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TRANSFER OF THE UNITARY K_1-FUNCTOR UNDER POLYNOMIAL EXTENSIONS

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Abstract. Transfer of the unitary K_1-functor under polynomial extensions of unitary rings is constructed and composition of this transfer with the natural homomorphism induced by embedding of polynomial rings is computed. As an application of the composition formula, unitary K_1-analogs of Springer and Farrell theorems are proved.

§1. Introduction

In a letter to the author dated 20 April 2015, Professor Ravi Rao from the Tata Institute of Fundamental Research, Mumbai, India, posed the following problem: construct the transfer $K_1 \operatorname{Sp}(R[X]) \to K_1 \operatorname{Sp}(R[X^2])$ and compute the composition of the natural homomorphism $K_1 \operatorname{Sp}(R[X^2]) \to K_1 \operatorname{Sp}(R[X])$ induced by the embedding $R[X^2] \to R[X]$ with the above transfer, with the hypothesis that it coincides with multiplication by 2. As is well known (see, for instance, [1, Chapter 9, Proposition 1.8]), in the linear case such a transfer exists and the composition $K_1(R[X^2]) \to K_1(R[X]) \to K_1(R[X^2])$ coincides with multiplication by $2 = [R^2]$, when we view $K_1(R[X^2])$ as a $K_0(R)$-module.

Our goal in the present paper is to construct, for an arbitrary unitary ring R with a form parameter Λ and a symmetry λ, and for an arbitrary integer $n \geq 2$, a transfer $(i_n)_*\colon K_1U^\lambda(R[X], \Lambda[X]) \to K_1U^\lambda(R[X^n], \Lambda[X^n])$, where i_n denotes the canonical embedding $R[X^n] \to R[X]$, and to calculate the composition $(i_n)_* \circ (i_n)^*$, where $(i_n)^*$ denotes the natural homomorphism $K_1U^\lambda(R[X^n], \Lambda[X^n]) \to K_1U^\lambda(R[X], \Lambda[X])$ induced by the embedding i_n. In Theorem 1 we construct such a transfer and in Theorem 2 we show that the composition $(i_n)_* \circ (i_n)^*$ coincides with kH if $n = 2k$, and with $\mathrm{id} + kH$ if $n = 2k+1$, where id denotes the identity map and kH denotes the k-multiple of the hyperbolic homomorphism H. In particular, when R is a commutative ring with trivial involution, $\Lambda = R$, and $\lambda = -1$, we get the usual symplectic K_1-functor, for which by Theorem 2 the composition $(i_2)_* \circ (i_2)^*\colon K_1 \operatorname{Sp}(R[X^2]) \to K_1 \operatorname{Sp}(R[X]) \to K_1 \operatorname{Sp}(R[X^2])$ coincides with the hyperbolic homomorphism $H\colon [\alpha] \to [\alpha \oplus (\alpha^t)^{-1}]$, which gives a complete answer to the problem posed by Rao.

Our results can be viewed as unitary K_1-analogs of the classical results by Scharlau in the algebraic theory of quadratic forms on the existence and properties of transfer under finite field extensions, for polynomial extensions of unitary rings. Observe that, in the construction of transfer, Scharlau assumed the existence of a trace map (Scharlau trace).

As corollaries to Theorem 2 in the present paper, we prove that for any odd n the restriction of the natural homomorphism $(i_n)^*$ to the Witt cogroup is a split monomorphism (Corollary 1), and, furthermore, the homomorphism induced by $(i_n)^*$ on the Witt group is a split monomorphism (Corollary 2).

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The above corollaries can be viewed as unitary K_1-analogs for the polynomial extensions of unitary rings of the classical theorem by Springer in the algebraic theory of quadratic forms, concerning the properties of quadratic forms under finite field extensions of odd degree.

As another application of the composition formula mentioned above we prove unitary K_1-analogs of Farrell's theorem for unitary Witt cogroups (Theorem 3) and for unitary Witt groups (Theorem 4).

The author is grateful to Prof. Rao for proposing the above problem.

§2. Hyperbolic unitary group

In the present paper we follow the standard setting and notation, see [2]. Let us reproduce some basic definitions and results used in the sequel.

We fix a unitary ring (R, λ, Λ), where R is an associative ring with 1, supporting an involution $x \to \bar{x}$. Further, let λ be a central element of R such that $\lambda \cdot \bar{\lambda} = 1$, and let Λ be an additive subgroup of R such that $\Lambda_{\min} \leq \Lambda \leq \Lambda_{\max}$, and, moreover, $\bar{x}\Lambda x \subseteq \Lambda$ for any $x \in R$. Here $\Lambda_{\min} = \{x - \lambda\bar{x}, x \in R\}$ and $\Lambda_{\max} = \{x \in R : x = -\lambda\bar{x}\}$. Note that $(R, \bar{\lambda}, \bar{\Lambda})$, where $\bar{\Lambda} = \{\bar{x}, x \in \Lambda\}$, is also a unitary ring. In the literature, the unitary rings (R, λ, Λ) are also called form rings with form parameter Λ and symmetry λ.

Extend the involution to the matrix ring $M_r(R)$ by setting $(a_{ij})^* = (\overline{a_{ji}})$. We say that a matrix $a = (a_{ij})$ is λ-anti-Hermitian if it satisfies the condition $a = -\lambda a^*$. If, furthermore, all diagonal entries of a belong to Λ, the matrix a is said to be Λ-Hermitian. A matrix is $\bar{\Lambda}$-Hermitian if it is $\bar{\lambda}$-anti-Hermitian and its diagonal entries belong to $\bar{\Lambda}$.

In the present paper we write matrices in a block form. Namely, $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2r}(R)$ means that $a, b, c, d \in M_r(R)$. To shorten the notation, we denote the block diagonal matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ by $a \oplus d$, and the matrix $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ by $b \oslash c$.

For a natural r we set

$$q_r = \begin{pmatrix} 0 & e_r \\ 0 & 0 \end{pmatrix} = e_r \oslash 0, \quad I_r^\lambda = \begin{pmatrix} 0 & e_r \\ \lambda e_r & 0 \end{pmatrix} = q_r + \lambda q_r^* = e_r \oslash \lambda e_r,$$

where e_r (respectively, 0) denotes the identity (respectively, zero) matrix of degree r. Observe that the matrix I_r^λ is invertible and $(I_r^\lambda)^{-1} = (I_r^\lambda)^*$. In the sequel the index r or even both r and λ in the matrices e_r, q_r, I_r^λ will be omitted if they are clear from the context.

A matrix $\alpha \in M_{2r}(R)$ is called Λ-unitary if the matrix $\alpha^* q\alpha - q$ is Λ-Hermitian. The fact that $\alpha^* q\alpha - q$ is λ-anti-Hermitian implies that $\alpha^* I^\lambda \alpha = I^\lambda$. The matrices subject to this condition will be called unitary. In particular, each Λ-unitary matrix $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible and

$$\alpha^{-1} = I^* \alpha^* I = \begin{pmatrix} d^* & \bar{\lambda} b* \\ \lambda c* & a* \end{pmatrix}.$$

The set of all Λ-unitary matrices of degree $2r$ forms a group denoted by $U_{2r}^\lambda(R, \Lambda)$ and is called the (hyperbolic) Λ-unitary group. The group $U_{2r}^\lambda(R, \Lambda_{\max})$ consisting of all unitary matrices is denoted by $U_{2r}^\lambda(R)$ and is called the (hyperbolic) unitary group. In the sequel we usually drop the indication "hyperbolic".

Observe that in the case where $\lambda = -1$ and $\Lambda = \Lambda_{\max}$, the group $U_{2r}^\lambda(R, \Lambda)$ is the classical unitary group $U_{2r}(R)$. Furthermore, when R is a commutative ring with trivial involution, $\lambda = -1$, and $\Lambda = \Lambda_{\max} = R$, we get the usual symplectic group $U_{2r}(R) = \mathrm{Sp}_{2r}(R)$. Similarly, when $\lambda = 1$ and $\Lambda = \Lambda_{\min} = 0$, we get the orthogonal group $U_{2r}^\lambda(R, \Lambda) = O_{2r}(R)$. Thus, passing to unitary rings with an arbitrary symmetry λ and an arbitrary form parameter Λ, one can unify the study of classical groups over rings with involution.

We reproduce several well-known equivalent definitions and some standard examples of Λ-unitary matrices (see, e.g., [2, Chapter 2]).

Proposition 1. *For a matrix* $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2r}(R)$ *the following conditions are equivalent*:

1) α *is* Λ*-unitary*;
2) α *is unitary and the diagonal entries of the matrices* a^*c, b^*d *are contained* Λ;
3) $a^*d + \lambda c^*b = e_r$ *and the matrices* a^*c, b^*d *are* Λ*-Hermitian.*

The identity $\alpha^* I^\lambda \alpha = I^\lambda$ implies that $\alpha^* = I^\lambda \alpha^{-1}(I^\lambda)^*$ and, in particular, $\alpha \in U_{2r}^\lambda(R,\Lambda)$ if and only if $\alpha^* \in U_{2r}^\lambda(R,\Lambda)$. Now, item 3) of Proposition 1 applied to the matrix α^* implies the following.

Corollary 1. *In the notation of Proposition* 1*, a matrix* α *is* Λ*-unitary if and only if* $ad^* + \lambda bc^* = e_r$ *and the matrices* ab^*, cd^* *are* Λ*-Hermitian.*

Corollary 2. *Let* a, $d \in M_r(R)$; *then* $a \oplus d \in U_{2r}^\lambda(R,\Lambda)$ *if and only if* a *is invertible and* $d = (a^*)^{-1}$.

A matrix of the form $a \oplus (a^*)^{-1}$, where $a \in \mathrm{GL}_r(R)$, is denoted by $H(a)$ and is said to be hyperbolic. The assignment $a \to H(a)$ gives rise to the hyperbolic mapping $H\colon \mathrm{GL}_r(R) \to U_{2r}^\lambda(R,\Lambda)$, which is a group homomorphism.

Corollary 3. *We have* $X_+(b) = \left(\begin{smallmatrix} e & b \\ 0 & e \end{smallmatrix}\right) \in U_{2r}^\lambda(R,\Lambda)$ *if and only if the matrix* b *is* $\bar\Lambda$*-Hermitian.*

Corollary 4. *We have* $X_-(c) = \left(\begin{smallmatrix} e & 0 \\ c & e \end{smallmatrix}\right) \in U_{2r}^\lambda(R,\Lambda)$ *if and only if the matrix* c *is* Λ*-Hermitian.*

Denote by $\mathrm{EU}_{2r}^\lambda(R,\Lambda)$ the subgroup of $U_{2r}^\lambda(R,\Lambda)$ generated by the matrices of the form $H(\alpha)$, $X_+(b)$, $X_-(c)$, where $\alpha \in E_r(A)$, b is $\bar\Lambda$-Hermitian, and c is Λ-Hermitian. The group $\mathrm{EU}_{2r}^\lambda(R,\Lambda)$ is called the elementary (hyperbolic) Λ-unitary group.

We introduce the following operation on matrices. Suppose $\alpha = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right) \in M_{2r}(R)$, $\beta = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right) \in M_{2s}(R)$. Denote

$$\alpha \perp \beta = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix} \in M_{2(r+s)}(R).$$

We list some properties of the above operation that will be used in the sequel.

1. $I_r^\lambda \perp I_s^\lambda = I_{r+s}^\lambda$ for all natural r, s. In particular, $\alpha \perp \beta \in U_{2(r+s)}^\lambda(R,\Lambda)$ if and only if $\alpha \in U_{2r}^\lambda(R,\Lambda)$, $\beta \in U_{2s}^\lambda(R,\Lambda)$.

2. $H(\alpha \oplus \beta) = H(\alpha) \perp H(\beta)$ for all $\alpha, \beta \in \mathrm{GL}(R)$.

Define the embedding $U_{2r}^\lambda(R,\Lambda) \to U_{2r+2}^\lambda(R,\Lambda)\colon \alpha \to \alpha \perp e_2$ and set $U^\lambda(R,\Lambda) = \cup U_{2r}^\lambda(R,\Lambda)$, $\mathrm{EU}^\lambda(R,\Lambda) = \cup\, \mathrm{EU}_{2r}^\lambda(R,\Lambda)$.

We recall the unitary analog of the Whitehead lemma, stated by Bass in [2, Chapter 2, Proposition 3.7] and proved by Vaserstein in [3, Lemma 1].

Proposition 2. *Let* $\alpha, \beta \in U_{2r}^\lambda(R,\Lambda)$*. Then the matrices* $\alpha \perp \alpha^{-1}$ *and* $[\alpha, \beta] \perp e_{2r}$ *are contained in* $\mathrm{EU}_{4r}^\lambda(R,\Lambda)$*, where* $[\alpha,\beta]$ *denotes the commutator.*

By Proposition 2, the group $\mathrm{EU}^\lambda(R,\Lambda)$ coincides with the commutator subgroup of the group $U^\lambda(R,\Lambda)$, so that the (Abelian) group $K_1U^\lambda(R,\Lambda) = U^\lambda(R,\Lambda)/\,\mathrm{EU}^\lambda(R,\Lambda)$ is well defined. The class of a matrix $\alpha \in U^\lambda(R,\Lambda)$ in the group $K_1U^\lambda(R,\Lambda)$ is denoted by $[\alpha]$.

We state some corollaries to Proposition 2 that will be used in the sequel.

Corollary 1. *Under the conditions of Proposition* 2, *in the group* $K_1U^\lambda(R,\Lambda)$ *we have* $[\alpha \perp \beta] = [\alpha\beta]$.

Proof. By Proposition 2, we have $\alpha \perp \beta \equiv (\alpha \perp \beta)(\beta \perp \beta^{-1}) = \alpha\beta \perp e_{2r} \bmod \mathrm{EU}_{4r}^\lambda(R,\Lambda)$, and, thus, $\alpha \perp \beta \equiv \alpha\beta \bmod \mathrm{EU}^\lambda(R,\Lambda)$. □

Corollary 2. *For any matrix* $\alpha \in U^\lambda(R,\Lambda)$ *in the group* $K_1U^\lambda(R,\Lambda)$ *we have* $[H(\alpha)] = [H((\alpha^*)^{-1})]$.

Proof. To prove the corollary, it suffices to verify that $\alpha^*\alpha \in \mathrm{EU}^\lambda(R,\Lambda)$. Let $\alpha \in U_{2r}^\lambda(R,\Lambda)$. By assumption, $\alpha^* I_r^\lambda \alpha = I_r^\lambda$ and, thus, $e = I^{-1}\alpha^* I^\lambda \alpha = [I,(\alpha^*)^{-1}](\alpha^*\alpha)$. By Proposition 2, we have $\alpha^*\alpha = [I,(\alpha^*)^{-1}]^{-1} = [(\alpha^*)^{-1}, I] \in \mathrm{EU}^\lambda(R,\Lambda)$. □

Below, for a natural k we denote by kH the kth multiple $H \perp H \perp \cdots \perp H$ of the hyperbolic mapping H.

Corollary 3. *Under the assumptions of Corollary* 2, *for any natural* k *in the group* $K_1U^\lambda(R,\Lambda)$ *we have* $[kH(\alpha)] = [H(\alpha^k)]$.

Proof. To use induction, it suffices to prove that $2H(\alpha) \equiv H(\alpha^2) \bmod \mathrm{EU}^\lambda(R,\Lambda)$. Let $\alpha \in U_{2r}^\lambda(R,\Lambda)$. We have $2H(\alpha) = H(\alpha) \perp H(\alpha) = H(\alpha \oplus \alpha)$. But by the Whitehead lemma (see, e.g., Proposition 1.7 in [1, Chapter 5]) we have $\alpha \oplus \alpha \equiv \alpha^2 \oplus e_{2r} \bmod E_{4r}(R)$, whence $\alpha \oplus \alpha \equiv \alpha^2 \bmod E(R)$. □

§3. Construction of transfer

Let (R,λ,Λ) be a unitary ring. We extend the involution to the polynomial ring $R[X]$ by setting $\bar X = X$. Then $(R[X^n],\lambda,\Lambda[X^n])$ is a unitary ring for any natural n. Below we assume that $n \geq 2$. Denote by i_n the canonical embedding of the unitary rings $(R[X^n],\lambda,\Lambda[X^n]) \to (R[X],\lambda,\Lambda[X])$ that induces the natural homomorphism $(i_n)^*\colon K_1U^\lambda(R[X^n],\Lambda[X^n]) \to K_1U^\lambda(R[X],\Lambda[X])$.

Take an arbitrary matrix $\alpha = \alpha(X) \in M_{2r}(R[X])$. Since $R[X]$ is a free (left) $R[X^n]$-module of rank n with the base $\{1, X, \dots, X^{n-1}\}$, the matrix α can uniquely be written in the form $\alpha = a_0 + a_1X + \cdots + a_{n-1}X^{n-1}$. Let $a_s = a_s(X^n) = \left(\begin{smallmatrix} a_{1s} & a_{2s} \\ a_{3s} & a_{4s}\end{smallmatrix}\right)$, where $a_{ks} = a_{ks}(X^n) \in M_r(R[X^n])$ for $k = 1,\dots,4$, $s = 0,\dots,n-1$.

Proposition 3. *In the above notation and under the above assumptions we have* $\alpha = \alpha(X) \in U_{2r}^\lambda(R[X],\Lambda[X])$ *if and only if*

1) *the following identities are true*:

1_0 : $a_0^* I a_0 + \sum_{s=1}^{n-1} a_s^* I a_{n-s} X^n = I$;

1_m : $\sum_{s=0}^{m} a_s^* I a_{m-s} + \sum_{s=m+1}^{n-1} a_s^* I a_{m+n-s} X^n = 0$ *for all* $m = 1,\dots,n-2$;

1_{n-1} : $\sum_{s=0}^{n-1} a_s^* I a_{n-1-s} = 0$;

2) *the diagonal entries of the following matrices are contained in* $\Lambda[X^n]$, *where* t *takes the values* 1 *or* 2:

2_0 : $a_{t0}^* a_{t+20} + \sum_{s=1}^{n-1} a_{ts}^* a_{t+2,n-s} X^n$;

2_m : $\sum_{s=0}^{m} a_{ts}^* a_{t+2,m-s} + \sum_{s=m+1}^{n-1} a_{ts}^* a_{t+2,m+n-s} X^n$ *for all* $m = 1,\dots,n-2$;

2_{n-1} : $\sum_{s=0}^{n-1} a_{ts}^* a_{t+2,n-1-s}$.

Proof. The claim follows from the equivalence of Conditions 1) and 2) of Proposition 1. Indeed, if a matrix α is $\Lambda[X]$-unitary, then it is unitary, and, thus,

$$\Big(\sum_{s=0}^{n-1} a_s X^s\Big)^* \cdot I \cdot \sum_{s=0}^{n-1} a_s X^s = I.$$

Equating the coefficients of the corresponding degrees of X on the left and on the right, we get identities 1_s for all $s = 0,\dots,n-1$.

Moreover, by item 2) of Proposition 1, the diagonal entries of the matrices

$$\Big(\sum_{s=0}^{n-1} a_{ts}X^s\Big)^* \cdot \sum_{s=0}^{n-1} a_{t+2,s}X^s$$

are contained in $\Lambda[X^n]$ for t equal to 1 or 2. This implies conditions 2_s for all $s = 0, \dots, n-1$. □

We state one of the main results of the present paper.

Theorem 1. *Let (R, λ, Λ) be a unitary ring. Then for any integer $n \geq 2$ there exists the transfer*

$$(i_n)_*\colon K_1U^\lambda(R[X], \Lambda[X]) \to K_1U^\lambda(R[X^n], \Lambda[X^n]),$$

where i_n denotes the canonical embedding $R[X^n] \to R[X]$.

Proof. We introduce some auxiliary notation. For an integer $n \geq 2$ we denote by J_n the matrix of degree $2rn$, written in the block form with blocks of degree $2r$, that has matrices I_r^λ at the secondary diagonal, whereas all other blocks are zero matrices of degree $2r$. Thus, $J_n = I_r^\lambda \oslash I_r^\lambda \oslash \dots \oslash I_r^\lambda$, with n copies of the matrix I_r^λ. Furthermore, set $\sigma_n = \sigma_n^0 \oplus e_{nr}$, where $\sigma_n^0 = J_k$ if $n = 2k$, and $\sigma_n^0 = J_k \oslash e_r$ if $n = 2k+1$. Straightforward verification shows that $J_n^* = \bar{\lambda} J_n$, $J_n J_n^* = J_n^* J_n = e_{2rn}$, $\sigma_n \sigma_n^* = \sigma_n^* \sigma_n = e_{2rn}$, and $\sigma_n^* J_n \sigma_n = I_{nr}^\lambda$.

1. Construction of transfer. Taking an arbitrary matrix

$$\alpha = \alpha(X) \in U_{2r}^\lambda(R[X], \Lambda[X]),$$

we write $\alpha = \alpha(X)$ as in Proposition 3. Set

$$\widehat{\alpha} = \begin{pmatrix} a_0 & X^n a_{n-1} & X^n a_{n-2} & \dots & X^n a_2 & X^n a_1 \\ a_1 & a_0 & X^n a_{n-1} & \dots & X^n a_3 & X^n a_2 \\ a_2 & a_1 & a_0 & \dots & X^n a_4 & X^n a_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-2} & a_{n-3} & a_{n-4} & \dots & a_0 & X^n a_{n-1} \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_1 & a_0 \end{pmatrix} \in M_{2nr}(R[X^n]).$$

We show that $\widehat{\alpha}^* J_n \widehat{\alpha} = J_n$. Let $\widehat{\alpha}^* J_n \widehat{\alpha} = (\alpha_{ij}) \in M_{2rn}(R[X^n])$, where $\alpha_{ij} = \alpha_{ij}(X^n) \in M_{2r}(R[X^n])$ for all $1 \leq i, j \leq n$. The first horizontal strip of the matrix $\widehat{\alpha}^* J_n$ has the form $(a_{n-1}^* I, a_{n-2}^* I, \dots, a_1^* I, a_0^* I)$ and, thus, by formula 1_{n-1} in Proposition 3, we have $\alpha_{11} = \sum_{s=0}^{n-1} a_s^* I a_{n-1-s} = 0$. Furthermore, formula 1_0 in Proposition 3 shows that $\alpha_{1n} = a_0^* I a_0 + X^n \sum_{s=1}^{n-1} a_s^* I a_{n-s} = I$. Finally, let $2 \leq k \leq n-1$. Then by formula 1_{n-k} in Proposition 3 we have

$$\alpha_{1k} = \sum_{s=0}^{n-k} a_s^* I a_{n-k-s} + X^n \sum_{s=n-k+1}^{n-1} a_s^* I a_{2n-k-s} = 0.$$

Thus, we have shown that the first horizontal strip of the matrix $\widehat{\alpha}^* J_n \widehat{\alpha}$ has the form $(0, 0, \dots, 0, I)$.

Let $0 \leq m \leq n-2$. Then $2 \leq n-m \leq n$, and the $(n-m)$th horizontal strip of the matrix $(\widehat{\alpha})^* J_n$ has the form

$$\big(a_m^* I, a_{m-1}^* I, \dots, a_1^* I, a_0^* I, X^n a_{n-1}^* I, \dots, X^n a_{m+1}^* I\big).$$

For $1 \leq k \leq m+1$ we have

$$\alpha_{n-m,k} = \sum_{s=0}^{m+1-k} a_s^* I a_{m+1-k-s} + X^n \sum_{s=m+2-k}^{n-1} a_s^* I a_{n+m+1-k-s}.$$

Thus, by formula 1_0 in Proposition 3 we have $\alpha_{n-m,m+1} = I$, and for $1 \le k \le m$ by formula 1_{m+1-k} in Proposition 3 we have $\alpha_{n-m,k} = 0$. Finally, by formula 1_{n-1} in Proposition 3 we have

$$\alpha_{n-m,m+2} = X^n \sum_{s=0}^{n-1} a_s^* I a_{n-1-s} = 0.$$

For $m+3 \le k \le n$, formula $1_{n+m+1-k}$ in Proposition 3 yields

$$\alpha_{n-m,k} = X^n \sum_{s=0}^{n+m+1-k} a_s^* I a_{n+m+1-k-s} + X^{2n} \sum_{s=n+m+2-k}^{n-1} a_s^* I a_{2n+m+1-k-s} = 0.$$

This concludes the proof of the identity $\widehat{\alpha}^* J_n \widehat{\alpha} = J_n$.

Set $\Gamma_n(\alpha) = \sigma_n^* \widehat{\alpha} \sigma_n \in M_{2rn}(R[X^n])$. Then combining the above identity with those at the beginning of the proof of the theorem, we see that

$$\Gamma_n(\alpha)^* I_{nr} \Gamma_n(\alpha) = (\sigma_n^* \widehat{\alpha}^* \sigma_n)(\sigma_n^* J_n \sigma_n)(\sigma_n^* \widehat{\alpha} \sigma_n) = \sigma_n^* \widehat{\alpha}^* J_n \widehat{\alpha} \sigma_n = \sigma_n^* J_n \sigma_n = I_{nr}.$$

It follows that the matrix $\Gamma_n(\alpha)$ is unitary.

Now we prove that the matrix $\Gamma_n(\alpha)$ is in fact $\Lambda[X^n]$-unitary. Since the shape of the matrix σ_n depends on the parity of n, we consider two cases separately.

2. The case of even n. Let n be even, $n = 2k$. In this case the matrix $\widehat{\alpha}$ can be written in the form $\widehat{\alpha} = \left(\begin{smallmatrix} \alpha_1 & X^n \alpha_2 \\ \alpha_2 & \alpha_1 \end{smallmatrix}\right) \in M_{2rn}(R[X^n])$, where

$$\alpha_1 = \begin{pmatrix} a_0 & X^n a_{n-1} & \dots & X^n a_{k+2} & X^n a_{k+1} \\ a_1 & a_0 & \dots & X^n a_{k+3} & X^n a_{k+2} \\ \dots & \dots & \dots & \dots & \dots \\ a_{k-2} & a_{k-3} & \dots & a_0 & X^n a_{n-1} \\ a_{k-1} & a_{k-2} & \dots & a_1 & a_0 \end{pmatrix} \in M_{rn}(R[X^n]),$$

$$\alpha_2 = \begin{pmatrix} a_k & a_{k-1} & \dots & a_2 & a_1 \\ a_{k+1} & a_k & \dots & a_3 & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-2} & a_{n-3} & \dots & a_k & a_{k-1} \\ a_{n-1} & a_{n-2} & \dots & a_{k+1} & a_k \end{pmatrix} \in M_{rn}(R[X^n]).$$

By item 2) of Proposition 1, to prove that the matrix

$$\Gamma_n(\alpha) = \sigma_n^* \widehat{\alpha} \sigma_n = \begin{pmatrix} J_k^* \alpha_1 J_k & X^n J_k^* \alpha_2 \\ \alpha_2 J_k & \alpha_1 \end{pmatrix}$$

is $\Lambda[X^n]$-unitary, it suffices to check that the diagonal entries of $(J_k^* \alpha_1 J_k)^* \alpha_2 J_k$ and $(J_k^* \alpha_2)^* \alpha_1$ are contained in $\Lambda[X^n]$. Below, to simplify the notation, we denote the matrix J_k simply by J.

2.1. Verification of the first condition. We prove that the diagonal entries of the matrix $(J^* \alpha_1 J)^* \alpha_2 J = (J^* \alpha_1^* J)(\alpha_2 J) = (J \alpha_1^*)(J^* \alpha_2 J)$ are contained in $\Lambda[X^n]$. To prove this we need some subsidiary facts.

Lemma 1. *For an arbitrary λ in Λ and any matrix $a = (a_{ij}) \in M_r(R)$, the matrix $a - \lambda a^*$ is Λ-Hermitian with the diagonal entries $a_{ii} - \lambda \overline{a_{ii}}$ contained in $\Lambda_{\min} (\subseteq \Lambda)$.*

Lemma 2. *In the notation and under the conditions of Proposition 3, the product $I a_s^* I^* a_m I$ is a block matrix $\left(\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right)$, where $c_1 = a_{2s}^* a_{4m} + \lambda a_{4s}^* a_{2m}$, $c_4 = a_{1s}^* a_{3m} + \lambda a_{3s}^* a_{1m}$ and $0 \le s, m \le n-1$.*

The claim of Lemma 1 is obvious, whereas Lemma 2 can be verified by a straightforward calculation.

Now, we pass to the verification of the first condition. Let $b_1, b_2, \dots, b_k$ be the diagonal blocks of the matrix $(J\alpha_1^*)(J^*\alpha_2 J)$ and let $b_i = \left(\begin{smallmatrix} b_{1i} & b_{2i} \\ b_{3i} & b_{4i} \end{smallmatrix}\right)$, where $b_{1i}, b_{2i}, b_{3i}, b_{4i} \in M_r(R[X^n])$. We need to prove that the diagonal entries of the matrices b_{1i}, b_{4i} are contained in $\Lambda[X^n]$ for all $i = 1, \dots, k$.

For $1 \le m \le k-1$, the matrix $b_m(X^n)$ equals the product of the mth horizontal strip $(X^n I a_{k+m}^*, X^n I a_{k+m+1}^*, \dots, X^n I a_{n-1}^*, I a_0^*, I a_1^*, \dots, I a_{m-1}^*)$ of $J\alpha_1^*$ by the mth vertical strip $[I^* a_{m+k-1}, I^* a_{m+k-2}, \dots, I^* a_{m+1}, I^* a_m]$ of $J^*\alpha_2 J$. Here and below $[c_1, c_2, \dots, c_k]$ denotes the vertical strip

$$\begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_k \end{pmatrix}.$$

Consequently,

$$b_m = \sum_{s=0}^{m-1} I a_s^* I^* a_{2m-1-s} + X^n \sum_{s=k+m}^{n-1} I a_s^* I^* a_{n+2m-1-s},$$

and thus by Lemma 2,

$$\begin{aligned} b_{1m}(X^n) &= \sum_{s=0}^{m-1} (a_{2s}^* a_{4,2m-1-s} + \lambda a_{4s}^* a_{2,2m-1-s}) \\ &\quad + X^n \sum_{s=k+m}^{n-1} (a_{2s}^* a_{4,n+2m-1-s} + \lambda a_{4s}^* a_{2,n+2m-1-s}) \\ &= \left(\sum_{s=0}^{2m-1} a_{2s}^* a_{4,2m-1-s} + X^n \sum_{s=2m}^{n-1} a_{2s}^* a_{4,n+2m-1-s} \right) \\ &\quad - \left(\sum_{s=m}^{2m-1} a_{2s}^* a_{4,2m-1-s} - \lambda \sum_{s=0}^{m-1} a_{4s}^* a_{2,2m-1-s} \right) \\ &\quad - X^n \left(\sum_{s=2m}^{k+m-1} a_{2s}^* a_{4,n+2m-1-s} - \lambda \sum_{s=k+m}^{n-1} a_{4s}^* a_{2,n+2m-1-s} \right). \end{aligned}$$

By condition 2_{2m-1} in Proposition 3 for $t = 2$, the diagonal entries of the matrix in the first parentheses are contained in $\Lambda[X^n]$. Plugging $s := 2m-1-s$ in the sum with coefficient λ in the second parentheses, we get the matrix

$$\sum_{s=m}^{2m-1} \big(a_{2s}^* a_{4,2m-1-s} - \lambda a_{4,2m-1-s}^* a_{2,s} \big),$$

whose diagonal entries are contained in $\Lambda_{\min}[X^n] (\subseteq \Lambda[X^n])$ by Lemma 1. Similarly, substituting $s := n + 2m - 1 - s$ in the sum multiplied by λ in the third parentheses we get the matrix

$$\sum_{s=2m}^{k+m-1} \big(a_{2s}^* a_{4,n+2m-1-s} - \lambda a_{4,n+2m-1-s}^* a_{2,s} \big),$$

whose diagonal entries are contained in $\Lambda_{\min}[X^n] (\subseteq \Lambda[X^n])$ by Lemma 1.

Thus, we see that the diagonal entries of the matrix $b_{1m}(X^n)$ are contained in $\Lambda[X^n]$. Similarly, we get

$$\begin{aligned} b_{4m}(X^n) = {} & \Big(\sum_{s=0}^{2m-1} a_{1s}^* a_{3,2m-1-s} + X^n \sum_{s=2m}^{n-1} a_{1s}^* a_{3,n+2m-1-s} \Big) \\ & - \sum_{s=m}^{2m-1} (a_{1s}^* a_{3,2m-1-s} - \lambda a_{3,2m-1-s}^* a_{1,s}) \\ & - X^n \Big(\sum_{s=2m}^{k+m-1} (a_{1s}^* a_{3,n+2m-1-s} - \lambda a_{3,n+2m-1-s}^* a_{1,s} \Big). \end{aligned}$$

By condition 2_{2m-1} in Proposition 3 for $t = 1$, the diagonal entries of the matrix in the first parentheses are contained in $\Lambda[X^n]$, whereas the diagonal entries of the matrices in the second and the third parentheses are contained in $\Lambda_{\min}[X^n] (\subseteq \Lambda[X^n])$ by Lemma 1.

It follows that the diagonal entries of the matrix $b_{4m}(X^n)$ are contained in $\Lambda[X^n]$ and, consequently, the diagonal entries of the matrix $b_m(X^n)$ are contained in $\Lambda[X^n]$ for all $1 \le m \le k-1$.

Consider the diagonal block $b_k(X^n) = \sum_{s=0}^{k-1} I a_s^* I^* a_{n-1-s}$ of the matrix $(J^* \alpha_1 J)^* \alpha_2 J$. Then by Lemma 2 we have

$$\begin{aligned} b_{1k}(X^n) &= \sum_{s=0}^{k-1} (a_{2s}^* a_{4,n-1-s} + \lambda a_{4s}^* a_{2,n-1-s}) \\ &= \sum_{s=0}^{n-1} a_{2s}^* a_{4,n-1-s} - \Big(\sum_{s=k}^{n-1} a_{2s}^* a_{4,n-1-s} - \lambda \sum_{s=0}^{k-1} a_{4s}^* a_{2,n-1-s} \Big). \end{aligned}$$

By condition 2_{n-1} in Proposition 3 for $t = 2$, the diagonal entries of the matrix in the first sum are contained in $\Lambda[X^n]$. Substituting $s := n-1-s$ in the sum with the coefficient λ in the parentheses, we get the matrix $\sum_{s=k}^{n-1} (a_{2s}^* a_{4,n-1-s} - \lambda a_{4,n-1-s}^* a_{2,s})$, whose diagonal entries are contained in $\Lambda_{\min}[X^n] (\subseteq \Lambda[X^n])$ by Lemma 1.

Thus, the diagonal entries of the matrix $b_{1k}(X^n)$ are contained in $\Lambda[X^n]$. Similarly, one can prove that the diagonal entries of the matrix $b_{4k}(X^n)$ are contained in $\Lambda[X^n]$ and, thus, the diagonal entries of the matrix $b_k(X^n)$ are contained in $\Lambda[X^n]$. It follows that the diagonal entries of the matrix $(J^* \alpha_1 J)^* \alpha_2 J$ are contained in $\Lambda[X^n]$.

2.2. Verification of the second condition. Let us prove that the diagonal entries of the matrix $\alpha_2^* J \alpha_1$ are contained in $\Lambda[X^n]$. For the proof we need the following subsidiary statement.

Lemma 3. *In the notation and under the conditions of Proposition* 3, *the product* $a_s^* I a_m$ *is the block matrix* $\left(\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right)$, *where* $c_1 = a_{1s}^* a_{3m} + \lambda a_{3s}^* a_{1m}$, $c_4 = a_{2s}^* a_{4m} + \lambda a_{4s}^* a_{2m}$, *and* $0 \le s, m \le n-1$.

The lemma is verified by a straightforward computation.

Now, we pass to the verification of the second condition. Let $b_1, b_2, \dots, b_k$ be the diagonal blocks of the matrix $\alpha_2^* J \alpha_1$, and let $b_i = \left(\begin{smallmatrix} b_{1i} & b_{2i} \\ b_{3i} & b_{4i} \end{smallmatrix}\right)$, where $b_{1i}, b_{2i}, b_{3i}, b_{4i} \in M_r(R[X^n])$. We must prove that the diagonal entries of the matrix b_{1i}, b_{4i} are contained in $\Lambda[X^n]$ for all $i = 1, \dots, k$.

The matrix $b_1(X^n)$ equals the product of the first horizontal strip

$$(a_{n-1}^* I, a_{n-2}^* I, \dots, a_{k+1}^* I, a_k^* I)$$

of the matrix $\alpha_2^* J$ by the first vertical strip $[a_0, a_1, \dots, a_{k-2}, a_{k-1}]$ of the matrix α_1. It follows that $b_1 = \sum_{s=k}^{n-1} a_s^* I a_{n-1-s}$, whence, by Lemma 3, we have

$$\begin{aligned} b_{11}(X^n) &= \sum_{s=k}^{n-1} (a_{1s}^* a_{3,n-1-s} + \lambda a_{3s}^* a_{1,n-1-s}) \\ &= \sum_{s=0}^{n-1} a_{1s}^* a_{3,n-1-s} - \biggl(\sum_{s=0}^{k-1} a_{1s}^* a_{3,n-1-s} - \lambda \sum_{s=k}^{n-1} a_{3s}^* a_{1,n-1-s} \biggr). \end{aligned}$$

By condition 2_{n-1} in Proposition 3 for $t = 1$, the diagonal entries of the matrix in the first parentheses are contained in $\Lambda[X^n]$. Substituting $s := n-1-s$ in the sum inside the parentheses with the coefficient λ, we get the matrix $\sum_{s=0}^{k-1} (a_{1s}^* a_{3,n-1-s} - \lambda a_{3,n-1-s}^* a_{1,s})$, whose diagonal entries are contained in $\Lambda_{\min}[X^n] (\subseteq \Lambda[X^n])$ by Lemma 1.

Thus, the diagonal entries of the matrix $b_{11}(X^n)$ are contained in $\Lambda[X^n]$. Similarly, one can prove that the diagonal entries of the matrix $b_{41}(X^n)$ are contained in $\Lambda[X^n]$, and, thus, the diagonal entries of the matrix $b_1(X^n)$ are contained in $\Lambda[X^n]$.

For $2 \le m \le k$, the matrix $b_m(X^n)$ is equal to the product of the mth horizontal strip $(a_{n-m}^* I, a_{n-m-1}^* I, \dots, a_{k-m+1}^* I)$ of the matrix $\alpha_2^* J$ by the mth vertical strip $[X^n a_{n-m+1}, X^n a_{n-m+2}, \dots, X^n a_{n-1}, a_0, a_1, \dots, a_{k-m}]$ of the matrix α_1. Thus,

$$b_m(X^n) = \sum_{s=0}^{k-m} a_{n-2m+1-s}^* I a_s + X^n \sum_{s=n-m+1}^{n-1} a_{2n-2m+1-s}^* I a_s.$$

Substituting $s := n - 2m + 1 - s$ in the first sum and $s := 2n - 2m + 1 - s$ in the second sum, we get

$$b_m(X^n) = \sum_{s=k-m+1}^{n-2m+1} a_s^* I a_{n-2m+1-s} + X^n \sum_{s=n-2m+2}^{n-m} a_s^* I a_{2n-2m+1-s}.$$

Therefore, by Lemma 3,

$$\begin{aligned} b_{1m}(X^n) = &\sum_{s=k-m+1}^{n-2m+1} (a_{1s}^* a_{3,n-2m+1-s} + \lambda a_{3s}^* a_{1,n-2m+1-s}) \\ &+ X^n \sum_{s=n-2m+2}^{n-m} (a_{1s}^* a_{3,2n-2m+1-s} + \lambda a_{3s}^* a_{1,2n-2m+1-s}) \\ = &\biggl(\sum_{s=0}^{n-2m+1} a_{1s}^* a_{3,n-2m+1-s} + X^n \sum_{s=n-2m+2}^{n-1} a_{1s}^* a_{3,2n-2m+1-s} \biggr) \\ &- \biggl(\sum_{s=0}^{k-m} a_{1s}^* a_{3,n-2m+1-s} - \lambda \sum_{s=k-m+1}^{n-2m+1} a_{3s}^* a_{1,n-2m+1-s} \biggr) \\ &- X^n \biggl(\sum_{s=n-m+1}^{n-1} a_{1s}^* a_{3,2n-2m+1-s} - \lambda \sum_{s=n-2m+2}^{n-m} a_{3s}^* a_{1,2n-2m+1-s} \biggr). \end{aligned}$$

By condition 2_{n-2m+1} in Proposition 3 for $t = 1$, the diagonal entries of the matrix in the first parentheses are contained in $\Lambda[X^n]$. Substituting $s := n - 2m + 1 - s$ in the sum with coefficient λ in the second parentheses, we get the matrix

$$\sum_{s=0}^{k-m} \bigl(a_{1s}^* a_{3,n-2m+1-s} - \lambda a_{3,n-2m+1-s}^* a_{2,s} \bigr),$$

whose diagonal entries are contained in $\Lambda_{\min}[X^n](\subseteq \Lambda[X^n])$ by Lemma 1. Similarly, substituting $s := 2n-2m+1-s$ in the sum with coefficient λ in the third parentheses, we get the matrix

$$\sum_{s=n-m+1}^{n-1} \big(a_{1s}^* a_{3,2n-2m+1-s} - \lambda a_{3,2n-2m+1-s}^* a_{1,s}\big),$$

whose diagonal entries are contained in $\Lambda_{\min}[X^n](\subseteq \Lambda[X^n])$ by Lemma 1. Thus, the diagonal entries of the matrix $b_{1m}(X^n)$ are contained in $\Lambda[X^n]$ for all $2 \le m \le k$.

Similarly one can prove that the diagonal entries of the matrix $b_{4m}(X^n)$ are contained in $\Lambda[X^n]$ for all $2 \le m \le k$. Consequently, the diagonal entries of the matrix $b_m(X^n)$ are contained in $\Lambda[X^n]$ for all $2 \le m \le k$. Thus, the diagonal entries of the matrix $\alpha_2^* J\alpha_1$ are also contained in $\Lambda[X^n]$, and, if n is even, the matrix $\Gamma_n(\alpha)$ is $\Lambda[X^n]$-unitary.

3. The case of odd n. Let n be odd, $n = 2k+1$, with $k \ge 1$. Below for two matrices a, b of degree r we denote by a/b the vertical strip $\substack{a\\b}$. In the case where n is odd, the matrix $\widehat{\alpha}$ can be written as $\widehat{\alpha} = \left(\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\right) \in M_{2rn}(R[X^n])$, where the matrices $\alpha_1, \alpha_2, \alpha_3, \alpha_4 (\in M_{rn}(R[X^n]))$ look like this:

$$\alpha_1 = \begin{pmatrix} a_0 & X^n a_{n-1} & \dots & X^n a_{k+2} & X^n a_{1,k+1}/X^n a_{3,k+1} \\ a_1 & a_0 & \dots & X^n a_{k+3} & X^n a_{1,k+2}/X^n a_{3,k+2} \\ \dots & \dots & \dots & \dots & \dots \\ a_{k-1} & a_{k-2} & \dots & a_0 & X^n a_{1,n-1}/X^n a_{3,n-1} \\ a_{1,k}a_{2,k} & a_{1,k-1}a_{2,k-1} & \dots & a_{11}a_{21} & a_{10} \end{pmatrix},$$

$$\alpha_2 = \begin{pmatrix} X^n a_{2,k+1}/X^n a_{4,k+1} & X^n a_k & \dots & X^n a_1 \\ X^n a_{2,k+2}/X^n a_{4,k+2} & X^n a_{k+1} & \dots & X^n a_2 \\ \dots & \dots & \dots & \dots \\ X^n a_{2,n-1}/X^n a_{4,n-1} & X^n a_{n-2} & \dots & X^n a_k \\ a_{2,0} & X^n a_{1,n-1} X^n a_{2,n-1} & \dots & X^n a_{1,k+1} X^n a_{2,k+1} \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} a_{3,k}a_{4,k} & a_{3,k-1}a_{4,k-1} & \dots & a_{31}a_{41} & a_{30} \\ a_{k+1} & a_k & \dots & a_2 & a_{11}/a_{31} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-2} & a_{n-3} & \dots & a_k & a_{1,k-1}/a_{3,k-1} \\ a_{n-1} & a_{n-2} & \dots & a_{k+1} & a_{1,k}/a_{3,k} \end{pmatrix},$$

$$\alpha_4 = \begin{pmatrix} a_{40} & X^n a_{3,n-1} X^n a_{4,n-1} & \dots & X^n a_{3,k+1} X^n a_{4,k+1} \\ a_{21}/a_{41} & a_0 & \dots & X^n a_{k+2} \\ \dots & \dots & \dots & \dots \\ a_{2,k-1}/a_{4,k-1} & a_{k-2} & \dots & X^n a_{n-1} \\ a_{2,k}/a_{4,k} & a_{k-1} & \dots & a_0 \end{pmatrix}.$$

Consequently,

$$\Gamma_n(\alpha) = \sigma_n^* \widehat{\alpha} \sigma_n = \begin{pmatrix} (\sigma_n^0)^* \alpha_1 \sigma_n^0 & (\sigma_n^0)^* \alpha_2 \\ \alpha_3 \sigma_n^0 & \alpha_4 \end{pmatrix}.$$

Above, we have verified that the matrix $\Gamma_n(\alpha)$ is unitary. By item 2 of Proposition 1, to prove that the matrix $\Gamma_n(\alpha)$ is $\Lambda[X^n]$-unitary it suffices to check that the diagonal entries of the matrices $((\sigma_n^0)^*\alpha_1\sigma_n^0)^*(\alpha_3\sigma_n^0)$ and $((\sigma_n^0)^*\alpha_2)^*\alpha_4$ are contained in $\Lambda[X^n]$.

3.1. Verification of the first condition. We prove that the diagonal entries of the matrix $((\sigma_n^0)^*\alpha_1\sigma_n^0)^*(\alpha_3\sigma_n^0) = ((\sigma_n^0)^*\alpha_1^*\sigma_n^0)(\alpha_3\sigma_n^0)$ are contained in $\Lambda[X^n]$. For this, it suffices to verify that the diagonal entries of the diagonal blocks $b_1, b_2, \dots, b_{2k}, b_{2k+1}$ of the matrix $((\sigma_n^0)^*\alpha_1^*\sigma_n^0)(\alpha_3\sigma_n^0)$, where $b_i \in M_r(R[X^n])$, are contained in $\Lambda[X^n]$.

The matrix $b_1(X^n)$ is equal to the product of the first horizontal strip

$$\left(a_{10}^*,\ \lambda X^n a_{3,n-1}^*, X^n a_{1,n-1}^*, \lambda X^n a_{3,n-2}^*, \dots, X^n a_{1,k+2}^*, \lambda X^n a_{3,k+1}^*, X^n a_{1,k+1}^*\right)$$

of the matrix $(\sigma_n^0)^*\alpha_1^*\sigma_n^0$ by the first vertical strip $[a_{30}, a_{11}, a_{31}, a_{12}, \dots, a_{1k}, a_{3k}]$ of the matrix $\alpha_3\sigma_n^0$. It follows that

$$b_1 = a_{10}^* a_{30} + X^n \sum_{s=1}^{k} a_{1,n-s}^* a_{3,s} + \lambda X^n \sum_{s=1}^{k} a_{3,n-s}^* a_{1,s}.$$

Substituting $s := n - s$ in the first sum, we get

$$\begin{aligned} b_1 &= a_{10}^* a_{30} + X^n \sum_{s=k+1}^{n-1} a_{1,s}^* a_{3,n-s} + \lambda X^n \sum_{s=1}^{k} a_{3,n-s}^* a_{1,s} \\ &= \left(a_{10}^* a_{30} + X^n \sum_{s=1}^{n-1} a_{1,s}^* a_{3,n-s}\right) - X^n \sum_{s=1}^{n-1} \left(a_{1,s}^* a_{3,n-s} - \lambda a_{3,n-s}^* a_{1,s}\right). \end{aligned}$$

By condition 2_0 in Proposition 3 for $t = 1$, the diagonal entries of the matrix in the first parentheses are contained in $\Lambda[X^n]$, whereas the diagonal entries of the matrices in the second sum are contained in $\Lambda_{\min}[X^n](\subseteq \Lambda[X^n])$ by Lemma 1. Consequently, the diagonal entries of the matrix $b_1(X^n)$ are contained in $\Lambda[X^n]$.

Let $2 \le m \le (k-1)$, and consider the diagonal blocks $b_{2m}(X^n)$ with even indices. Then the matrix $b_{2m}(X^n)$ equals the product of the $2m$th horizontal strip

$$(\bar\lambda a_{2,m}^*, a_{4,m-1}^*, \bar\lambda a_{2,m-1}^*, \dots, \bar\lambda a_{2,0}^*, X^n a_{4,n-1}^*, \bar\lambda X^n a_{2,n-1}^*, \dots, \bar\lambda X^n a_{2,k+m+1}^*)$$

of the matrix $(\sigma_n^0)^*\alpha_1^*\sigma_n^0$ by the $2m$th vertical strip

$$\left[\lambda a_{4,m}, \lambda a_{2,m+1}, \lambda a_{4,m+1}, \dots, \lambda a_{2,m+k}, \lambda a_{4,m+k}\right]$$

of the matrix $\alpha_3\sigma_n^0$. It follows that

$$\begin{aligned} b_{2m}(X^n) &= \sum_{s=0}^{m} a_{2,s}^* a_{4,2m-s} + \lambda \sum_{s=0}^{m-1} a_{4,s}^* a_{2,2m-s} \\ &\quad + X^n \sum_{s=k+m+1}^{n-1} a_{2,s}^* a_{4,2m+n-s} + \lambda X^n \sum_{s=k+m+1}^{n-1} a_{4,s}^* a_{2,2m+n-s} \\ &= \left(\sum_{s=0}^{2m} a_{2,s}^* a_{4,2m-s} + X^n \sum_{s=2m+1}^{n-1} a_{2,s}^* a_{4,2m+n-s}\right) \\ &\quad - \left(\sum_{s=m+1}^{2m} a_{2,s}^* a_{4,2m-s} - \lambda \sum_{s=0}^{m-1} a_{4,s}^* a_{2,2m-s}\right) \\ &\quad - X^n\left(\sum_{s=2m+1}^{k+m} a_{2,s}^* a_{4,2m+n-s} - \lambda \sum_{s=k+m+1}^{n-1} a_{4,s}^* a_{2,2m+n-s}\right). \end{aligned}$$

By condition 2_{2m} in Proposition 3 for $t = 2$, the diagonal entries of the matrix in the first parentheses are contained in $\Lambda[X^n]$. Substituting $s := 2m - s$ in the sum with coefficient λ in the second parentheses, we get the matrix $\sum_{s=m+1}^{2m}(a_{2,s}^* a_{4,2m-s} - \lambda a_{4,2m-s}^* a_{2,s})$, whose diagonal entries are contained in $\Lambda_{\min}[X^n](\subseteq \Lambda[X^n])$ by Lemma 1. Similarly, substituting $s := n + 2m - s$ in the sum with coefficient λ in the third parentheses, we get the matrix $\sum_{s=2m+1}^{k+m}(a_{2,s}^* a_{4,n+2m-s} - \lambda a_{4,n+2m-s}^* a_{2,s})$, whose diagonal entries are contained in $\Lambda_{\min}[X^n](\subseteq \Lambda[X^n])$ by Lemma 1. Consequently, the diagonal entries of the matrix $b_{2m}(X^n)$ are contained in $\Lambda[X^n]$ for all $2 \le m \le (k-1)$.

Similarly, when $2 \leq m \leq (k-1)$ for the diagonal blocks $b_{2m+1}(X^n)$ with odd indices we get

$$\begin{aligned} b_{2m+1}(X^n) = \Big(& \sum_{s=0}^{2m} a_{1,s}^* a_{3,2m-s} + X^n \sum_{s=2m+1}^{n-1} a_{1,s}^* a_{3,2m+n-s} \Big) \\ & - \sum_{s=m+1}^{2m} \big(a_{1,s}^* a_{3,2m-s} - \lambda a_{3,2m-s}^* a_{1,s} \big) \\ & - \lambda \sum_{s=2m+1}^{k+m} \big(a_{1,s}^* a_{3,n+2m-s} - \lambda a_{3,n+2m-s}^* a_{1,s} \big). \end{aligned}$$

By condition 2_{2m} in Proposition 3 for $t = 1$, the diagonal entries of the matrix in the first parentheses are contained in $\Lambda[X^n]$, whereas the diagonal entries of the matrices in the second and the third parentheses are contained in $\Lambda_{\min}[X^n](\subseteq \Lambda[X^n])$ by Lemma 1. It follows that the diagonal entries of the matrix $b_{2m+1}(X^n)$ are contained in $\Lambda[X^n]$ for all $2 \leq m \leq (k-1)$.

The matrix $b_{2k}(X^n)$ is equal to the product of the $2k$th horizontal strip

$$\big(\bar{\lambda} a_{2,k}^*, a_{4,k-1}^*, \bar{\lambda} a_{2,k-1}^*, \dots, \bar{\lambda} a_{2,1}^*, a_{4,0}^*, \bar{\lambda} a_{2,0}^* \big)$$

of $(\sigma_n^0)^* \alpha_1^* \sigma_n^0$ by the $2k$th vertical strip $[\lambda a_{4,k}, \lambda a_{2,k+1}, \lambda a_{4,k+1}, \dots, \lambda a_{2,n-1}, \lambda a_{4,n-1}]$ of $\alpha_3 \sigma_n^0$. It follows that

$$\begin{aligned} b_{2k}(X^n) &= \sum_{s=0}^{k} a_{2,s}^* a_{4,n-1-s} + \lambda \sum_{s=0}^{k-1} a_{4,s}^* a_{2,n-1-s} \\ &= \sum_{s=0}^{n-1} a_{2,s}^* a_{4,n-1-s} - \Big(\sum_{s=k+1}^{n-1} a_{2,s}^* a_{4,n-1-s} - \lambda \sum_{s=0}^{k-1} a_{4,s}^* a_{2,n-1-s} \Big). \end{aligned}$$

By condition 2_{n-1} in Proposition 3 for $t = 2$, the diagonal entries of the matrices in the first sum are contained in $\Lambda[X^n]$. Substituting $s := n-1-s$ in the sum with the coefficient λ within the parentheses, we get the matrix

$$\sum_{s=k+1}^{n-1} \big(a_{2,s}^* a_{4,n-1-s} - \lambda a_{4,n-1-s}^* a_{2,s} \big),$$

whose diagonal entries are contained in $\Lambda_{\min}[X^n](\subseteq \Lambda[X^n])$ by Lemma 1. Consequently, the diagonal entries of the matrix $b_{2k}(X^n)$ are contained in $\Lambda[X^n]$.

Similarly,

$$b_n(X^n) = b_{2k+1}(X^n) = \sum_{s=0}^{n-1} a_{1,s}^* a_{3,n-1-s} - \sum_{s=k+1}^{n-1} \big(a_{1,s}^* a_{3,n-1-s} - \lambda a_{3,n-1-s}^* a_{1,s} \big).$$

By condition 2_{n-1} in Proposition 3 for $t = 1$, the diagonal entries of the matrix in the first sum are contained in $\Lambda[X^n]$, whereas the diagonal entries of the matrices in the second sum within the parentheses are contained in $\Lambda_{\min}[X^n](\subseteq \Lambda[X^n])$ by Lemma 1. It follows that the diagonal entries of the matrix $b_m(X^n)$ are contained in $\Lambda[X^n]$ for all $1 \leq m \leq n$. This concludes the proof of the fact that the diagonal entries of the matrix $((\sigma_n^0)^* \alpha_1 \sigma_n^0)^*)(\alpha_3 \sigma_n^0)$ belong to $\Lambda[X^n]$.

3.2. Verification of the second condition. Next, we prove that the diagonal entries of the matrix $((\sigma_n^0)^*\alpha_2)^*\alpha_4 = (\alpha_2^*\sigma_n^0)\alpha_4$ are contained in $\Lambda[X^n]$. For this, it suffices to verify that the diagonal entries of the diagonal blocks $b_1, b_2, \dots, b_{2k}, b_{2k+1} = b_n$ of the matrix $(\alpha_2^*\sigma_n^0)\alpha_4$, where $b_i \in M_r(R[X^n])$, are contained in $\Lambda[X^n]$.

The matrix $b_1(X^n)$ is equal to the product of the first horizontal strip

$$\left(a_{20}^*, \lambda X^n a_{4,n-1}^*, X^n a_{2,n-1}^*, \dots, \lambda X^n a_{4,k+1}^*, X^n a_{2,k+1}^*\right)$$

of the matrix $\alpha_2^*\sigma_n^0$ by the first vertical strip $[a_{40}, a_{21}, a_{41}, \dots, a_{2k}, a_{4k}]$ of the matrix α_4. It follows that

$$\begin{aligned} b_1 &= a_{20}^* a_{40} + X^n \sum_{s=k+1}^{n-1} a_{2,s}^* a_{4,n-s} + \lambda \sum_{s=k+1}^{n-1} a_{4,s}^* a_{2,n-s} \\ &= \left(a_{20}^* a_{40} + X^n \sum_{s=1}^{n-1} a_{2,s}^* a_{4,n-s}\right) - X^n \left(\sum_{s=1}^{k} a_{2,s}^* a_{4,n-s} - \lambda \sum_{s=k+1}^{n-1} a_{4,s}^* a_{1,n-s}\right). \end{aligned}$$

By condition 2_{n-2} in Proposition 3 for $t = 2$, the diagonal entries of the matrix in the first parentheses are contained in $\Lambda[X^n]$. Substituting $s := n - s$ in the sum with the coefficient λ in the second parentheses, we get the matrix $\sum_{s=1}^{k}(a_{2,s}^* a_{4,n-s} - \lambda a_{4,n-s}^* a_{2,s})$, whose diagonal entries are contained in $\Lambda_{\min}[X^n](\subseteq \Lambda[X^n])$ by Lemma 1. It follows that the diagonal entries of the matrix $b_1(X^n)$ are contained in $\Lambda[X^n]$.

Let $1 \le m \le k$. Consider the diagonal blocks $b_{2m}(X^n)$ with even indices. The matrix $b_{2m}(X^n)$ is equal to the product of the $2m$th horizontal strip

$$\left(X^n a_{1,n-m}^*, \lambda X^n a_{3,n-m-1}^*, X^n a_{1,n-m-1}^*, \dots, \lambda X^n a_{3,k-m+1}^*, X^n a_{1,k-m+1}^*\right)$$

of the matrix $\alpha_2^*\sigma_n^0$ by the $2m$th vertical strip

$$[X^n a_{3,n-m}, X^n a_{1,n-m+1}, X^n a_{3,n-m+1}, \dots, X^n a_{3,n-1}, a_{10}, a_{30}, \dots, a_{1,k-m}, a_{3,k-m}]$$

of the matrix α_4. Consequently,

$$\begin{aligned} b_{2m}(X^n) = X^n &\sum_{s=0}^{k-m} \left(a_{1,n-2m-s}^* a_{3,s} + \lambda a_{3,n-2m-s}^* a_{1,s}\right) \\ &+ X^{2n} \sum_{s=n-m}^{n-1} \left(a_{1,2n-2m-s}^* a_{3,s} + \lambda a_{3,2n-2m-s}^* a_{1,s}\right). \end{aligned}$$

Substituting $s := n - 2m - s$ in the first parentheses and $s := 2n - 2m - s$ in the next one, we get

$$\begin{aligned} b_{2m}(X^n) = X^n &\sum_{s=k-m+1}^{n-2m} \left(a_{1,s}^* a_{3,n-2m-s} + \lambda a_{3,s}^* a_{1,n-2m-s}\right) \\ &+ X^{2n}\left(\sum_{s=n-2m+1}^{n-m} a_{1,s}^* a_{3,2n-2m-s} + \lambda \sum_{s=n-2m+1}^{n-m-1} a_{3,s}^* a_{1,2n-2m-s}\right) \\ = X^n &\left(\sum_{s=0}^{n-2m} a_{1,s}^* a_{3,n-2m-s} + X^n \sum_{s=n-2m+1}^{n-1} a_{1,s}^* a_{3,2n-2m-s}\right) \\ &- X^n\left(\sum_{s=0}^{k-m} a_{1,s}^* a_{3,n-2m-s} - \lambda \sum_{s=k-m+1}^{n-2m} a_{3,s}^* a_{1,n-2m-s}\right) \\ &- X^{2n}\left(\sum_{s=n-m+1}^{n-1} a_{1,s}^* a_{3,2n-2m-s} - \lambda \sum_{s=n-2m+1}^{n-m-1} a_{3,s}^* a_{1,2n-2m-s}\right). \end{aligned}$$

By condition 2_{n-2m} in Proposition 3 for $t = 1$, the diagonal entries of the matrix in the first parentheses are contained in $\Lambda[X^n]$. Substituting $s := n - 2m - s$ in the sum with the coefficient λ, we get the matrix $\sum_{s=0}^{k-m}(a_{1,s}^* a_{3,n-2m-s} - \lambda a_{3,n-2m-s}^* a_{1,s})$, whose diagonal entries are contained in $\Lambda_{\min}[X^n](\subseteq \Lambda[X^n])$ by Lemma 1. Similarly, substituting $s := 2n - 2m - s$ in the sum with the coefficient λ in the third parentheses, we get the matrix $\sum_{s=n-m+1}^{n-1}(a_{1,s}^* a_{3,2n-2m-s} - \lambda a_{3,2n-2m-s}^* a_{1,s})$, whose diagonal entries are contained in $\Lambda_{\min}[X^n](\subseteq \Lambda[X^n])$ by Lemma 1. It follows that the diagonal entries of the matrix $b_{2m}(X^n)$ are contained in $\Lambda[X^n]$ for all $1 \le m \le k$.

Similarly, when $1 \le m \le k$, for the diagonal blocks $b_{2m+1}(X^n)$ with odd indices we get

$$
\begin{aligned}
b_{2m+1}(X^n) = X^n\Bigg(&\sum_{s=0}^{n-2m} a_{2,s}^* a_{4,n-2m-s} + X^n \sum_{s=n-2m+1}^{n-1} a_{2,s}^* a_{4,2n-2m-s}\Bigg) \\
&- X^n \sum_{s=0}^{k-m}\big(a_{2,s}^* a_{4,n-2m-s} - \lambda a_{4,n-2m-s}^* a_{2,s}\big) \\
&- X^{2n} \sum_{s=n-m+1}^{n-1}\big(a_{2,s}^* a_{4,2n-2m-s} - \lambda a_{4,2n-2m-s}^* a_{2,s}\big).
\end{aligned}
$$

By condition 2_{2m} in Proposition 3 for $t = 2$, the diagonal entries of the matrix in the first parentheses are contained in $\Lambda[X^n]$, whereas the diagonal entries of the matrices in the second and the third parentheses are contained in $\Lambda_{\min}[X^n](\subseteq \Lambda[X^n])$ by Lemma 1. Consequently, the diagonal entries of the matrix $b_{2m+1}(X^n)$ are contained in $\Lambda[X^n]$ for all $1 \le m \le k$ and, thus, the diagonal entries of the matrix $b_m(X^n)$ are contained in $\Lambda[X^n]$ for all $1 \le m \le n$. It follows that the diagonal entries of the matrix $((\sigma_n^0)^*\alpha_2)^*\alpha_4$ are contained in $\Lambda[X^n]$, which concludes the verification of conditions in item 2 of Proposition 1. This means that for odd n the matrix $\Gamma_n(\alpha)$ is $\Lambda[X^n]$-unitary.

4. Consistency of the definition. We prove that if $\alpha \in \mathrm{EU}^\lambda(R[X], \Lambda[X])$, then $\Gamma_n(\alpha) \in \mathrm{EU}^\lambda(R[X^n], \Lambda[X^n])$. Since we consider the stable group $\mathrm{EU}^\lambda(R[X], \Lambda[X])$, and for $r \ge 3$ the group $\mathrm{EU}_{2r}^\lambda(R[X], \Lambda[X])$ is generated by the matrices $X_+(b)$, $X_-(c)$, where b is $\bar\Lambda[X]$-Hermitian and c is $\Lambda[X]$-Hermitian (by Proposition 5.1 [2, Chapter 2]), it suffices to prove that the matrices of the form $\Gamma_n(X_+(b)), \Gamma_n(X_-(c))$ are contained in $\mathrm{EU}^\lambda(R[X^n], \Lambda[X^n])$. We reproduce detailed calculations for matrices $X_+(b)$, where b is $\bar\Lambda[X]$-Hermitian. For matrices $X_-(c)$, where c is $\Lambda[X]$-Hermitian, the proof is similar.

We take an arbitrary $\bar\Lambda[X]$-Hermitian matrix

$$
b = b(X) = b_0 + b_1X + \cdots + b_{n-1}X^{n-1} (\in M_r(R[X])),
$$

where $r \ge 3$ and the matrices $b_k = b_k(X^n) \in M_r(R[X^n])$ are $\bar\Lambda[X^n]$-Hermitian for all $0 \le k \le (n-1)$. Then

$$
X_+(b) = \begin{pmatrix} e_r & b_0 \\ 0 & e_r \end{pmatrix} + \sum_{k=1}^{n-1} \begin{pmatrix} 0 & b_r \\ 0 & 0 \end{pmatrix} X^k.
$$

We prove that the matrix $\Gamma_n(X_+(b))$ is contained in $\mathrm{EU}^\lambda(R[X^n], \Lambda[X^n])$.

First, let n be even, $n = 2k$ with $k \ge 1$. In this case

$$
\widehat{X_+(b)} = \begin{pmatrix} \alpha_1 & X^n\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix} \in M_{2rn}(R[X^n]),
$$

where the matrices α_1, α_2 are as described above in the proof of Theorem 1, in the subsection concerning the case where n is even. Then

$$\Gamma_n(X_+(b)) = \sigma_n^* \widehat{X_+(b)} \sigma_n = \begin{pmatrix} J_k^*\alpha_1 J_k & X^n J_k^* \alpha_2 \\ \alpha_2 J_k & \alpha_1 \end{pmatrix}.$$

Applying the above description of α_1 to the matrix $\alpha = X_+(b)$, we see that

$$J_k^*\alpha_1 J_k = \begin{pmatrix} e & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \lambda b_0 & e & \lambda b_1 & 0 & \dots & \lambda b_{k-2} & 0 & \lambda b_{k-1} & 0 \\ 0 & 0 & e & 0 & \dots & 0 & 0 & 0 & 0 \\ \lambda X^n b_{n-1} & 0 & \lambda b_0 & e & \dots & \lambda b_{k-3} & 0 & \lambda b_{k-2} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \lambda X^n b_{k+2} & 0 & \lambda X^n b_{k+3} & 0 & \dots & \lambda b_0 & e & \lambda b_1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & e & 0 \\ \lambda X^n b_{k+1} & 0 & \lambda X^n b_{k+2} & 0 & \dots & \lambda X^n b_{n-1} & 0 & \lambda b_0 & e \end{pmatrix}.$$

Obviously, this matrix $J_k^*\alpha_1 J_k$ is contained in $E_{rn}(R[X^n])$ and, hence, $H(J_k^*\alpha_1 J_k) \in \mathrm{EU}_{2rn}^{\lambda}(R[X^n], \Lambda[X^n])$. Multiplying $\Gamma_n(X_+(b))$ on the left by the matrix $H(J_k^*\alpha_1 J_k)^{-1}$, we get a $\Lambda[X^n]$-unitary matrix of the shape $\left(\begin{smallmatrix} e_{rn} & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. It follows that β is $\bar{\Lambda}[X^n]$-Hermitian, γ is $\Lambda[X^n]$-Hermitian and, finally, $\Gamma_n(X_+(b)) = H(J_k^*\alpha_1 J_k) X_-(\gamma) X_+(\beta)$. This means that for an even n the matrix $\Gamma_n(X_+(b))$ is contained in $\mathrm{EU}^{\lambda}(R[X^n], \Lambda[X^n])$.

Now, consider the case of an odd n, $n = 2k+1$, where $k \geq 1$. In this case,

$$\widehat{X_+(b)} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in M_{2rn}(R[X^n]),$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the matrices described above in the proof of Theorem 1, in the subsection concerning the case of an odd n. Then

$$\Gamma_n(X_+(b)) = \begin{pmatrix} (\sigma_n^0)^*\alpha_1\sigma_n^0 & (\sigma_n^0)^*\alpha_2 \\ \alpha_3\sigma_n^0 & \alpha_4 \end{pmatrix}.$$

Applying the above description of α_1 to the matrix $\alpha = X_+(b)$, we see that

$$(\sigma_n^0)^*\alpha_1\sigma_n^0 = \begin{pmatrix} e & \lambda b_1 & 0 & \lambda b_2 & \dots & \lambda b_{k-1} & 0 & \lambda b_k & 0 \\ 0 & e & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \lambda b_0 & e & \lambda b_1 & \dots & \lambda b_{k-2} & 0 & \lambda b_{k-1} & 0 \\ 0 & 0 & 0 & e & \dots & 0 & 0 & 0 & 0 \\ 0 & \lambda X^n b_{n-1} & 0 & \lambda b_0 & \dots & \lambda b_{k-3} & 0 & \lambda b_{k-2} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \lambda X^n b_{k+3} & 0 & \lambda X^n b_{k+4} & \dots & \lambda b_0 & e & \lambda b_1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & e & 0 \\ 0 & \lambda X^n b_{k+2} & 0 & \lambda X^n b_{k+3} & \dots & \lambda X^n b_{n-1} & 0 & \lambda b_0 & e \end{pmatrix}.$$

Obviously, the above matrix $(\sigma_n^0)^*\alpha_1\sigma_n^0$ is contained in $E_{rn}(R[X^n])$ and, consequently, $H((\sigma_n^0)^*\alpha_1\sigma_n^0) \in \mathrm{EU}_{2rn}^{\lambda}(R[X^n], \Lambda[X^n])$. From this point, the proof is completed in exactly the same way as for the case where n is even.

Summarizing, we can conclude that for any integer $n \geq 2$ we get a well-defined map $(i_n)_*\colon K_1U^{\lambda}(R[X], \Lambda[X]) \to K_1U^{\lambda}(R[X^n], \Lambda[X^n])$, which is a group homomorphism. Indeed, take arbitrary $\alpha, \beta \in U^{\lambda}(R[X], \Lambda[X])$. Let $\alpha, \beta \in U_{2r}^{\lambda}(R[X], \Lambda[X])$ for some r. Then, invoking Corollary 1 to Proposition 2, we get

$$\begin{aligned} \Gamma_n(\alpha\beta \perp e_{2r}) &\equiv \Gamma_n(\alpha \perp \beta) = \Gamma_n(\alpha) \perp \Gamma_n(\beta) \\ &\equiv \Gamma_n(\alpha)\Gamma_n(\beta) \perp e_{2r} \bmod \mathrm{EU}_{4r}^{\lambda}(R[X^n], \Lambda[X^n]). \end{aligned}$$

This amounts to $\Gamma_n(\alpha\beta) \equiv \Gamma_n(\alpha)\Gamma_n(\beta) \bmod \mathrm{EU}^\lambda(R[X^n], \Lambda[X^n])$, which completes the proof of our theorem. □

§4. Computation of the composition $(i_n)_* \circ (i_n)^*$

In this section, for any unitary ring (R, λ, Λ) and any integer $n \geq 2$, we compute the composition $(i_n)_* \circ (i_n)^*$, and also derive some consequences of this result.

Theorem 2. *For any integer $n \geq 2$, the composition $(i_n)_* \circ (i_n)^*$ of the natural homomorphism $(i_n)^* : K_1U^\lambda(R[X^n], \Lambda[X^n]) \to K_1U^\lambda(R[X], \Lambda[X])$ induced by the embedding, and the transfer $(i_n)_*\colon K_1U^\lambda(R[X], \Lambda[X]) \to K_1U^\lambda(R[X^n], \Lambda[X^n])$ constructed in Theorem 1 is equal to kH when $n = 2k$, and to $\mathrm{id} + kH$, when $n = 2k + 1$.*

In this theorem id denotes the identity map, while kH denotes the kth multiple of the hyperbolic homomorphism H.

Proof. Since the above construction of the transfer $(i_n)_*$ depends on the parity of n, we need to consider two cases separately.

First, let n be even, $n = 2k$, where $k \geq 1$. Take an arbitrary matrix $\alpha = a(X^n) \in U_{2r}^\lambda(R[X^n], \Lambda[X^n])$. The image of $[a]$ under the action of $(i_n)^*$ will be also denoted by $[a]$. Then $\widehat{\alpha} = a \oplus a \oplus \cdots \oplus a (\in \mathrm{GL}_{2rn} R[X])$, and, consequently, $\Gamma_n(\alpha) = \sigma_n^* \widehat{\alpha} \sigma_n = \left(\begin{smallmatrix} J_k^* \alpha_1 J_k & 0 \\ 0 & \alpha_1 \end{smallmatrix}\right)$, where $\alpha_1 = a \oplus a \oplus \cdots \oplus a (\in \mathrm{GL}_{rn} R[X])$. It follows that

$$J_k^* \alpha_1 J_k = I^* a I \oplus I^* a I \oplus \cdots \oplus I^* a I = (a^*)^{-1} \oplus (a^*)^{-1} \oplus \cdots \oplus (a^*)^{-1}.$$

Invoking Corollaries 2 and 3 to Proposition 2, we see that

$$\begin{aligned} \Gamma_n(\alpha) &= (a^*)^{-1} \oplus (a^*)^{-1} \oplus \cdots \oplus (a^*)^{-1} \oplus a \oplus a \oplus \cdots \oplus a \\ &= H((a^*)^{-1}) \perp \cdots \perp H((a^*)^{-1}) \equiv H(a) \perp \cdots \perp H(a) \\ &\equiv H(a^k) \bmod \mathrm{EU}^\lambda(R[X^n], \Lambda[X^n]). \end{aligned}$$

It follows that $(i_n)_* \circ (i_n)^*([a]) = [H(a^k)] = k[H(a)]$ for all $[a] \in K_1U^\lambda(R[X^n], \Lambda[X^n])$ and all even n.

Next, let n be odd, $n = 2k + 1$, where $k \geq 1$. Take an arbitrary matrix $\alpha = a(X^n) \in U_{2r}^\lambda(R[X^n], \Lambda[X^n])$, and let $a = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$, where $a_i \in M_r(R[X^n])$ for $i = 1, \ldots, 4$. The image of $[a]$ under the action of $(i_n)^*$ will be also denoted by $[a]$. Then $\widehat{\alpha} = \left(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix}\right)$, where

$$\begin{aligned} \alpha_1 &= a \oplus a \oplus \cdots \oplus a \oplus a_1 (\in M_{rn} R[X]), \\ \alpha_2 &= 0 \oslash 0 \oslash \cdots \oslash 0 \oslash a_2, \\ \alpha_3 &= a_3 \oslash 0 \oslash \cdots \oslash 0 \oslash 0, \\ \alpha_4 &= a_4 \oplus a \oplus \cdots \oplus a \oplus a (\in M_{rn} R[X]), \end{aligned}$$

while 0 denotes the zero matrix of degree $2r$. Hence,

$$\Gamma_n(\alpha) = \sigma_n^* \widehat{\alpha} \sigma_n = \begin{pmatrix} (\sigma_n^0)^* \alpha_1 \sigma_n^0 & (\sigma_n^0)^* \alpha_2 \\ \alpha_3 \sigma_n^0 & \alpha_4 \end{pmatrix},$$

where

$$(\sigma_n^0)^* \alpha_1 \sigma_n^0 = a_1 \oplus (a^*)^{-1} \oplus \cdots \oplus (a^*)^{-1}, \quad (\sigma_n^0)^* \alpha_2 = a_2 \oplus 0 \oplus \cdots \oplus 0,$$
$$\alpha_3 \sigma_n^0 = a_3 \oplus 0 \oplus \cdots \oplus 0.$$

Invoking Corollaries 2 and 3 to Proposition 2, we get

$$\begin{aligned} \Gamma_n(\alpha) &= a \perp H((a^*)^{-1}) \perp \cdots \perp H((a^*)^{-1}) \\ &\equiv a \perp H(a) \perp \cdots \perp H(a) \equiv a \perp H(a^k) \bmod \mathrm{EU}^\lambda(R[X^n], \Lambda[X^n]). \end{aligned}$$

It follows that for all $[a] \in K_1U^\lambda(R[X^n], \Lambda[X^n])$ and all odd n we have

$$(i_n)_* \circ (i_n)^*([a]) = [a] + [H(a^k)] = [a] + k[H(a)],$$

which completes the proof of our theorem. □

The above statement can be regarded as a unitary K_1-analog of the Scharlau theorem (see, e.g., [4, Chapter 7, Theorems 1.6 and 1.7]) on transfer of quadratic forms under finite field extensions.

Recall that for an arbitrary unitary ring (R, λ, Λ) the kernel of the homomorphism

$$K_1U^\lambda(R, \Lambda) \to K_1(R), \qquad \alpha \bmod \mathrm{EU}^\lambda(R, \Lambda) \to \alpha \bmod E(R),$$

induced by the forgetful functor, is denoted by $W_1'U^\lambda(R, \Lambda)$ and is called the unitary Witt 1-cogroup. Similarly, the cokernel of the hyperbolic homomorphism

$$H\colon K_1(R) \to K_1U^\lambda(R, \Lambda), \qquad \alpha \bmod E(R) \to H(\alpha) \bmod \mathrm{EU}^\lambda(R, \Lambda)$$

is denoted by $W_1U^\lambda(R, \Lambda)$ and is called the unitary Witt 1-group. In the sequel, we usually talk simply of the Witt group and the Witt cogroup.

Corollary 1. *Under the conditions of Theorem* 2*, the restriction of the composition* $(i_n)_* \circ (i_n)^*$ *to the Witt cogroup* $W_1'U^\lambda(R[X^n], \Lambda[X^n])$ *is equal to*

1) *the identity automorphism if* n *is odd*;
2) *the zero homomorphism if* n *is even.*

In particular, for n *odd the restriction of the natural homomorphism*

$$(i_n)^*\colon W_1'U^\lambda(R[X^n], \Lambda[X^n]) \to W_1'U^\lambda(R[X], \Lambda[X])$$

is a split monomorphism.

Proof. To prove this, it suffices to note that if $\alpha = \alpha(X^n) \in W_1'U^\lambda(R[X^n], \Lambda[X^n])$, then, by definition, $\alpha \in E(R[X^n])$ and, thus, $H(\alpha^k) \in \mathrm{EU}^\lambda(R[X^n], \Lambda[X^n])$ for all natural numbers k. □

Corollary 2. *Under the conditions of Theorem* 2*, the composition* $(i_n)_* \circ (i_n)^*$ *induces a well-defined endomorphism of the Witt group* $W_1U^\lambda(R[X^n], \Lambda[X^n])$*, which is*

1) *the identity automorphism if* n *is odd*;
2) *the zero homomorphism if* n *is even.*

In particular, for n *odd the homomorphism*

$$W_1U^\lambda(R[X^n], \Lambda[X^n]) \to W_1U^\lambda(R[X], \Lambda[X]),$$

induced by the natural homomorphism $(i_n)^*$ *is a split monomorphism.*

Proof. Indeed, if $\alpha = H(a(X^n))$ for some $a(X^n) \in \mathrm{GL}(R[X^n])$, then, obviously, $(i_n)^*(\alpha)$ and $(i_n)_* \circ (i_n)^*(\alpha)$ are contained in the image of the hyperbolic mapping H, so that $(i_n)^*$ induces a well-defined homomorphism

$$W_1U^\lambda(R[X^n], \Lambda[X^n]) \to W_1U^\lambda(R[X], \Lambda[X]),$$

whereas $(i_n)_* \circ (i_n)^*$ induces a well-defined endomorphism

$$W_1U^\lambda(R[X^n], \Lambda[X^n]) \to W_1U^\lambda(R[X^n], \Lambda[X^n]).$$

The remaining claims immediately follow from the above theorem. □

If n is odd, Corollaries 1 and 2 can be viewed as unitary K_1-analogs of the Springer theorem (see, e.g., [4, Chapter 7, Corollary 2.2 and Theorem 2.3]) on the properties of quadratic forms under odd degree field extensions.

Corollary 3. *Under the conditions of Theorem 2, assume that R is a commutative ring. Then for n odd the kernel of the natural homomorphism*

$$(i_n)^*\colon K_1U^\lambda(R[X^n],\Lambda[X^n]) \to K_1U^\lambda(R[X],\Lambda[X])$$

is annihilated by n.

Proof. Take an arbitrary $[\alpha] = [\alpha(X^n)] \in \operatorname{Ker}(i_n)^*$. Then $\alpha \in \mathrm{EU}^\lambda(R[X],\Lambda[X])$ and, in particular, $\alpha \in E(R[X])$. But in accordance with Proposition 1.8 in [1, Chapter 9] the kernel of the natural homomorphism $(i_n)^*\colon K_1(R[X^n]) \to K_1(R[X])$ is annihilated by n and, thus, $\alpha^n \in E(R[X^n])$. It follows that $n[\alpha] = [\alpha^n] \in W_1'U^\lambda(R[X^n],\Lambda[X^n])$. Since $(i_n)^*(n[\alpha]) = 0$ and n is odd, we have $n[\alpha] = 0$ by Corollary 1. □

§5. Unitary Bass' nilpotent K_1-group

In this section we establish unitary K_1-analogs of Farrell's theorem [5] in the algebraic K-theory.

For any unitary ring (R,λ,Λ) and a natural n we denote by $\mathrm{NK}_{1n}\,U^\lambda(R,\Lambda)$ the kernel of the homomorphism

$$K_1U^\lambda(R[X^n],\Lambda[X^n]) \to K_1U^\lambda(R,\Lambda)$$

induced by the split epimorphism $R[X^n] \to R\colon X^n \to 0$. In particular,

$$K_1U^\lambda(R[X^n],\Lambda[X^n]) = \mathrm{NK}_{1n}\,U^\lambda(R,\Lambda) \oplus K_1U^\lambda(R,\Lambda).$$

The above group $\mathrm{NK}_{1n}\,U^\lambda(R,\Lambda)$ is called the (Bass) nilpotent unitary K_1-group of the ring $(R[X^n],\Lambda[X^n])$. We denote the group $\mathrm{NK}_{11}\,U^\lambda(R,\Lambda)$ by $\mathrm{NK}_1\,U^\lambda(R,\Lambda)$. Obviously, $(i_n)^*(\mathrm{NK}_{1n}\,U^\lambda(R,\Lambda)) \subseteq \mathrm{NK}_1\,U^\lambda(R,\Lambda)$ for all natural n.

In the sequel, to simplify the notation, we denote the class $[\alpha]$ of a matrix α in some K_1-group simply by α.

Proposition 4. *Let $\alpha = \alpha(X) \in \mathrm{NK}_1\,U^\lambda(R,\Lambda)$. If $n \geq \max(3, 2\deg\alpha(X))$, then $(i_n)_*(\alpha(X)) = 0$.*

For the proof of this proposition we need the following lemma.

Lemma 4. *Let $\alpha = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in U_{2r}^\lambda(R,\Lambda)$. If $a \in \mathrm{GL}_r(R)$, then*

$$\alpha \equiv H(a) \bmod \mathrm{EU}_{4r}^\lambda(R,\Lambda).$$

Moreover, if $a \in E_r(R)$, then $\alpha \in \mathrm{EU}_{2r}^\lambda(R,\Lambda)$.

Proof. First, we prove the lemma. Since α is Λ-unitary, then by Proposition 1 the matrix a^*c is Λ-Hermitian, whereas the matrix $a^{-1}b$ is $\bar\Lambda$-Hermitian. Now, Proposition 2 shows that the first claim follows from the decomposition $\alpha = X_-(a^*c)H(a)X_+(a^{-1}b) = H(a)[H(a),X_-(-a^*c)]X_-(-a^*c)X_+(a^{-1}b)$. If, moreover, $a \in E_r(R)$, then $H(a) \in \mathrm{EU}_{2r}^\lambda(R,\Lambda)$, and, thus, $\alpha = X_-(a^*c)H(a)X_+(a^{-1}b) \in \mathrm{EU}_{2r}^\lambda(R,\Lambda)$. The lemma is proved.

Next, we prove the proposition. Suppose $\deg\alpha(X) = m$ and let $\alpha(X) = a_0 + a_1X + \dots + a_mX^m$, where $a_s \in M_{2r}(R)$ for all $0 \le s \le m$, and $a_0 = e_{2r}$, $a_m \ne 0$. Since our construction of the transfer $(i_n)_*$ depends on the parity of n, we consider two cases separately.

First, consider the case where $n = 2k$ is even. Then the condition on n implies that $a_s = 0$ for all $s \geq k+1$. Using the definition of transfer for even n reproduced in Theorem 1 and the notation of Theorem 1, we see that $(i_n)_*(\alpha(X))$ is the class of the matrix $\Gamma_n(\alpha) = \left(\begin{smallmatrix} J_k^*\alpha_1 J_k & X^n J_k^*\alpha_2 \\ \alpha_2 J_k & \alpha_1 \end{smallmatrix}\right)$ in the group $K_1U^\lambda(R[X^n], \Lambda[X^n])$; here the matrix α_1 has the form

$$\begin{pmatrix} e_{2r} & 0 & 0 & \dots & 0 & 0 \\ a_1 & e_{2r} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{k-2} & a_{k-3} & a_{k-4} & \dots & e_{2r} & 0 \\ a_{k-1} & a_{k-2} & a_{k-3} & \dots & a_1 & e_{2r} \end{pmatrix}.$$

Since α_1 is a lower unitriangular matrix over R, the matrix $J_k^*\alpha_1 J_k$ is an upper unitriangular matrix and, in particular, $J_k^*\alpha_1 J_k \in E_{nr}(R)$. By Lemma 4, it follows that $\Gamma_n(\alpha) \in \mathrm{EU}_{2nr}^\lambda(R[X^n], \Lambda[X^n])$ and thus $(i_n)_*(\alpha(X)) = 0$, which concludes the proof of the proposition for the case where n is even.

Now, consider the case where $n = 2k+1$ is odd. Then the condition on n implies that $a_s = 0$ for all $s \geq k+1$. Using the definition of transfer for odd n reproduced in Theorem 1 and the notation of Theorem 1, we see that $(i_n)_*(\alpha(X))$ is the class of the matrix $\Gamma_n(\alpha) = \left(\begin{smallmatrix} J_k^*\alpha_1 J_k & X^n J_k^*\alpha_2 \\ \alpha_3 J_k & \alpha_4 \end{smallmatrix}\right)$ in the group $K_1U^\lambda(R[X^n], \Lambda[X^n])$; here the matrix α_1 has the form

$$\begin{pmatrix} e_{2r} & 0 & 0 & \dots & 0 & 0 \\ a_1 & e_{2r} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{k-1} & a_{k-2} & a_{k-3} & \dots & e_{2r} & 0 \\ a_{1k}a_{2k} & a_{1,k-1}a_{2,k-1} & a_{1,k-2}a_{2,k-2} & \dots & a_{11}a_{21} & e_r \end{pmatrix}.$$

Now, the matrix α_1 is a lower unitriangular matrix over R and, thus, in the case where n is odd the proof of the proposition can be completed in exactly the same way as in the case where n is even. □

Now we prove an analog of Farrell's theorem for the unitary Witt 1-cogroup.

Theorem 3. *If $W_1'U^\lambda(R[X], \Lambda[X]) \cap \mathrm{NK}_1\, U^\lambda(R, \Lambda) \neq 0$, then the group*

$$W_1'U^\lambda(R[X], \Lambda[X]) \cap \mathrm{NK}_1\, U^\lambda(R, \Lambda)$$

is not finitely generated.

Corollary. *Under the assumptions of Theorem 3, the groups*

$$W_1'U^\lambda(R[X], \Lambda[X]),\ \mathrm{NK}_1\, U^\lambda(R, \Lambda),\ K_1U^\lambda(R[X], \Lambda[X])$$

are not finitely generated.

Proof. We argue by contradiction. Assume that $W_1'U^\lambda(R[X], \Lambda[X]) \cap \mathrm{NK}_1\, U^\lambda(R, \Lambda) \neq 0$, but the group $W_1'U^\lambda(R[X], \Lambda[X]) \cap \mathrm{NK}_1\, U^\lambda(R, \Lambda)$ is finitely generated. Then Proposition 4 implies that there exists a natural number N such that $(i_n)_*(\alpha(X)) = 0$ for all $\alpha(X) \in W_1'U^\lambda(R[X], \Lambda[X]) \cap \mathrm{NK}_1\, U^\lambda(R, \Lambda)$ and all $n > N$. Fix an odd $n = 2k+1 > N$ and a nonzero $\alpha(X)$ in $W_1'U^\lambda(R[X], \Lambda[X]) \cap \mathrm{NK}_1\, U^\lambda(R, \Lambda)$. Then the element $\alpha(X^n)$ is nonzero in the group $W_1'U^\lambda(R[X^n], \Lambda[X^n]) \cap \mathrm{NK}_{1n}\, U^\lambda(R, \Lambda)$. Since by Corollary 1 to Theorem 2 the natural homomorphism $(i_n)^*$ is a (split) monomorphism, we have $(i_n)^*(\alpha(X^n)) \neq 0$. Now, Proposition 4 and our choice of n imply that $(i_n)_* \circ (i_n)^*(\alpha(X^n)) = 0$. On the other hand, since $\alpha(X^n) \in W_1'U^\lambda(R[X^n], \Lambda[X^n])$, we have $\alpha(X^n) \in E(R[X^n])$, so that $H(\alpha(X^n)) \in \mathrm{EU}^\lambda(R[X^n], \Lambda[X^n])$. By Theorem 2 we get $(i_n)_* \circ (i_n)^*(\alpha(X^n)) = \alpha(X^n) + kH(\alpha(X^n)) = \alpha(X^n) \neq 0$. This contradiction proves the theorem. □

We prove an analog of Farrell's theorem for the unitary Witt 1-group.

Theorem 4. *Let $p\colon K_1U^\lambda(R[X], \Lambda[X]) \to W_1U^\lambda(R[X], \Lambda[X])$ be the canonical projection. If $p(\mathrm{NK}_1\, U^\lambda(R, \Lambda)) \neq 0$, then $p(\mathrm{NK}_1\, U^\lambda(R, \Lambda))$ is not finitely generated.*

Proof. In this proof we denote by $\overline{\alpha}$ the image $p(\alpha)$ of an element $\alpha \in K_1U^\lambda(R,\Lambda)$. Let $G' = p(\mathrm{NK}_1\,U^\lambda(R,\Lambda))$. As above, we prove the theorem by contradiction. Namely, we suppose that $G' \neq 0$, but the group G' is finitely generated. Then there exist (nonzero) $\alpha_1(X),\dots,\alpha_k(X) \in \mathrm{NK}_1\,U^\lambda(R,\Lambda])$ such that $\overline{\alpha_1(X)},\dots,\overline{\alpha_k(X)}$ generate the group G'. Denote by G the subgroup of $\mathrm{NK}_1\,U^\lambda(R,\Lambda)$ generated by $\alpha_1(X),\dots,\alpha_k(X)$. Since $G' \neq 0$ and $p(G) = G'$, we have $G \neq 0$. Then by Proposition 4 there exists a natural number N such that $(i_n)_*(G) = 0$ for all $n > N$. Fix an odd $n = 2k+1 > N$ and take an arbitrary $\alpha(X) \neq 0$ in the group G such that $\overline{\alpha(X)} \neq 0$. Then $\alpha(X^n)$ is a nonzero element of the group $\mathrm{NK}_{1n}\,U^\lambda(R,\Lambda)$, and $\overline{\alpha(X^n)} \neq 0$. Since by Corollary 1 to Theorem 2 the natural homomorphism $(i_n)^*$ is a (split) monomorphism, the image $(i_n)^*(\alpha(X^n))$ is a nonzero element of the group $\mathrm{NK}_1\,U^\lambda(R,\Lambda)$. By Proposition 4 and the choice of n, we have $(i_n)_*\circ(i_n)^*(\alpha(X^n)) = 0$, so that $\overline{(i_n)_*\circ(i_n)^*}(\overline{\alpha(X^n)}) = 0$. On the other hand, by Theorem 2 we have $(i_n)_*\circ(i_n)^*(\alpha(X^n)) = \alpha(X^n) + kH(\alpha(X^n))$, whence $\overline{(i_n)_*\circ(i_n)^*}(\overline{\alpha(X^n)}) = \overline{\alpha(X^n)} \neq 0$. This contradiction proves the theorem. □

Corollary. *Under the assumptions of Theorem* 4, *the groups*

$$W_1U^\lambda(R[X],\Lambda[X]), \quad \mathrm{NK}_1\,U^\lambda(R,\Lambda), \quad K_1U^\lambda(R[X],\Lambda[X])$$

are not finitely generated.

In conclusion, we mention that the Bass nilpotent unitary group $\mathrm{NK}_1\,U^\lambda(R,\Lambda)$ was introduced by the author in [6] (it was denoted by $HU^\lambda(R,\Lambda)$). There it was used to solve the homotopization problem for the unitary K_1-functor. There it was shown (see [6, Proposition 2]) that any element of the group $\mathrm{NK}_1\,U^\lambda(R,\Lambda)$ has a representative of the form

$$[a;b,c]_n = \begin{pmatrix} e_r - aX & bX \\ -cX^{n-1} & e_r + a^*X + \dots + (a^*)^{n-1}X^{n-1} \end{pmatrix}$$

for some natural r, n and some $a, b, c \in M_r(R)$ satisfying the following conditions:

1) the matrices b and $a \cdot b$ are $\bar{\Lambda}$-Hermitian;
2) the matrices c and $a^* \cdot c$ are Λ-Hermitian;
3) $b \cdot c = a^n$.

It is well known (see, e.g., [1, Chapter 12, Corollary 5.3]) that any element of the Bass nilpotent group $\mathrm{NK}_1(R)$ has a unipotent representative $e_r - aX$ for some natural r, where $a(\in M_r(R))$ is a nilpotent matrix. For $n = 2$ the above element $[a;b,c]_n$ of the group $\mathrm{NK}_1\,U^\lambda(R,\Lambda)$ is unipotent for all a, b, c satisfying conditions 1)–3). At the same time, it is easy to check that for $n \geq 3$ the matrix $[a;b,c]_n$ is unipotent if and only if so is the matrix $e_r - aX$, and in this case Lemma 4 of the present section implies that

$$[a;b,c]_n \equiv H(e_r - aX) \bmod \mathrm{EU}^\lambda_{4r}(R[X],\Lambda[X]).$$

Thus, we get a complete description of the unipotent part of the group $\mathrm{NK}_1\,U^\lambda(R,\Lambda)$.

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