Алгебра и анализ
Том 29 (2017), № 3

St. Petersburg Math. J.
Vol. 29 (2018), No. 3, Pages 439–445
http://dx.doi.org/10.1090/spmj/1501
Article electronically published on March 30, 2018

RATIONALITY IN MAP AND HYPERMAP ENUMERATION BY GENUS

P. ZOGRAF AND M. KAZARIAN

Abstract. Generating functions for a fixed genus map and hypermap enumeration become rational after a simple explicit change of variables. Their numerators are polynomials with integral coefficients that obey a differential recursion, and the denominators are products of powers of explicit linear functions.

§1. Introduction

By a *map* or a *ribbon graph* we understand a finite connected graph with prescribed cyclic orders of half-edges at each vertex. It also can be realized as the 1-skeleton of a polygonal partition of a closed orientable surface. The genus g of a map (ribbon graph) satisfies the Euler formula

$$2-2g=\#v-\#e+\#f\,,$$

where $\#v, \#e, \#f$ are the numbers of vertices, edges and faces of the map respectively. By a *hypermap* we understand a bicolored map, i.e., a map whose faces are properly colored in two colors (say, white and black) so that no adjacent faces have the same color. The dual graph to a hypermap is a bipartite ribbon graph, or *Grothendieck's "design d'enfant"*[1].

We are interested in the weighted count of maps and hypermaps, where the weights are reciprocal to the orders of the corresponding automorphism groups. This is equivalent to counting *rooted* maps and hypermaps (i.e., those with a marked half-edge). The passage from the rooted count to the unrooted one is known, cf. [9, 10].

Denote by $\widetilde{c}_{g,n}$ (respectively, $c_{g,n}$) the number of rooted maps (respectively, hypermaps) of genus g with n edges (darts), and consider the genus g generating functions

$$\widetilde{C}_g(s)=\sum_{n=2g}^{\infty}\widetilde{c}_{g,n}s^n \quad g\ge 0, \tag{1}$$

$$C_g(s)=\sum_{n=2g+1}^{\infty}c_{g,n}s^n \quad g\ge 0. \tag{2}$$

The classical problem that goes back to Tutte [11] (or even earlier) is to compute the numbers $c_{g,n}$ and $\widetilde{c}_{g,n}$. Effective algorithms for computing these numbers first appeared

---

2010 *Mathematics Subject Classification.* Primary 05C30, 37K10.

Key words and phrases. Enumeration of maps (ribbon graphs), enumeration of hypermaps (Grothendieck's "dessins d'enfants").

Supported by the Russian Science Foundation: Theorem 1 by grant no. 14-21-00035, and Theorem 2 by grant no. 16-11-10316.

[1] As was observed by Grothendieck, the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ naturally acts on dessins (hypermaps); we refer the reader to [8] for the details.

in [13] (for maps) and in [14] (for hypermaps)[2]. Recursions for the numbers $c_{g,n}$ and $\widetilde{c}_{g,n}$, and differential equations for the generating functions $C_g(s)$ and $\widetilde{C}_g(s)$ were first obtained in [7] (cf. also [2] for an alternative approach to map enumeration).

In this note we show that the generating functions $C_g(s)$ and $\widetilde{C}_g(s)$ become rational functions after simple explicit changes of the variable s. Their numerators are then polynomials with integer coefficients that obey a differential recursion, and the denominators are products of powers of explicit linear functions.

§2. Main results

We start with the case of hypermaps (Grothendieck's dessins d'enfants).

Theorem 1. *Under the substitution $s = t(1-2t)$ we have*

$$\begin{aligned} C_0(t(1-2t)) &= \frac{t(1-3t)}{(1-2t)^2}, \\ C_1(t(1-2t)) &= \frac{t^3}{(1-t)(1-4t)^2}, \qquad (3)\\ C_g(t(1-2t)) &= \frac{P_g(t)}{(1-t)^{4g-3}(1-4t)^{5g-3}}, \quad g \geq 2, \end{aligned}$$

where $P_g(t) = \sum_{i=2g+1}^{9g-7} p_{g,i}\, t^i$ is a polynomial with integral coefficients and $p_{g,2g+1} = \frac{(2g)!}{g+1}$. The polynomials $P_g(t)$ can be computed recursively by formula (7).

Remark 1. The polynomials $P_g(t)$ for $g = 2, 3$ are:

$$\begin{aligned} P_2(t) &= 8t^5 - 92t^6 + 464t^7 - 1316t^8 + 2204t^9 - 2048t^{10} + 816t^{11}, \\ P_3(t) &= 180t^7 - 3648t^8 + 35424t^9 - 218944t^{10} + 958160t^{11} - 3102528t^{12} \\ &\quad + 7503664t^{13} - 13310768t^{14} + 16365216t^{15} - 11823680t^{16} + 117916t^{17} \\ &\quad + 6614784t^{18} - 6008320t^{19} + 1823744t^{20}. \end{aligned}$$

In principle, they can be computed for much larger values of g.

Remark 2. A similar result was independently obtained in [3] by a different (more complicated) method.

Proof. To prove the theorem, we recall a specialization of the Kadomtsev–Petviashvili (KP) equation for the hypermap count derived in [7]:[3]

$$(sC_g)' = 3\big(2s^2C_g' + sC_g\big) + 3s^3C_g' + s^3\big(s(sC_{g-1})'\big)'' + s^3\sum_{i=0}^{g}\big(4C_i + 6sC_i'\big)C_{g-i}' + 2s\delta_{g,0}, \qquad (4)$$

where the prime $'$ stands for the derivative $\frac{d}{ds}$. This equation is just the differential form of the recursion (11) in [7] for $t = u = v = 1$:

[2] An effective enumeration of 1-vertex maps was obtained in [4], and of 1-vertex hypermaps in [5] and, independently, in [1]. Enumeration of 1-vertex maps (or genus g gluings of a $2n$-gon) was a crucial point in computing the Euler characteristic of the moduli space of algebraic curves in [4].

[3] The validity of the equations of KP hierarchy and other integrable equations of mathematical physics for generating functions is a common feature of a wide range of problems of enumerative combinatorics. In [6] a survey of a collection of such problems was given, including, among others, the enumeration of maps and hypermaps.

$$(n+1)c_{g,n} = 3(2n-1)\,c_{g,n-1} + (n-2)\,c_{g,n-2} + (n-1)^2(n-2)\,c_{g-1,n-2} + \sum_{i=0}^{g}\sum_{j=1}^{n-3}(4+6j)(n-2-j)\,c_{i,j}\,c_{g-i,n-2-j}.$$

For $g \geq 1$ we can further rewrite (4) as the differential recursion

$$\begin{aligned}&\big(s-6s^2-3s^3-4s^3C_0-12s^4C_0'\big)\,C_g' + \big(1-3s-4s^3C_0'\big)C_g \\ &\qquad = s^5C_{g-1}''' + 5s^4C_{g-1}'' + 4s^3C_{g-1}' + s^3\sum_{i=1}^{g-1}(4C_i + 6sC_i')C_{g-i}'.\end{aligned} \tag{5}$$

For $g = 0$ we get an ordinary differential equation that can be solved explicitly:

$$C_0(s) = \frac{-1+12s-24s^2+(1-8s)^{3/2}}{32s^2}\,.$$

It is easily seen that the substitution $s = t(1-2t)$ considerably simplifies C_0 and makes it a rational function, namely

$$C_0(t(1-2t)) = \frac{t(1-3t)}{(1-2t)^2}$$

(cf. [12]). Substituting $s = t(1-2t)$ in (5), we get

$$\begin{aligned}&t(1-t)^2(1-2t)\,\dot{C}_g + (1-t)(1-2t+4t^2)\,C_g \\ &\qquad = t^3(1-2t)^3\left(D_tC_{g-1} + \frac{1}{(1-4t)^2}\sum_{i=1}^{g-1}\big(4(1-4t)C_i + 6t(1-2t)\dot{C}_i\big)\dot{C}_{g-i}\right),\end{aligned} \tag{6}$$

where $\dot{C}_g = \frac{d}{dt}C_g(t(1-2t))$ and

$$D_t = \frac{t^2(1-2t)^2}{(1-4t)^3}\cdot\frac{d^3}{dt^3} + \frac{t(1-2t)(5-28t+56t^2)}{(1-4t)^4}\cdot\frac{d^2}{dt^2} + \frac{4(1-11t+58t^2-144t^3+144t^4)}{(1-4t)^5}\cdot\frac{d}{dt}.$$

Assuming that $C_0,\dots,C_{g-1}$ are known, we can think of (6) as an ODE for C_g. The integrating factor for this equation is $\frac{1-t}{(1-2t)^3}$, so that we get from (6)

$$\frac{d}{dt}\left(\frac{t(1-t)^3}{(1-2t)^2}\,C_g\right) = t^3(1-t)\left(D_tC_{g-1} + \frac{1}{(1-4t)^2}\sum_{i=1}^{g-1}\big(4(1-4t)C_i + 6t(1-2t)\dot{C}_i\big)\dot{C}_{g-i}\right),$$

or, equivalently,

$$\begin{aligned}&C_g(t(1-2t) \\ &= \frac{(1-2t)^2}{t(1-t)^3}\int t^3(1-t)\left(D_tC_{g-1} + \frac{1}{(1-4t)^2}\sum_{i=1}^{g-1}\big(4(1-4t)C_i + 6t(1-2t)\dot{C}_i\big)\dot{C}_{g-i}\right)dt.\end{aligned} \tag{7}$$

Since by definition $C_g(0) = 0$ for all $g \geq 0$, equation (7) determines C_g uniquely in terms of $C_0,\dots,C_{g-1}$. In particular, this equation immediately yields

$$C_1(t(1-2t)) = \frac{t^3}{(1-t)(1-4t)^2}\,,$$

$$C_2(t(1-2t)) = \frac{8t^5-92t^6+464t^7-1316t^8+2204t^9-2048t^{10}+816t^{11}}{(1-t)^5(1-4t)^7}.$$

Let us show that $C_g(t(1-2t))$ has the form (3) for any $g \geq 3$. We will use the elementary formula

$$\frac{d}{dt}\left(\frac{t^\alpha}{(1-t)^\beta(1-4t)^\gamma}\right) = \frac{\alpha t^{\alpha-1} + (-5\alpha+\beta+4\gamma)t^\alpha + 4(\alpha-\beta-\gamma)t^{\alpha+1}}{(1-t)^{\beta+1}(1-4t)^{\gamma+1}}\,. \tag{8}$$

Then we have

$$D_t C_{g-1} = \frac{(2g-1)(2g)^2 p_{g-1,2g-1}t^{2g-2} + \cdots - 256 p_{g-1,9g-16}t^{9g-9}}{(1-t)^{4g-4}(1-4t)^{5g-2}} \tag{9}$$

and

$$\frac{1}{(1-4t)^2}\sum_{i=1}^{g-1}\big(4(1-4t)C_i + 6t(1-2t)\dot{C}_i\big)\dot{C}_{g-i} = \frac{r_g t^{2g+1} + \ldots + 256 p_{g-1,9g-16}t^{9g-9}}{(1-t)^{4g-4}(1-4t)^{5g-2}}, \tag{10}$$

where r_g is some constant. Notice that the top degree term in the numerator on the right-hand side of (10) comes entirely from the product $C_1\dot{C}_{g-1}$. Multiplying both sides of (9) and (10) by $t^3(1-t)$ and taking their sum, we see that the integrand in (7) has the form

$$\frac{Q_g(t)}{(1-t)^{4g-5}(1-4t)^{5g-2}}, \tag{11}$$

where $Q_g(t) = \sum_{i=2g+1}^{9g-7} q_{g,i}t^i$ is a polynomial with $q_{g,2g+1} = (2g-1)(2g)^2 p_{g-1,2g-1}$. Therefore, we can rewrite (7) as

$$C_g(t(1-2t)) = \frac{(1-2t)^2}{t(1-t)^3}\int \frac{Q_g(t)}{(1-t)^{4g-5}(1-4t)^{5g-2}}\,dt. \tag{12}$$

To perform integration in (12), we decompose the integrand in the sum

$$\frac{Q_g(t)}{(1-t)^{4g-5}(1-4t)^{5g-2}} = a + \sum_{i=2}^{4g-5}\frac{a_i}{(1-t)^i} + \sum_{j=2}^{5g-2}\frac{b_j}{(1-4t)^j}\,. \tag{13}$$

Note that no terms of the form $\frac{a_1}{1-t}$ or $\frac{b_1}{1-4t}$ can appear on the right hand side of (13) because the Taylor series expansion of the left-hand side of (12) has integral coefficients[4]. Integrating, we obtain

$$\int \frac{Q_g(t)}{(1-t)^{4g-5}(1-4t)^{5g-2}}\,dt = at + b + \sum_{i=1}^{4g-6}\frac{a_{i+1}}{i}\cdot\frac{1}{(1-t)^i} + \sum_{j=1}^{5g-3}\frac{b_{j+1}}{4j}\cdot\frac{1}{(1-4t)^j}, \tag{14}$$

where the condition $C_g(0) = 0$ implies

$$b = -\sum_{i=1}^{4g-6}\frac{a_i}{i} - \sum_{j=2}^{5g-3}\frac{b_{j+1}}{4j}\,.$$

Multiplying the right-hand side of (14) by $(1-t)^{4g-6}(1-4t)^{5g-3}$, we get a polynomial of the form $R_g(t) = \sum_{i=2g+2}^{9g-8} r_{g,i}t^i$. To complete the proof, we put $P_g(t) = \frac{(1-2t)^2}{t}R_g(t)$ and notice that $p_{g,2g+1} = \frac{(2g-1)(2g)^2}{2g+2}p_{g-1,2g-1}$. Moreover, we see that $t = 1/2$ is a root of $P_g(t)$ of multiplicity 2 provided $g \geq 2$.[5] □

Now we continue with map enumeration.

[4]We owe this observation to F. Petrov.

[5]Numerically, we also have $p_{g,9g-7} \neq 0$, $P_g(1) \neq 0$, $P_g(1/4) \neq 0$. In principle, this can be verified along the same lines as above, but computations become too cumbersome to reproduce them here.

Theorem 2. *Under the substitution $s = t(1-3t)$ we have*

$$\begin{aligned} \widetilde{C}_0(t(1-3t)) &= \frac{1-4t}{(1-3t)^2}, \\ \widetilde{C}_1(t(1-3t)) &= \frac{t^2}{(1-2t)(1-6t)^2}, \\ \widetilde{C}_g(t(1-3t)) &= \frac{\widetilde{P}_g(t)}{(1-2t)^{3g-2}(1-6t)^{5g-3}}, \quad g \geq 2, \end{aligned} \tag{15}$$

where $\widetilde{P}_g(t) = \sum_{i=2g}^{8g-6} \widetilde{p}_{g,i}\, t^i$ with $\widetilde{p}_{g,2g} = \frac{(4g-1)!!}{2g+1}$. The polynomials $\widetilde{P}_g(t)$ can be computed recursively by (20).

Remark 3. The polynomials $\widetilde{P}_g(t)$ for $g = 2, 3$ are:

$$\begin{aligned} \widetilde{P}_2(t) &= 21t^4 - 336t^5 + 2334t^6 - 9108t^7 + 21177t^8 - 27756t^9 + 15876t^{10}, \\ \widetilde{P}_3(t) &= 1485t^6 - 41184t^7 + 539073t^8 - 4483458t^9 + 26893989t^{10} - 124232004t^{11} \\ &\quad + 453861279t^{12} - 1307353122t^{13} + 2897271774t^{14} - 4737605112t^{15} \\ &\quad + 5355443952t^{16} - 3723895296t^{17} + 1197496224t^{18}. \end{aligned}$$

Like in the case of hypermaps, they can be computed for much larger values of g.

Proof. The proof of Theorem 2 is quite similar to that of Theorem 1, so we shall only outline its main steps. We recall a specialization of the Kadomtsev–Petviashvili (KP) equation for the map count, derived in [7]:

$$\begin{aligned} (s\widetilde{C}_g)' = 4\big(2s^2\widetilde{C}_g' + s\widetilde{C}_g\big) + 2s^3\big(2s(s\widetilde{C}_{g-1})' + s\widetilde{C}\big)'' + s^2\big(2s(s\widetilde{C}_{g-1})' + s\widetilde{C}\big)' \\ + 3s^2 \sum_{i=0}^{g} \big(\widetilde{C}_i + 2s\widetilde{C}_i'\big)\big(\widetilde{C}_{g-i} + 2s\widetilde{C}_{g-i}'\big) + \delta_{g,0}, \end{aligned} \tag{16}$$

where the prime $'$ stands for the derivative $\frac{d}{ds}$. This equation is merely a differential form of the recursion (16) in [7] for $t = u = 1$:

$$\begin{aligned} (n+1)\widetilde{c}_{g,n} = 4(2n-1)\,\widetilde{c}_{g,n-1} + (2n-1)(2n-3)(n-1)\,\widetilde{c}_{g-1,n-2} \\ + 3\sum_{i=0}^{g}\sum_{j=0}^{n-2}(2j+1)(2(n-2-j)+1)\,\widetilde{c}_{i,j}\,\widetilde{c}_{g-i,n-2-j}. \end{aligned} \tag{17}$$

For $g \geq 1$, we can further rewrite formula (16) as the differential recursion

$$\begin{aligned} &\big(s - 8s^2 - 12s^3\widetilde{C}_0 - 24s^4\widetilde{C}_0'\big)\widetilde{C}_g' + \big(1 - 4s - 6s^2\widetilde{C}_0 - 12s^3\widetilde{C}_0'\big)\widetilde{C}_g \\ &= 4s^5\widetilde{C}_{g-1}''' + 24s^4\widetilde{C}_{g-1}'' + 27s^3\widetilde{C}_{g-1}' + 3s^2\widetilde{C}_{g-1} + 3s^2\sum_{i=0}^{g}(\widetilde{C}_i + 2s\widetilde{C}_i')(\widetilde{C}_{g-i} + 2s\widetilde{C}_{g-i}'). \end{aligned} \tag{18}$$

For $g = 0$ we get an ordinary differential equation that can be solved explicitly:

$$\widetilde{C}_0(s) = \frac{-1 + 18s + (1-12s)^{3/2}}{54s^2}.$$

It is easily seen that the substitution $s = t(1-3t)$ simplifies $\widetilde{C}_0$ considerably and makes it a rational function, namely,

$$\widetilde{C}_0(t(1-3t)) = \frac{1-4t}{(1-3t)^2}$$

(cf. [11]). Substituting $s = t(1-3t)$ in (18), we get

$$\begin{aligned} &t(1-2t)(1-3t)\,\dot{\widetilde{C}}_g + (1-4t+6t^2)\widetilde{C}_g \\ &\quad = t^2(1-3t)^2\left(\widetilde{D}_t\widetilde{C}_{g-1} + 3\sum_{i=1}^{g-1}\left(\widetilde{C}_i + \frac{t(1-3t)}{1-6t}\dot{\widetilde{C}}_i\right)\left(\widetilde{C}_{g-i} + \frac{t(1-3t)}{1-6t}\dot{\widetilde{C}}_{g-i}\right)\right), \end{aligned} \tag{19}$$

where $\dot{\widetilde{C}}_g = \frac{d}{dt}\widetilde{C}_g(t(1-3t))$ and

$$\begin{aligned} \widetilde{D}_t = \frac{4t^3(1-3t)^3}{(1-6t)^3}\cdot\frac{d^3}{dt^3} &+ \frac{24t^2(1-3t)^2(1-9t+27t^2)}{(1-6t)^4}\cdot\frac{d^2}{dt^2} \\ &+ \frac{9t(1-3t)(3-56t+456t^2-1728t^3+2592t^4)}{(1-6t)^5}\cdot\frac{d}{dt} + 3. \end{aligned}$$

Assuming that $\widetilde{C}_0, \dots, \widetilde{C}_{g-1}$ are known, we can think of (19) as an ODE for $\widetilde{C}_g$. The integrating factor for this equation is $\frac{t(1-2t)}{1-3t}$, so that from (19) we get

$$\begin{aligned} &\widetilde{C}_g(t(1-3t) \\ &\quad = \frac{1-3t}{t(1-2t)}\int t^2\left(\widetilde{D}_t\widetilde{C}_{g-1} + 3\sum_{i=1}^{g-1}\left(\widetilde{C}_i + \frac{t(1-3t)}{1-6t}\dot{\widetilde{C}}_i\right)\left(\widetilde{C}_{g-i} + \frac{t(1-3t)}{1-6t}\dot{\widetilde{C}}_{g-i}\right)\right)dt. \end{aligned} \tag{20}$$

Since by definition $\widetilde{C}_g(0) = 0$ for all $g \geq 1$, formula (20) determines $\widetilde{C}_g$ uniquely in terms of $\widetilde{C}_0, \dots, \widetilde{C}_{g-1}$. In particular, this formula immediately yields

$$\widetilde{C}_1(t(1-3t)) = \frac{t^2}{(1-2t)(1-6t)^2}\,.$$

We show that $\widetilde{C}_g(t(1-3t))$ has the form (15) for any $g \geq 2$. Using an analog of (8), we deduce, after some cancellations, that the integrand in (20) has the form

$$\frac{\widetilde{Q}_g(t)}{(1-2t)^{3g-2}(1-6t)^{5g-2}}\,, \tag{21}$$

where $\widetilde{Q}_g(t) = \sum_{i=2g}^{8g-6}\widetilde{q}_{g,i}t^i$ is a polynomial with $\widetilde{q}_{g,2g} = (2g-1)(4g-1)(4g-3)\,p_{g-1,2g-2}$. It can be further decomposed into the sum

$$\frac{\widetilde{Q}_g(t)}{(1-2t)^{3g-2}(1-6t)^{5g-2}} = \sum_{i=2}^{3g-2}\frac{\widetilde{a}_i}{(1-2t)^i} + \sum_{j=2}^{5g-2}\frac{\widetilde{b}_j}{(1-6t)^j}\,. \tag{22}$$

Integrating it, we obtain

$$\int\frac{\widetilde{Q}_g(t)}{(1-2t)^{3g-2}(1-6t)^{5g-2}}\,dt = \widetilde{b} + \sum_{i=1}^{3g-3}\frac{\widetilde{a}_{i+1}}{2i}\cdot\frac{1}{(1-2t)^i} + \sum_{j=2}^{5g-3}\frac{\widetilde{b}_{j+1}}{6j}\cdot\frac{1}{(1-6t)^j}, \tag{23}$$

where the condition $\widetilde{C}_g(0) = 0$ implies

$$\widetilde{b} = -\sum_{i=1}^{3g-3}\frac{\widetilde{a}_{i+1}}{2i} - \sum_{j=1}^{5g-3}\frac{\widetilde{b}_{j+1}}{6j}\,.$$

Multiplying the right-hand side of (23) by $(1-2t)^{3g-3}(1-6t)^{5g-3}$, we get a polynomial of the form $\widetilde{R}_g(t) = \sum_{i=2g+1}^{8g-6}\widetilde{r}_{g,i}t^i$. To complete the proof, we put $\widetilde{P}_g(t) = \frac{1-3t}{t}\widetilde{R}_g(t)$ and notice that $\widetilde{p}_{g,2g} = \frac{(2g-1)(4g-1)(4g-3)}{2g+1}\,\widetilde{p}_{g-1,2g-2}$. Moreover, we see that $t = 1/3$ is a root of $\widetilde{P}_g(t)$.[6] ☐

[6]Numerically, we also have $\widetilde{p}_{g,8g-6} \neq 0$, $\widetilde{P}_g(1/2) \neq 0$, $\widetilde{P}_g(1/6) \neq 0$.

Acknowledgments

We are grateful to F. Petrov for a useful suggestion. We thank A. Giorgetti, A. Mednykh, R. Nedela, and T. R. S. Walsh for discussions.

References

[1] N. M. Adrianov, *An analogue of the Harer–Zagier formula for unicellular two-color maps*, Funktsional. Anal. i Prilozhen. **31** (1997), no. 3, 1–9; English transl., Funct. Anal. Appl. **31** (1997), no. 3, 149–155. MR1479071

[2] S. R. Carrell and G. Chapuy, *Simple recurrence formulas to count maps on orientable surfaces*, J. Combin. Theory Ser. A **133** (2015), 58–75. MR3325628

[3] A. Giorgetti and T. R. S. Walsh, *Enumeration of hypermaps of a given genus*, `arXiv:1510.09019 (2015).`

[4] J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986), no. 3, 457–485. MR848681

[5] D. M. Jackson, *Counting cycles in permutations by group characters, with an application to a topological problem*, Trans. Amer. Math. Soc. **299** (1987), no. 2, 785–801. MR869231

[6] M. E. Kazaryan and S. K. Lando, *Combinatorial solutions to integrable hierarchies*, Uspekhi Mat. Nauk **70** (2015), no. 3, 77–106; English transl., Russian Math. Surveys **70** (2015), no. 3, 453–482. MR3400565

[7] M. Kazarian and P. Zograf, *Virasoro constraints and topological recursion for Grothendieck's dessin counting*, Lett. Math. Phys. **105** (2015), no. 8, 1057–1084. MR3366120

[8] S. K. Lando and A. K. Zvonkin, *Graphs on surfaces and their applications*, MCCME, Moscow, 2010. (Russian).

[9] A. Mednykh and R. Nedela, *Enumeration of unrooted maps of a given genus*, J. Combin. Theory Ser. B **96** (2006), no. 5, 706–729. MR2236507

[10] ______, *Enumeration of unrooted hypermaps of a given genus*, Discrete Math. **310** (2010), no. 3, 518–526. MR2564806

[11] W. T. Tutte, *A census of planar maps*, Canad. J. Math. **15** (1963), no. 2, 249–271. MR0146823

[12] T. R. S. Walsh, *Hypermaps versus bipartite maps*, J. Combin. Theory Ser. B **18** (1975), no. 2, 155–163. MR0360328

[13] T. R. S. Walsh and A. B. Lehman, *Counting rooted maps by genus*. I, J. Combin. Theory Ser. B **13** (1972), 192–218. MR0314686

[14] P. Zograf, *Enumeration of Grothendieck's dessins and KP hierarchy*, Int. Math. Res. Not. IMRH **2015**, no. 24, 13533–13544. MR3436154

St. Petersburg Branch, Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191023, Russia

Chebyshev Laboratory, St. Petersburg State University, 14th Line V.O. 29B, St. Petersburg 199178, Russia
Email address: `zograf@pdmi.ras.ru`

Steklov Mathematical Institute, Gubkin St. 8, Moscow 119991, Russia

Department of Mathematics, National Research University Higher School of Economics, Usacheva Str. 6, Moscow 119048 Russia
Email address: `kazarian@mccme.ru`

Received 10/NOV/2016

Translated by P. ZOGRAF