}_0 = \varepsilon^p(\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)|_{\mathcal{O}}$:

$$\|\mathbf{v}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le \varepsilon^p \|\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le \mathcal{C}_5 c(\varphi) \varepsilon^p |\zeta|^{-1/2} \|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{5.36}$$

Combining this with (5.35), for $\varepsilon > 0$ we get

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le \widetilde{\mathcal{C}}_7\big(c(\varphi)^4 \varepsilon |\zeta|^{-1+1/2p} + c(\varphi)^3 \varepsilon^p |\zeta|^{-1/2}\big) \|\mathbf{F}\|_{L_2(\mathcal{O})} + \|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}, \tag{5.37}$$

where $\widetilde{\mathcal{C}}_7 = \mathcal{C}_7 + \mathcal{C}_5$. Therefore, in order to prove Theorem 5.1, we need to obtain a suitable estimate for the norm $\|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}$.

§6. Estimates for the correction term $\mathbf{w}_\varepsilon$. Proof of Theorems 5.1 and 5.2

First, we estimate the norm of $\mathbf{w}_\varepsilon$ in $H^p(\mathcal{O})$ and complete the proof of Theorem 5.2. Next, using the already proved Theorem 5.2 and duality arguments, we estimate the L_2-norm of the correction term and prove Theorem 5.1.

6.1. Localization near the boundary. Suppose that the number $\varepsilon_0 \in (0,1]$ is subject to Condition 4.4. Let $0 < \varepsilon \le \varepsilon_0$. We fix a smooth cut-off function $\theta_\varepsilon(\mathbf{x})$ in $\mathbb{R}^d$ such that

$$\begin{aligned} &\theta_\varepsilon \in C_0^\infty(\mathbb{R}^d), \quad \operatorname{supp}\theta_\varepsilon \subset (\partial\mathcal{O})_\varepsilon, \quad 0 \le \theta_\varepsilon(\mathbf{x}) \le 1, \\ &\theta_\varepsilon(\mathbf{x}) = 1 \text{ for } \mathbf{x} \in (\partial\mathcal{O})_{\varepsilon/2}, \quad \varepsilon^l |\mathbf{D}^l \theta_\varepsilon(\mathbf{x})| \le \varkappa, \ l = 1,\dots,p. \end{aligned} \tag{6.1}$$

The constant $\varkappa$ depends only on the domain $\mathcal{O}$. Consider the following function on $\mathbb{R}^d$:

$$\boldsymbol{\phi}_\varepsilon(\mathbf{x}) = \varepsilon^p \theta_\varepsilon(\mathbf{x}) \Lambda^\varepsilon(\mathbf{x}) (S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)(\mathbf{x}). \tag{6.2}$$

Lemma 6.1. *Suppose that $\mathbf{w}_\varepsilon \in H^p(\mathcal{O};\mathbb{C}^n)$ satisfies conditions* (5.23) *and* (5.24). *Let $\boldsymbol{\phi}_\varepsilon$ be defined by* (6.2). *Then for $0 < \varepsilon \le \varepsilon_0$ and $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$, $|\zeta| \ge 1$, we have*

$$\|\mathbf{D}^p \mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathcal{C}_{10} c(\varphi) \big(|\zeta|^{1/2} \|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)} + \|\mathbf{D}^p \boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)}\big), \tag{6.3}$$

$$\|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathcal{C}_{11} c(\varphi) \big(\|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)} + |\zeta|^{-1/2} \|\mathbf{D}^p \boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)}\big). \tag{6.4}$$

The constants $\mathcal{C}_{10}$ and $\mathcal{C}_{11}$ depend only on d, p, α_0, α_1, $\|g\|_{L_\infty}$, and $\|g^{-1}\|_{L_\infty}$.

Proof. By (5.23), (5.24), (6.1), and (6.2), the function $\boldsymbol{\rho}_\varepsilon := \mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon \in H^p_0(\mathcal{O};\mathbb{C}^n)$ satisfies the identity

$$\begin{aligned}(6.5)\qquad &\big(g^\varepsilon b(\mathbf{D})\boldsymbol{\rho}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta}\big)_{L_2(\mathcal{O})} - \zeta(\boldsymbol{\rho}_\varepsilon,\boldsymbol{\eta})_{L_2(\mathcal{O})}\\ &\quad = -\big(g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta}\big)_{L_2(\mathcal{O})} + \zeta(\boldsymbol{\phi}_\varepsilon,\boldsymbol{\eta})_{L_2(\mathcal{O})},\quad \boldsymbol{\eta}\in H^p_0(\mathcal{O};\mathbb{C}^n).\end{aligned}$$

We substitute $\boldsymbol{\eta} = \boldsymbol{\rho}_\varepsilon$ in (6.5) and take the imaginary part in the corresponding identity. Then

$$(6.6)\quad |\operatorname{Im}\zeta|\|\boldsymbol{\rho}_\varepsilon\|^2_{L_2(\mathcal{O})} \le \mathcal{C}_{12}\|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)}\|(g^\varepsilon)^{1/2}b(\mathbf{D})\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})} + |\zeta|\|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})}\|\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})},$$

where $\mathcal{C}_{12} = \mathfrak{c}_p^{1/2}\alpha_1^{1/2}\|g\|^{1/2}_{L_\infty}$. We have taken (1.4) and (1.7) into account. For $\operatorname{Re}\zeta \ge 0$ (in this case $\operatorname{Im}\zeta \ne 0$) we deduce the estimate

$$(6.7)\quad |\zeta|\,\|\boldsymbol{\rho}_\varepsilon\|^2_{L_2(\mathcal{O})} \le 2\mathcal{C}_{12}c(\varphi)\|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)}\|(g^\varepsilon)^{1/2}b(\mathbf{D})\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})} + c(\varphi)^2|\zeta|\|\boldsymbol{\phi}_\varepsilon\|^2_{L_2(\mathcal{O})}.$$

If $\operatorname{Re}\zeta < 0$, we take the real part of the corresponding identity, obtaining

$$(6.8)\quad |\operatorname{Re}\zeta|\|\boldsymbol{\rho}_\varepsilon\|^2_{L_2(\mathcal{O})} \le \mathcal{C}_{12}\|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)}\|(g^\varepsilon)^{1/2}b(\mathbf{D})\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})} + |\zeta|\|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})}\|\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})}.$$

Adding (6.6) and (6.8), we deduce an analog of (6.7). As a result, for all values of ζ under consideration we have

$$(6.9)\quad |\zeta|\|\boldsymbol{\rho}_\varepsilon\|^2_{L_2(\mathcal{O})} \le 4\mathcal{C}_{12}c(\varphi)\|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)}\|(g^\varepsilon)^{1/2}b(\mathbf{D})\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})} + 4c(\varphi)^2|\zeta|\|\boldsymbol{\phi}_\varepsilon\|^2_{L_2(\mathcal{O})}.$$

Now, from (6.5) with $\boldsymbol{\eta} = \boldsymbol{\rho}_\varepsilon$ and (6.9) it follows that

$$a_{D,\varepsilon}[\boldsymbol{\rho}_\varepsilon,\boldsymbol{\rho}_\varepsilon] \le 9c(\varphi)^2|\zeta|\|\boldsymbol{\phi}_\varepsilon\|^2_{L_2(\mathcal{O})} + 9\mathcal{C}_{12}c(\varphi)\|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)}\|(g^\varepsilon)^{1/2}b(\mathbf{D})\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})}.$$

We see that

$$(6.10)\qquad a_{D,\varepsilon}[\boldsymbol{\rho}_\varepsilon,\boldsymbol{\rho}_\varepsilon] \le 18c(\varphi)^2|\zeta|\|\boldsymbol{\phi}_\varepsilon\|^2_{L_2(\mathcal{O})} + 81\mathcal{C}_{12}^2c(\varphi)^2\|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|^2_{L_2(\mathbb{R}^d)}.$$

By (6.10) and (4.2),

$$\|\mathbf{D}^p\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})} \le c_0^{-1/2}c(\varphi)\big(\sqrt{18}|\zeta|^{1/2}\|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} + 9\mathcal{C}_{12}\|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)}\big).$$

This implies (6.3) with the constant $\mathcal{C}_{10} = \max\{\sqrt{18}c_0^{-1/2}, 9\mathcal{C}_{12}c_0^{-1/2}+1\}$. Next, from (6.9) and (6.10) it follows that

$$\|\boldsymbol{\rho}_\varepsilon\|_{L_2(\mathcal{O})} \le c(\varphi)\big(\sqrt{22}\|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} + \sqrt{85}\mathcal{C}_{12}|\zeta|^{-1/2}\|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)}\big),$$

which yields (6.4) with the constant $\mathcal{C}_{11} = \max\{\sqrt{22}+1, \sqrt{85}\mathcal{C}_{12}\}$. $\square$

6.2. Estimate of the function $\boldsymbol{\phi}_\varepsilon$. Now, we estimate the function (6.2).

Lemma 6.2. *Suppose that ε_1 is subject to Condition 4.4. Let $\boldsymbol{\phi}_\varepsilon$ be defined by (6.2). Then for $0<\varepsilon\le\varepsilon_1$ and $\zeta\in\mathbb{C}\setminus\mathbb{R}_+$, $|\zeta|\ge 1$, we have*

$$(6.11)\qquad \|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)} \le \mathcal{C}_5c(\varphi)|\zeta|^{-1/2}\varepsilon^p\|\mathbf{F}\|_{L_2(\mathcal{O})},$$

$$(6.12)\qquad \|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)} \le \mathcal{C}_{13}c(\varphi)\big(\varepsilon^p + \varepsilon^{1/2}|\zeta|^{-1/2+1/4p}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

The constants $\mathcal{C}_5$ and $\mathcal{C}_{13}$ depend only on d, p, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

Proof. Estimate (6.11) follows from (5.21) and (6.1).

Consider the derivatives of $\boldsymbol{\phi}_\varepsilon$ for $|\alpha| = p$:

$$(6.13)\qquad \partial^\alpha\boldsymbol{\phi}_\varepsilon = \varepsilon^p\sum_{\beta\le\alpha}\sum_{\gamma\le\alpha-\beta} C_\alpha^\beta C_{\alpha-\beta}^\gamma(\partial^\gamma\theta_\varepsilon)(\partial^{\alpha-\beta-\gamma}\Lambda^\varepsilon)(S_\varepsilon b(\mathbf{D})\partial^\beta\tilde{\mathbf{u}}_0).$$

If $k = |\beta| \geq 1$, we use (6.1), Proposition 1.8, (1.4), (2.3), and (5.14):

$$\begin{aligned}(6.14)\qquad &\varepsilon^p \big\|(\partial^\gamma \theta_\varepsilon)(\partial^{\alpha-\beta-\gamma}\Lambda^\varepsilon)(S_\varepsilon b(\mathbf{D})\partial^\beta \widetilde{\mathbf{u}}_0)\big\|_{L_2(\mathbb{R}^d)} \\ &\leq \varkappa \varepsilon^k \big\|(\partial^{\alpha-\beta-\gamma}\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D})\partial^\beta \widetilde{\mathbf{u}}_0\big\|_{L_2(\mathbb{R}^d)} \\ &\leq \varkappa \varepsilon^k C_\Lambda \alpha_1^{1/2} \|\widetilde{\mathbf{u}}_0\|_{H^{p+k}(\mathbb{R}^d)} \leq \mathcal{C}^{(k)} \varepsilon^k c(\varphi)|\zeta|^{-1/2+k/2p}\|\mathbf{F}\|_{L_2(\mathcal{O})}.\end{aligned}$$

Here $\mathcal{C}^{(k)} = \varkappa C_\Lambda \alpha_1^{1/2} C^{(p+k)}$.

If $\beta = 0$, we apply Lemma 4.6. Let $0 < \varepsilon \leq \varepsilon_1$. Taking (6.1) into account, we have

$$\begin{aligned}&\varepsilon^p \big\|(\partial^\gamma \theta_\varepsilon)(\partial^{\alpha-\gamma}\Lambda^\varepsilon)(S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)\big\|_{L_2(\mathbb{R}^d)} \\ &\leq \varkappa \big\|(\partial^{\alpha-\gamma}\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\big\|_{L_2((\partial\mathcal{O})_\varepsilon)} \\ &\leq \varepsilon^{1/2}\varkappa \beta_*^{1/2}|\Omega|^{-1/2}\|\partial^{\alpha-\gamma}\Lambda\|_{L_2(\Omega)}\|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|^{1/2}_{H^1(\mathbb{R}^d)}\|b(\mathbf{D})\widetilde{\mathbf{u}}_0\|^{1/2}_{L_2(\mathbb{R}^d)}.\end{aligned}$$

Combining this with (1.4) and (2.3), we obtain

$$\begin{aligned}(6.15)\qquad &\varepsilon^p \big\|(\partial^\gamma \theta_\varepsilon)(\partial^{\alpha-\gamma}\Lambda^\varepsilon)(S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)\big\|_{L_2(\mathbb{R}^d)} \\ &\leq \varepsilon^{1/2}\varkappa \beta_*^{1/2} C_\Lambda \alpha_1^{1/2}\|\widetilde{\mathbf{u}}_0\|^{1/2}_{H^{p+1}(\mathbb{R}^d)}\|\widetilde{\mathbf{u}}_0\|^{1/2}_{H^p(\mathbb{R}^d)}.\end{aligned}$$

Now, relations (5.12), (5.14), and (6.15) imply that

$$(6.16)\qquad \varepsilon^p \big\|(\partial^\gamma \theta_\varepsilon)(\partial^{\alpha-\gamma}\Lambda^\varepsilon)(S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)\big\|_{L_2(\mathbb{R}^d)} \leq \mathcal{C}_{14}\varepsilon^{1/2}c(\varphi)|\zeta|^{-1/2+1/4p}\|\mathbf{F}\|_{L_2(\mathcal{O})},$$

where $\mathcal{C}_{14} = \varkappa \beta_*^{1/2} C_\Lambda \alpha_1^{1/2}(C^{(p)}C^{(p+1)})^{1/2}$.

Estimating the summands in (6.13) with $k = |\beta| \geq 1$ with the help of (6.14), and the summands with $\beta = 0$ by (6.16), we arrive at the inequality

$$(6.17)\qquad \|\partial^\alpha \boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)} \leq \mathcal{C}_{15}c(\varphi)\left(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \sum_{k=1}^{p}\varepsilon^k|\zeta|^{-1/2+k/2p}\right)\|\mathbf{F}\|_{L_2(\mathcal{O})},$$

where $\mathcal{C}_{15} = \kappa_7(d,p)\max\{\mathcal{C}_{14}, \mathcal{C}^{(1)}, \dots, \mathcal{C}^{(p)}\}$. It is easily seen that the expression in parentheses does not exceed $2p(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p)$. Then (6.17) implies (6.12) with the constant $\mathcal{C}_{13} = \kappa_8(d,p)\mathcal{C}_{15}$. □

6.3. Completion of the proof of Theorem 5.2. From Lemmas 6.1 and 6.2 it follows that

$$\begin{aligned}\|\mathbf{D}^p \mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} &\leq \mathcal{C}_{16}c(\varphi)^2\big(\varepsilon^p + \varepsilon^{1/2}|\zeta|^{-1/2+1/4p}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \\ \|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} &\leq \mathcal{C}_{17}c(\varphi)^2\big(\varepsilon^p|\zeta|^{-1/2} + \varepsilon^{1/2}|\zeta|^{-1+1/4p}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})},\end{aligned}$$

where $\mathcal{C}_{16} = \mathcal{C}_{10}(\mathcal{C}_5 + \mathcal{C}_{13})$ and $\mathcal{C}_{17} = \mathcal{C}_{11}(\mathcal{C}_5 + \mathcal{C}_{13})$. Together with (5.34) and (5.35), for $0 < \varepsilon \leq \varepsilon_1$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, this yields

$$\begin{aligned}(6.18)\qquad \|\mathbf{D}^p(\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon)\|_{L_2(\mathcal{O})} &\leq \mathcal{C}_6\big(c(\varphi)^4\varepsilon|\zeta|^{-1/2+1/2p} + c(\varphi)^3\varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} \\ &\quad + \mathcal{C}_{16}c(\varphi)^2\big(\varepsilon^p + \varepsilon^{1/2}|\zeta|^{-1/2+1/4p}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} \\ &\leq \mathcal{C}_{18}\big(c(\varphi)^2\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + c(\varphi)^4\varepsilon|\zeta|^{-1/2+1/2p} + c(\varphi)^3\varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} \\ &\leq 2\mathcal{C}_{18}c(\varphi)^4\big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})},\end{aligned}$$

$$\begin{aligned}(6.19)\qquad \|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{L_2(\mathcal{O})} &\leq \mathcal{C}_7\big(c(\varphi)^4\varepsilon|\zeta|^{-1+1/2p} + c(\varphi)^3\varepsilon^p|\zeta|^{-1/2}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} \\ &\quad + \mathcal{C}_{17}c(\varphi)^2\big(\varepsilon^p|\zeta|^{-1/2} + \varepsilon^{1/2}|\zeta|^{-1+1/4p}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} \\ &\leq \mathcal{C}_{19}\big(c(\varphi)^2\varepsilon^{1/2}|\zeta|^{-1+1/4p} + c(\varphi)^4\varepsilon|\zeta|^{-1+1/2p} + c(\varphi)^3\varepsilon^p|\zeta|^{-1/2}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})} \\ &\leq 2\mathcal{C}_{19}c(\varphi)^4\big(\varepsilon^{1/2}|\zeta|^{-1+1/4p} + \varepsilon^p|\zeta|^{-1/2}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})},\end{aligned}$$

where $\mathcal{C}_{18} = \mathcal{C}_6 + \mathcal{C}_{16}$ and $\mathcal{C}_{19} = \mathcal{C}_7 + \mathcal{C}_{17}$. Since

$$\|\mathbf{u}\|_{H^p(\mathcal{O})} \le C(p;\mathcal{O})\left(\|\mathbf{D}^p\mathbf{u}\|_{L_2(\mathcal{O})} + \|\mathbf{u}\|_{L_2(\mathcal{O})}\right), \quad \mathbf{u} \in H^p(\mathcal{O};\mathbb{C}^n), \tag{6.20}$$

where the constant $C(p;\mathcal{O})$ depends only on p and $\mathcal{O}$, inequalities (6.18) and (6.19) imply (5.8) with the constant $\mathcal{C}_2 = 2C(p;\mathcal{O})(\mathcal{C}_{18} + \mathcal{C}_{19})$.

It remains to check (5.10). From (5.8) and (1.3), (1.5) it follows that

$$\|\mathbf{p}_\varepsilon - g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathcal{C}_{20} c(\varphi)^4\big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{6.21}$$

where $\mathcal{C}_{20} = \kappa_9(d,p)\|g\|_{L_\infty}\alpha_1^{1/2}\mathcal{C}_2$. By (1.3) and (5.5), (5.6), we have

$$\begin{aligned} g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon &= g^\varepsilon b(\mathbf{D})\mathbf{u}_0 + g^\varepsilon (b(\mathbf{D})\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0 \\ &\quad + \sum_{|\alpha|=p}\sum_{\beta\le\alpha:|\beta|\ge1} g^\varepsilon b_\alpha C_\alpha^\beta \varepsilon^{|\beta|}(\mathbf{D}^{\alpha-\beta}\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D})\mathbf{D}^\beta\tilde{\mathbf{u}}_0. \end{aligned} \tag{6.22}$$

Proposition 1.7 implies that

$$\begin{aligned} \left\|g^\varepsilon b(\mathbf{D})\mathbf{u}_0 - g^\varepsilon S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0\right\|_{L_2(\mathcal{O})} &\le \|g^\varepsilon (I - S_\varepsilon) b(\mathbf{D})\tilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \\ &\le \varepsilon\|g\|_{L_\infty} r_1 \|\mathbf{D}b(\mathbf{D})\tilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)}. \end{aligned} \tag{6.23}$$

The third term on the right-hand side of (6.22) is estimated by using (2.3) and Proposition 1.8. Taking (1.5) into account, we obtain

$$\begin{aligned} &\left\|\sum_{|\alpha|=p}\sum_{\beta\le\alpha\,:\,|\beta|\ge1} g^\varepsilon b_\alpha C_\alpha^\beta \varepsilon^{|\beta|}(\mathbf{D}^{\alpha-\beta}\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D})\mathbf{D}^\beta\tilde{\mathbf{u}}_0\right\|_{L_2(\mathbb{R}^d)} \\ &\qquad \le \mathcal{C}_{21}\sum_{l=1}^{p}\varepsilon^l\|\mathbf{D}^l b(\mathbf{D})\tilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)}, \end{aligned} \tag{6.24}$$

where $\mathcal{C}_{21} = \kappa_{10}(d,p)\|g\|_{L_\infty}\alpha_1^{1/2}C_\Lambda$.

From (1.4) and (5.14) it follows that

$$\|\mathbf{D}^l b(\mathbf{D})\tilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le \alpha_1^{1/2}C^{(p+l)}c(\varphi)|\zeta|^{-1/2+l/2p}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad l = 1,\dots,p. \tag{6.25}$$

Comparing (1.12) and (6.22)–(6.25), we arrive at

$$\left\|g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0\right\|_{L_2(\mathcal{O})} \le \mathcal{C}_{22}c(\varphi)\big(\varepsilon|\zeta|^{-1/2+1/2p} + \varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{6.26}$$

where $\mathcal{C}_{22} = \alpha_1^{1/2}\big(\|g\|_{L_\infty} r_1 C^{(p+1)} + p\,\mathcal{C}_{21}\max_{1\le l\le p} C^{(p+l)}\big)$.

As a result, relations (6.21) and (6.26) imply the required inequality (5.10) with the constant $\mathcal{C}_3 = \mathcal{C}_{20} + 2\mathcal{C}_{22}$. □

6.4. Proof of Theorem 5.1. We estimate the L_2-norm of the correction term $\mathbf{w}_\varepsilon$.

Lemma 6.3. *Suppose that $\mathbf{w}_\varepsilon \in H^p(\mathcal{O};\mathbb{C}^n)$ satisfies* (5.23) *and* (5.24)*. If ε_1 is subject to Condition* 4.4*, then for $0 < \varepsilon \le \varepsilon_1$ and $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$, $|\zeta| \ge 1$, we have*

$$\|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathcal{C}_{23}c(\varphi)^5\big(\varepsilon|\zeta|^{-1+1/2p} + \varepsilon^{2p}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{6.27}$$

The constant $\mathcal{C}_{23}$ depends only on d, p, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

Proof. In identity (6.5), we substitute a test function of the form $\boldsymbol{\eta} = \boldsymbol{\eta}_\varepsilon = (A_{D,\varepsilon} - \bar{\zeta}I)^{-1}\boldsymbol{\Phi}$, where $\boldsymbol{\Phi} \in L_2(\mathcal{O};\mathbb{C}^n)$. Then the left-hand side of (6.5) can be written as $(\mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon, \boldsymbol{\Phi})_{L_2(\mathcal{O})}$. Hence,

$$(\mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon, \boldsymbol{\Phi})_{L_2(\mathcal{O})} = -(g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta}_\varepsilon)_{L_2(\mathcal{O})} + \zeta(\boldsymbol{\phi}_\varepsilon, \boldsymbol{\eta}_\varepsilon)_{L_2(\mathcal{O})}. \tag{6.28}$$

To approximate the function $\boldsymbol{\eta}_\varepsilon$ in $H^p(\mathcal{O};\mathbb{C}^n)$, we apply the already proved Theorem 5.2. We put $\boldsymbol{\eta}_0 = (A_D^0 - \overline{\zeta} I)^{-1}\boldsymbol{\Phi}$ and $\widetilde{\boldsymbol{\eta}}_0 = P_{\mathcal{O}}\boldsymbol{\eta}_0$. An approximation for the function $\boldsymbol{\eta}_\varepsilon$ is given by $\boldsymbol{\eta}_0 + \varepsilon^p \Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0$. We rewrite (6.28) as

$$\begin{aligned}(6.29)\quad &(\mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon, \boldsymbol{\Phi})_{L_2(\mathcal{O})}\\ &= -\,(g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, b(\mathbf{D})(\boldsymbol{\eta}_\varepsilon - \boldsymbol{\eta}_0 - \varepsilon^p\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0))_{L_2(\mathcal{O})} - (g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta}_0)_{L_2(\mathcal{O})}\\ &\quad - (g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, b(\mathbf{D})(\varepsilon^p\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0))_{L_2(\mathcal{O})} + \zeta(\boldsymbol{\phi}_\varepsilon, \boldsymbol{\eta}_\varepsilon)_{L_2(\mathcal{O})}.\end{aligned}$$

Denote the consecutive terms on the right-hand side of (6.29) by $\mathcal{I}_j(\varepsilon,\zeta)$, $j=1,2,3,4$.

The term $\mathcal{I}_4(\varepsilon,\zeta)$ can easily be estimated by using Lemma 4.1 and (6.11):

$$(6.30)\qquad |\mathcal{I}_4(\varepsilon,\zeta)| \le |\zeta| \|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \|\boldsymbol{\eta}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathcal{C}_5 c(\varphi)^2 \varepsilon^p |\zeta|^{-1/2} \|\mathbf{F}\|_{L_2(\mathcal{O})} \|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}.$$

Now, we estimate $\mathcal{I}_1(\varepsilon,\zeta)$. By (1.3) and (1.5),

$$|\mathcal{I}_1(\varepsilon,\zeta)| \le \|g\|_{L_\infty} \kappa_{11}(d,p)\alpha_1 \|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \big\|\mathbf{D}^p(\boldsymbol{\eta}_\varepsilon - \boldsymbol{\eta}_0 - \varepsilon^p\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0)\big\|_{L_2(\mathcal{O})}.$$

Applying Theorem 5.2 (precisely, the analog of estimate (6.18) for $\boldsymbol{\eta}_\varepsilon$) and (6.12), we arrive at

$$|\mathcal{I}_1(\varepsilon,\zeta)| \le 2\|g\|_{L_\infty} \kappa_{11}(d,p)\alpha_1 \mathcal{C}_{13}\mathcal{C}_{18} c(\varphi)^5 \big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big)^2 \|\mathbf{F}\|_{L_2(\mathcal{O})} \|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}.$$

Consequently,

$$(6.31)\qquad |\mathcal{I}_1(\varepsilon,\zeta)| \le \gamma_1 c(\varphi)^5 \big(\varepsilon|\zeta|^{-1+1/2p} + \varepsilon^{2p}\big) \|\mathbf{F}\|_{L_2(\mathcal{O})} \|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})},$$

where $\gamma_1 = 4\|g\|_{L_\infty}\kappa_{11}(d,p)\alpha_1\mathcal{C}_{13}\mathcal{C}_{18}$.

To estimate $\mathcal{I}_2(\varepsilon,\zeta)$, we recall that the function $\boldsymbol{\phi}_\varepsilon$ is supported in the ε-neighborhood of the boundary and apply Lemma 4.5. By (1.3) and (1.5), we have

$$\begin{aligned}(6.32)\quad |\mathcal{I}_2(\varepsilon,\zeta)| &\le \|g\|_{L_\infty}\kappa_{11}(d,p)\alpha_1 \|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \|\mathbf{D}^p\boldsymbol{\eta}_0\|_{L_2(B_\varepsilon)}\\ &\le \|g\|_{L_\infty}\kappa_{11}(d,p)\alpha_1 \|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \beta_0^{1/2}\varepsilon^{1/2} \|\mathbf{D}^p\boldsymbol{\eta}_0\|^{1/2}_{H^1(\mathcal{O})} \|\mathbf{D}^p\boldsymbol{\eta}_0\|^{1/2}_{L_2(\mathcal{O})}.\end{aligned}$$

To estimate $\|\mathbf{D}^p\boldsymbol{\eta}_0\|_{L_2(\mathcal{O})}$, we use Lemma 4.3:

$$(6.33)\qquad \|\mathbf{D}^p\boldsymbol{\eta}_0\|_{L_2(\mathcal{O})} \le \mathcal{C}_0 c(\varphi)|\zeta|^{-1/2}\|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}.$$

Next, from (4.15) it follows that

$$\|\mathbf{D}^p\boldsymbol{\eta}_0\|_{H^p(\mathcal{O})} \le \widehat{c} c(\varphi)\|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}.$$

Since $H^1(\mathcal{O})$ coincides with the interpolational space $[L_2(\mathcal{O}), H^p(\mathcal{O})]_{1/p}$ and the corresponding norms are equivalent, we use interpolation to obtain

$$(6.34)\qquad \|\mathbf{D}^p\boldsymbol{\eta}_0\|_{H^1(\mathcal{O})} \le \mathcal{C}_{24} c(\varphi)|\zeta|^{-1/2+1/2p}\|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})},$$

where $\mathcal{C}_{24} = \check{C}(p;\mathcal{O})\mathcal{C}_0^{1-1/p}\widehat{c}^{1/p}$. Now, relations (6.32)–(6.34) and (6.12) imply that

$$\begin{aligned}|\mathcal{I}_2(\varepsilon,\zeta)| \le{}& \|g\|_{L_\infty}\kappa_{11}(d,p)\alpha_1\mathcal{C}_{13} c(\varphi)\big(\varepsilon^p + \varepsilon^{1/2}|\zeta|^{-1/2+1/4p}\big)\\ &\times \beta_0^{1/2}\varepsilon^{1/2}(\mathcal{C}_0\mathcal{C}_{24})^{1/2} c(\varphi)|\zeta|^{-1/2+1/4p}\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}.\end{aligned}$$

Hence,

$$(6.35)\qquad |\mathcal{I}_2(\varepsilon,\zeta)| \le \gamma_2 c(\varphi)^2 \big(\varepsilon|\zeta|^{-1+1/2p} + \varepsilon^{2p}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})},$$

where $\gamma_2 = 2\|g\|_{L_\infty}\kappa_{11}(d,p)\alpha_1\beta_0^{1/2}\mathcal{C}_{13}(\mathcal{C}_0\mathcal{C}_{24})^{1/2}$.

It remains to estimate the term $\mathcal{I}_3(\varepsilon,\zeta)$. By (1.3), we have

$$\mathcal{I}_3(\varepsilon,\zeta) = -\mathcal{I}_3^{(1)}(\varepsilon,\zeta) - \mathcal{I}_3^{(2)}(\varepsilon,\zeta), \tag{6.36}$$

$$\mathcal{I}_3^{(1)}(\varepsilon,\zeta) = \sum_{|\alpha|=p}\sum_{\beta\le\alpha:|\beta|\ge 1} C_\alpha^\beta \varepsilon^{|\beta|}\big(g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, b_\alpha(\mathbf{D}^{\alpha-\beta}\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D})\mathbf{D}^\beta\widetilde{\boldsymbol{\eta}}_0\big)_{L_2(\mathcal{O})}, \tag{6.37}$$

$$\mathcal{I}_3^{(2)}(\varepsilon,\zeta) = \big(g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, (b(\mathbf{D})\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0\big)_{L_2(\mathcal{O})}. \tag{6.38}$$

To estimate the term (6.37), we use (1.4), (1.5), Proposition 1.8, and (2.3):

$$|\mathcal{I}_3^{(1)}(\varepsilon,\zeta)| \le \kappa_{12}(d,p)\|g\|_{L_\infty}\alpha_1^{3/2}C_\Lambda\|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)}\left(\sum_{k=1}^p \varepsilon^k\|\widetilde{\boldsymbol{\eta}}_0\|_{H^{p+k}(\mathbb{R}^d)}\right).$$

Combined with (6.12) and an analog of estimate (5.14) for $\widetilde{\boldsymbol{\eta}}_0$, this implies

$$\begin{aligned}|\mathcal{I}_3^{(1)}(\varepsilon,\zeta)| &\le \kappa_{12}(d,p)\|g\|_{L_\infty}\alpha_1^{3/2}C_\Lambda\mathcal{C}_{13}c(\varphi)\big(\varepsilon^p+\varepsilon^{1/2}|\zeta|^{-1/2+1/4p}\big)\\ &\quad\times\left(\sum_{k=1}^p\varepsilon^k C^{(p+k)}c(\varphi)|\zeta|^{-1/2+k/2p}\right)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}.\end{aligned}$$

Then it is easy to deduce that

$$|\mathcal{I}_3^{(1)}(\varepsilon,\zeta)| \le \gamma_3^{(1)}c(\varphi)^2\big(\varepsilon|\zeta|^{-1+1/2p}+\varepsilon^{2p}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}, \tag{6.39}$$

where $\gamma_3^{(1)} = 2p\kappa_{12}(d,p)\|g\|_{L_\infty}\alpha_1^{3/2}C_\Lambda\mathcal{C}_{13}\max_{1\le k\le p}C^{(p+k)}$.

The term (6.38) is estimated with the help of (1.4), (1.7), Lemma 4.6, and (2.2):

$$\begin{aligned}|\mathcal{I}_3^{(2)}(\varepsilon,\zeta)| &\le \|g\|_{L_\infty}(\mathfrak{c}_p\alpha_1)^{1/2}\|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)}\left(\int_{(\partial\mathcal{O})_\varepsilon}|(b(\mathbf{D})\Lambda)^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0|^2\,d\mathbf{x}\right)^{1/2}\\ &\le \|g\|_{L_\infty}(\mathfrak{c}_p\alpha_1)^{1/2}\|\mathbf{D}^p\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathbb{R}^d)}\beta_*^{1/2}\varepsilon^{1/2}C_\Lambda^{(2)}\|b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0\|^{1/2}_{H^1(\mathbb{R}^d)}\|b(\mathbf{D})\widetilde{\boldsymbol{\eta}}_0\|^{1/2}_{L_2(\mathbb{R}^d)}.\end{aligned} \tag{6.40}$$

Like in (5.12) and (5.14), we have

$$\|\widetilde{\boldsymbol{\eta}}_0\|_{H^p(\mathbb{R}^d)} \le C^{(p)}c(\varphi)|\zeta|^{-1/2}\|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}, \tag{6.41}$$

$$\|\widetilde{\boldsymbol{\eta}}_0\|_{H^{p+1}(\mathbb{R}^d)} \le C^{(p+1)}c(\varphi)|\zeta|^{-1/2+1/2p}\|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}. \tag{6.42}$$

Relations (6.40)–(6.42), (1.4), and (6.12) yield

$$|\mathcal{I}_3^{(2)}(\varepsilon,\zeta)| \le \mathcal{C}_{25}c(\varphi)^2\big(\varepsilon^p+\varepsilon^{1/2}|\zeta|^{-1/2+1/4p}\big)\varepsilon^{1/2}|\zeta|^{-1/2+1/4p}\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})},$$

where $\mathcal{C}_{25} = \|g\|_{L_\infty}\mathfrak{c}_p^{1/2}\alpha_1\beta_*^{1/2}\mathcal{C}_{13}C_\Lambda^{(2)}(C^{(p)}C^{(p+1)})^{1/2}$. Hence,

$$|\mathcal{I}_3^{(2)}(\varepsilon,\zeta)| \le \gamma_3^{(2)}c(\varphi)^2\big(\varepsilon|\zeta|^{-1+1/2p}+\varepsilon^{2p}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}, \tag{6.43}$$

where $\gamma_3^{(2)} = 2\mathcal{C}_{25}$.

Finally, combining (6.29), (6.30), (6.31), (6.35), (6.36), (6.39), and (6.43), we see that

$$|(\mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon, \boldsymbol{\Phi})_{L_2(\mathcal{O})}| \le \gamma c(\varphi)^5\big(\varepsilon|\zeta|^{-1+1/2p}+\varepsilon^{2p}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}\|\boldsymbol{\Phi}\|_{L_2(\mathcal{O})}$$

for any $\boldsymbol{\Phi} \in L_2(\mathcal{O};\mathbb{C}^n)$, where $\gamma = \mathcal{C}_5+\gamma_1+\gamma_2+\gamma_3^{(1)}+\gamma_3^{(2)}$. Hence,

$$\|\mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \le \gamma c(\varphi)^5\big(\varepsilon|\zeta|^{-1+1/2p}+\varepsilon^{2p}\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{6.44}$$

Now, relations (6.44) and (6.11) directly imply the required estimate (6.27) with the constant $\mathcal{C}_{23} = \gamma + \mathcal{C}_5$. □

Completion of the proof of Theorem 5.1. By (5.37) and (6.27),

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le \widetilde{\mathcal{C}}_7 \big(c(\varphi)^4 \varepsilon |\zeta|^{-1+1/2p} + c(\varphi)^3 \varepsilon^p |\zeta|^{-1/2}\big) \|\mathbf{F}\|_{L_2(\mathcal{O})} + \mathcal{C}_{23} c(\varphi)^5 \big(\varepsilon |\zeta|^{-1+1/2p} + \varepsilon^{2p}\big) \|\mathbf{F}\|_{L_2(\mathcal{O})}$$

for $0 < \varepsilon \le \varepsilon_1$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \ge 1$. This implies estimate (5.1) with the constant $\mathcal{C}_1 = 2\widetilde{\mathcal{C}}_7 + \mathcal{C}_{23}$. □

§7. Removal of the smoothing operator. Special cases

7.1. Removal of the smoothing operator. Under Condition 3.6, instead of the corrector (5.4) one can use the standard corrector

$$K_D^0(\zeta;\varepsilon) := \Lambda^\varepsilon b(\mathbf{D})(A_D^0 - \zeta I)^{-1}, \tag{7.1}$$

which in this case is a continuous mapping of $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^p(\mathcal{O};\mathbb{C}^n)$. Accordingly, instead of the function (5.7), one can take the function

$$\mathbf{v}_\varepsilon^0 := (A_D^0 - \zeta I)^{-1}\mathbf{F} + \varepsilon^p K_D^0(\zeta;\varepsilon)\mathbf{F} = \mathbf{u}_0 + \varepsilon^p \Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0 \tag{7.2}$$

as an approximation to the solution of problem (4.4).

Theorem 7.1. *Suppose that the assumptions of Theorem* 5.1 *and Condition* 3.6 *are satisfied. Let* $K_D^0(\zeta;\varepsilon)$ *be the operator* (7.1), *and let* $\mathbf{v}_\varepsilon^0$ *be the function* (7.2). *Then for* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon^0\|_{H^p(\mathcal{O})} \le \widetilde{\mathcal{C}}_2 c(\varphi)^4 \big(\varepsilon^{1/2} |\zeta|^{-1/2+1/4p} + \varepsilon^p\big) \|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{7.3}$$

or, in operator terms,

$$\big\|(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1} - \varepsilon^p K_D^0(\zeta;\varepsilon)\big\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})} \le \widetilde{\mathcal{C}}_2 c(\varphi)^4 \big(\varepsilon^{1/2} |\zeta|^{-1/2+1/4p} + \varepsilon^p\big). \tag{7.4}$$

For $0 < \varepsilon \le \varepsilon_1$, *the flux* $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ *satisfies*

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \widetilde{\mathcal{C}}_3 c(\varphi)^4 \big(\varepsilon^{1/2} |\zeta|^{-1/2+1/4p} + \varepsilon^p\big) \|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{7.5}$$

The constants $\widetilde{\mathcal{C}}_2$ *and* $\widetilde{\mathcal{C}}_3$ *depend only on* m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *the domain* $\mathcal{O}$, *and also on* $\|\Lambda\|_{L_\infty}$ *and* M_Λ.

To prove Theorem 7.1, we need the following lemma.

Lemma 7.2.

1°. *Suppose that* Λ *is a multiplier from* $H^p(\mathbb{R}^d;\mathbb{C}^m)$ *to* $H^p(\mathbb{R}^d;\mathbb{C}^n)$ *and* M_Λ *is the norm of this multiplier. For any* $\mathbf{u} \in H^p(\mathbb{R}^d;\mathbb{C}^m)$ *and* $\varepsilon > 0$, *we have*

$$\varepsilon^{2p} \int_{\mathbb{R}^d} |\mathbf{D}^p(\Lambda^\varepsilon(\mathbf{x})\mathbf{u}(\mathbf{x}))|^2 \, d\mathbf{x} \le \check{\mathfrak{c}}_p M_\Lambda^2 \int_{\mathbb{R}^d} \big(|\mathbf{u}(\mathbf{x})|^2 + \varepsilon^{2p} |\mathbf{D}^p \mathbf{u}(\mathbf{x})|^2\big) \, d\mathbf{x}. \tag{7.6}$$

2°. *Suppose that Condition* 3.6 *is fulfilled. Then the matrix-valued function* (1.12) *is a multiplier from* $H^p(\mathbb{R}^d;\mathbb{C}^m)$ *to* $L_2(\mathbb{R}^d;\mathbb{C}^m)$, *and the norm of this multiplier does not exceed a constant* $M_{\widetilde{g}}$ *depending only on* d, p, $\|g\|_{L_\infty}$, α_1, $\|\Lambda\|_{L_\infty}$, *and* M_Λ. *Moreover, for any* $\mathbf{u} \in H^p(\mathbb{R}^d;\mathbb{C}^m)$ *and* $\varepsilon > 0$ *we have*

$$\int_{\mathbb{R}^d} |\widetilde{g}^\varepsilon(\mathbf{x})\mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x} \le \check{\mathfrak{c}}_p M_{\widetilde{g}}^2 \int_{\mathbb{R}^d} \big(|\mathbf{u}(\mathbf{x})|^2 + \varepsilon^{2p} |\mathbf{D}^p \mathbf{u}(\mathbf{x})|^2\big) \, d\mathbf{x}. \tag{7.7}$$

Proof. Let $\mathbf{u} \in H^p(\mathbb{R}^d;\mathbb{C}^m)$. The change of variables $\mathbf{x} = \varepsilon\mathbf{y}$, $\mathbf{u}(\mathbf{x}) = \mathbf{v}(\mathbf{y})$ yields

$$\varepsilon^{2p}\int_{\mathbb{R}^d} |\mathbf{D}^p_{\mathbf{x}}(\Lambda^\varepsilon(\mathbf{x})\mathbf{u}(\mathbf{x}))|^2\,d\mathbf{x} = \varepsilon^d\int_{\mathbb{R}^d} |\mathbf{D}^p_{\mathbf{y}}(\Lambda(\mathbf{y})\mathbf{v}(\mathbf{y}))|^2\,d\mathbf{y} \le M_\Lambda^2\varepsilon^d\|\mathbf{v}\|^2_{H^p(\mathbb{R}^d)}. \tag{7.8}$$

By (3.9),

$$\varepsilon^d\|\mathbf{v}\|^2_{H^p(\mathbb{R}^d)} \le \check{\mathfrak{c}}_p\varepsilon^d\int_{\mathbb{R}^d}\left(|\mathbf{v}(\mathbf{y})|^2 + |\mathbf{D}^p_{\mathbf{y}}\mathbf{v}(\mathbf{y})|^2\right)d\mathbf{y} = \check{\mathfrak{c}}_p\int_{\mathbb{R}^d}\left(|\mathbf{u}(\mathbf{x})|^2 + \varepsilon^{2p}|\mathbf{D}^p_{\mathbf{x}}\mathbf{u}(\mathbf{x})|^2\right)d\mathbf{x}.$$

Together with (7.8), this implies (7.6).

Now, we prove assertion 2°. By Lemma 1 in Subsection 1.3.2 of the book [MSh], Condition 3.6 implies that $\mathbf{D}^\alpha\Lambda$ with $|\alpha| = p$ is a multiplier from $H^p(\mathbb{R}^d;\mathbb{C}^m)$ to $L_2(\mathbb{R}^d;\mathbb{C}^n)$, and the norm of this multiplier is controlled in terms of $\|\Lambda\|_{L_\infty}$ and M_Λ. Then, by (1.3) and (1.5), the matrix-valued function $\widetilde{g} = g(b(\mathbf{D})\Lambda + \mathbf{1}_m)$ is a multiplier from $H^p(\mathbb{R}^d;\mathbb{C}^m)$ to $L_2(\mathbb{R}^d;\mathbb{C}^m)$, and the norm of this multiplier does not exceed a constant $M_{\widetilde{g}}$ depending only on d, p, $\|g\|_{L_\infty}$, α_1, $\|\Lambda\|_{L_\infty}$, and M_Λ. Inequality (7.7) is proved by the changes $\mathbf{x} = \varepsilon\mathbf{y}$, $\mathbf{u}(\mathbf{x}) = \mathbf{v}(\mathbf{y})$ (as in the proof of estimate (7.6)). □

Proof of Theorem 7.1. Let the functions $\mathbf{v}_\varepsilon$ and $\mathbf{v}^0_\varepsilon$ be defined by (5.7) and (7.2), respectively. We estimate their difference in the $H^p(\mathcal{O};\mathbb{C}^n)$-norm. By (3.9), we have

$$\begin{aligned}\|\mathbf{v}_\varepsilon - \mathbf{v}^0_\varepsilon\|^2_{H^p(\mathcal{O})} &\le \varepsilon^{2p}\|\Lambda^\varepsilon(I - S_\varepsilon)b(\mathbf{D})\widetilde{\mathbf{u}}_0\|^2_{H^p(\mathbb{R}^d)}\\ &\le \check{\mathfrak{c}}_p\varepsilon^{2p}\left(\left\|\Lambda^\varepsilon(I - S_\varepsilon)b(\mathbf{D})\widetilde{\mathbf{u}}_0\right\|^2_{L_2(\mathbb{R}^d)} + \left\|\mathbf{D}^p(\Lambda^\varepsilon(I - S_\varepsilon)b(\mathbf{D})\widetilde{\mathbf{u}}_0)\right\|^2_{L_2(\mathbb{R}^d)}\right).\end{aligned}\tag{7.9}$$

Combining Condition 3.6, inequality $\|S_\varepsilon\|_{L_2\to L_2} \le 1$, and (1.4), we obtain

$$\left\|\Lambda^\varepsilon(I - S_\varepsilon)b(\mathbf{D})\widetilde{\mathbf{u}}_0\right\|_{L_2(\mathbb{R}^d)} \le 2\|\Lambda\|_{L_\infty}\alpha_1^{1/2}\|\widetilde{\mathbf{u}}_0\|_{H^p(\mathbb{R}^d)}. \tag{7.10}$$

Next, Lemma 7.2 implies that

$$\begin{aligned}&\varepsilon^{2p}\left\|\mathbf{D}^p(\Lambda^\varepsilon(I - S_\varepsilon)b(\mathbf{D})\widetilde{\mathbf{u}}_0)\right\|^2_{L_2(\mathbb{R}^d)}\\ &\qquad\le \check{\mathfrak{c}}_p M_\Lambda^2\left(\|(I - S_\varepsilon)b(\mathbf{D})\widetilde{\mathbf{u}}_0\|^2_{L_2(\mathbb{R}^d)} + \varepsilon^{2p}\|(I - S_\varepsilon)\mathbf{D}^p b(\mathbf{D})\widetilde{\mathbf{u}}_0\|^2_{L_2(\mathbb{R}^d)}\right).\end{aligned}\tag{7.11}$$

By Proposition 1.7 and (1.4),

$$\|(I - S_\varepsilon)b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le \varepsilon r_1\alpha_1^{1/2}\|\widetilde{\mathbf{u}}_0\|_{H^{p+1}(\mathbb{R}^d)}. \tag{7.12}$$

From the inequality $\|S_\varepsilon\|_{L_2\to L_2} \le 1$ and (1.4) it follows that

$$\|(I - S_\varepsilon)\mathbf{D}^p b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \le 2\alpha_1^{1/2}\|\widetilde{\mathbf{u}}_0\|_{H^{2p}(\mathbb{R}^d)}. \tag{7.13}$$

Combining (7.9)–(7.13) and (5.12)–(5.14), we arrive at

$$\|\mathbf{v}_\varepsilon - \mathbf{v}^0_\varepsilon\|_{H^p(\mathcal{O})} \le \mathcal{C}_{26}c(\varphi)\left(\varepsilon|\zeta|^{-1/2+1/2p} + \varepsilon^p\right)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{7.14}$$

where $\mathcal{C}_{26} = \alpha_1^{1/2}\max\{\check{\mathfrak{c}}_p M_\Lambda r_1 C^{(p+1)}, 2\check{\mathfrak{c}}_p^{1/2}\|\Lambda\|_{L_\infty}C^{(p)} + 2\check{\mathfrak{c}}_p M_\Lambda C^{(2p)}\}$.

Now, (5.8) and (7.14) imply the required estimate (7.3) with the constant $\widetilde{\mathcal{C}}_2 = \mathcal{C}_2 + 2\mathcal{C}_{26}$.

We proceed to the proof of inequality (7.5). By (7.7),

$$\begin{aligned}\left\|\widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0 - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\right\|^2_{L_2(\mathcal{O})} &\le \left\|\widetilde{g}^\varepsilon(I - S_\varepsilon)b(\mathbf{D})\widetilde{\mathbf{u}}_0\right\|^2_{L_2(\mathbb{R}^d)}\\ &\le \check{\mathfrak{c}}_p M_{\widetilde{g}}^2\left(\|(I - S_\varepsilon)b(\mathbf{D})\widetilde{\mathbf{u}}_0\|^2_{L_2(\mathbb{R}^d)} + \varepsilon^{2p}\|(I - S_\varepsilon)\mathbf{D}^p b(\mathbf{D})\widetilde{\mathbf{u}}_0\|^2_{L_2(\mathbb{R}^d)}\right).\end{aligned}\tag{7.15}$$

Together with (7.12), (7.13), (5.13), and (5.14), this yields

$$\|\widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0 - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \le \mathcal{C}_{27}c(\varphi)\left(\varepsilon|\zeta|^{-1/2+1/2p} + \varepsilon^p\right)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{7.16}$$

where $\mathcal{C}_{27} = \check{\mathfrak{c}}_p^{1/2}M_{\widetilde{g}}\alpha_1^{1/2}\max\{r_1C^{(p+1)}, 2C^{(2p)}\}$.

Now, (5.10) and (7.16) imply the required inequality (7.5) with the constant $\widetilde{\mathcal{C}}_3 = \mathcal{C}_3 + 2\mathcal{C}_{27}$. $\square$

Comparing Theorem 7.1 and Proposition 3.7, we arrive at the following statement.

Corollary 7.3. *Under the assumptions of Theorem* 5.1, *let* $K_D^0(\zeta;\varepsilon)$ *be the operator* (7.1), *and let* $\mathbf{v}_\varepsilon^0$ *be given by* (7.2). *Let* $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$, *and let* $\widetilde{g}(\mathbf{x})$ *be the matrix-valued function* (1.12). *Suppose that at least one of the following conditions is fulfilled*:

1°. $2p > d$;

2°. $g^0 = \underline{g}$ (*i.e., the representations* (1.18) *are valid*).

Then for $0 < \varepsilon \le \varepsilon_1$ *we have estimates* (7.3)–(7.5), *and the constants* $\widetilde{\mathcal{C}}_2$ *and* $\widetilde{\mathcal{C}}_3$ *depend only on* m, n, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

Remark 7.4.

1) For fixed $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \ge 1$, the estimates of Theorem 7.1 are of order $O(\varepsilon^{1/2})$. The error becomes smaller as $|\zeta|$ grows.

2) The estimates of Theorem 7.1 are uniform with respect to the angle φ in a domain $\{\zeta = |\zeta|e^{i\varphi} : |\zeta| \ge 1, \varphi_0 \le \varphi \le 2\pi - \varphi_0\}$ with arbitrarily small $\varphi_0 > 0$.

3) The assumptions of Corollary 7.3 are valid in the following cases, which are of interest for applications: a) if $p = 2$ and $d = 2$ or $d = 3$, we have $2p > d$; b) if $m = n$, then $g^0 = \underline{g}$. For instance, this condition is fulfilled for the operator $A_\varepsilon = \Delta g^\varepsilon(\mathbf{x})\Delta$ in $L_2(\mathbb{R}^d)$ for arbitrary dimension.

7.2. Special cases. If $g^0 = \overline{g}$ (i.e., relations (1.17) are valid), then the Γ-periodic solution of problem (1.10) is equal to zero: $\Lambda(\mathbf{x}) = 0$. In this case, we have $\mathbf{v}_\varepsilon = \mathbf{u}_0$ and $\mathbf{w}_\varepsilon = 0$. Lemma 5.4 together with (3.9) implies the following result.

Proposition 7.5. *Under the assumptions of Theorem* 5.1, *if* $g^0 = \overline{g}$ (*i.e., relations* (1.17) *are valid*), *then for* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^p(\mathcal{O})} \le \widetilde{\mathcal{C}}_6\big(c(\varphi)^4\varepsilon|\zeta|^{-1/2+1/2p} + c(\varphi)^3\varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})},$$

where $\widetilde{\mathcal{C}}_6 = \check{\mathfrak{c}}_p^{1/2}(\mathcal{C}_6 + \mathcal{C}_7)$.

By Remark 1.4, if $g^0 = \underline{g}$, then the matrix-valued function (1.12) is constant: $\widetilde{g}(\mathbf{x}) = g^0 = \underline{g}$. Applying the statement of Corollary 7.3 concerning fluxes, we arrive at the following result.

Proposition 7.6. *Under the assumptions of Theorem* 5.1, *if* $g^0 = \underline{g}$ (*i.e., the representations* (1.18) *are valid*), *then for* $0 < \varepsilon \le \varepsilon_1$ *the flux* $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ *satisfies*

$$\|\mathbf{p}_\varepsilon - g^0 b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \widetilde{\mathcal{C}}_3 c(\varphi)^4\big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

§8. Approximation of the resolvent $(A_{D,\varepsilon} - \zeta I)^{-1}$ for $\zeta \in \mathbb{C} \setminus [c_*, \infty)$

8.1. The general case. In the theorems of §5 and §7, it was assumed that $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$ and $|\zeta| \ge 1$. In the present section, we obtain a result on approximation for the resolvent $(A_{D,\varepsilon} - \zeta I)^{-1}$ valid in a wider set of the parameter ζ. For bounded values of $|\zeta|$ and for points ζ with small φ or $2\pi - \varphi$, this result may be preferable.

Theorem 8.1. *Let* $\mathcal{O}$ *be a bounded domain of class* C^{2p}. *Let* $\zeta \in \mathbb{C} \setminus [c_*, \infty)$, *where* $c_* > 0$ *is a common lower bound of the operators* $A_{D,\varepsilon}$ *and* A_D^0. *Let* $\zeta - c_* = |\zeta - c_*|e^{i\psi}$. *Denote*

$$\rho_*(\zeta) = \begin{cases} c(\psi)^2|\zeta - c_*|^{-2}, & |\zeta - c_*| < 1, \\ c(\psi)^2, & |\zeta - c_*| \ge 1, \end{cases}$$

where $c(\psi)$ is given by (3.1)*. Let $\mathbf{u}_\varepsilon$ be the solution of problem* (4.4)*, and let $\mathbf{u}_0$ be the solution of problem* (4.11) *with $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. Suppose that the operator $K_D(\zeta;\varepsilon)$ is given by* (5.4)*, and the function $\mathbf{v}_\varepsilon$ is defined by* (5.7)*. Let ε_1 be subject to Condition* 4.4*. Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\begin{aligned} &\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le \mathfrak{C}_1 \varepsilon \rho_*(\zeta) \|\mathbf{F}\|_{L_2(\mathcal{O})}, \\ &\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon\|_{H^p(\mathcal{O})} \le \mathfrak{C}_2 \varepsilon^{1/2} \rho_*(\zeta) \|\mathbf{F}\|_{L_2(\mathcal{O})}. \end{aligned} \tag{8.1}$$

In operator terms,

$$\left\|(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1}\right\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le \mathfrak{C}_1 \varepsilon \rho_*(\zeta), \tag{8.2}$$

$$\left\|(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1} - \varepsilon^p K_D(\zeta;\varepsilon)\right\|_{L_2(\mathcal{O}) \to H^p(\mathcal{O})} \le \mathfrak{C}_2 \varepsilon^{1/2} \rho_*(\zeta). \tag{8.3}$$

For $0 < \varepsilon \le \varepsilon_1$ the flux $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ satisfies

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D}) \widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \le \mathfrak{C}_3 \varepsilon^{1/2} \rho_*(\zeta) \|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{8.4}$$

where $\widetilde{g}(\mathbf{x})$ is the matrix-valued function (1.12)*. The constants $\mathfrak{C}_1$, $\mathfrak{C}_2$, and $\mathfrak{C}_3$ depend only on m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.*

Remark 8.2.

1) The quantity $c(\psi)^2|\zeta - c_*|^{-2}$ is inverse to the square of the distance from the point ζ to $[c_*, \infty)$.

2) One may take $c_* = c_2$, where c_2 is defined by (4.3).

3) Let $\nu > 0$ be an arbitrarily small number. If ε is sufficiently small, then one may take $c_* = \lambda_1^0 - \nu$, where λ_1^0 is the first eigenvalue of the operator A_D^0.

4) It is easy to give an upper bound for c_*: from (4.2) it is seen that $c_* \le c_1 \mu_1^0$, where μ_1^0 is the first eigenvalue of the operator $B_p = \sum_{|\alpha|=p} \mathbf{D}^{2\alpha}$ with the Dirichlet conditions. Therefore, c_* does not exceed a number depending only on d, p, $\|g\|_{L_\infty}$, α_1, and the domain $\mathcal{O}$.

Proof. We apply Theorem 5.1 with $\zeta = -1$. By (5.2),

$$\left\|(A_{D,\varepsilon} + I)^{-1} - (A_D^0 + I)^{-1}\right\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le 2\mathcal{C}_1 \varepsilon, \quad 0 < \varepsilon \le \varepsilon_1. \tag{8.5}$$

We use the identity

$$\begin{aligned} &(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1} \\ &\quad = (A_{D,\varepsilon} + I)(A_{D,\varepsilon} - \zeta I)^{-1}\big((A_{D,\varepsilon} + I)^{-1} - (A_D^0 + I)^{-1}\big)(A_D^0 + I)(A_D^0 - \zeta I)^{-1}. \end{aligned} \tag{8.6}$$

From (8.5) and (8.6) it follows that

$$\left\|(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1}\right\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le 2\mathcal{C}_1 \varepsilon \sup_{x \ge c_*} (x+1)^2 |x - \zeta|^{-2} \tag{8.7}$$

for $0 < \varepsilon \le \varepsilon_1$. A calculation shows that

$$\sup_{x \ge c_*} (x+1)^2 |x - \zeta|^{-2} \le \check{c} \rho_*(\zeta), \tag{8.8}$$

where $\check{c} = (c_* + 2)^2$. By Remark 8.2(4), $\check{c}$ does not exceed a number depending only on d, p, α_1, $\|g\|_{L_\infty}$, and the domain $\mathcal{O}$. Relations (8.7) and (8.8) imply the required estimate (8.2) with the constant $\mathfrak{C}_1 = 2\mathcal{C}_1 \check{c}$.

Now, we apply Theorem 5.2 with $\zeta = -1$. By (5.9), for $0 < \varepsilon \le \varepsilon_1$ we have

$$\left\|(A_{D,\varepsilon} + I)^{-1} - (A_D^0 + I)^{-1} - \varepsilon^p K_D(-1;\varepsilon)\right\|_{L_2(\mathcal{O}) \to H^p(\mathcal{O})} \le 2\mathcal{C}_2 \varepsilon^{1/2}. \tag{8.9}$$

Using Lemma 6.2 with $\zeta = -1$ and (6.20), we see that

$$\|\varepsilon^p \theta_\varepsilon K_D(-1;\varepsilon)\|_{L_2(\mathcal{O}) \to H^p(\mathcal{O})} \le \mathfrak{C}_4 \varepsilon^{1/2} \tag{8.10}$$

for $0<\varepsilon\le\varepsilon_1$, where $\mathfrak{C}_4 = C(p;\mathcal{O})(\mathcal{C}_5+2\mathcal{C}_{13})$. From (8.9) and (8.10) it follows that

$$\big\|(A_{D,\varepsilon}+I)^{-1}-(A_D^0+I)^{-1}-\varepsilon^p(1-\theta_\varepsilon)K_D(-1;\varepsilon)\big\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})}\le\mathfrak{C}_5\varepsilon^{1/2} \tag{8.11}$$

for $0<\varepsilon\le\varepsilon_1$, where $\mathfrak{C}_5 = 2\mathcal{C}_2+\mathfrak{C}_4$. We use the identity

$$\begin{aligned}&(A_{D,\varepsilon}-\zeta I)^{-1}-(A_D^0-\zeta I)^{-1}-\varepsilon^p(1-\theta_\varepsilon)K_D(\zeta;\varepsilon)\\ &\quad=(A_{D,\varepsilon}+I)(A_{D,\varepsilon}-\zeta I)^{-1}\big((A_{D,\varepsilon}+I)^{-1}-(A_D^0+I)^{-1}-\varepsilon^p(1-\theta_\varepsilon)K_D(-1;\varepsilon)\big)\\ &\qquad\times(A_D^0+I)(A_D^0-\zeta I)^{-1}+\varepsilon^p(\zeta+1)(A_{D,\varepsilon}-\zeta I)^{-1}(1-\theta_\varepsilon)K_D(\zeta;\varepsilon).\end{aligned} \tag{8.12}$$

Since the range of the operators in (8.12) is contained in $H_0^p(\mathcal{O};\mathbb{C}^n)$, we can multiply by $A_{D,\varepsilon}^{1/2}$ from the left. Taking (8.8) into account, we obtain

$$\begin{aligned}&\big\|A_{D,\varepsilon}^{1/2}\big((A_{D,\varepsilon}-\zeta I)^{-1}-(A_D^0-\zeta I)^{-1}-\varepsilon^p(1-\theta_\varepsilon)K_D(\zeta;\varepsilon)\big)\big\|_{L_2\to L_2}\\ &\quad\le\check{c}\rho_*(\zeta)\big\|A_{D,\varepsilon}^{1/2}\big((A_{D,\varepsilon}+I)^{-1}-(A_D^0+I)^{-1}-\varepsilon^p(1-\theta_\varepsilon)K_D(-1;\varepsilon)\big)\big\|_{L_2\to L_2}\\ &\qquad+\varepsilon^p|\zeta+1|\sup_{x\ge c_*}x^{1/2}|x-\zeta|^{-1}\|(1-\theta_\varepsilon)K_D(\zeta;\varepsilon)\|_{L_2\to L_2}.\end{aligned} \tag{8.13}$$

Denote the summands on the right-hand side of (8.13) by $\mathcal{L}_1(\zeta;\varepsilon)$ and $\mathcal{L}_2(\zeta;\varepsilon)$. Relations (4.2) and (8.11) imply the following estimate for the first term:

$$\mathcal{L}_1(\zeta;\varepsilon)\le\check{c}\mathfrak{C}_5c_1^{1/2}\varepsilon^{1/2}\rho_*(\zeta),\quad 0<\varepsilon\le\varepsilon_1. \tag{8.14}$$

Since $K_D(\zeta;\varepsilon)=R_{\mathcal{O}}[\Lambda^\varepsilon]S_\varepsilon b(\mathbf{D})P_{\mathcal{O}}(A_D^0)^{-1/2}(A_D^0)^{1/2}(A_D^0-\zeta I)^{-1}$, we can use (1.4), (5.3), (5.20), and (6.1) to show that

$$\begin{aligned}&\|(1-\theta_\varepsilon)K_D(\zeta;\varepsilon)\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})}\\ &\quad\le C_\Lambda^{(1)}\alpha_1^{1/2}C_{\mathcal{O}}^{(p)}\|(A_D^0)^{-1/2}\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})}\sup_{x\ge c_*}x^{1/2}|x-\zeta|^{-1}.\end{aligned} \tag{8.15}$$

Combining (3.9) with analogs of estimates (4.2) and (4.3) for the operator A_D^0, we get

$$\|(A_D^0)^{-1/2}\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})}\le\check{\mathfrak{c}}_p^{1/2}(c_0^{-1/2}+c_2^{-1/2}). \tag{8.16}$$

By (8.15) and (8.16), the second term on the right-hand side of (8.13) admits the estimate

$$\mathcal{L}_2(\zeta;\varepsilon)\le\mathfrak{C}_6\varepsilon^p|\zeta+1|\sup_{x\ge c_*}x|x-\zeta|^{-2}, \tag{8.17}$$

where $\mathfrak{C}_6=C_\Lambda^{(1)}\alpha_1^{1/2}C_{\mathcal{O}}^{(p)}\check{\mathfrak{c}}_p^{1/2}(c_0^{-1/2}+c_2^{-1/2})$. In accordance with [Su5, (8.17)],

$$|\zeta+1|\sup_{x\ge c_*}x|x-\zeta|^{-2}\le(c_*+2)(c_*+1)\rho_*(\zeta). \tag{8.18}$$

Now, from (8.17) and (8.18) it follows that

$$\mathcal{L}_2(\zeta;\varepsilon)\le\mathfrak{C}_6(c_*+2)(c_*+1)\varepsilon^p\rho_*(\zeta). \tag{8.19}$$

As a result, inequalities (8.13), (8.14), and (8.19) imply

$$\big\|A_{D,\varepsilon}^{1/2}\big((A_{D,\varepsilon}-\zeta I)^{-1}-(A_D^0-\zeta I)^{-1}-\varepsilon^p(1-\theta_\varepsilon)K_D(\zeta;\varepsilon)\big)\big\|_{L_2\to L_2}\le\mathfrak{C}_7\varepsilon^{1/2}\rho_*(\zeta),\quad 0<\varepsilon\le\varepsilon_1,$$

where $\mathfrak{C}_7=\check{c}\mathfrak{C}_5c_1^{1/2}+\mathfrak{C}_6(c_*+2)(c_*+1)$. Combining this with (4.2), (4.3), and (3.9), we obtain

$$\big\|(A_{D,\varepsilon}-\zeta I)^{-1}-(A_D^0-\zeta I)^{-1}-\varepsilon^p(1-\theta_\varepsilon)K_D(\zeta;\varepsilon)\big\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})}\le\mathfrak{C}_8\varepsilon^{1/2}\rho_*(\zeta),\quad 0<\varepsilon\le\varepsilon_1, \tag{8.20}$$

where $\mathfrak{C}_8 = \check{\mathfrak{c}}_p^{1/2}(c_0^{-1/2} + c_2^{-1/2})\mathfrak{C}_7$. Finally, by (8.8) and (8.10),

$$\begin{aligned}
&\|\varepsilon^p \theta_\varepsilon K_D(\zeta;\varepsilon)\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})} \\
&\quad \le \|\varepsilon^p \theta_\varepsilon K_D(-1;\varepsilon)\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})} \big\|(A_D^0 + I)(A_D^0 - \zeta I)^{-1}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \\
&\quad \le \check{c}^{1/2}\mathfrak{C}_4 \varepsilon^{1/2}\rho_*(\zeta)^{1/2}, \quad 0<\varepsilon\le\varepsilon_1.
\end{aligned}$$

Together with (8.20), this yields the required inequality (8.3) with the constant $\mathfrak{C}_2 = \mathfrak{C}_8 + \check{c}^{1/2}\mathfrak{C}_4$.

It remains to check (8.4). From (8.1), (1.3), and (1.5) it follows that

$$\|\mathbf{p}_\varepsilon - g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon\|_{L_2(\mathcal{O})} \le \kappa_9(d,p)\|g\|_{L_\infty}\alpha_1^{1/2}\mathfrak{C}_2\varepsilon^{1/2}\rho_*(\zeta)\|\mathbf{F}\|_{L_2(\mathcal{O})} \tag{8.21}$$

for $0<\varepsilon\le\varepsilon_1$. Next, like in (6.22)–(6.24), we have

$$\begin{aligned}
&\big\|g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\big\|_{L_2(\mathcal{O})} \\
&\quad \le \varepsilon\|g\|_{L_\infty} r_1 \|\mathbf{D}b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} + \mathcal{C}_{21}\sum_{l=1}^{p}\varepsilon^l\|\mathbf{D}^l b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathbb{R}^d)} \\
&\quad \le \mathfrak{C}_9\varepsilon\|\widetilde{\mathbf{u}}_0\|_{H^{2p}(\mathbb{R}^d)},
\end{aligned} \tag{8.22}$$

where $\mathfrak{C}_9 = \|g\|_{L_\infty} r_1\alpha_1^{1/2} + p\,\mathcal{C}_{21}\alpha_1^{1/2}$.

From (4.10) and (8.8) it follows that

$$\|(A_D^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to H^{2p}(\mathcal{O})} \le \widehat{c}\sup_{x\ge c_*} x|x-\zeta|^{-1} \le \check{c}^{1/2}\widehat{c}\rho_*(\zeta)^{1/2}.$$

Hence, by (5.3),

$$\|\widetilde{\mathbf{u}}_0\|_{H^{2p}(\mathbb{R}^d)} \le C_{\mathcal{O}}^{(2p)}\|\mathbf{u}_0\|_{H^{2p}(\mathcal{O})} \le C_{\mathcal{O}}^{(2p)}\check{c}^{1/2}\widehat{c}\rho_*(\zeta)^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{8.23}$$

Combining this with (8.22), we obtain

$$\|g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \le \mathfrak{C}_{10}\varepsilon\rho_*(\zeta)^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})},$$

where $\mathfrak{C}_{10} = \mathfrak{C}_9 C_{\mathcal{O}}^{(2p)}\check{c}^{1/2}\widehat{c}$. Together with (8.21), this yields (8.4) with the constant $\mathfrak{C}_3 = \kappa_9(d,p)\|g\|_{L_\infty}\alpha_1^{1/2}\mathfrak{C}_2 + \mathfrak{C}_{10}$. ☐

8.2. Removal of the smoothing operator.

Theorem 8.3. *Under Condition 3.6 and the assumptions of Theorem 8.1, suppose that the operator $K_D^0(\zeta;\varepsilon)$ is given by (7.1), and the function $\mathbf{v}_\varepsilon^0$ is given by (7.2). Then for $0<\varepsilon\le\varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon^0\|_{H^p(\mathcal{O})} \le \widetilde{\mathfrak{C}}_2\varepsilon^{1/2}\rho_*(\zeta)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{8.24}$$

or, in operator terms,

$$\big\|(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1} - \varepsilon^p K_D^0(\zeta;\varepsilon)\big\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})} \le \widetilde{\mathfrak{C}}_2\varepsilon^{1/2}\rho_*(\zeta). \tag{8.25}$$

For $0<\varepsilon\le\varepsilon_1$ the flux $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ satisfies

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \widetilde{\mathfrak{C}}_3\varepsilon^{1/2}\rho_*(\zeta)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{8.26}$$

The constants $\widetilde{\mathfrak{C}}_2$ and $\widetilde{\mathfrak{C}}_3$ depend only on m, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, the domain $\mathcal{O}$, and also on $\|\Lambda\|_{L_\infty}$ and M_Λ.

Proof. As in (7.9)–(7.13), we have

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_\varepsilon^0\|_{H^p(\mathcal{O})} \le \mathfrak{C}_{11}\varepsilon\|\widetilde{\mathbf{u}}_0\|_{H^{2p}(\mathbb{R}^d)}, \tag{8.27}$$

where $\mathfrak{C}_{11} = \alpha_1^{1/2}(2\check{\mathfrak{c}}_p^{1/2}\|\Lambda\|_{L_\infty} + \check{\mathfrak{c}}_p M_\Lambda(r_1+2))$. From (8.23) and (8.27) it follows that

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_\varepsilon^0\|_{H^p(\mathcal{O})} \le \mathfrak{C}_{12}\varepsilon\rho_*(\zeta)^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{8.28}$$

where $\mathfrak{C}_{12} = C_{\mathcal{O}}^{(2p)}\check{c}^{1/2}\widehat{c}\,\mathfrak{C}_{11}$.

Inequalities (8.1) and (8.28) imply the required estimate (8.24) with the constant $\widetilde{\mathfrak{C}}_2 = \mathfrak{C}_2 + \mathfrak{C}_{12}$.

It remains to check (8.26). As in (7.12), (7.13), and (7.15), we have

$$\|\widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0 - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \le \mathfrak{C}_{13}\varepsilon\|\widetilde{\mathbf{u}}_0\|_{H^{2p}(\mathbb{R}^d)}, \tag{8.29}$$

where $\mathfrak{C}_{13} = \check{\mathfrak{c}}_p^{1/2}M_{\widetilde{g}}\alpha_1^{1/2}(r_1+2)$. From (8.23) and (8.29) it follows that

$$\|\widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0 - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \le \mathfrak{C}_{14}\varepsilon\rho_*(\zeta)^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{8.30}$$

where $\mathfrak{C}_{14} = C_{\mathcal{O}}^{(2p)}\check{c}^{1/2}\widehat{c}\,\mathfrak{C}_{13}$. Comparing (8.4) and (8.30), we arrive at the required estimate (8.26) with the constant $\widetilde{\mathfrak{C}}_3 = \mathfrak{C}_3 + \mathfrak{C}_{14}$. □

Theorem 8.3 and Proposition 3.7 directly imply the following statement.

Corollary 8.4. *Under the assumptions of Theorem* 8.1, *let the operator* $K_D^0(\zeta;\varepsilon)$ *be given by* (7.1) *and the function* $\mathbf{v}_\varepsilon^0$ *by* (7.2). *Let* $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$, *and let* $\widetilde{g}(\mathbf{x})$ *be the matrix-valued function* (1.12). *Suppose that at least one of the following conditions is fulfilled*:

1°. $2p > d$;

2°. $g^0 = \underline{g}$ (*i.e., the representations* (1.18) *are valid*).

Then for $0 < \varepsilon \le \varepsilon_1$ *we have estimates* (8.24)–(8.26), *and the constants* $\widetilde{\mathfrak{C}}_2$ *and* $\widetilde{\mathfrak{C}}_3$ *depend only on* m, n, d, p, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

8.3. Special cases. The following statement easily follows from Proposition 7.5 and identity (8.6).

Proposition 8.5. *Under the assumptions of Theorem* 8.1, *if* $g^0 = \overline{g}$ (*i.e., relations* (1.17) *are valid*), *then*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^p(\mathcal{O})} \le \mathfrak{C}_{15}\varepsilon\rho_*(\zeta)\|\mathbf{F}\|_{L_2(\mathcal{O})}$$

for $0 < \varepsilon \le \varepsilon_1$. *Here* $\mathfrak{C}_{15} = 2\check{\mathfrak{c}}_p^{1/2}c_1^{1/2}\check{c}(c_0^{-1/2} + c_2^{-1/2})\widetilde{\mathcal{C}}_6$.

The proof of following statement is similar to that of Proposition 7.6.

Proposition 8.6. *Under the assumptions of Theorem* 8.1, *if* $g^0 = \underline{g}$ (*i.e., the representations* (1.18) *are valid*), *then for* $0 < \varepsilon \le \varepsilon_1$ *the flux* $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ *satisfies*

$$\|\mathbf{p}_\varepsilon - g^0 b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \widetilde{\mathfrak{C}}_3\varepsilon^{1/2}\rho_*(\zeta)\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

§9. Added in proof

In the present section, we improve estimates of Theorems 5.1, 5.2, 7.1 and Propositions 7.5, 7.6, refining the dependence on the angle $\varphi = \arg\zeta$. For this, we use arguments suggested in the paper [MeSu, §10].

Theorem 9.1. *Under the assumptions of Theorem* 5.1, *for* $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$, $|\zeta| \ge 1$, *and* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\left\|(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1}\right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le \widehat{\mathcal{C}}_1 c(\varphi)^2\varepsilon|\zeta|^{-1+1/2p}. \tag{9.1}$$

The constant $\widehat{\mathcal{C}}_1$ *depends only on* d, p, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

Proof. We apply estimate (5.2) at a point $\widehat{\zeta} \in \mathbb{C}$, $|\widehat{\zeta}| \geq 1$, $\operatorname{Re}\widehat{\zeta} \leq 0$:

$$\big\|(A_{D,\varepsilon} - \widehat{\zeta} I)^{-1} - (A_D^0 - \widehat{\zeta} I)^{-1}\big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq \mathcal{C}_1\big(\varepsilon |\widehat{\zeta}|^{-1+1/2p} + \varepsilon^{2p}\big). \tag{9.2}$$

If $\varepsilon \leq |\widehat{\zeta}|^{-1/2p}$, then $\varepsilon^{2p} \leq \varepsilon|\widehat{\zeta}|^{-1+1/2p}$, whence the right-hand side of (9.2) does not exceed $2\mathcal{C}_1\varepsilon|\widehat{\zeta}|^{-1+1/2p}$. In the case where $\varepsilon > |\widehat{\zeta}|^{-1/2p}$, we apply Lemmas 4.1 and 4.3. Then the left-hand side of (9.2) does not exceed $2|\widehat{\zeta}|^{-1} \leq 2\varepsilon|\widehat{\zeta}|^{-1+1/2p}$. As a result, we obtain

$$\big\|(A_{D,\varepsilon} - \widehat{\zeta} I)^{-1} - (A_D^0 - \widehat{\zeta} I)^{-1}\big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq \mathcal{C}_1'\varepsilon |\widehat{\zeta}|^{-1+1/2p}, \tag{9.3}$$

where $\mathcal{C}_1' = 2\max\{1, \mathcal{C}_1\}$. This proves estimate (9.1) in the case where the point $\widehat{\zeta}$ lies in the left half-plane.

Now, suppose that $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, and $\operatorname{Re}\zeta > 0$. We put $\widehat{\zeta} = -\operatorname{Re}\zeta + i\operatorname{Im}\zeta$. Note that $|\zeta| = |\widehat{\zeta}|$. As in (8.6), we have

$$\begin{aligned}(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1} &= (A_{D,\varepsilon} - \widehat{\zeta} I)(A_{D,\varepsilon} - \zeta I)^{-1} \\ &\times \big((A_{D,\varepsilon} - \widehat{\zeta} I)^{-1} - (A_D^0 - \widehat{\zeta} I)^{-1}\big)(A_D^0 - \widehat{\zeta} I)(A_D^0 - \zeta I)^{-1}.\end{aligned} \tag{9.4}$$

From (9.3) and (9.4) it follows that

$$\big\|(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1}\big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq \mathcal{C}_1'\varepsilon|\zeta|^{-1+1/2p} \sup_{x \geq 0} \frac{|x - \widehat{\zeta}|^2}{|x - \zeta|^2}. \tag{9.5}$$

A calculation shows that

$$\sup_{x \geq 0} \frac{|x - \widehat{\zeta}|^2}{|x - \zeta|^2} \leq 4c(\varphi)^2. \tag{9.6}$$

As a result, (9.5) and (9.6) imply estimate (9.1) with the constant $\widehat{\mathcal{C}}_1 = 4\mathcal{C}_1'$. □

Theorem 9.2. *Under the assumptions of Theorem* 5.2, *for* $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, *and* $0 < \varepsilon \leq \varepsilon_1$ *we have*

$$\begin{aligned}&\big\|(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1} - \varepsilon^p K_D(\zeta;\varepsilon)\big\|_{L_2(\mathcal{O}) \to H^p(\mathcal{O})} \\ &\qquad \leq \widehat{\mathcal{C}}_2 c(\varphi)^2\big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big).\end{aligned} \tag{9.7}$$

The flux $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ *satisfies*

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \leq \widehat{\mathcal{C}}_3 c(\varphi)^2\big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{9.8}$$

The constants $\widehat{\mathcal{C}}_2$ *and* $\widehat{\mathcal{C}}_3$ *depend only on* d, p, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

Proof. We apply estimate (5.9) at a point $\widehat{\zeta} \in \mathbb{C}$, $|\widehat{\zeta}| \geq 1$, $\operatorname{Re}\widehat{\zeta} \leq 0$:

$$\begin{aligned}&\big\|(A_{D,\varepsilon} - \widehat{\zeta} I)^{-1} - (A_D^0 - \widehat{\zeta} I)^{-1} - \varepsilon^p K_D(\widehat{\zeta};\varepsilon)\big\|_{L_2(\mathcal{O}) \to H^p(\mathcal{O})} \\ &\qquad \leq \mathcal{C}_2\big(\varepsilon^{1/2}|\widehat{\zeta}|^{-1/2+1/4p} + \varepsilon^p\big).\end{aligned} \tag{9.9}$$

Now, suppose that $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, and $\operatorname{Re}\zeta > 0$, and let $\widehat{\zeta} = -\operatorname{Re}\zeta + i\operatorname{Im}\zeta$. Recall that $\theta_\varepsilon(\mathbf{x})$ is a cut-off function satisfying (6.1). Denote

$$T(\zeta;\varepsilon) := (A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1} - \varepsilon^p(1 - \theta_\varepsilon)K_D(\zeta;\varepsilon).$$

By (4.2), Lemma 6.2, and (9.9), we have

$$\big\|A_{D,\varepsilon}^{1/2} T(\widehat{\zeta};\varepsilon)\big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq \mathcal{C}_2'\big(\varepsilon^{1/2}|\widehat{\zeta}|^{-1/2+1/4p} + \varepsilon^p\big), \tag{9.10}$$

where $\mathcal{C}_2' = c_1^{1/2}\big(\mathcal{C}_2 + \mathcal{C}_{13}\big)$. The following identity is similar to (8.12):

$$\begin{aligned} T(\zeta;\varepsilon) = (A_{D,\varepsilon} - \widehat{\zeta} I)(A_{D,\varepsilon} - \zeta I)^{-1} T(\widehat{\zeta};\varepsilon)(A_D^0 - \widehat{\zeta} I)(A_D^0 - \zeta I)^{-1} \\ + \varepsilon^p (\zeta - \widehat{\zeta})(A_{D,\varepsilon} - \zeta I)^{-1}(1-\theta_\varepsilon) K_D(\zeta;\varepsilon). \end{aligned} \tag{9.11}$$

Applying the operator $A_{D,\varepsilon}^{1/2}$ to the two sides of (9.11) and using (9.6) and (9.10), we obtain

$$\begin{aligned} \big\| A_{D,\varepsilon}^{1/2} T(\zeta;\varepsilon)\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le 4c(\varphi)^2 \mathcal{C}_2' \big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big) \\ + \varepsilon^p |\zeta - \widehat{\zeta}| \big\| A_{D,\varepsilon}^{1/2}(A_{D,\varepsilon} - \zeta I)^{-1}\big\|_{L_2\to L_2} \big\|(1-\theta_\varepsilon)K_D(\zeta;\varepsilon)\big\|_{L_2\to L_2}. \end{aligned} \tag{9.12}$$

As in (8.15) and (8.16), we have

$$\|(1-\theta_\varepsilon)K_D(\zeta;\varepsilon)\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le \mathfrak{C}_6 \sup_{x\ge 0} \frac{\sqrt{x}}{|x-\zeta|}.$$

A calculation shows that

$$\sup_{x\ge 0} \frac{x}{|x-\zeta|^2} \le c(\varphi)^2 |\zeta|^{-1}.$$

Since $|\zeta - \widehat{\zeta}| \le 2|\zeta|$, we see that the second term on the right-hand side of (9.12) does not exceed $2\mathfrak{C}_6 c(\varphi)^2 \varepsilon^p$. Combining this with (9.12), (3.9), (4.2), and (4.3), we obtain

$$\|T(\zeta;\varepsilon)\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})} \le \mathcal{C}_2'' c(\varphi)^2 \big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big), \tag{9.13}$$

where $\mathcal{C}_2'' = \check{\mathfrak{c}}_p^{1/2}\big(c_0^{-1/2} + c_2^{-1/2}\big)(4\mathcal{C}_2' + 2\mathfrak{C}_6)$. It remains to use the inequality

$$\varepsilon^p \|\theta_\varepsilon K_D(\zeta;\varepsilon)\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})} \le C(p;\mathcal{O})(\mathcal{C}_5 + \mathcal{C}_{13}) c(\varphi)\big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big),$$

which follows from Lemma 6.2 and (6.20). Together with (9.13), this yields the required estimate (9.7) with the constant $\widehat{\mathcal{C}}_2 = \mathcal{C}_2'' + C(p;\mathcal{O})(\mathcal{C}_5 + \mathcal{C}_{13})$.

Next, as in (6.21), inequality (9.7) implies that

$$\|\mathbf{p}_\varepsilon - g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon\|_{L_2(\mathcal{O})} \le \mathcal{C}_3' c(\varphi)^2 \big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{9.14}$$

where $\mathcal{C}_3' = \kappa_9(d,p)\|g\|_{L_\infty}\alpha_1^{1/2}\widehat{\mathcal{C}}_2$. Relations (6.26) and (9.14) yield (9.8) with the constant $\widehat{\mathcal{C}}_3 = \mathcal{C}_3' + \mathcal{C}_{22}$. $\square$

Theorem 9.3. *Under the assumptions of Theorem* 7.1, *for* $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$, $|\zeta| \ge 1$, *and* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\begin{aligned} \big\|(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1} - \varepsilon^p K_D^0(\zeta;\varepsilon)\big\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})} \\ \le \mathcal{C}_2^\circ c(\varphi)^2\big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big). \end{aligned} \tag{9.15}$$

The flux $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ *satisfies*

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \le \mathcal{C}_3^\circ c(\varphi)^2\big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{9.16}$$

The constants $\mathcal{C}_2^\circ$ *and* $\mathcal{C}_3^\circ$ *depend only on* d, p, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *the domain* $\mathcal{O}$, *and also on* $\|\Lambda\|_{L_\infty}$ *and* M_Λ.

Proof. Estimate (9.15) with the constant $\mathcal{C}_2^\circ = \widehat{\mathcal{C}}_2 + \mathcal{C}_{26}$ directly follows from (9.7) and (7.14). Inequality (9.16) with the constant $\mathcal{C}_3^\circ = \widehat{\mathcal{C}}_3 + \mathcal{C}_{27}$ is a consequence of (9.8) and (7.16). $\square$

Proposition 9.4. *Under the assumptions of Proposition* 7.5, *for* $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$, $|\zeta| \ge 1$, *and* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^p(\mathcal{O})} \le \mathcal{C}_6^\circ c(\varphi)^2 \varepsilon |\zeta|^{-1/2+1/2p}\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{9.17}$$

The constant $\mathcal{C}_6^\circ$ *depends only on* d, p, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

Proof. Suppose that $\widehat{\zeta} \in \mathbb{C}$, $|\widehat{\zeta}| \geq 1$, and $\operatorname{Re}\widehat{\zeta} \leq 0$. By Proposition 7.5, under our assumptions we have

$$\big\|(A_{D,\varepsilon} - \widehat{\zeta}I)^{-1} - (A_D^0 - \widehat{\zeta}I)^{-1}\big\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})} \leq \widetilde{\mathcal{C}}_6\big(\varepsilon|\widehat{\zeta}|^{-1/2+1/2p} + \varepsilon^p\big). \tag{9.18}$$

If $\varepsilon \leq |\widehat{\zeta}|^{-1/2p}$, then $\varepsilon^p \leq \varepsilon|\widehat{\zeta}|^{-1/2+1/2p}$, whence the right-hand side of (9.18) does not exceed $2\widetilde{\mathcal{C}}_6\varepsilon|\widehat{\zeta}|^{-1/2+1/2p}$. In the case where $\varepsilon > |\widehat{\zeta}|^{-1/2p}$, we apply Lemmas 4.1 and 4.3. Then the left-hand side of (9.18) does not exceed $4\widetilde{\mathcal{C}}_0|\widehat{\zeta}|^{-1/2} \leq 4\widetilde{\mathcal{C}}_0\varepsilon|\widehat{\zeta}|^{-1/2+1/2p}$. Hence,

$$\big\|(A_{D,\varepsilon} - \widehat{\zeta}I)^{-1} - (A_D^0 - \widehat{\zeta}I)^{-1}\big\|_{L_2(\mathcal{O})\to H^p(\mathcal{O})} \leq \mathcal{C}_6'\varepsilon|\widehat{\zeta}|^{-1/2+1/2p}, \tag{9.19}$$

where $\mathcal{C}_6' = \max\{2\widetilde{\mathcal{C}}_6, 4\widetilde{\mathcal{C}}_0\}$. This proves estimate (9.17) in the case where the point $\widehat{\zeta}$ lies in the left half-plane.

For $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$, $|\zeta| \geq 1$, $\operatorname{Re}\zeta > 0$, we put $\widehat{\zeta} = -\operatorname{Re}\zeta + i\operatorname{Im}\zeta$ and use identity (9.4). Applying the operator $A_{D,\varepsilon}^{1/2}$ to the two sides of (9.4) and taking (4.2), (9.6), and (9.19) into account, we arrive at

$$\big\|A_{D,\varepsilon}^{1/2}\big((A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1}\big)\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq 4c_1^{1/2}\mathcal{C}_6' c(\varphi)^2\varepsilon|\zeta|^{-1/2+1/2p}.$$

Combining this with (3.9), (4.2), and (4.3), we obtain estimate (9.17) with the constant $\mathcal{C}_6^\circ = 4c_1^{1/2}\check{\mathfrak{c}}_p^{1/2}(c_0^{-1/2} + c_2^{-1/2})\mathcal{C}_6'$. $\square$

The following statement is deduced from Theorem 9.3; the proof is similar to that of Proposition 7.6.

Proposition 9.5. *Under the assumptions of Proposition* 7.6, *for* $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$, $|\zeta| \geq 1$, *and* $0 < \varepsilon \leq \varepsilon_1$ *we have*

$$\|\mathbf{p}_\varepsilon - g^0 b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathcal{O})} \leq \mathcal{C}_3^\circ c(\varphi)^2\big(\varepsilon^{1/2}|\zeta|^{-1/2+1/4p} + \varepsilon^p\big)\|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

References

[BaPan] N. S. Bakhvalov and G. P. Panasenko, *Homogenization: averaging processes in periodic media. Mathematical problems in mechanics of composite materials*, Nauka, Moscow, 1984. MR0797571 (86m:73049); English transl., Math. Appl. (Soviet Ser.), vol. 36, Kluwer Acad. Publ. Group, Dordrecht, 1989. MR1112788

[BeLPa] A. Bensoussan, J. L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, Stud. Math. Appl., vol. 5, North-Holland Publ. Co., Amsterdam–New York, 1978. MR503330

[BSu1] M. Sh. Birman ans T. A. Suslina, *Threshold effects near the lower edge of the spectrum for periodic differential operators of mathematical physics*, Systems, Approximation, Singular Integral Operators, and Related Topics (Bordeaux, 2000), Oper. Theory Adv. Appl., vol. 129, Birkhäuser, Basel, 2001, pp. 71–107. MR1882692

[BSu2] ———, *Second order periodic differential operators. Threshold properties and homogenization*, Algebra i Analiz **15** (2003), no. 5, 1–108; English transl., St. Petersburg Math. J. **15** (2004), no. 5, 639–714. MR2068790

[BSu3] ———, *Homogenization with corrector term for periodic elliptic differential operators*, Algebra i Analiz **17** (2005), no. 6, 1–104; English transl., St. Petersburg Math. J. **17** (2006), no. 6, 897–973. MR2202045

[BSu4] ———, *Homogenization with corrector for periodic differential operators. Approximation of solutions in the Sobolev class* $H^1(\mathbb{R}^d)$, Algebra i Analiz **18** (2006), no. 6, 1–130; Englishs transl., St. Petersburg Math. J. **18** (2007), no. 6, 857–955. MR2307356

[V] N. A. Veniaminov, *Homogenization of higher-order periodic differential operators*, Algebra i Analiz **22** (2010), no. 5, 69–103; English transl., St. Petersburg Math. J. **22** (2011), no. 5, 751–775. MR2828827

[Gr1] G. Griso, *Error estimate and unfolding for periodic homogenization*, Asymptot. Anal. **40** (2004), no. 3–4, 269–286. MR2107633

[Gr2] ______, *Interior error estimate for periodic homogenization*, Anal. Appl. (Singap.) **4** (2006), no. 1, 61–79. MR2199793

[Zh] V. V. Zhikov, *On operator estimates in homogenization theory*, Dokl. Akad. Nauk **403** (2005), no. 3, 305–308; English transl., Dokl. Math. **72** (2005), no. 1, 534–538. MR2164541

[ZhKO] V. V. Zhikov, S. M. Kozlov, and O. A. Oleĭnik, *Homogenization of differential operators*, Nauka, Moscow, 1993; English transl., Homogenization of differential operators and integral functions, Springer-Verlag, Berlin, 1994. MR1329546 (96h:35003b)

[ZhPas1] V. V. Zhikov ans S. E. Pastukhova, *On operator estimates for some problems in homogenization theory*, Russ. J. Math. Phys. **12** (2005), no. 4, 515–524. MR2201316

[ZhPas2] ______, *On operator estimates on homogenization theory*, Uspekhi Mat. Nauk **71** (2016), no. 3, 27–122; English transl., Russian Math. Surveys **71** (2016), no. 3, 471–511. MR3535364

[KeLiS] C. E. Kenig, F. Lin, Z. Shen, *Convergence rates in L^2 for elliptic homogenization problems*, Arch. Ration. Mech. Anal. **203** (2012), no. 3, 1009–1036. MR2928140

[KoE] V. A. Kondrat′ev and S. D. Eidel′man, *Boundary-surface conditions in the theory of elliptic boundary value problems*, Dokl. Akad. Nauk SSSR **246** (1979), no. 4, 812–815; English transl., Sov. Math. Dokl. **20** (1979), 561–563. MR543538

[KuSu] A. A. Kukushkin and T. A. Suslina, *Homogenization of high-order elliptic operators with periodic coefficients*, Algebra i Analiz **28** (2016), no. 1, 89–149; English transl., St. Petersburg Math. J. **28** (2017), no. 1, 65–108. MR3591067

[MSh] V. G. Maz′ya and T. O. Shaposhnikova, *Multipliers in spaces of differentiable functions*, Leningrad. Univ., Leningrad, 1986. (Russian) MR881055

[MeSu] Yu. M. Meshkova and T. A. Suslina, *Homogenization of the Dirichlet problem for elliptic systems: Two-parametric error estimates*, Preprint (2017); `arXiv:1702.00550v2.`

[Pas1] S. E. Pastukhova, *Homogenization estimates of operator type for fourth order elliptic equations*, Algebra i Analiz **28** (2016), no. 2, 204–226; English transl., St. Petersburg Math. J. **28** (2017), no. 2, 273–289. MR3593009

[Pas2] ______, *Estimates in homogenization of higher-order elliptic operators*, Appl. Anal. **95** (2016), no. 7, 1449–1466. MR3499669

[PSu1] M. A. Pakhnin and T. A. Suslina, *Homogenization of the Dirichlet problem: error estimates in the $(L_2 \to H^1)$-norm*, Funktsional. Anal. i Prilozhen. **46** (2012), no. 2, 92–96; English transl., Funct. Anal. Appl. **46** (2012), no. 2, 155–159. MR2978064

[PSu2] ______, *Operator error estimates for the homogenization of the elliptic Dirichlet problem in a bounded domain*, Algebra i Analiz **24** (2012), no. 6, 139–177; English transl., St. Petersburg Math. J. **24** (2013), no. 6, 949–976. MR3097556

[So] V. A. Solonnikov, *General boundary value problems for systems elliptic in the sense of A. Douglas and L. Nirenberg*. II, Tr. Mat. Inst. Steklova **92** (1966), 233–297; English transl., Proc. Steklov Inst. Math. **92** (1968), 269–339. MR0211071

[St] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math., Ser., vol. 30, Princeton Univ. Press, Princeton, NJ, 1970. MR0290095 (44:7280)

[Su1] T. A. Suslina, *Operator error estimates in L_2 for the homogenization of an elliptic Dirichlet problem*, Funktsional. Anal. i Prilozen. **46** (2012), no. 3, 91–96; English transl., Funct. Anal. Appl. **46** (2012), no. 3, 234–238. MR3075045

[Su2] ______, *Homogenization of the Dirichlet problem for elliptic systems: L_2-operator error estimates*, Mathematika **59** (2013), no. 2, 463–476. MR3081781

[Su3] ______, *Homogenization of the Neumann problem for elliptic systems with periodic coefficients*, SIAM J. Math. Anal. **45** (2013), no. 6, 3453–3493. MR3131481

[Su4] ______, *Homogenization of elliptic problems depending on a spectral parameter*, Funktsional. Anal. i Prilozhen. **48** (2014), no. 4, 88–94; English transl., Funct. Anal. Appl. **48** (2014), no. 4, 309–313. MR3372745

[Su5] ______, *Homogenization of elliptic operators with periodic coefficients depending on the spectral parameter*, Algebra i Analiz **27** (2015), no. 4, 87–166; English transl., St. Petersburg Math. J. **27** (2016), no. 4, 651–708. MR3580194

St. Petersburg State University, Universitetskaya nab. 7/9, 199034 St. Petersburg, Russia
Email address: `t.suslina@spbu.ru`

Received 24/AUG/2016

Translated by T. A. SUSLINA