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RECTANGULAR LATTICES OF CYLINDRICAL QUANTUM WAVEGUIDES. I. SPECTRAL PROBLEMS ON A FINITE CROSS

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Abstract. The spectrum of truncated cross-shaped waveguides is studied under the Dirichlet conditions on the lateral surface and various boundary conditions on the ends of the column and the cross bar. The monotonicity and asymptotics of the eigenvalues are discussed in dependence on the size of a cross whose section may be fairly arbitrary. In the case of a round section, the estimates found for the second eigenvalue agree with the asymptotic formulas obtained. Such information is needed for the spectral analysis of thin periodic lattices of quantum waveguides.

§1. Motivation

In the past 20 years, great attention (see the surveys [1, 2] and other publications) has been paid to modeling of thin (with diameter $O(h)$, $h \ll 1$) lattices of acoustic and quantum waveguides via one-dimensional problems on graphs. Under this approach, the spectral Neumann and Dirichlet problems for the Laplace operator are studied, and their spectra are described as $h \to +0$. Although the deduction of ordinary differential equations on the graph's edges is quite easy, the choice of transmission conditions at the nodes requires a detailed analysis. However, in the case of the Neumann conditions (acoustic waveguides with rigid walls) the desired transmission conditions are none other than the classical Kirchhoff transmission rules, meaning that the solution is continuous and the total node flow is zero. The corresponding results for finite lattices were obtained in [3, 4, 5] in a fairly general setting.

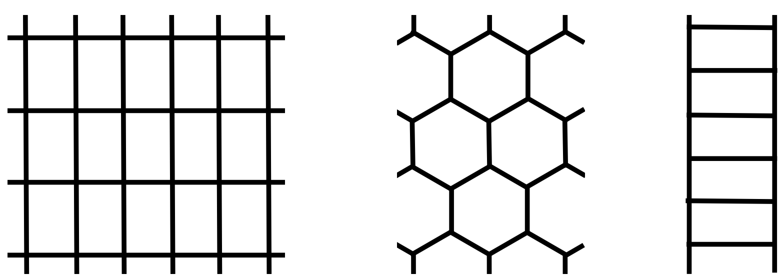

Figure 1. Rectangular and hexagonal lattices, and a ladder.

The Dirichlet problem on the junction of thin domains (quantum waveguides or acoustic waveguides with soft walls) is much less studied. In [5] it was shown that the type

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of the boundary condition is determined by the boundary layer phenomenon near the multidimensional nodes of the lattice at the threshold value of the spectral parameter. While for the Neumann problem the threshold is always zero, for the Dirichlet problem it is positive, and most of the methods available in the first case cease to work. Precisely for this reason, the number of waveguides' shapes that have been studied completely is small. We mention the complete analysis in [6, 7, 8, 9] of the planar rectangular and hexagonal lattices and the Dirichlet ladder (Figure 1); the last object was explored even for the two-dimensional equations of isotropic elasticity theory, see [10]. However, all results mentioned above pertain to the case of planar quantum waveguides, and practically nothing is known about the spectrum of the Dirichlet problems for multi-dimensional lattices. The sole exception are the works [11, 12] of the present authors, which are devoted to a cross-shaped waveguide formed by two mutually orthogonal circular cylinders of equal radii and with intersecting axes. It was established that the discrete spectrum of such a waveguide consists of a single simple eigenvalue $\Lambda_1^\infty \in (0, \Lambda_\dagger)$, and that the homogeneous problem with the threshold value $\Lambda_\dagger > 0$ of the spectral parameter has no bounded solutions. This information makes it possible to apply the general results of [5] and to draw conclusions on the nature of the spectrum of the Dirichlet problem on a square[1] lattice of thin circular quantum waveguides. Namely, the low-frequency range of the spectrum consists of one spectral segment of length $O(e^{-\delta/h})$, $\delta > 0$, located near the point $h^{-2}\Lambda_1^\infty$, and the mid-range spectrum is formed by spectral segments of width $O(h)$ near the points $h^{-2}\Lambda_\dagger + \pi^2 n^2$, $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. These facts ensure a rich family of gaps with width $O(h^{-2})$ in the low-frequency and with width $O(1)$ in the mid-frequency ranges. However, the description of the spectral segments turns out to be incomplete, because the asymptotics for their endpoints remain unknown, requiring construction of lower terms in the expansions for eigenvalues, and, thus, also for eigenfunctions of the model problem on the periodicity cell. The spectral analysis conducted in [5] does not allow one to judge about the behavior of eigenfunctions as $h \to +0$, and we are going to fill these gaps.

The formal asymptotic analysis of the spectral problem on a periodicity cell formed by junction of thin domains, is to a large extent traditional. The main difficulty in the deduction of refined asymptotic formulas is the justification procedure, which requires thorough work, because the behavior of the first (concentrated near the crossing) eigenfunction differs much from that of the other eigenfunctions, which are distributed over all the cross. The necessary preparatory work reduces to verification of auxiliary but substantial inequalities and goes beyond the scope of general methods of constructing the higher-order asymptotic terms. These inequalities, together with some additional properties of eigenvalues for various boundary problems in a finite cross with long arms, constitute the content of the first part of our work.

§2. Statement of the problem and description of results

Consider the spectral Dirichlet problem

$$-\Delta u(x) = \lambda u(x),\ \ x \in \Pi^\infty, \qquad u(x) = 0,\ \ x \in \partial\Pi^\infty, \tag{1}$$

for the Laplace operator on a cross-shaped waveguide Π^∞ composed of two circular cylinders of unit diameter,

$$\Pi^\infty = Q_1 \cup Q_2, \tag{2}$$

$$Q_j = \{x = (x_1, x_2, x_3) : x_{3-j}^2 + x_3^2 < 1/4\}, \quad j = 1, 2. \tag{3}$$

[1]The results remain valid for rectangular lattices; for simplicity, in this paper we assume that the lattices under consideration are square with ligaments of length 1.

The positive definite selfadjoint operator $\mathcal{L}$ of problem (1) is generated (see [13, Chapter 10]) by the nonnegative closed sesquilinear form

$$\ell^\infty[u_1, u_2] = (\nabla u_1, \nabla u_2)_{\Pi^\infty},$$

defined on the subspace

$$\mathcal{H} = \{u \in H^1(\Pi^\infty) \,:\, u(x) = 0,\ x \in \partial\Pi^\infty\}$$

of the Sobolev space $H^1(\Pi^\infty)$; here $(\,\cdot\,,\,\cdot\,)_\Omega$ is the scalar product in the Lebesgue space $L_2(\Omega)$.

The continuous spectrum of problem (1) occupies the half-line $[\Lambda_\dagger, +\infty)$ with the cut-off point $\Lambda_\dagger < 2.35\pi^2$, the first eigenvalue of the Dirichlet problem on the circular cross-section ω_j of the cylinders Q_j,

$$\omega_j = \{|x_{3-j}|^2 + |x_3|^2 < 1/4\}, \quad j = 1, 2. \tag{4}$$

The discrete spectrum of problem (1) is formed (see [11, 12]) by a unique simple eigenvalue $\Lambda_1^\infty \in (0, \Lambda_\dagger)$. Separation of variables allows us to show that a unique eigenfunction w_1^∞ and its gradient decay exponentially at infinity, namely, with $\beta = \sqrt{\Lambda_\dagger - \Lambda_1^\infty}$ we have

$$w_1^\infty(x) = O(e^{-\beta|x|}) \quad \text{and} \quad |\nabla w_1^\infty(x)| = O(e^{-\beta|x|}) \quad \text{as} \quad |x| \to \infty. \tag{5}$$

Also, we consider spectral problems for the Laplace operator on the truncated waveguide

$$\Pi^R = Q_1^R \cup Q_2^R, \tag{6}$$

$$Q_j^R = \{x \in Q_j,\, |x_j| < R\}, \quad j = 1, 2, \tag{7}$$

with various boundary conditions on its ends. In the corresponding variational problems

$$(\nabla u, \nabla v)_{\Pi^R} = \Lambda(u, v)_{\Pi^R}, \quad v \in \mathcal{H}^R \subset L_2(\Pi^R), \tag{8}$$

the function spaces $\mathcal{H}^R$ for eigen- and test-functions are different, but a positive definite selfadjoint operator is constructed for each of the boundary conditions under consideration:

- the subspace

$$\mathcal{H}_D^R := \{u \in H^1(\Pi^R) \,:\, u(x) = 0,\ x \in \partial\Pi^R\}$$

corresponds to the Laplacian $\mathcal{L}_D^R$ with the Dirichlet boundary conditions;
- the subspace

$$\mathcal{H}_N^R := \{u \in H^1(\Pi^R) \,:\, u(x) = 0,\ x \in \partial\Pi^R \cap \partial\Pi^\infty\} \tag{9}$$

corresponds to the Laplacian $\mathcal{L}_N^R$ with the Neumann boundary conditions;
- the subspace

$$\mathcal{H}_\eta^R := \{u \in \mathcal{H}_N^R \,:\, u(x)\big|_{x_j=R} = e^{i\eta_j R} u(x)\big|_{x_j=-R},\ j = 1, 2\}$$

corresponds to the Laplacian $\mathcal{L}_\eta^R$ with quasiperiodic conditions at the ends that involve the Floquet parameter $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$.

In what follows, by $\mathcal{L}_\sharp^R$ we mean any of these three operators. Since the domain Π^R is bounded, the subspace $\mathcal{H}_\sharp^R$ (again, $\sharp = D, N$ or $\sharp = \eta$) compactly embeds in $L_2(\Pi^R)$; consequently, the spectrum of the operator $\mathcal{L}_\sharp^R$ is discrete and forms an unbounded monotone sequence $\{\Lambda_{k,\sharp}^R\}_{k\in\mathbb{N}}$. The corresponding eigenfunctions $w_{k,\sharp}^R$, $k \in \mathbb{N}$, will be orthonormalized in $L_2(\Pi^R)$.

Remark 1. Since $\mathcal{H}^R_D \subset \mathcal{H}^R_\eta \subset \mathcal{H}^R_N$, the max-min principle (see, e.g., [13, Theorem 10.2.2], [14, Theorem XIII.2]) shows that the eigenvalues satisfy

$$\Lambda^R_{k,N} \leq \Lambda^R_{k,\eta} \leq \Lambda^R_{k,D}, \quad k \geq 1.$$

The main result of the paper, to be proved in §3, is the following theorem, which gives a lower bound for the eigenvalues $\Lambda^R_{k,N}$ for $k \geq 2$.

Theorem 1. *There exist numbers $R_* > 1/2$ and $c_N > 0$ such that, for all $R > R_*$, we have*

$$\Lambda^R_{k,N} > \Lambda^R_{2,N} \geq \Lambda_\dagger + c_N R^{-2}, \quad k \geq 2. \tag{10}$$

Theorem 1, which will play a central role in the asymptotic analysis in the second part of the paper, serves only the waveguide (6) formed of the circular cylinders (7); we consciously ignore the simplest generalizations: the arms of the waveguide may have different length in all four directions, and the circular cross-sections may be somewhat deformed (e.g., they may be ellipses with small eccentricity). The remaining sections of our work are devoted to derivation of asymptotic formulas for the near-threshold eigenvalues

$$\Lambda^R_{k,\sharp} = \Lambda_\dagger + o(1), \quad R \to +\infty, \quad k \geq 2, \tag{11}$$

for which the possible generalizations become substantial. For instance, in accordance with [19, 20], in an infinite cross (2) formed by the cylinders

$$Q_j = \{x : x_j \in \mathbb{R}, (x_{3-j}, x_3) \in \omega\}, \quad j = 1, 2, \tag{12}$$

with a cross-section $\omega \subset \mathbb{R}^2$ bounded by a piecewise smooth contour, bounded (stabilizing or decaying) solutions of problem (1) may occur at the threshold $\lambda = \Lambda_\dagger$ of the continuous spectrum. The arising of such solutions considerably affects the one-dimensional model (cf. the general situation treated in [5] and the specific result (see [11, 12]) about the absence of such solutions in the case of the waveguide (2), (3)). The identical cross-section ω for all four arms of the waveguide (2), (12) and the mirror symmetry

$$(x_j, x_3) \in \omega \quad \Leftrightarrow \quad (-x_j, x_3) \in \omega \tag{13}$$

with respect to the ordinate axis are assumed in order to simplify the presentation: e.g., the threshold $\Lambda_\dagger$ in (11), the first eigenvalue M_1 of the Dirichlet problem in the domain ω, corresponds to each of the arms. The first eigenvalue M_1 is simple, and the corresponding eigenfunction Φ_1 can be fixed to be positive in ω; in what follows we shall need the notation $M_2 > M_1$ and Φ_2 for the second eigenpair.

In principle, many assumptions made in the paper can be removed.

It should be noted that in Remark 5 we correct an inaccuracy occurring in the proof of Theorem 3 in [12].

§3. Proof of Theorem 1

The collection of eigenvalues of the Dirichlet problem for the Laplace operator in Q^R_1 with the Dirichlet boundary conditions includes the sequence $\Lambda_\dagger + \pi^2 n^2 (2R)^{-2}$, $n \in \mathbb{N}$. The corresponding eigenfunctions, extended by zero outside Q^R_1, fall into the space $\mathcal{H}^R_D$. Plugging them into the max-min principle for $\mathcal{L}^R_D$, we get the inequality

$$\Lambda^R_{k,D} \leq \Lambda_\dagger + \pi^2 k^2 (2R)^{-2}, \quad k \in \mathbb{N}. \tag{14}$$

Since this argument does not involve the shape of the cross-section, an upper estimate similar to (14) admits clear generalizations.

Estimate (10) is much harder to verify; to prove it we shall use the geometric specifics of the waveguide (6), (7). We start with an auxiliary statement.

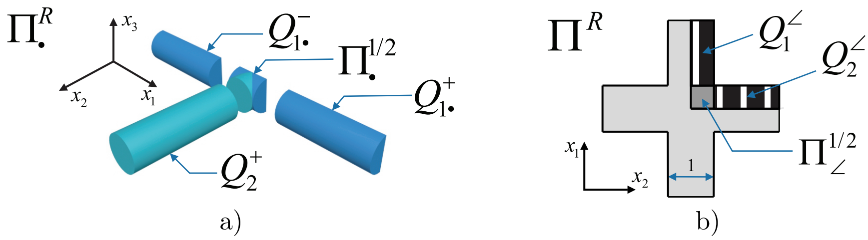

FIGURE 2. Splitting of the truncated waveguide $\Pi^R_\bullet$ (a), and a planar image of the waveguide Π^R and of its quarter (b).

Lemma 3.1. *There exist numbers $R_* > 1/2$ and $\mathbf{c} > 0$ such that for $R > R_*$ and any eigenfunction v of $\mathcal{L}^R_N$ that is odd in x_2 (or x_1) we have*

$$\|\nabla v; L_2(\Pi^R)\|^2 \geq \Lambda_\dagger \|v; L_2(\Pi^R)\|^2 + \mathbf{c}\Big(\|v; L_2(\Pi^{1/2})\|^2 + \sum_{j=1,2} \Big\|\frac{\partial v}{\partial x_j}; L_2(Q^R_j)\Big\|^2\Big).$$

Proof. If an eigenfunction v is odd in x_2, then it is simultaneously an eigenfunction for the Laplace operator in the upper half of the truncated waveguide

$$\Pi^R_\bullet = \{(x_1, x_2, x_3) \in \Pi^R \,:\, x_2 > 0\},$$

which is associated with the sesquilinear form

$$\ell^R_\bullet[u_1, u_2] = (\nabla u_1, \nabla u_2)_{L_2(\Pi^R_\bullet)}$$

on the domain

$$\mathcal{H}^R_{\bullet N} = \{u \in H^1(\Pi^R_\bullet) \,:\, u(x) = 0 \text{ for } x \in \partial\Pi^R_\bullet,\ |x_j| \neq R,\ j = 1, 2\}.$$

The two-dimensional Friedrichs inequalities on the disk and the half-disk (see Figure 2, a) imply the relations

$$\|\nabla v; L_2(Q^+_2)\|^2 \geq \Lambda_\dagger \|v; L_2(Q^+_2)\|^2 + \|\partial_2 v; L_2(Q^+_2)\|^2, \tag{15}$$

$$\|\partial_2 v; L_2(Q^\pm_{1\bullet})\|^2 + \|\partial_3 v; L_2(Q^\pm_{1\bullet})\|^2 \geq \Lambda_\bullet \|v; L_2(Q^\pm_{1\bullet})\|^2, \tag{16}$$

where $\partial_j = \partial/\partial x_j$, $\Lambda_\bullet > 5.95\pi^2$ is the first eigenvalue of the Dirichlet problem in the half-disk of unit diameter, and

$$Q^+_2 = \{x \in Q^R_2 \,:\, \tfrac{1}{2} < x_2 < R\}, \quad Q^\pm_{1\bullet} = \{x \in Q^R_1 \,:\, x_2 > 0,\ \tfrac{1}{2} < \pm x_1 < R\}.$$

Here and in what follows in this section we omit the index R to make formulas shorter.

To obtain estimates on the central part $\Pi^{1/2}_\bullet = \{x \in \Pi^R_\bullet \,:\, |x_1| \leq 1/2,\ x_2 \leq 1/2\}$, we use the Dirichlet conditions on a part of its boundary and apply the Friedrichs inequalities with respect to x_2 and x_3. This yields the estimates

$$\|\partial_2 v; L_2(\Pi^{1/2}_\bullet)\|^2 \geq \pi^2(1 - \mathbf{c})\|v; L_2(\Pi^{1/2}_\bullet)\|^2 + \mathbf{c}\|\partial_2 v; L_2(\Pi^{1/2}_\bullet)\|^2, \tag{17}$$

$$\|\partial_3 v; L_2(\Pi^{1/2}_\bullet)\|^2 \geq \pi^2\|v; L_2(\Pi^{1/2}_\bullet)\|^2, \tag{18}$$

where $\mathbf{c}$ is a constant to be specified below.

The set $\Pi^{1/2}_\bullet \setminus Q_1$ is a union of segments parallel to the x_1-axis and having ends on the upper part of the boundary of the cylinder ∂Q_2. Using the one-dimensional Friedrichs inequalities on these segments, we deduce the estimates

$$\|\partial_1 v; L_2(\Pi^{1/2}_\bullet \setminus Q_1)\|^2 \geq \pi^2(1 - \mathbf{c})\|v; L_2(\Pi^{1/2}_\bullet \setminus Q_1)\|^2 + \mathbf{c}\|\partial_1 v; L_2(\Pi^{1/2}_\bullet \setminus Q_1)\|^2. \tag{19}$$

The surplus of the norm of the squared gradient of the function v on the “half-arms” can be employed to compensate for the lack of it on the central part $\Pi_\bullet^{1/2} \cap Q_1$. In order to implement this idea, we invoke the following inequality established in [12],

$$\int_{Q_{1\bullet}} |\partial_1 v(x)|^2\,dx + a^2 \int_{Q_{1\bullet}^- \cup Q_{1\bullet}^+} |v(x)|^2\,dx \geq \mu(R,a) \int_{Q_{1\bullet}\cap \Pi^{1/2}} |v(x)|^2\,dx, \tag{20}$$

and moreover, $\lim_{R\to+\infty} \mu(R,a) = \mu_\infty(a)$, where $\mu_\infty(a)$ is the smallest positive root of the transcendental equation

$$\sqrt{\mu}\tan\left(\frac{\sqrt{\mu}}{2}\right) = a. \tag{21}$$

The notation $Q_{1\bullet}$ means the set $\Pi_\bullet^R \cap Q_1$. Direct calculations show that $\mu_\infty(\Lambda_\bullet - \Lambda_\dagger) > \Lambda_\dagger - 2\pi^2$. Consequently, there exist numbers $R_* > 1/2$ and $\mathbf{c} > 0$ such that, for all $R > R_*$, we have

$$\begin{aligned}(1-\mathbf{c})\|\partial_1 v; L_2(Q_{1\bullet})\|^2 + (\Lambda_\bullet - \Lambda_\dagger)\|v; L_2(Q_{1\bullet}^- \cup Q_{1\bullet}^+)\|^2 \\ \geq (\Lambda_\dagger - 2\pi^2 + \mathbf{c}\pi^2 + \mathbf{c})\|v; L_2(Q_{1\bullet} \cap \Pi^{1/2})\|^2.\end{aligned}$$

Adding this to estimates (15)–(19) and taking $\mathbf{c}$ sufficiently small, we get the required inequality. □

Remark 2. The first eigenfunction $w_{1,N}^R$ of the operator $\mathcal{L}_N^R$ (see formulas (8) and (9)) with $R > R_*$ is even in x_1 and x_2, because the first eigenvalue is less than $\Lambda_\dagger$, and the first eigenfunction does not vanish identically on the central part by the unique continuation theorem (see, e.g., the book [15, §4.3]).

Remark 3. The quantities $\Lambda_\dagger$ and $\Lambda_\bullet$ can be expressed in terms of roots of Bessel functions and their derivatives (see, e.g., [16]). Moreover, it is not difficult to calculate the roots of equation (21).

Lemma 3.2. *For $R > 1/2$ and any $f \in H^1(-R;R)$, we have*

$$\int_{-R}^{R} |f(z)|^2\,dz \leq 4R\int_{-1/2}^{1/2} |f(z)|^2\,dz + 2R^2 \int_{-R}^{R} |\partial_z f(z)|^2\,dz.$$

Proof. We check the corresponding inequality for a function restricted to $(0,R)$ (the passage to $(-R,R)$ is obvious). We write the Newton–Leibniz formula

$$f(z) - f(t) = \int_t^z \partial_z f(\tau)\,d\tau.$$

The Young and the Cauchy–Schwarz inequalities yield

$$|f(z)|^2 \leq 2|f(t)|^2 + 2\left|\int_t^z \partial_z f(\tau)\,d\tau\right|^2 \leq 2|f(t)|^2 + 2\Big(\int_0^R |\partial_z f(\tau)|\,d\tau\Big)^2$$

and

$$|f(z)|^2 \leq 2|f(t)|^2 + 2R\int_0^R |\partial_z f(\tau)|^2\,d\tau.$$

Now, integration in t and z gives the desired result. □

Proof of Theorem 1. Every eigenfunction w of the Laplace operator $\mathcal{L}_N^R$ can be written as a sum

$$w(x) = v_1(x) + v_2(x) + v(x)$$

with functions v_j odd in x_j and a function v even both in x_1 and in x_2, and

$$\begin{aligned}
v_1(x) &= \frac{1}{2}\big(w(x_1,x_2,x_3) - w(-x_1,x_2,x_3)\big),\\
v_2(x) &= \frac{1}{4}\big(w(x_1,x_2,x_3) + w(-x_1,x_2,x_3) - w(x_1,-x_2,x_3) - w(-x_1,-x_2,x_3)\big),\\
v(x) &= \frac{1}{4}\big(w(x_1,x_2,x_3) + w(-x_1,x_2,x_3) + w(x_1,-x_2,x_3) + w(-x_1,-x_2,x_3)\big).
\end{aligned}$$

Since the waveguide Π^R has mirror symmetry, the functions v_1, v_2, and v, whenever nontrivial, are also eigenfunctions of $\mathcal{L}_N^R$. Moreover, they are mutually orthogonal in the space $L_2(\Pi^R)$. Thus, for each eigenvalue we can find an eigenfunction that is either odd in one of the variables x_1 and x_2, or even in both.

If $v_1 \neq 0$, we apply Lemma 3.2 to the functions $f_1(z) = v_1(z, x_2, x_3)$ and $f_2(z) = v_1(x_1, z, x_3)$ and integrate the resulting inequalities over the circular sections (4). As a result, we obtain

$$\|\partial_j v_1; L_2(Q_j^R)\|^2 + \frac{2}{R}\|v_1; L_2(Q_j^{1/2})\|^2 \geq \frac{1}{2R^2}\|v_1; L_2(Q_j^R)\|^2, \quad j = 1, 2.$$

Therefore,

$$\|\partial_1 v_1; L_2(Q_1^R)\|^2 + \|\partial_2 v_1; L_2(Q_2^R)\|^2 + \frac{4}{R}\|v_1; L_2(\Pi^{1/2})\|^2 \geq \frac{1}{2R^2}\|v_1; L_2(\Pi^R)\|^2.$$

For $R > 2$, combining this inequality with Lemma 3.1, we get

$$\|\nabla v_1; L_2(\Pi^R)\|^2 \geq \Lambda_\dagger \|v_1; L_2(\Pi^R)\|^2 + \frac{\mathbf{c}}{2R^2}\|v_1; L_2(\Pi^R)\|^2.$$

Consequently, the min-max principle ensures that, for $k \geq 2$,

$$\Lambda_{k,N}^R > \Lambda_\dagger + \frac{\mathbf{c}}{2R^2}.$$

In the case where $v_2 \neq 0$, the argument is similar. Thus, it suffices to assume that $v \neq 0$. Observe that v is also an eigenfunction (corresponding to the same eigenvalue $\Lambda_{k,N}^R$) of the Laplacian $\mathcal{L}_\angle^R$ with the mixed boundary conditions on the quarter

$$\Pi_\angle^R = \big\{x \in \Pi^R \,:\, x_j > 0,\ j = 1, 2\big\}$$

of the waveguide (its projection is shown by hatching and by black color in Figure 2, b). The operator $\mathcal{L}_\angle^R$ is generated by the sesquilinear form

$$\ell_\angle^R[u_1, u_2] = (\nabla u_1, \nabla u_2)_{\Pi_\angle^R}$$

defined in the space

$$\mathcal{H}_\angle^R = \big\{u \in H^1(\Pi_\angle^R) \,:\, u(x) = 0,\ x \in \partial\Pi_\angle^R \cap \partial\Pi^R\big\}.$$

The first eigenfunction of the operator $\mathcal{L}_N^R$ is even in both variables x_1 and x_2 (see Remark 2). Consequently, its restriction to $\Pi_\angle^R$ is an eigenfunction (being positive, it is the first eigenfunction) for $\mathcal{L}_\angle^R$. As a result, for $k \geq 2$ the max-min principle yields the relation

$$\Lambda_{k,N}^R = \frac{\|\nabla w_{k,N}^R; L_2(\Pi_\angle^R)\|^2}{\|w_{k,N}^R; L_2(\Pi_\angle^R)\|^2} \geq \max_E \inf_{u \in E\setminus\{0\}} \frac{\|\nabla u; L_2(\Pi_\angle^R)\|^2}{\|u; L_2(\Pi_\angle^R)\|^2}, \tag{22}$$

where the maximum is calculated over all subspaces $E \subset \mathcal{H}_\angle^R$ of codimension 1.

To estimate the Rayleigh ratio on the left-hand side of (22), we consider a function V that minimizes the ratio

$$\frac{\|\nabla V; L_2(\Pi_\angle^R)\|^2}{\|V; L_2(\Pi_\angle^R)\|^2}$$

on the specific subspace

$$E_\perp = \left\{ w \in \mathcal{H}^R_\angle : \int_{\Pi^{1/2}_\angle} w(x)\cos(\pi x_3)\,dx = 0 \right\}, \quad \operatorname{codim} E_\perp = 1.$$

Here, $\Pi^{1/2}_\angle = \{x \in \Pi^R_\angle : x_j \le 1/2,\ j = 1, 2\}$ (the projection of $\Pi^{1/2}_\angle$ is shown by black color in Figure 2, b). Since the subspace $E_\perp$ is weakly closed, the minimizer V exists (see, e.g., [17]).

Consider the spectral problem for the Laplace operator on the rectangular parallelepiped $\Sigma = \{x : 0 \le x_j \le 1/2,\ j = 1, 2,\ |x_3| \le 1/2\}$ with the Dirichlet conditions on its bases $\{x \in \partial\Sigma : |x_3| = 1/2\}$. The first eigenfunction $u_1(x) = \cos(\pi x_3)$ corresponds to the eigenvalue π^2, and the second eigenfunction $u_2(x) = \cos(2\pi x_3)$ corresponds to the eigenvalue $4\pi^2$. Note that $\Pi^{1/2}_\angle \subset \Sigma$, and that the function V extended by zero outside of $\Pi^{1/2}_\angle$ is an element of the set

$$E^0_\perp = \left\{ w \in H^1(\Sigma) : w(x) = 0 \ \text{ for } \ |x_3| = 1/2, \ \int_\Sigma w(x)\cos(\pi x_3)\,dx = 0 \right\}.$$

Consequently, by the Poincaré inequality,

$$\|\nabla V; L_2(\Pi^{1/2}_\angle)\|^2 \ge 4\pi^2 \|V; L_2(\Pi^{1/2}_\angle)\|^2,$$

we have

$$\|\nabla V; L_2(\Pi^{1/2}_\angle)\|^2 \ge \frac{4\pi^2 + \Lambda^\infty_\dagger}{2} \|V; L_2(\Pi^{1/2}_\angle)\|^2 + \frac{4\pi^2 - \Lambda_\dagger}{8\pi^2} \|\nabla V; L_2(\Pi^{1/2}_\angle)\|^2. \tag{23}$$

Denoting $Q^\angle_j = \{x \in \Pi^R_\angle : x_j > 1/2\}$, $j = 1, 2$, we deduce the estimate

$$\|\partial_{3-j} V; L_2(Q^\angle_j)\|^2 + \|\partial_3 V; L_2(Q^\angle_j)\|^2 \ge \Lambda_\dagger \|V; L_2(Q^\angle_j)\|^2. \tag{24}$$

Summing (23) and (24), $j = 1, 2$, we get

$$\begin{aligned} \|\nabla V; L_2(\Pi^R_\angle)\|^2 \ge \Lambda_\dagger \|V; L_2(\Pi^R_\angle)\|^2 &+ \frac{4\pi^2 - \Lambda_\dagger}{2} \|V; L_2(\Pi^{1/2}_\angle)\|^2 \\ &+ \frac{4\pi^2 - \Lambda_\dagger}{8\pi^2} \|\nabla V; L_2(\Pi^{1/2}_\angle)\|^2 + \sum_{j=1,2} \|\partial_j V; L_2(Q^\angle_j)\|^2. \end{aligned}$$

Lemma 3.2 applied to V results in the inequality

$$\|\nabla V; L_2(\Pi^R_\angle)\|^2 \ge \left(\Lambda_\dagger + \frac{4\pi^2 - \Lambda_\dagger}{8\pi^2 R^2} \right) \|V; L_2(\Pi^R_\angle)\|^2.$$

Now to complete the proof it remains to recall (22). □

§4. Estimates for asymptotic remainders

Theorem 1 will help us to find the asymptotic behavior of the first eigenvalue $\Lambda^R_{1,\sharp}$ and the first eigenfunction $w^R_{1,\sharp}$. The next two statements will be checked in the case of the circular cylinders (3), but they mainly remain true also for the cylinders (12) with arbitrary cross-sections. The needed modifications in the formulations and proofs are almost obvious and we do not present them for brevity.

Theorem 2. *There exists a constant $C^\Lambda_1 > 0$ such that, for $R > 1/2$, we have*

$$|\Lambda^R_{1,\sharp} - \Lambda^\infty_1| \le C^\Lambda_1 e^{-\beta R}. \tag{25}$$

Theorem 3. *There exists a constant $C^w_1 > 0$ such that, for $R > 1/2$, we have*

$$\|w^\infty_1 - w^R_{1,\sharp}; H^1(\Pi^R)\| \le C^w_1 e^{-\beta R/2}.$$

For the proof of Theorem 2, we explore how close are the functions $w^R_{1,\sharp}$ in the $L_2(\Pi^R)$-norm to the function w^∞_1 restricted to the truncated waveguide. Since w^∞_1 belongs neither to $\mathcal{H}^R_D$ nor to $\mathcal{H}^R_\eta$, we multiply it by a smooth cut-off function χ^R such that

$$\begin{aligned}\chi^R(x) = 1 \quad \text{for } |x_j| \le R-1, \quad \chi^R(x) = 0 \quad \text{for } |x_j| > R, \quad j = 1, 2,\\ 0 \le \chi^R \le 1, \;\; |\nabla \chi^R| \le c_\chi.\end{aligned} \tag{26}$$

Consider the orthogonal sum

$$\chi^R w^\infty_1 = a_R w^R_{1,\sharp} + b_R v, \tag{27}$$

where $\|v; L_2(\Pi^R)\| = 1$ and $(w^R_{1,\sharp}, v)_{\Pi^R} = 0$.

Lemma 4.1. *There exists R_* such that, for $R > R_*$, the coefficient a_R is uniformly separated away from zero.*

Proof. The integral identity implies

$$(\nabla w^R_{1,\sharp}, \nabla v)_{\Pi^R} = 0,$$

whence

$$\|\nabla(\chi^R w^\infty_1); L_2(\Pi^R)\|^2 = |a_R|^2 \Lambda^R_{1,\sharp} + |b_R|^2 \|\nabla v; L_2(\Pi^R)\|^2. \tag{28}$$

By the max-min principle we have $\|\nabla v; L_2(\Pi^R)\|^2 \ge \Lambda^R_{2,\sharp}$, and Theorem 1 shows that the inequality $\Lambda^R_{2,\sharp} > \Lambda_\dagger$ is fulfilled for $R > R_*$. Therefore,

$$\|\nabla(\chi^R w^\infty_1); L_2(\Pi^R)\|^2 \ge |a_R|^2 \Lambda^R_{1,\sharp} + |b_R|^2 \Lambda_\dagger. \tag{29}$$

The exponential decay of the eigenfunction w^∞_1 (see (5)) implies that

$$\|\chi^R w^\infty_1; L_2(\Pi^R)\|^2 = 1 - O(e^{-2\beta R})$$

as $R \to \infty$. Consequently,

$$|a_R|^2 + |b_R|^2 = 1 - O(e^{-2\beta R}). \tag{30}$$

Moreover, we have the asymptotic formula

$$\|\nabla w^\infty_1; L_2(\Pi^R)\|^2 = \Lambda^\infty_1 + O(e^{-2\beta R}). \tag{31}$$

Relations (29)–(31) and the inequality $\Lambda_\dagger > \Lambda^\infty_1$ guarantee that the quantity $|a_R|$ is separated away from zero. □

Proof of Theorem 2. It suffices to check inequality (25) for sufficiently large R. In order to estimate the difference between Λ^∞_1 and $\Lambda^R_{1,\sharp}$, we substitute the product $\chi^R w^\infty_1$ along with the first eigenfunction of the operator $\mathcal{L}^R_\sharp$ in the form (8) with $\Lambda = \Lambda^R_{1,\sharp}$, obtaining

$$(\nabla w^R_{1,\sharp}, \nabla(\chi^R w^\infty_1))_{\Pi^R} = \Lambda^R_{1,\sharp} (w^R_{1,\sharp}, \chi^R w^\infty_1)_{\Pi^R}.$$

Integration by parts on the left-hand side results in the identity

$$\big(\nabla w^R_{1,\sharp}, \nabla(\chi^R w^\infty_1)\big)_{\Pi^R} = \Lambda^\infty_1 (w^R_{1,\sharp}, \chi^R w^\infty_1)_{\Pi^R} - \big(w^R_{1,\sharp}, [\Delta, \chi^R] w^\infty_1\big)_{\Pi^R},$$

where $[\Delta, \chi^R]$ is the commutator of the operator Δ with the cut-off function (26). Consequently,

$$\Lambda^\infty_1 - \Lambda^R_{1,\sharp} = \frac{(w^R_{1,\sharp}, [\Delta, \chi^R] w^\infty_1)_{\Pi^R}}{(w^R_{1,\sharp}, \chi^R w^\infty_1)_{\Pi^R}}. \tag{32}$$

Since $\operatorname{supp}[\Delta, \chi^R] \subset \{x : |x_j| \in [R, R+1]\}$ and the function w^∞_1 decays exponentially at infinity, the Cauchy inequality shows that the numerator on the right-hand side in (32) is $O(e^{-\beta R})$. At the same time, by Lemma 4.1, the denominator is separated away from zero, which completes the verification of estimate (25). □

Lemma 4.2. *For $R > 1/2$, the coefficients in* (28) *satisfy the inequalities*

$$|a_R| = \left|(w^R_{1,\sharp}, \chi^R w^\infty_1)_{\Pi^R}\right| > 1 - Ce^{-\beta R} \quad \textit{and} \quad |b_R| < Ce^{-\beta R/2},$$

where C is a positive constant.

Proof. Combining estimate (29) with formulas (30), (31), we see that

$$|a_R|^2(\Lambda^R_{1,\sharp} - \Lambda_\dagger) \leq \Lambda^\infty_1 - \Lambda_\dagger - ce^{-2\beta R}.$$

Since the inequality $\Lambda^R_{1,\sharp} > \Lambda^\infty_1 - C^\Lambda_1 e^{-\beta R}$ is ensured by Theorem 2, we easily get the claim. □

Remark 4. Similar arguments provide an inequality for the coefficients of the orthogonal decomposition of the function w^∞_1 restricted to the truncated waveguide, namely

$$|(w^R_{1,\sharp}, w^\infty_1)_{\Pi^R}| > 1 - Ce^{-\beta R}.$$

Proof of Theorem 3. The "good" behavior (5) of the function w^∞_1 at infinity shows that

$$\|w^\infty_1 - \chi^R w^\infty_1; H^1(\Pi^R)\| < C_1 e^{-\beta R}. \tag{33}$$

The orthogonal decomposition (27) of $\chi^R w^\infty_1$ implies the identity

$$\chi^R w^\infty_1 - w^R_{1,\sharp} = b_R v + (a_R - 1) w^R_{1,\sharp}.$$

Therefore, using the estimates of the quantities $|a_R|$ and $|b_R|$ obtained in Lemma 4.2, we conclude that

$$\|\chi^R w^\infty_1 - w^R_{1,\sharp}; L_2(\Pi^R)\| < C_2 e^{-\beta R/2}. \tag{34}$$

Invoking (5) once again and calculating directly we show that

$$\begin{aligned} &\|\chi^R w^\infty_1 - w^R_{1,\sharp}; L_2(\Pi^R)\|^2 = 2 - 2\,\mathrm{Re}\left(\chi^R w^\infty_1, w^R_{1,\sharp}\right)_{\Pi^R} + O(e^{-2\beta R}), \\ &\|\nabla(\chi^R w^\infty_1 - w^R_{1,\sharp}); L_2(\Pi^R)\|^2 = \Lambda^\infty_1 + \Lambda^R_{1,\sharp} - 2\,\mathrm{Re}\left(\nabla(\chi^R w^\infty_1), \nabla w^R_{1,\sharp}\right)_{\Pi^R} + O(e^{-2\beta R}). \end{aligned} \tag{35}$$

Since the product $\chi^R w^\infty_1$ belongs to $\mathcal{H}^R_\sharp$ and $w^R_{1,\sharp}$ is an eigenfunction of $\mathcal{L}^R_\sharp$, we have

$$\left(\nabla(\chi^R w^\infty_1), \nabla w^R_{1,\sharp}\right)_{\Pi^R} = \Lambda^R_{1,\sharp}\left(\chi^R w^\infty_1, w^R_{1,\sharp}\right)_{\Pi^R}.$$

Relations (34) and (35) ensure the inequality

$$\mathrm{Re}(\chi^R w^\infty_1, w^R_{1,\sharp}) > 1 - C_3 e^{-\beta R}.$$

Therefore,

$$\|\nabla(\chi^R w^\infty_1 - w^R_{1,\sharp}); L_2(\Pi^R)\|^2 < C_4 e^{-\beta R}.$$

Combining this with (33) and (34), we complete the proof. □

Remark 5. We eliminate an inaccuracy in the verification of Theorem 3 in [12]. In fact, in [12, Proposition 3], the exponential closeness of the eigenvalue $\Lambda^R_{2,N}$ to the threshold $\Lambda_\dagger$ was only proved under the condition that at the threshold values of the spectral parameter there exists a bounded solution. Inequality (10) in Theorem 1 in the present paper shows that such closeness is impossible for large R, so that, indeed, problem (1) with $\lambda = \Lambda_\dagger$ does not admit any bounded solutions.

§5. Remarks on monotonicity

The eigenvalues of the Dirichlet problem on the truncated waveguide decrease monotonically as R grows:

$$\Lambda_{k,D}^{R_1} > \Lambda_{k,D}^{R_2} \text{ for } R_1 < R_2 \quad \text{whenever} \quad k \in \mathbb{N}.$$

This follows from the max-min principle and the embedding $\mathcal{H}_D^{R_1} \subset \mathcal{H}_D^{R_2}$.

The question about the behavior of eigenvalues of the Neumann problem is much more involved. We give a partial answer in the case of an arbitrary section ω of the cylinders (12) forming an infinite cross-shaped waveguide (2) (the restriction (13) is not needed). Let R_ω denote a length for which

$$\{x \in \Pi^\infty \,:\, \pm x_j > R_\omega\} = \{x \in Q_j \,:\, \pm x_j > R_\omega\}, \quad j = 1, 2.$$

Proposition 1. *Suppose that $\Lambda_{k,N}^{R_0} < \Lambda_\dagger$ for some $R_0 > R_\omega$, $k \in \mathbb{N}$. Then there exists $\rho_{0k} > 0$ such that $\Lambda_{k,N}^{R_0} < \Lambda_{k,N}^{R} < \Lambda_\dagger$ for $R \in (R_0, R_0 + \rho_{0k})$.*

Proof. Let us regard the operator $\mathcal{L}_N^{R+\rho}$ as a perturbation of $\mathcal{L}_N^R$ for small $\rho > 0$. Assuming that Λ is a simple eigenvalue, we normalize the corresponding eigenfunction in the space $L_2(\Pi^R)$ (for brevity, we omit the indices R, N and k). We assume the simplest asymptotic *Ansätze*

$$\Lambda^{R+\rho} = \Lambda + \rho\Lambda' + \dots, \tag{36}$$

$$w^{R+\rho} = w + \rho w' + \dots, \tag{37}$$

where the correction terms Λ' and w' are to be determined, and the small remainders are hidden in dots. The functions w and w' are given initially on Π^R but are extended to $\Pi^\infty \supset \Pi^{R+\rho}$ with preservation of smoothness. Plugging (36) and (37) in the equation for $w^{R+\rho}$ on Π^R and collecting the coefficients of ρ, we get the equation

$$\Delta w'(x) + \Lambda w'(x) = -\Lambda' w(x), \quad x \in \Pi^R. \tag{38}$$

Imposing the Dirichlet conditions

$$w'(x) = 0, \quad x \in \partial\Pi^R \setminus \Gamma^R, \tag{39}$$

outside of the union Γ^R of the ends of the truncated cross-shaped waveguide is quite clear. The Neumann boundary condition $\partial_n w^{R+\rho}(x) = 0$, $x \in \Gamma^{R+\rho}$, will be carried over to the surface Γ^R with the help of the Taylor formula with respect to the variable n equal to the oriented distance to Γ^R, with $n < 0$ inside Π^R. We have

$$\partial_n w^{R+\rho}\big|_{n=\rho} = \partial_n w\big|_{n=\rho} + \rho\partial_n w'\big|_{n=\rho} + \dots = \partial_n w\big|_{n=0} + \rho\partial_n^2 w\big|_{n=0} + \rho\partial_n w'\big|_{n=0} + \dots$$

Thus,

$$\partial_n w'(x) = -\partial_n^2 w(x) = \Delta_\perp w(x) + \Lambda w(x), \quad x \in \Gamma^R. \tag{40}$$

Here, we have used the Helmholtz equation for w, and $\Delta_\perp = \nabla_\perp \cdot \nabla_\perp$ stands for the two-dimensional Laplace operator in the tangent variables on the ends ($\nabla_\perp$ is the similar gradient). The Fredholm alternative shows that the role of the compatibility condition for problem (38)–(40) can be played by the identity

$$\Lambda' = -(\Delta_\perp w + \Lambda w, w)_{\Gamma^R} = \|\nabla_\perp w; L_2(\Gamma^R)\|^2 - \Lambda\|w; L_2(\Gamma^R)\|^2. \tag{41}$$

The correction term (41) is positive, because, by the Friedrichs inequality on ω, we have

$$\Lambda' \geq (\Lambda_\dagger - \Lambda)\|w; L_2(\Gamma^R)\|^2 > 0.$$

Recall that w cannot vanish identically on Γ^R in view of the unique continuation theorem (see, e.g., [15, Chapter 4]).

In the case where the eigenvalue $\Lambda = \Lambda^R_{k,N}$ has multiplicity $\varkappa$, the calculations do not change, but the role of the principal term in (37) will be played by a linear combination of eigenfunctions corresponding to Λ and normalized in $L_2(\Pi^R)$, with a coefficient column $\mathbf{c} = (c_1, c_2, \dots, c_\varkappa)^\top$, and the correction terms $\Lambda'_1, \dots, \Lambda'_\varkappa$ satisfy the following system of linear algebraic equations:

$$\mathbf{M}\mathbf{c} = \Lambda'\mathbf{c}.$$

Its matrix $\mathbf{M}$ of size $\varkappa \times \varkappa$ acquires the entries

$$\mathbf{M}_{pq} = (\nabla_\perp w_p, \nabla_\perp w_q)_{\Gamma^R} - \Lambda(w_p, w_q)_{\Gamma^R}, \quad p, q = 1, \dots, \varkappa,$$

and is symmetric and positive definite for the same reason as before.

Thus, the correction term in the formal asymptotics (36) is positive in any case. The asymptotic remainder can easily be estimated by the classical method (see [18, Chapter 7, §6.5], because it is not difficult to build an "almost identical" diffeomorphism of the domain $\Pi^{R+\rho}$ onto Π^R in such a way that this diffeomorphism is identical near the middle part of the cross and coincides with a uniform shift near its ends. $\square$

§6. The eigenvalues generated by stabilizing and decaying solutions at the threshold

Suppose that, for the threshold value $\Lambda_\dagger$ of the parameter λ, problem (1) in the waveguide Π^∞ given by formulas (2), (12), and (13), has a bounded or stabilizing solution, i.e., an almost standing wave, or a solution decaying at infinity, i.e., respectively, a trapped wave. Recall that the possibility for such solutions to arise at the threshold when the shape of cross-section varies is in agreement with the results in [19, 20]. In what follows, under a simplifying assumption, we shall construct the asymptotics

$$\Lambda^R_\sharp = \Lambda_\dagger + \ell(R) + \dots \tag{42}$$

for the near-threshold eigenvalues of the operators $\mathcal{L}^R_D$ and $\mathcal{L}^R_N$ in the truncated waveguide Π^R; the behavior of the correction term as $R \to +\infty$ will also be determined. We do not take care of justifying the formal asymptotics (42), because it only requires repetition, with minor modifications, of the arguments used in the proof of Theorem 2 or in other papers, e.g., in [21].

Assuming that there are no other (linearly independent) solutions of problem (1) with $\lambda = \Lambda_\dagger$, we write the expansion

$$w^\infty(x) = \sum_{j=1,2} \sum_{\pm} \chi(\pm x_j) K_{j\pm} \Phi_1(x_{3-j}, x_3) + \widetilde{w}^\infty(x), \tag{43}$$

where Φ_1 is the first eigenfunction of the Dirichlet problem on the set ω normalized in $L_2(\omega)$, χ is a smooth cut-off function equal to 1 on $[1, +\infty)$ and to 0 on $(-\infty, 1/2]$, and $\widetilde{w}^\infty$ is a remainder decaying exponentially. Observe that uniqueness and the geometric symmetry of the waveguide show that two situations are possible:

$$K_{j\pm} = K \quad \text{or} \quad K_{j\pm} = (-1)^j K. \tag{44}$$

Of course, $K \neq 0$ for a stabilizing solution, and we shall need additional information about the eigenfunction $w^\infty \in \mathcal{H}$. The second case in (44) can be studied much as the first; in what follows it is assumed that $K_{j\pm} = K$, i.e., we deal with the solution (50) even with respect to the axes of the Cartesian system (x_1, x_2) and the bisectors of its quadrants.

First, we treat the Dirichlet problem in Π^R. Consider a solution of the auxiliary problem

$$-\Delta w'(x) - \Lambda_\dagger w'(x) = w^\infty(x), \;\; x \in \Pi^\infty, \quad w'(x) = 0, \;\; x \in \partial\Pi^\infty, \tag{45}$$

which, surely, is not unique but admits the representation

$$w'(x) = \sum_{j=1,2} \sum_{\pm} \chi_j(\pm x_j)\Big(- K\frac{x_j^2}{2} + K^1_{j\pm} x_j + K^0_{j\pm}\Big)\Phi_1(x_{3-j}, x_3) + \widetilde{w}'(x). \tag{46}$$

By symmetry, it may be assumed that $K^1_{j\pm} = \pm K^1$, $K^0_{j\pm} = K^0 = 0$. We shall not need these coefficients in the sequel.

On the ends of the truncated waveguide, the sum of two terms of the asymptotic *Ansatz* for the eigenfunction

$$w^R_D(x) = w^\infty(x) + \ell(R) w'(x) + \dots \tag{47}$$

turns into

$$K\Phi_1 + \ell(R)\Big(- \frac{1}{2} R^2 K \pm RK^1\Big)\Phi_1,$$

Therefore, keeping the Dirichlet condition in the main, we put

$$\ell(R) = 2R^{-2}. \tag{48}$$

Thus, the ansatz (42) takes the form

$$\Lambda^{R,st}_D = \Lambda_\dagger + 2R^{-2} + o(R^{-2}). \tag{49}$$

The stabilizing solution (43) leaves an exponentially small discrepancy in the Neumann conditions at the ends. Accordingly, we refine its expansion:

$$w^\infty(x) = \sum_{j=1,2} \sum_{\pm} \chi(\pm x_j)\big(K\Phi_1(x_{3-j}, x_3) + e^{\mp\beta_1 x_j}\Phi_2(x_{3-j}, x_3)\big) + \widehat{w}^\infty(x). \tag{50}$$

Here $\beta_1 = \sqrt{M_2 - M_1}$, $\widehat{w}^\infty$ is a rapidly decaying remainder, and Φ_2 is the second (non-normalized) eigenfunction of the Dirichlet problem on the cross-section. The function Φ_2 depends on the solution w^∞; without any normalization, it is a linear combination of orthonormalized eigenfunctions corresponding to the second eigenvalue $M_2 > M_1 = \Lambda_\dagger$. In what follows we assume that $\Phi_2 \neq 0$. We return to problem (45), this time seeking a solution of it such that

$$w'(x) = \sum_{j=1,2} \sum_{\pm} \chi(\pm x_j)\Big(\mathbf{K} e^{\pm\beta_1 x_j}\Phi_2(x_{3-j}, x_3) + \Big(- K\frac{x_j^2}{2} \pm K^1 x_j\Big)\Phi_1(x_{3-j}, x_3)\Big) + \widetilde{w}'(x), \tag{51}$$

where K^1 and $\mathbf{K}$ are unknown coefficients. Substituting w' and w^∞ in the Green formula on Π^R, we see that

$$\int_{\Pi^R} |w^\infty(x)|^2\, dx = - \sum_{j=1,2} \sum_{\pm} \pm \int_\omega \big(w^\infty(x)\partial_j w'(x) - w'(x)\partial_j w^\infty(x)\big)\Big|_{x_j = \pm R} dx_{3-j}\, dx_3.$$

On the right-hand side, we replace w' and w^∞ with their representations (51) and (50) and then pass to the limit as $R \to +\infty$. As a result, we get

$$A^\infty := \lim_{R\to+\infty} \left(\int_{\Pi^R} |w^\infty(x)|\, dx - 4RK^2|\omega| \right) = -8\beta_1 \|\Phi_2; L_2(\omega)\|^2 \mathbf{K}. \tag{52}$$

Therefore, on the ends of Π^R, the normal derivative of the sum of the terms written out in (47) will vanish in the main whenever

$$-\beta_1 K e^{-\beta_1 R}\Phi_2 + \ell(R)\beta_1 \mathbf{K} e^{\beta_1 R}\Phi_2 = 0,$$

i.e.,

$$\ell(R) = e^{-2\beta_1 R}\frac{KA^\infty}{8\beta_1}\|\Phi_2; L_2(\omega)\|^{-2}. \tag{53}$$

Now, suppose that the solution (43) is a trapped wave, i.e., in (50) we have $K = 0$. Despite the fact that $\Lambda_\dagger$ is an eigenvalue, problem (45) admits the solution (51) with exponential growth at infinity. Formula (52) for the coefficient $\mathbf{K}$ in the corresponding expansion remains valid, but now

$$A^\infty = \|w^\infty; L_2(\Pi^\infty)\|^2,$$

because $K = 0$. As a result, the eigenvalue (42) of the operator $\mathcal{L}_N^R$ takes the form

$$\Lambda_N^R = \Lambda_\dagger - \ell(R) + o(e^{-2\beta_1 R}), \tag{54}$$

$$\ell(R) = -\frac{1}{8\beta_1} e^{-2\beta_1 R} \frac{\|w^\infty; L_2(\Pi^\infty)\|^2}{\|\Phi_2; L_2(\omega)\|^2}. \tag{55}$$

Similar calculations show that the eigenvalue of $\mathcal{L}_D^R$ looks like this:

$$\Lambda_D^R = \Lambda_\dagger + \ell(R) + o(e^{-2\beta_1 R}). \tag{56}$$

For clear reasons, the eigenvalues (49) and (56) of the Dirichlet problem are located above the threshold, namely, $\Lambda_D^R > \Lambda_\dagger$, under the restrictions imposed. The eigenvalue (54) "goes down" the threshold, and its monotone growth is predicted in Proposition 1. We cannot conclude the same about the eigenvalue Λ_N^R in the case of perturbation of a stabilizing solution, because the limit in the middle part in (52) exists due to the expansion (50), but the sign of A^∞ remains unknown. Even the assumption that $w^\infty(x) \geq K > 0$ everywhere in Π^∞ does not guarantee the inequality $A^\infty > 0$, because the factor $4R|\omega|$ in the subtrahend under the limit sign is larger than the volume of Π^R. However, we have $A^\infty < 0$ in the hypothetical case where $|w^\infty(x)| \leq K$.

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