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ORIENTED AREA IS A PERFECT MORSE FUNCTION

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Abstract. An appropriate generalization of the *oriented area function* is a perfect Morse function on the space of three-dimensional configurations of an equilateral polygonal linkage with odd number of edges. Therefore, the cyclic equilateral polygons (which appear as Morse points) can be viewed as independent generators of the homology groups of the (decorated) configuration space.

§1. Introduction

A Morse function on a smooth manifold is said to be *perfect* if the number of critical points equals the sum of Betti numbers. Not every manifold possesses a perfect Morse function. The homological spheres (that are not spheres) do not possess it; the manifolds with torsions in homologies do not possess it, etc. On the other hand, the celebrated Millnor–Smale theorem on cancellation of critical points with neighbor indices (or equivalently, cancellation of handles) provides a series of existence-type theorems [1, 4], which are the key tool in Smale's proof of the generalized Poincaré conjecture [7].

In this paper we focus on one particular example of a perfect Morse function and discuss some related problems. Namely, we restrict ourselves to the space of configuration of the equilateral polygonal linkage with odd number $n = 2k+1$ of edges. As the ambient space, it makes sense to take either $\mathbb{R}^2$ or $\mathbb{R}^3$, which gives us the spaces $M_2(n)$ and $M_3(n)$. In larger dimension the configuration space is not a manifold. The number n is chosen to be odd for the same reason: for even n, the configuration space of the equilateral polygonal linkage has singular points.

We are interested in finding a "natural" perfect Morse function, that is, a function that has a transparent physical or geometrical meaning. Our first candidate for a "natural" Morse function on $M_2(n)$ is the oriented area function A. Indeed, it is a Morse function with an easy description of its critical points as cyclic polygons (that is, polygons with a superscribed circle), and with a simple formula for the Morse index of a critical point, see [3]. However, for $M_2(n)$ A is not perfect. In particular, for the equilateral pentagonal linkage, A has one additional local maximum (except for the global maximum) and one additional local minimum, see Example 1. For the equilateral heptagonal linkage the number of Morse points greatly exceeds the sum of Betti numbers of the configuration space.

To build up a perfect Morse function, we take the space $M_3(n)$ and *decorate* it. The decorated space $\widetilde{M}_3(n)$ is well adjusted for an appropriate generalization S of the area function A. Its critical points (loosely speaking) are again cyclic polygons. Surprisingly, the function S is a perfect Morse function. As a direct corollary, we interpret the cyclic equilateral polygons as independent generators of the homology groups of the configuration space $\widetilde{M}_3(n)$.

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§2. Preliminaries and notation

For an odd $n = 2k + 1$, an equilateral *polygonal n-linkage* should be interpreted as a collection of rigid bars of lengths 1 joined consecutively by revolving joints in a chain.

A *configuration* of the polygonal n-linkage in the Euclidean space $\mathbb{R}^d$, $d = 2,\ 3$, is a sequence of points $R = (p_1, \dots, p_{n+1})$, $p_i \in \mathbb{R}^d$, such that $|p_i, p_{i+1}| = 1$ for all i and $|p_n, p_1| = 1$ modulo the action of orientation preserving isometries of the space $\mathbb{R}^d$. We also call P a *polygon.* A configuration carries a natural orientation, which we indicate in figures by an arrow.

The space $M_d(n)$ of all configurations up to an orientation-preserving isometry of the ambient space is the *moduli space*, or the *configuration space of the polygonal linkage L.*

For $d = 2, 3$ the space $M_d(n)$ is a smooth manifold.

Below in this section we explain what is known about planar configurations and the signed area function as the Morse function on the configuration space.

Definition 1. The *signed area* of a polygon $P \in M_2(n)$ with the vertices $p_i = (x_i, y_i)$ is defined by

$$2A(P) = (x_1y_2 - x_2y_1) + \dots + (x_ny_1 - x_1y_n).$$

Definition 2. A polygon P is *cyclic* if all its vertices p_i lie on a circle.

A polygon P is a critical point of the signed area function A if and only if P is cyclic. The Morse indices of cyclic polygons were calculated in [3, 9].

Example 1 (see [6])**.** The equilateral pentagonal linkage has 14 cyclic configurations depicted in Figure 1.

(1) The convex regular pentagon and its mirror image are the global maximum and minimum of the signed area A. Their Morse indices are 2 and 0, respectively.

(2) The starlike configurations are a local maximum and a local minimum of A.

(3) There are ten configurations that have a fold of three consecutive edges. Their Morse indices are equal to 1.

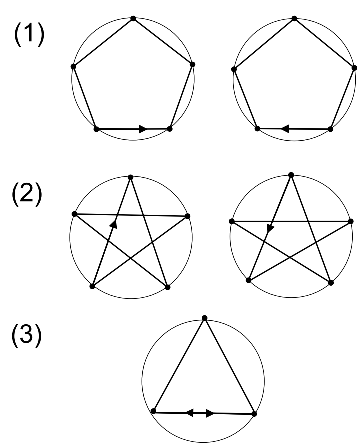


Figure 1. Cyclic configurations of the equilateral pentagonal linkage

§3. The decorated configuration space $\widetilde{\mathcal{M}}_3(n)$ and the area function S

Definition 3. The *decorated configuration space* is defined as the space of pairs

$$\widetilde{M}_3(n) = \{(P,\xi) \mid P \text{ is a polygon in } \mathbb{R}^3 \text{ with the sidelengths } 1;\ \xi \in S^2\},$$

factorized by the diagonal action of the orientation preserving isometries of $\mathbb{R}^3$.

Here $S^2 \in \mathbb{R}^3$ is the unit sphere centered at the origin O.

Lemma 1.

(1) *The space $\widetilde{M}_3(n)$ is an orientable fibration over $M_3(n)$ whose fiber is S^2.*
(2) *The Euler class of this fibration equals zero.*

Proof. (1) It is known that the space of all polygons with fixed sidelengths (before factorization by isometries) is orientable. Therefore, the space of the pairs (a polygon, a vector) is also orientable as the total space of the trivial fibration. Since we factorize by the action of orientation preserving isometries, the factor space is also orientable.

(2) $s(P) := \frac{\overrightarrow{p_1p_2}}{|p_1p_2|}$ determines a nonzero section. □

The Gysin sequence [8] implies the following.

Corollary 1.

(1) *We have the short exact sequence*

$$0 \to H^m(M(n)) \to H^m(\widetilde{M}(n)) \to H^{m-2}(M(n)) \to 0.$$

(2) *The homology groups $H_m(\widetilde{M}(n))$ are free Abelian. For the Betti numbers we have*

$$\beta^m(\widetilde{M}(n)) = \beta^m(M(n)) + \beta^{m-2}(M(n)).$$

Definition 4. Let $(P,\xi) \in \widetilde{M}_3(n)$, and let p_i be the vertices of P. The *area* of the pair (P,ξ) is defined as the scalar product

$$S(P,\xi) = \frac{1}{2}(p_1 \times p_2 + p_2 \times p_3 + \cdots + p_n \times p_1, \xi).$$

An alternative equivalent definition is as follows:

$$S(P,\xi) = A(pr_{\xi^\perp}(P)),$$

where $pr_{\xi^\perp}$ is the plane orthogonal to ξ and cooriented by ξ.

Proposition 1. *For an equilateral polygon with odd number of edges, the critical points (P,ξ) of the function S are pairs (P,ξ) such that P is a planar cyclic polygon, and ξ is orthogonal to the affine hull of P. If (P,ξ) is a critical point, then $(P,-\xi)$ is also critical.*

Proof. The paper [5] contains a characterization of all critical points for a generic (not necessarily equilateral) polygonal linkage. In our particular case, the critical points (P,ξ) of the function S fall into two classes.

(1) **Planar cyclic configurations.** These are pairs (P,ξ) such that P is a planar cyclic polygon, and ξ is orthogonal to the affine hull of P.
(2) **Nonplanar configurations.** They are characterized by the following conditions:
   (a) the vectors ξ and $2\overrightarrow{S} = p_1 \times p_2 + p_2 \times p_3 + \cdots + p_n \times p_1$ are parallel (but they can have opposite directions);
   (b) the orthogonal projection of P onto the plane $\overrightarrow{S(P)}^\perp$ is a cyclic polygon;
   (c) for every i, the vectors $\overrightarrow{T_i}$, $\overrightarrow{S}$, and $\overrightarrow{d_i}$ are coplanar.

   Here $\overrightarrow{d_i}$ is the ith short diagonal, and $\overrightarrow{T_i}$ is the vector area of the triangle $p_{i-1}p_ip_{i+1}$.

Let us show that the second class (nonplanar configurations) is empty. Indeed, given a nonplanar critical configuration, introduce a Cartesian system with the z-axes parallel to ξ. The three conditions (a), (b), and (c) imply that the absolute value of the slope of an edge with respect to the plane $(x, y) = \xi^{\perp}$ does not depend on the edge. This implies a contradiction with the closing condition: $\sum_{i=o}^{n}(z(p_i) - z(p_{i-1})) = 0$, where the indices are modulo n. □

Theorem 1. (1) *For an equilateral linkage with odd number of edges, the function S is a perfect Morse function on the decorated configuration space $\widetilde{M}_3(L)$.*

(2) *Each critical point of the function S is a pair (P, ξ), where P is a planar cyclic configuration and ξ is a unit vector orthogonal to P. Each planar cyclic configuration P gives two critical points of the function S (with two different choices of the normal vector ξ).*

(3) *The Morse index of a critical point (P, ξ) is*

$$m(P, \xi) = 2e - 2\omega - 2,$$

where ω is the winding number of P around the center of the circumscribed circle, and e is the number of edges that go counterclockwise.[1]

Proof. (i) The second statement has already been proved. We show that the number of critical points equals the sum of Betti numbers. These are already known due to A. Klyachko [2]:

$$\beta^{2p}(M_3(n)) = \sum_{0\leq i\leq p} \binom{2k}{i}, \quad p < k.$$

By Corollary 1, we have

$$\beta^{2p}(\widetilde{M}_3(n)) = \sum_{0\leq i\leq p} \binom{n}{i}, \quad p < k.$$

Each equilateral cyclic n-gon with an orthogonal vector ξ is determined by its winding number ω and by the set of edges that go clockwise. Assume that the winding number is positive (the negative values are treated by symmetry). If the number of edges that go clockwise is e, then the winding number ranges from 1 to $(k - e)$.

For $p = 0, 1, \ldots, k$, denote by N_n^p the number of cyclic equilateral polygons for which $e - \omega - 1 = p$. Then $\widetilde{\beta}_n^{2p} = N_n^p$.

(iii) Straightforward analysis of the Hesian matrix is very complicated (probably impossible), so we use a combinatorial approach. We prove the formula for Morse index by induction on n. The base is given by the equilateral pentagon. By symmetry, we assume that we have a critical point (P, ξ) such that $S(P, \xi) > 0$, or equivalently, with winding number $\omega > 0$.

The proof is based on two observations.

(1) There is a natural embedding of a neighborhood of P in the space $M_2(n)$ to $\widetilde{M}_3(n)$. It maps a configuration P to (P, ξ) with the same P and with ξ orthogonal to the affine hull of P. The direction of ξ is chosen so that S is positive. We choose a basis of the tangent space $T_{(P,\xi)}\widetilde{M}_3(n)$ that starts with the pushforward of a basis of $T_P M_2(n)$ and ends with the coordinates of ξ. The Hessian matrix related to this basis is a block matrix:

$$\mathrm{HESS}(P, \xi) = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & -E \end{pmatrix},$$

[1]The vector ξ sets an orientation on the plane of the polygon, so it makes sense to talk of "edges going clockwise" and "edges going counterclockwise".

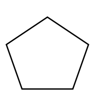
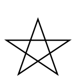
where H_1 is the $(n-3) \times (n-3)$ Hessian matrix of the planar polygon P related to the area function A and the space $M_2(n)$, and E is the the unit matrix of size 2×2.

(2) For an $(n+2)$-gon and a number $1 \le i \le n$, consider the embedding $\varphi_i \colon \widetilde{M}_3(n) \to \widetilde{M}_3(n+2)$ that keeps ξ and replaces the edge number i by a fold of three edges (see Figure 1, (3) for an equilateral triangle with an edge replaced by a three-fold). Each critical point $(P,\xi) \in \widetilde{M}_3(n)$ induces a critical point $\varphi_i(P,\xi) \in \widetilde{M}_3(n+2)$. Since this embedding has codimension two, and all Morse indices can only be even, we have either $m(\varphi_i(P,\xi)) = m(P,\xi)$, $m(\varphi_i(P,\xi)) = 2 + m(P,\xi)$, or $m(\varphi_i(P,\xi)) = 4 + m(P,\xi)$. More precisely, replacing an edge by a three-fold adds two extra columns to H_1 and two extra columns to H_2. From [3] we know that the Morse index of P related to $M_2(n)$ and the oriented area A (that is, the number of negative eigenvalues of H_1) equals $e - 2\omega - 1$. It increases by one after replacement of an edge by a three-fold. This only leaves the case $m(\varphi_i(P,\xi)) = 2 + m(P,\xi)$.

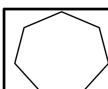
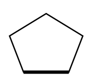
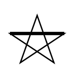
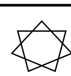
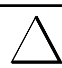
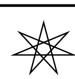
These arguments allow us to make an induction step $n \to n+2$, thus proving 3 for all cyclic configurations with triple edges.

It remains to prove the formula for configurations without triple edges, that is, with all the edges going counterclockwise. We keep assuming that $S(P,\xi) > 0$, so that there are exactly k such polygons: with $\omega = 1, 2, 3, \dots, k$. Their Morse indices should be $2n-4$, $2n-6$, $2n-8$, etc. The only question is which configuration has one or other index. We know that H_1 contributes $n-3$, $n-5$, $n-7$, etc. to each of the Morse indices. The block H_2 contributes at most $n-3$, and $-E$ contributes exactly two. The statement follows. $\square$

As an illustration, below we list all cyclic equilateral pentagons and heptagons. The first column depicts a combinatorial type, the third tells the number of configurations of this type, and the last column tells the Morse index.

	orientation	number	Morse index
	1	1	6
	-1	1	0
	1	5	4
	-1	5	2
	1	1	4
	-1	1	2

(a) Critical equilateral pentagons

polygon	orientation	number	Morse index
	1	1	10
	-1	1	0
	1	7	8
	-1	7	2
	1	7	6
	-1	7	4
	1	1	8
	-1	1	2
	1	21	6
	-1	21	4
	1	1	6
	-1	1	4

(b) Critical equilateral heptagons

FIGURE 2

Concluding remarks

The decorated configuration space and the function S can be defined for a polygonal linkage that is not necessarily equilateral. However, generically, the function S is not a perfect Morse function.

In the paper we omit the discussion of the nondegeneracy of critical points, because it appears to be somewhat technical. However, the following arguments work: one may replace the equilateral linkage $(1, 1, \ldots, 1)$ by its perturbation $(1+\epsilon_1, 1+\epsilon_2, \ldots, 1+\epsilon_n)$. The latter has nondegenerate critical configurations that are close to equilateral ones described above.

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