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HEAT TRACES AND SPECTRAL ZETA FUNCTIONS FOR p-ADIC LAPLACIANS

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Abstract. The study of the heat traces and spectral zeta functions for certain p-adic Laplacians is initiated. It is shown that the heat traces are given by p-adic integrals of Laplace type, and that the spectral zeta functions are p-adic integrals of Igusa type. Good estimates are found for the behavior of the heat traces when the time tends to infinity, and for the asymptotics of the function counting the eigenvalues less than or equal to a given quantity.

§1. Introduction

The p-adic heat equation is defined as

$$\frac{\partial u(x,t)}{\partial t} + D^{\beta} u(x,t) = 0, \quad x \in \mathbb{Q}_p, \ t \geq 0, \tag{1.1}$$

where

$$\big(D^{\beta}\varphi\big)(x) = \mathcal{F}^{-1}_{\xi\to x}(|\xi|_p^{\beta}\mathcal{F}_{x\to\xi}\varphi), \quad \beta > 0,$$

is the Vladimirov operator (a p-adic Laplacian), and $\mathcal{F}$ denotes the p-adic Fourier transform. This equation is the p-adic counterpart of the classical fractional heat equation, which describes a random motion (the fractional Brownian motion) of a particle; a "similar" statement is valid for the p-adic heat equation. More precisely, the fundamental solution of (1.1) is the transition density of a bounded right-continuous Markov process without second kind discontinuities. The family of non-Archimedean heat-type equations is very large, and it has a deep relationship with mathematical physics. For instance, in [4, 5], Avetisov et al. introduced a new class of models for complex systems based on p-adic analysis. From a mathematical point of view, in these models the time-evolution of a complex system is described by a p-adic master equation (a parabolic-type pseudodifferential equation) that controls the time-evolution of a transition function of a Markov process on an ultrametric space. The simplest type of a master equation is the one-dimensional p-adic heat equation. This equation was introduced in the book [35] by Vladimirov, Volovich, and Zelenov. It should be mentioned here that the p-adic heat equation also appeared in certain works connected with the Riemann hypothesis [23]. In recent years, the non-Archimedean heat-type equations and their associated Markov processes have been studied intensively, see, e.g., [19, 35, 7, 9, 10, 15, 30, 34, 38, 39] and the references therein. On the other hand, the study of pseudo-differential operators on complex-valued functions defined on the field of p-adic numbers or its subsets was

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started around 1990 by several authors (Haran, Vladimirov, Volovich, Kochubei, Khrennikov, Kozyrev, and others). By this time, there are well-developed theories of elliptic and parabolic equations on p-adics, some results on hyperbolic equations and constructions appearing as p-adic counterparts of quantum mechanics and quantum field theory, see, e.g., [2, 15, 17, 19, 21, 20, 35].

The links between the Archimedean heat equations and number theory and geometry are well known and deep. Here we mention the relationship with the Riemann zeta function, which drives naturally to trace-type formulas, see, e.g., [3] and the references therein, and the relationship with the Atiyah–Singer index theorem, see, e.g., [16] and the references therein. The study of non-Archimedean counterparts of the matters mentioned above is quite relevant, especially if we take into account that the Connes and Deninger programs to attack the Riemann hypothesis drive naturally to these matters, see, e.g., [12, 13, 24] and the references therein. For instance, several types of p-adic trace formulas have been studied, see, e.g., [1, 6, 37] and the references therein.

Nowadays there is no theory of pseudodifferential operators over p-adic manifolds comparable to the classical theory, see, e.g., [31] and the references therein. The n-dimensional unit ball is the simplest p-adic compact manifold possible. From a topological point of view, this ball is a fractal, more precisely, it is topologically equivalent to a Cantor-like subset of the real line, see, e.g., [2, 35]. Currently, there is a lot of interest to spectral zeta functions attached to fractals see, e.g., [22, 32].

In this paper we initiate the study of heat traces and spectral zeta functions attached to certain p-adic Laplacians, denoted by $\boldsymbol{A}_\beta$, which are generalizations of the p-adic Laplacians introduced by the authors in [9], see also [10]. By using an approach inspired by the work of Minakshisundaram and Pleijel, see [25, 26, 27], we find a formula for the trace of the semigroup $e^{-t\boldsymbol{A}_\beta}$ acting on the space of square integrable functions supported on the unit ball with average zero, see Theorem 6.7. The trace of $e^{-t\boldsymbol{A}_\beta}$ is a p-adic oscillatory integral of Laplace type; we do not know the exact asymptotics of this integral as t tends to infinity, however, we obtain a good estimate for its behavior at infinity, see Theorem 6.7 (ii). Several unexpected mathematical situations occur in the p-adic setting. For instance, the spectral zeta functions are p-adic Igusa-type integrals, see Theorem 7.5. The p-adic spectral zeta functions studied here may have infinitely many poles on the boundary of its domain of holomorphy, then, to the best of our knowledge, the standard Ikehara Tauberian theorems cannot be applied to obtain the asymptotic behavior for the function counting the eigenvalues of $\boldsymbol{A}_\beta$ less than or equal to $T \geq 0$. However, we are still able to find good estimates for this function, see Theorem 7.5, Remark 7.6, and Conjecture 7.7. The proofs require several results on certain "boundary value problems" attached to p-adic heat equations associated with operators $\boldsymbol{A}_\beta$, see Proposition 5.3, Theorem 6.5, and Proposition 6.6. Finally, we mention that our results and techniques are completely different from those presented in [1, 6, 37].

§2. Preliminaries

In this section we fix the notation and collect some basic results on p-adic analysis that we shall use through the paper. For a detailed exposition on p-adic analysis the reader may consult [2, 33, 35].

2.1. The field of p-adic numbers. Throughout, p will denote a prime number. The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if } \ x = 0, \\ p^{-\gamma} & \text{if } \ x = p^\gamma \frac{a}{b}, \end{cases}$$

where a and b are integers coprime to p. The integer $\gamma = \operatorname{ord}_p(x) := \operatorname{ord}(x)$, with $\operatorname{ord}(0) := +\infty$, is called the *$p$-adic order of x*. We extend the p-adic norm to $\mathbb{Q}_p^n$ by taking

$$\|x\|_p := \max_{1\leq i\leq n} |x_i|_p \quad \text{for} \quad x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n.$$

We define $\operatorname{ord}(x) = \min_{1\leq i\leq n}\{\operatorname{ord}(x_i)\}$; then $\|x\|_p = p^{-\operatorname{ord}(x)}$. The metric space $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is a complete ultrametric space. As a topological space, $\mathbb{Q}_p$ is homeomorphic to a Cantor-like subset of the real line, see, e.g., [2, 35].

Any p-adic number $x \neq 0$ has a unique expansion of the form

$$x = p^{\operatorname{ord}(x)} \sum_{j=0}^{\infty} x_j p^j,$$

where $x_j \in \{0, 1, 2, \dots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define *the fractional part* $\{x\}_p$ *of* $x \in \mathbb{Q}_p$ as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } \operatorname{ord}(x) \geq 0, \\ p^{\operatorname{ord}(x)} \sum_{j=0}^{-\operatorname{ord}(x)-1} x_j p^j & \text{if } \operatorname{ord}(x) < 0. \end{cases}$$

Any $x \in \mathbb{Q}_p^n \setminus \{0\}$ can be represented uniquely as $x = p^{\operatorname{ord}(x)} v(x)$, where $\|v(x)\|_p = 1$.

2.2. Additive characters. Set $\chi_p(y) = \exp(2\pi i\{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e., a continuous map from $(\mathbb{Q}_p, +)$ into S (the unit circle viewed as a multiplicative group) satisfying $\chi_p(x_0+x_1) = \chi_p(x_0)\chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$. The additive characters of $\mathbb{Q}_p$ form an Abelian group isomorphic to $(\mathbb{Q}_p, +)$, with isomorphism given by $\xi \to \chi_p(\xi x)$, see, e.g., [2, Subsection 2.3].

2.3. Topology of $\mathbb{Q}_p^n$. For $r \in \mathbb{Z}$, we denote by $B_r^n(a) = \{x \in \mathbb{Q}_p^n; \|x-a\|_p \leq p^r\}$ *the ball of radius p^r with center at $a = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$*, and take $B_r^n(0) := B_r^n$. Note that $B_r^n(a) = B_r(a_1) \times \cdots \times B_r(a_n)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p; |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0 = \mathbb{Z}_p$, *the ring of p-adic integers*. We also denote by $S_r^n(a) = \{x \in \mathbb{Q}_p^n; \|x-a\|_p = p^r\}$ *the sphere of radius p^r with center at $a = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$*, and take $S_r^n(0) := S_r^n$. Observe that $S_0^1 = \mathbb{Z}_p^\times$ (the group of units of $\mathbb{Z}_p$), but $(\mathbb{Z}_p^\times)^n \subsetneq S_0^n$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$. Moreover, either two balls in $\mathbb{Q}_p^n$ are disjoint, or one is contained in the other.

As a topological space $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is totally disconnected, i.e., the only connected subsets of $\mathbb{Q}_p^n$ are the empty set and the points. A subset of $\mathbb{Q}_p^n$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^n$, see, e.g., [35, Subsection 1.3], or [2, Subsection 1.8]. The balls and spheres are compact subsets. Thus, $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is a locally compact topological space.

We use $\Omega(p^{-r}\|x-a\|_p)$ to denote the characteristic function of the ball $B_r^n(a)$. For more general sets, we shall use the notation $\mathbb{1}_A$ for the characteristic function of a set A.

§3. The Bruhat–Schwartz space and the Fourier transform

A complex-valued function φ defined on $\mathbb{Q}_p^n$ is said to be *locally constant* if for any $x \in \mathbb{Q}_p^n$ there exist an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \quad \text{for} \quad x' \in B_{l(x)}^n. \tag{3.1}$$

A function $\varphi\colon \mathbb{Q}_p^n \to \mathbb{C}$ is called a *Bruhat–Schwartz function* (or a *test function*) if it is locally constant with compact support. Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls. The $\mathbb{C}$-vector space of Bruhat–Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p^n)$. For $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, the largest number $l = l(\varphi)$ satisfying (3.1) is called the *exponent of local constancy* (or the *parameter of constancy*) *of* φ.

If U is an open subset of $\mathbb{Q}_p^n$, we write $\mathcal{D}(U)$ for the space of test functions with supports contained in U; then $\mathcal{D}(U)$ is dense in

$$L^\rho(U) = \left\{\varphi\colon U \to \mathbb{C}; \|\varphi\|_\rho = \left\{\int_U |\varphi(x)|^\rho \, d^n x\right\}^{\frac{1}{\rho}} < \infty\right\},$$

where $d^n x$ is the Haar measure on $\mathbb{Q}_p^n$ normalized by the condition $vol(B_0^n) = 1$, for $1 \leq \rho < \infty$, see, e.g., [2, Subsection 4.3].

3.1. The Fourier transform of test functions. Given $\xi = (\xi_1, \dots, \xi_n)$ and $y = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$, we set $\xi \cdot x := \sum_{j=1}^n \xi_j x_j$. The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x)\varphi(x) \, d^n x \quad \text{for} \quad \xi \in \mathbb{Q}_p^n,$$

where $d^n x$ is the normalized Haar measure on $\mathbb{Q}_p^n$. The Fourier transform is a linear isomorphism from $\mathcal{D}(\mathbb{Q}_p^n)$ onto itself satisfying $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$, see, e.g., [2, Subsection 4.8]. We shall also use the notation $\mathcal{F}_{x\to\xi}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ.

§4. p-adic Laplacians

Let $\mathbb{R}_+ := \{x \in \mathbb{R}; x \geq 0\}$; we fix a function

$$A\colon \mathbb{Q}_p^n \to \mathbb{R}_+$$

satisfying the following properties:
(i) $A(\xi)$ is a radial function, i.e., $A(\xi) = g(\|\xi\|_p)$ for some $g\colon \mathbb{R}_+ \to \mathbb{R}_+$, for simplicity we use the notation $A(\xi) = A(\|\xi\|_p)$;
(ii) there exist constants C_0, $C_1 > 0$ and $\beta > 0$ such that

$$C_0\|\xi\|_p^\beta \leq A(\xi) \leq C_1\|\xi\|_p^\beta \quad \text{for} \quad x \in \mathbb{Q}_p^n. \tag{4.1}$$

Since β in (4.1) is unique, we use the notation $A_\beta(\|\xi\|_p) = A(\|\xi\|_p)$.

We define a pseudodifferential operator $\boldsymbol{A}_\beta$ by

$$(\boldsymbol{A}_\beta\varphi)(x) = \mathcal{F}^{-1}_{\xi\to x}[A_\beta(\xi)\mathcal{F}_{x\to\xi}\varphi] \quad \text{for} \quad \varphi \in \mathcal{D}(\mathbb{Q}_p^n), \tag{4.2}$$

calling $A_\beta(\xi)$ *the symbol of* $\boldsymbol{A}_\beta$. The operator $\boldsymbol{A}_\beta$ extends to an unbounded and densely defined operator in $L^2(\mathbb{Q}_p^n)$ with the domain

$$\mathrm{Dom}(\boldsymbol{A}_\beta) = \{\varphi \in L^2; A_\beta(\xi)\mathcal{F}\varphi \in L^2\}. \tag{4.3}$$

Moreover:
(i) $(\boldsymbol{A}_\beta, \mathrm{Dom}(\boldsymbol{A}_\beta))$ is a selfadjoint and positive operator;
(ii) $-\boldsymbol{A}_\beta$ is the infinitesimal generator of a contraction C_0-semigroup, cf. [9, Proposition 3.3].

With the operator $\boldsymbol{A}_\beta$ we associate the following "heat equation":

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \boldsymbol{A}_\beta u(x,t) = 0, & x \in \mathbb{Q}_p^n, \ t \in [0,\infty), \\ u(x,0) = u_0(x), & u_0(x) \in \mathrm{Dom}(\boldsymbol{A}_\beta). \end{cases}$$

This initial-value problem has a unique solution given by

$$u(x,t) = \int_{\mathbb{Q}_p^n} Z(x-y,t)u_0(y) \, d^n y,$$

where

$$Z(x,t;\boldsymbol{A}_\beta) := Z(x,t) = \int_{\mathbb{Q}_p^n} \chi_p(-\xi\cdot x) e^{-tA_\beta(\xi)}\, d^n\xi, \quad t>0,\ x\in\mathbb{Q}_p^n,$$

cf. [9, Theorem 6.5]. The function $Z(x,t)$ is called the heat kernel associated with $\boldsymbol{A}_\beta$.

4.1. Operators $\boldsymbol{W}_\alpha$. The class of operators $\boldsymbol{A}_\beta$ includes the class of operators $\boldsymbol{W}_{\boldsymbol{\alpha}}$ studied by the authors in [9], see also [10]. In addition, most of the results on the $\boldsymbol{W}_{\boldsymbol{\alpha}}$ operators are valid for the $\boldsymbol{A}_\beta$ operators. We review briefly the definition of these operators. Fix a function

$$w_\alpha\colon \mathbb{Q}_p^n \to \mathbb{R}_+$$

with the following properties:
(i) $w_\alpha(y)$ is a radial, i.e., $w_\alpha(y) = w_\alpha(\|y\|_p)$;
(ii) $w_\alpha(\|y\|_p)$ is a continuous and monotone increasing function of $\|y\|_p$;
(iii) $w_\alpha(y)=0$ if and only if $y=0$;
(iv) there exist constants $C_0, C_1>0$ and $\alpha>n$ such that

$$C_0\|y\|_p^\alpha \le w_\alpha(\|y\|_p) \le C_1\|y\|_p^\alpha, \quad x\in\mathbb{Q}_p^n.$$

Now we define the operator

$$(\boldsymbol{W}_\alpha\varphi)(x) = \kappa\int_{\mathbb{Q}_p^n} \frac{\varphi(x-y)-\varphi(x)}{w_\alpha(\|y\|_p)}\, d^n y, \quad \varphi\in\mathcal{D}(\mathbb{Q}_p^n),$$

where κ is a positive constant. The operator $\boldsymbol{W}_\alpha$ is pseudodifferential, more precisely, if

$$A_{w_\alpha}(\xi) := \int_{\mathbb{Q}_p^n} \frac{1-\chi_p(y\cdot\xi)}{w_\alpha(\|y\|_p)}\, d^n y,$$

then

$$(\boldsymbol{W}_{\boldsymbol{\alpha}}\varphi)(x) = -\kappa\mathcal{F}^{-1}_{\xi\to x}[A_{w_\alpha}(\xi)\mathcal{F}_{x\to\xi}\varphi], \quad \varphi\in\mathcal{D}(\mathbb{Q}_p^n).$$

The function $A_{w_\alpha}(\xi)$ is radial (so we write $A_{w_\alpha}(\xi) = A_{w_\alpha}(\|\xi\|_p)$), continuous, nonnegative, and satisfies $A_{w_\alpha}(0)=0$, and it obeys the inequalities

$$C_0'\|\xi\|_p^{\alpha-n} \le A_{w_\alpha}(\|\xi\|_p) \le C_1'\|\xi\|_p^{\alpha-n}, \quad x\in\mathbb{Q}_p^n,$$

cf. [9, Lemmas 3.1, 3.2, 3.3]. The operator $\boldsymbol{W}_\alpha$ extends to an unbounded and densely defined operator in $L^2(\mathbb{Q}_p^n)$.

4.2. Examples.

Example 4.1. The Taibleson operator is defined as

$$(D_T^\beta\phi)(x) = \mathcal{F}^{-1}_{\xi\to x}(\|\xi\|_p^\beta\mathcal{F}_{x\to\xi}\phi), \quad \text{with}\quad \beta>0 \quad\text{and}\quad \phi\in\mathcal{D}(\mathbb{Q}_p^n).$$

cf. [30] and [2, Subsection 9.2.2].

Example 4.2. Take $A_\beta(\xi) = \|\xi\|_p^\beta\{B - Ae^{-\|\xi\|_p}\}$ with $B>A>0$. Then $A_\beta(\xi)$ satisfies all the requirements announced at the beginning of this section. In general, if $f:\mathbb{Q}_p^n\to\mathbb{R}_+$ is a radial function satisfying

$$0 < \inf_{\xi\in\mathbb{Q}_p^n} f(\|\xi\|_p) < \sup_{\xi\in\mathbb{Q}_p^n} f(\|\xi\|_p) < \infty,$$

then $A_\beta(\|\xi\|_p)f(\|\xi\|_p)$ satisfies all the requirements announced at the beginning of this section.

§5. Lizorkin spaces, eigenvalues, and eigenfunctions for the $\boldsymbol{A}_\beta$ operators

We set $\mathcal{L}_0(\mathbb{Q}_p^n) := \{\varphi \in \mathcal{D}(\mathbb{Q}_p^n); \widehat{\varphi}(0) = 0\}$. The $\mathbb{C}$-vector space $\mathcal{L}_0$ is called the p-adic Lizorkin space of second class. We recall that $\mathcal{L}_0$ is dense in L^2, see [2, Theorem 7.4.3], and that $\varphi \in \mathcal{L}_0(\mathbb{Q}_p^n)$ if and only if

$$\int_{\mathbb{Q}_p^n} \varphi(x)\, d^n x = 0. \tag{5.1}$$

Consider the operator $(\boldsymbol{A}_\beta\varphi)(x) = \mathcal{F}^{-1}_{\xi\to x}[A_\beta(\xi)\mathcal{F}_{x\to\xi}\varphi]$ on $\mathcal{L}_0(\mathbb{Q}_p^n)$; then $\boldsymbol{A}_\beta$ is densely defined on L^2, and $\boldsymbol{A}_\beta : \mathcal{L}_0(\mathbb{Q}_p^n) \to \mathcal{L}_0(\mathbb{Q}_p^n)$ is a well-defined linear operator.

We set $\mathcal{L}_0(\mathbb{Z}_p^n) := \{\varphi \in \mathcal{L}_0(\mathbb{Q}_p^n); \operatorname{supp}\varphi \subseteq \mathbb{Z}_p^n\}$ and define

$$L_0^2(\mathbb{Z}_p^n, d^n x) := L_0^2(\mathbb{Z}_p^n) = \left\{ f \in L^2(\mathbb{Z}_p^n, d^n x); \int_{\mathbb{Z}_p^n} f(x)\, d^n x = 0 \right\}.$$

Notice that, since $L_0^2(\mathbb{Z}_p^n)$ is the orthogonal complement in $L^2(\mathbb{Z}_p^n)$ of the space generated by the characteristic function of $\mathbb{Z}_p^n$, we see that $L_0^2(\mathbb{Z}_p^n)$ is a Hilbert space.

Then $\mathcal{L}_0(\mathbb{Z}_p^n)$ is dense in $L_0^2(\mathbb{Z}_p^n)$. Indeed, set

$$\delta_k(x) := p^{nk}\Omega(p^k\|x\|_p) \quad \text{for} \quad k \in \mathbb{N}.$$

Then $\int_{\mathbb{Q}_p^n} \delta_k(x) d^n x = 1$ for any k, and taking $f \in L_0^2(\mathbb{Z}_p^n)$, we get $f_k = f * \delta_k \in \mathcal{L}_0(\mathbb{Z}_p^n)$, and $f_k \xrightarrow{\|\cdot\|_{L^2}} f$.

Set

$$\omega_{\gamma bk}(x) := p^{-\frac{n\gamma}{2}}\chi_p(p^{-1}k\cdot(p^\gamma x - b))\Omega(\|p^\gamma x - b\|_p),$$

where $\gamma \in \mathbb{Z}$, $b \in (\mathbb{Q}_p/\mathbb{Z}_p)^n$, $k = (k_1, \ldots, k_n)$ with $k_i \in \{0, \ldots p-1\}$ for $i = 1, \ldots, n$, and $k \neq (0, \ldots, 0)$.

Lemma 5.1. *With the above notation,*

$$(\boldsymbol{A}_\beta\omega_{\gamma bk})(x) = \lambda_{\gamma bk}\omega_{\gamma bk}(x),$$

where

$$\lambda_{\gamma bk} = A_\beta(p^{1-\gamma}).$$

Moreover, $\int_{\mathbb{Q}_p^n} \omega_{\gamma bk}(x)\, d^n x = 0$ *and* $\{\omega_{\gamma bk}(x)\}_{\gamma bk}$ *forms a complete orthogonal basis of* $L^2(\mathbb{Q}_p^n, d^n x)$.

Proof. This follows from Theorems 9.4.5 and 8.9.3 in [2], by using the fact that A_β satisfies $A_\beta(\|p^\gamma(-p^{-1}k+\eta)\|_p) = A_\beta(\|p^{\gamma-1}k\|_p) = A_\beta(p^{1-\gamma})$ for all $\eta \in \mathbb{Z}_p^n$. □

Remark 5.2. (i) Notice that $\boldsymbol{A}_\beta$ has eigenvalues of infinite multiplicity. Now, if we consider only eigenfunctions satisfying supp $\omega_{\gamma bk}(x) \subset \mathbb{Z}_p^n$, then necessarily $\gamma \leq 0$ and $b \in p^\gamma\mathbb{Z}_p^n/\mathbb{Z}_p^n$. For γ fixed, there are only finitely many eigenfunctions $\omega_{\gamma bk}$ satisfying $\boldsymbol{A}_\beta\omega_{\gamma bk} = \lambda_{\gamma bk}\omega_{\gamma bk}$, i.e., the multiplicities of the $\lambda_{\gamma bk}$ are finite. Therefore, we can enumerate these eigenfunctions and eigenvalues in the form ω_m, λ_m with $m \in \mathbb{N}\backslash\{0\}$ so that $\lambda_m \leq \lambda_{m'}$ for $m \leq m'$.
(ii) Since any $\omega_m(x)$ is orthogonal to $\Omega(\|x\|_p)$, we see that $\{\omega_m(x)\}_{m\in\mathbb{N}\backslash\{0\}}$ is not a complete orthonormal basis of $L^2(\mathbb{Z}_p^n, d^n x)$. We now recall that $\mathcal{L}_0(\mathbb{Z}_p^n)$ is dense in $L_0^2(\mathbb{Z}_p^n)$, and since the algebraic span of $\{\omega_m(x)\}_{m\in\mathbb{N}\backslash\{0\}}$ contains $\mathcal{L}_0(\mathbb{Z}_p^n)$, it follows that $\{\omega_m(x)\}_{m\in\mathbb{N}\backslash\{0\}}$ is a complete orthonormal basis of $L_0^2(\mathbb{Z}_p^n)$.

Proposition 5.3. *Consider* $(\boldsymbol{A}_\beta, \mathcal{L}_0(\mathbb{Z}_p^n))$ *and the eigenvalue problem*

$$\boldsymbol{A}_\beta u = \lambda u, \quad \lambda > 0, \;\; u \in \mathcal{L}_0(\mathbb{Z}_p^n). \tag{5.2}$$

Then the function $u(x) = \omega_m(x)$ *is solves problem* (5.2) *corresponding to* $\lambda = \lambda_m$*, for* $m \in \mathbb{N} \setminus \{0\}$*. Moreover, the spectrum has the form*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \ldots \quad \textit{with} \quad \lambda_m \uparrow +\infty,$$

where all the eigenvalues have finite multiplicity, and $\{\omega_m(x)\}$ *with* $m \in \mathbb{N} \setminus \{0\}$ *is a complete orthonormal basis of* $L_0^2(\mathbb{Z}_p^n, d^n x)$.

Proof. The result follows from Lemma 5.1, Remark 5.2, and inequalities (4.1). □

Definition 5.4. We define the spectral zeta function associated with the eigenvalue problem (5.2) as

$$\zeta(s; \boldsymbol{A}_\beta, \mathcal{L}_0(\mathbb{Z}_p^n)) := \zeta(s; \boldsymbol{A}_\beta) = \sum_{m=1}^{\infty} \frac{1}{\lambda_m^s}, \quad s \in \mathbb{C}.$$

Later, it will be shown that $\zeta(s; \boldsymbol{A}_\beta)$ converges if $\mathrm{Re}(s)$ is sufficiently large, and it does not depend on the basis $\{\omega_m(x)\}$ used in its computation. By abuse of language (or following the classical literature, see [36]), we shall say that $\zeta(s; \boldsymbol{A}_\beta)$ is the spectral zeta function of the operator $\boldsymbol{A}_\beta$.

5.1. Example. We compute $\zeta(s; D_T^\beta)$. First, we note that

$$D_T^\beta \omega_{\gamma b k} = p^{-(\gamma-1)\beta} \omega_{\gamma b k}.$$

Recall that if $\operatorname{supp} \omega_{\gamma b k} \subset \mathbb{Z}_p^n$, then $\gamma \leq 0$ and $b \in p^\gamma \mathbb{Z}_p^n / \mathbb{Z}_p^n$. Now we take $-\gamma + 1 = m$ with $m \in \mathbb{N} \setminus \{0\}$. Then $b \in p^{-m+1} \mathbb{Z}_p^n / \mathbb{Z}_p^n$, $\lambda_m = p^{m\beta}$, and the multiplicity of λ_m is equal to $(p^n - 1)p^{n(m-1)} = p^{nm}(1 - p^{-n})$ for $m \in \mathbb{N} \setminus \{0\}$. Hence,

$$\zeta(s; D_T^\beta) = \sum_{m=1}^{\infty} \frac{p^{nm}(1 - p^{-n})}{p^{m\beta s}} = \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \frac{d^n \xi}{\|\xi\|_p^{\beta s}} = (1 - p^{-n}) \frac{p^{n - \beta s}}{1 - p^{n - \beta s}}$$

whenever $\mathrm{Re}(s) > \frac{n}{\beta}$. Then $\zeta(s; D_T^\beta)$ admits meromorphic continuation to the entire complex plane as a rational function of p^{-s} with poles in the set $\frac{n}{\beta} + \frac{2\pi i \mathbb{Z}}{\beta \ln p}$.

§6. Heat traces and p-adic heat equations on the unit ball

From now on, $(\boldsymbol{A}_\beta, \mathrm{Dom}(\boldsymbol{A}_\beta))$ is given by

$$(\boldsymbol{A}_\beta \varphi)(x) = \mathcal{F}_{\xi \to x}^{-1}(\boldsymbol{A}_\beta(\xi) \mathcal{F}_{x \to \xi} \varphi) \;\; \text{for} \;\; \varphi \in \mathrm{Dom}(\boldsymbol{A}_\beta) = \mathcal{L}_0(\mathbb{Z}_p^n). \tag{6.1}$$

6.1. p-adic heat equations on the unit ball. We introduce the following function:

$$K(x, t) = \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \chi_p(-x \cdot \xi) e^{-t A_\beta(\xi)} \, d^n \xi, \quad t > 0, \;\; x \in \mathbb{Q}_p^n.$$

We note that, by (4.1), $e^{-t A_\beta(\xi)} \leq e^{-t C_0 \|\xi\|_p^\beta} \in L^1$ for $t > 0$, which implies that $K(x, t)$ is well defined for $t > 0$ and $x \in \mathbb{Q}_p^n$.

Lemma 6.1. *With the above notation, the following formula holds true*:

$$K(x,t) = \begin{cases} \Omega(\|x\|_p) \big\{ (1 - p^{-n}) \sum_{j=1}^{\operatorname{ord}(x)} e^{-t A_\beta(p^j)} p^{nj} - p^{\operatorname{ord}(x) n} e^{-t A_\beta(p^{\operatorname{ord}(x)+1})} \big\} & \textit{if} \;\; x \neq 0, \\ (1 - p^{-n}) \sum_{j=1}^{\infty} e^{-t A_\beta(p^j)} p^{nj} & \textit{if} \;\; x = 0, \end{cases}$$

for any $t > 0$.

Proof. Take $x = p^{\operatorname{ord}(x)} x_0$ with $\|x_0\|_p = 1$, then

$$\begin{aligned} K(x,t) &= \sum_{j=1}^{\infty} e^{-tA_\beta(p^j)} \int_{\|\xi\|_p = p^j} \chi_p(-x\cdot\xi)\, d^n\xi \\ &= \sum_{j=1}^{\infty} e^{-tA_\beta(p^j)} p^{nj} \int_{\|y\|_p=1} \chi_p(-p^{-j+\operatorname{ord}(x)} x_0 \cdot y)\, d^n y \\ &= \sum_{j=1}^{\infty} e^{-tA_\beta(p^j)} p^{nj} \begin{cases} 1 - p^{-n} & \text{if } j \le \operatorname{ord}(x), \\ -p^{-n} & \text{if } j = \operatorname{ord}(x)+1, \\ 0 & \text{if } j \ge \operatorname{ord}(x)+2. \end{cases} \end{aligned}$$

Then $K(x,t) = 0$ for $\|x\|_p > 1$ and $t > 0$. Finally, we note that the announced formula is valid if $x = 0$. □

We identify $L_0^2(\mathbb{Z}_p^n)$ with an isometric subspace of $L^2(\mathbb{Q}_p^n)$ by extending the functions of $L_0^2(\mathbb{Z}_p^n)$ by zero outside of $\mathbb{Z}_p^n$. We define $\{T(t)\}_{t\ge 0}$ as the family of operators

$$\begin{array}{ccc} L_0^2(\mathbb{Z}_p^n) & \to & L_0^2(\mathbb{Z}_p^n), \\ f & \mapsto & T(t)f, \end{array}$$

with

$$(T(t)f)(x) = \begin{cases} f(x) & \text{if } t = 0, \\ (K(\,\cdot\,,t) * f)(x) & \text{if } t > 0. \end{cases}$$

Lemma 6.2. *With the above notation, the following assertions are true*:
(i) *the operator $T(t)$, $t \ge 0$, is a well-defined bounded linear operator*;
(ii) *for $t \ge 0$,*

$$(T(t)f)(x) = \mathcal{F}^{-1}_{\xi\to x}[\mathbb{1}_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n}(\xi) e^{-tA_\beta(\xi)} \widehat{f}(\xi)],$$

where $\widehat{f}(\xi)$ denotes the Fourier transform in $L^2(\mathbb{Q}_p^n)$ of $f \in L_0^2(\mathbb{Z}_p^n)$;
(iii) *for $t > 0$, $T(t)$ is a compact, selfadjoint, and nonnegative operator.*

Proof. (i) We recall that $K(\,\cdot\,,t) \in L^1(\mathbb{Q}_p^n)$ for $t > 0$. Therefore, if $f \in L_0^2(\mathbb{Z}_p^n) \subset L^2(\mathbb{Q}_p^n)$, then, by the Young inequality,

$$u(x,t) := (K(\,\cdot\,,t) * f)(x) \in L^2(\mathbb{Q}_p^n) \quad \text{for} \quad t > 0.$$

Now, by Lemma 6.1, supp $u(x,t) \subset \mathbb{Z}_p^n$ for $t > 0$, i.e., $u(x,t) \in L^2(\mathbb{Z}_p^n)$ for $t > 0$. Again by the Young inequality, for $t > 0$ we have

$$\|u(x,t)\|_{L_0^2(\mathbb{Z}_p^n)} = \|u(x,t)\|_{L^2(\mathbb{Q}_p^n)} \le \|K(x,t)\|_{L^1(\mathbb{Q}_p^n)} \|f(x)\|_{L^2(\mathbb{Q}_p^n)} = C(t)\|f(x)\|_{L_0^2(\mathbb{Z}_p^n)}.$$

Finally, we show that

$$\int_{\mathbb{Z}_p^n} u(x,t)\, d^n x = 0 \quad \text{for} \quad t > 0.$$

Indeed, for $t > 0$, Fubini's theorem yields

$$\begin{aligned} \int_{\mathbb{Z}_p^n} u(x,t)\, d^n x &= \int_{\mathbb{Z}_p^n} \left\{ \int_{\mathbb{Z}_p^n} K(y,t) f(x-y)\, d^n y \right\} d^n x \\ &= \int_{\mathbb{Z}_p^n} K(y,t) \left\{ \int_{\mathbb{Z}_p^n} f(x-y)\, d^n x \right\} d^n y \text{ (taking } z_1 = x-y, z_2 = y) \\ &= \int_{\mathbb{Z}_p^n} K(z_2,t) \left\{ \int_{\mathbb{Z}_p^n} f(z_1)\, d^n z_1 \right\} d^n z_2 = 0. \end{aligned}$$

(ii) Since $f(x)$, $u(x,t) \in L^1(\mathbb{Z}_p^n) \cap L^2(\mathbb{Z}_p^n)$ for $t > 0$, because $L^2(\mathbb{Z}_p^n) \subset L^1(\mathbb{Z}_p^n)$, we have

$$\mathcal{F}_{x\to\xi}(u(x,t)) = \mathbb{1}_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n}(\xi)e^{-tA_\beta(\xi)}\widehat{f}(\xi),$$

and this last function belongs to $L^1(\mathbb{Q}_p^n)$. Indeed, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\big\|\mathbb{1}_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n}(\xi)e^{-tA_\beta(\xi)}\widehat{f}(\xi)\big\|_{L^1(\mathbb{Q}_p^n)} &\le \big\|\mathbb{1}_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n}(\xi)e^{-tA_\beta(\xi)}\big\|_{L^2(\mathbb{Q}_p^n)}\|\widehat{f}(\xi)\|_{L^2(\mathbb{Q}_p^n)}\\ &\le \|e^{-tA_\beta(\xi)}\|_{L^2(\mathbb{Q}_p^n)}\|f(\xi)\|_{L^2(\mathbb{Q}_p^n)}\\ &= \|e^{-tA_\beta(\xi)}\|_{L^2(\mathbb{Q}_p^n)}\|f(\xi)\|_{L_0^2(\mathbb{Z}_p^n)} < \infty\end{aligned}$$

because $\int_{\mathbb{Q}_p^n} e^{-2tA_\beta(\xi)}d^n\xi \le \int_{\mathbb{Q}_p^n} e^{-2C_0t\|\xi\|_p^\beta}d^n\xi < \infty$, cf. (4.1). Finally,

$$\begin{aligned}(T(0)f)(x) &= \int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n}\chi_p(-\xi\cdot x)\widehat{f}(\xi)\,d^n\xi\\ &= \int_{\mathbb{Q}_p^n}\chi_p(-\xi\cdot x)\widehat{f}(\xi)\,d^n\xi - \int_{\mathbb{Z}_p^n}\chi_p(-\xi\cdot x)\widehat{f}(\xi)\,d^n\xi\\ &= f(x) - \mathcal{F}_{x\to\xi}^{-1}(\Omega(\|\xi\|_p)\widehat{f}(\xi)) = f(x) - \Omega(\|x\|_p) * f(x)\\ &= f(x) - \Omega(\|x\|_p)\int_{\mathbb{Z}_p^n} f(x)\,d^nx = f(x).\end{aligned}$$

(iii) Since $T(t)$, $t > 0$, is bounded and $\langle T(t)f, g\rangle = \langle f, T(t)g\rangle$ for $f, g \in L^2(\mathbb{Z}_p^n)$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the inner product of $L^2(\mathbb{Q}_p^n)$, the operator $T(t)$ is selfadjoint for $t > 0$. To prove compactness we show a sequence of bounded operators $T_l(t)$ with finite range such that $T_l(t) \xrightarrow{\|\cdot\|} T(t)$ for $t > 0$. For $l \in \mathbb{N}$, we set $G_l := (\mathbb{Z}_p/p^l\mathbb{Z}_p)^n$. We fix representatives, denoted by $\boldsymbol{i}$, of G_l in $\mathbb{Z}_p^n$. In particular $\|\boldsymbol{i}\|_p$ makes sense for $\boldsymbol{i} \in G_l$. Set $L(l)$ to be the $\mathbb{C}$-vector space spanned by $\{\Omega(p^l\|x - \boldsymbol{i}\|_p)\}_{\boldsymbol{i}\in G_l}$. Observe that $\varphi \in L(l)$ if and only if $\operatorname{supp}\varphi \subset B_0^n$ and $\varphi|_{\boldsymbol{i}+(p^l\mathbb{Z}_p)^n} = \varphi(\boldsymbol{i})$. On the other hand, by Lemma 6.1, $K(x,t) = \Omega(\|x\|_p)h(\|x\|_p, t)$, where $h(0,t)$ is defined and $h(\|x\|_p, t)$ is bounded on the unit ball for $t > 0$. For $l \in \mathbb{N}$ and $t > 0$, we set

$$K_l(x,t) := \sum_{\boldsymbol{i}\in G_l} h(\|\boldsymbol{i}\|_p, t)\Omega(p^l\|x - \boldsymbol{i}\|_p)$$

and $T_l(t)f := K_l(\,\cdot\,,t) * f$ for $f \in L_0^2(\mathbb{Z}_p^n)$. Then $T_l(t)\colon L_0^2(\mathbb{Z}_p^n) \to L(l) \subset L_0^2(\mathbb{Z}_p^n)$ is a bounded operator with finite range. Indeed, $\Omega(p^l\|x - \boldsymbol{i}\|_p) * f$ has support in B_0^n and $\Omega(p^l\|x - \boldsymbol{i}\|_p) * f$ is a constant function on the ball $\boldsymbol{i} + (p^l\mathbb{Z}_p)^n$. Finally, for $t > 0$ we have $\|T_l(t) - T(t)\| \le \|K_l(\,\cdot\,,t) - K(\,\cdot\,,t)\|_{L^1(\mathbb{Z}_p^n)} \to 0$ as $l \to \infty$, by the dominated convergence theorem and the fact that $K_l(x,t) \to K(x,t)$ as $l \to \infty$, and $\operatorname{supp} K_l(\,\cdot\,,t)$, $\operatorname{supp} K(\,\cdot\,,t) \subset B_0^n$. ☐

Lemma 6.3. *The one-parameter family $\{T(t)\}_{t\ge0}$ of bounded linear operators from $L_0^2(\mathbb{Z}_p^n)$ into itself is a contraction semigroup.*

Proof. The lemma is implied by the following claims.

Claim 1. $\|T(t)\|_{L_0^2(\mathbb{Z}_p^n)} \le 1$ for $t \ge 0$. Moreover, $\|T(t)\|_{L_0^2(\mathbb{Z}_p^n)} < 1$ for $t > 0$.

For $t > 0$, by Lemma 6.2 and (4.1) we have

$$\begin{aligned}\|T(t)f\|^2_{L^2_0(\mathbb{Z}_p^n)} &= \|T(t)f\|^2_{L^2(\mathbb{Q}_p^n)} = \|\widehat{T(t)f(\xi)}\|^2_{L^2(\mathbb{Q}_p^n)} \leq \int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n} e^{-2tA_\beta(\xi)}|\widehat{f}(\xi)|^2\, d^n\xi \\ &\leq \int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n} e^{-2C_0t\|\xi\|_p^\beta}|\widehat{f}(\xi)|^2\, d^n\xi \leq \sup_{\xi\in\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n} e^{-2C_0t\|\xi\|_p^\beta}\int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n} |\widehat{f}(\xi)|^2\, d^n\xi \\ &< \int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n} |\widehat{f}(\xi)|^2\, d^n\xi \leq \|f\|^2_{L^2(\mathbb{Q}_p^n)} = \|f\|^2_{L^2_0(\mathbb{Z}_p^n)},\end{aligned}$$

where we have used the inequality $\sup_{\xi\in\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n} e^{-2C_0t\|\xi\|_p^\beta} < 1$.

Claim 2. $T(0) = I$.

Claim 3. $T(t+s) = T(t)T(s)$ for t, $s \geq 0$.

This follows from Lemma 6.2(ii).

Claim 4. For $f \in L^2_0(\mathbb{Z}_p^n)$, the function $t \mapsto T(t)f$ belongs to $C\big([0,\infty), L^2_0(\mathbb{Z}_p^n)\big)$.

Notice that, since $\mathcal{L}^2_0(\mathbb{Z}_p^n)$ is dense in $L^2_0(\mathbb{Z}_p^n)$ for the $\|\cdot\|_{L^2}$-norm, it suffices to check Claim 4 for $f \in \mathcal{L}^2_0(\mathbb{Z}_p^n)$. Indeed, we have

$$\begin{aligned}&\lim_{t\to t_0}\|T(t)f - T(t_0)f\|^2_{L^2_0(\mathbb{Z}_p^n)} = \lim_{t\to t_0}\|T(t)f - T(t_0)f\|^2_{L^2(\mathbb{Q}_p^n)} \\ &\quad = \lim_{t\to t_0}\|\widehat{T(t)f} - \widehat{T(t_0)f}\|^2_{L^2(\mathbb{Q}_p^n)} = \lim_{t\to t_0}\int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n} |\widehat{f}(\xi)|^2|e^{-tA_\beta(\xi)} - e^{-t_0A_\beta(\xi)}|^2 d^n\xi.\end{aligned}$$

Now, since $1\!\!1_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n}(\xi)|\widehat{f}(\xi)|^2|e^{-tA_\beta(\xi)} - e^{-t_0A_\beta(\xi)}|^2 \leq 4|\widehat{f}(\xi)|^2$, which is an integrable function, we can apply the dominated convergence theorem to show that

$$\lim_{t\to t_0}\|T(t)f - T(t_0)f\|^2_{L^2_0(\mathbb{Z}_p^n)} = 0. \qquad \square$$

Lemma 6.4. *The infinitesimal generator of the semigroup $\{T(t)\}_{t\geq 0}$ restricted to $\mathcal{L}_0(\mathbb{Z}_p^n)$ agrees with $(-\boldsymbol{A}_\beta, \mathcal{L}_0(\mathbb{Z}_p^n))$.*

Proof. We show that

$$\lim_{t\to 0^+}\left\|\frac{T(t)f - f}{t} + \boldsymbol{A}_\beta f\right\|_{L^2_0(\mathbb{Z}_p^n)} = 0 \quad \text{for} \quad f \in \mathcal{L}_0(\mathbb{Z}_p^n).$$

Indeed, by Lemma 6.2(ii),

$$\begin{aligned}\left\|\frac{T(t)f - f}{t} + \boldsymbol{A}_\beta f\right\|_{L^2_0(\mathbb{Z}_p^n)} &= \left\|\frac{T(t)f - f}{t} + \boldsymbol{A}_\beta f\right\|_{L^2(\mathbb{Q}_p^n)} = \left\|\frac{\widehat{T(t)f} - \widehat{f}}{t} + \widehat{\boldsymbol{A}_\beta f}\right\|_{L^2(\mathbb{Q}_p^n)} \\ &= \left\|\left\{\frac{1\!\!1_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n}(\xi)e^{-tA_\beta(\xi)} - 1}{t} + A_\beta(\xi)\right\}\widehat{f}(\xi)\right\|_{L^2(\mathbb{Q}_p^n)}.\end{aligned}$$

Now we note that

$$\big\{1\!\!1_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n}(\xi)e^{-tA_\beta(\xi)} - 1\big\}\widehat{f}(\xi) = \widehat{f}(\xi)\{e^{-tA_\beta(\xi)} - 1\} - 1\!\!1_{\mathbb{Z}_p^n}(\xi)e^{-tA_\beta(\xi)}\widehat{f}(\xi),$$

and since supp $f \subset \mathbb{Z}_p^n$, we have $\widehat{f}(\xi+\xi_0) = \widehat{f}(\xi)$ for any $\xi_0 \in \mathbb{Z}_p^n$; this fact implies that $1_{\mathbb{Z}_p^n}(\xi)e^{-tA_\beta(\xi)}\widehat{f}(\xi) = e^{-tA_\beta(\xi)}\widehat{f}(0) = 0$ because $f \in \mathcal{L}_0(\mathbb{Z}_p^n)$. Hence,

$$\left\|\left\{\frac{1_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n}(\xi)e^{-tA_\beta(\xi)}-1}{t}+A_\beta(\xi)\right\}\widehat{f}(\xi)\right\|_{L^2(\mathbb{Q}_p^n)} = \left\|\frac{\{e^{-tA_\beta(\xi)}-1\}\widehat{f}(\xi)}{t}+A_\beta(\xi)\widehat{f}(\xi)\right\|_{L^2(\mathbb{Q}_p^n)}$$
$$= \left\|A_\beta(\xi)\widehat{f}(\xi)\{1-e^{-\tau A_\beta(\xi)}\}\right\|_{L^2(\mathbb{Q}_p^n)} \quad (\text{for some } \ \tau \in (0,t)).$$

Therefore, using the dominated convergence theorem, we get

$$\lim_{t\to 0^+}\left\|\frac{T(t)f-f}{t}+\boldsymbol{A}_\beta f\right\|_{L_0^2(\mathbb{Z}_p^n)} = \lim_{t\to 0^+}\left\|A_\beta(\xi)\widehat{f}(\xi)\{1-e^{-\tau A_\beta(\xi)}\}\right\|_{L^2(\mathbb{Q}_p^n)} = 0$$

because $A_\beta(\xi)\widehat{f}(\xi) \in \mathcal{D}(\mathbb{Q}_p^n)$. □

Theorem 6.5. *The initial value problem*

$$\begin{cases} u(x,t) \in C([0,\infty), \mathrm{Dom}\,(\boldsymbol{A}_\beta)) \cap C^1([0,\infty), L_0^2(\mathbb{Z}_p^n)), \\ \frac{\partial u(x,t)}{\partial t} + \boldsymbol{A}_\beta u(x,t) = 0, \ x \in \mathbb{Q}_p^n, \ t \in [0,\infty), \\ u(x,0) = \varphi(x) \in \mathrm{Dom}(\boldsymbol{A}_\beta), \end{cases} \tag{6.2}$$

where $(\boldsymbol{A}_\beta, \mathrm{Dom}(\boldsymbol{A}_\beta))$ *is given by* (6.1), *has a unique solution given by* $u(x,t) = T(t)\varphi(x)$.

Proof. By Lemmas 6.3–6.4 and the Hille–Yosida–Phillips theorem, see, e.g., [8, Theorem 3.4.4], the operator $(-\boldsymbol{A}_\beta, \mathrm{Dom}(\boldsymbol{A}_\beta))$ is m-dissipative with dense domain in $L_0^2(\mathbb{Z}_p^n)$. Therefore, the announced theorem follows from [8, Theorem 3.1.1 and Proposition 3.4.5]. □

6.2. Heat Traces.

Proposition 6.6. *Let* $\{\omega_m\}_{m\in\mathbb{N}\setminus\{0\}}$ *be the complete orthonormal basis of* $L_0^2(\mathbb{Z}_p^n)$ *as above. Then*

$$K(x-y,t) = \sum_{m=1}^{\infty} e^{-\lambda_m t}\omega_m(x)\overline{\omega_m(y)}$$

where the convergence is uniform on $\mathbb{Z}_p^n \times \mathbb{Z}_p^n \times [\epsilon,\infty)$, *for every* $\epsilon > 0$.

Proof. By applying the Hilbert–Schmidt theorem to $T(1)$, see, e.g., [28, Theorem VI.16], which is selfadjoint and compact, cf. Lemma 6.2 (iii), in $L_0^2(\mathbb{Z}_p^n)$ there exists a complete orthonormal basis $\{\phi_m\}$, $m \in \mathbb{N}\setminus\{0\}$, consisting of eigenfunctions of $T(1)$. Let $\{\mu_m\}$, $m \in \mathbb{N}\setminus\{0\}$, be the sequence of the corresponding eigenvalues. Moreover, $\mu_m \to 0$ as $m \to \infty$. Since the $\{T(t)\}_{t\geq 0}$ form a semigroup, we have $T(\frac{l}{k})\phi_m = \mu_m^{l/k}\phi_m$ for every positive rational number $\frac{l}{k}$. Using the continuity of $\{T(t)\}_{t\geq 0}$, we get

$$T(t)\phi_m = \mu_m^t\phi_m \quad \text{for} \quad t \in \mathbb{R}_+.$$

We note that $\mu_m > 0$ for every m, indeed,

$$\phi_m = \lim_{t\to 0^+} T(t)\phi_m = \phi_m \lim_{t\to 0^+}\mu_m^t$$

implies that $\lim_{t\to 0^+}\mu_m^t = 1$ because $\phi_m \neq 0$. Hence, $\mu_m = e^{-\lambda_m}$ with $\lambda_m > 0$, because $\|T(t)\|_{L_0^2(\mathbb{Z}_p^n)} < 1$ for $t > 0$, cf. Lemma 6.3 (i), implies that $\mu_m < 1$ and $\lim_{m\to\infty}\lambda_m = \infty$, because $\lim_{m\to\infty}\mu_m = 0$.

By using Mercer's theorem, see, e.g., [14, 29] and the references therein, we have

$$K(x-y,t) = \sum_{m=1}^{\infty} e^{-\lambda_m t}\phi_m(x)\overline{\phi_m(y)}. \tag{6.3}$$

Now, since $T(t)\phi_m(x) = e^{-\lambda_m t}\phi_m(x)$ is a solution of problem (6.2) with the initial data ϕ_m, cf. Theorem 6.5, and

$$-\lambda_m e^{-\lambda_m t}\phi_m(x) = \frac{\partial}{\partial t}(e^{-\lambda_m t}\phi_m(x)) = -\boldsymbol{A}_\beta(e^{-\lambda_m t}\phi_m(x)) = -e^{-\lambda_m t}\boldsymbol{A}_\beta\phi_m(x),$$

it follows that $\phi_m(x)$ is an eigenfunction of $\boldsymbol{A}_\beta$ with supp $\phi_m \subset \mathbb{Z}_p^n$. Now, since $\boldsymbol{A}_\beta\omega_m = \lambda_m\omega_m$, see Proposition 5.3, we see that $u = e^{-\lambda_m t}\omega_m$ solves the following boundary value problem:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = -\boldsymbol{A}_\beta u(x,t), & u(x,t) \in L_0^2(\mathbb{Z}_p^n), \ \ t \geq 0, \\ u(x,0) = \omega_m(x), & \omega_m(x) \in \mathcal{L}_0(\mathbb{Z}_p^n). \end{cases}$$

Then, by Theorem 6.5, the above problem has a unique solution, which implies

$$u(x,t) = T(t)\omega_m(x) = e^{-\lambda_m t}\omega_m,$$

so that we can replace $\{\phi_m\}$ by $\{\omega_m\}$ in (6.3). □

In the next result, we will use the classical notation $e^{-t\boldsymbol{A}_\beta}$ for the operator $T(t)$ to emphasize the dependence on the operator $\boldsymbol{A}_\beta$.

Theorem 6.7. *For $t > 0$, the operator $e^{-t\boldsymbol{A}_\beta}$ is of trace class and*
(i)

$$\operatorname{Tr}(e^{-t\boldsymbol{A}_\beta}) = \sum_{m=1}^{\infty} e^{-\lambda_m t} = \int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n} e^{-tA_\beta(\xi)}\, d^n\xi. \tag{6.4}$$

(ii) *there exist positive constants C, C' such that*

$$Ct^{-\frac{n}{\beta}} \leq \operatorname{Tr}(e^{-t\boldsymbol{A}_\beta}) \leq C't^{-\frac{n}{\beta}}$$

for $t > 0$.

Proof. By Proposition 6.6 and the definition of $K(x,t)$, for $t > 0$ we have

$$K(0,t) = \int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n} e^{-tA_\beta(\xi)}\, d^n\xi = \sum_{m=1}^{\infty} e^{-\lambda_m t}|\omega_m(x)|^2. \tag{6.5}$$

The dominated convergence theorem and the fact that $\sum_m e^{-\lambda_m t}$ converges for $t > 0$ allow us to integrate the two sides of (6.5) with respect to the variable x over $\mathbb{Z}_p^n$, obtaining

$$\int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n} e^{-tA_\beta(\xi)}\, d^n\xi = \sum_{m=1}^{\infty} e^{-\lambda_m t} \quad \text{for} \quad t > 0. \tag{6.6}$$

We recall that

$$e^{-C_1 t\|\xi\|_p^\beta} \leq e^{-tA_\beta(\xi)} \leq e^{-C_0 t\|\xi\|_p^\beta}, \tag{6.7}$$

see (4.1), and that $e^{-Ct\|\xi\|_p^\beta} \in L^1$ for $t > 0$ and any positive constant C; then the series on the right-hand side of (6.6) converges. Now,

$$\operatorname{Tr}(e^{-t\boldsymbol{A}_\beta}) = \sum_{m=1}^{\infty} \langle e^{-t\boldsymbol{A}_\beta}\omega_m, \omega_m\rangle = \sum_{m=1}^{\infty} e^{-\lambda_m t}\|\omega_m\|_{L^2}^2 = \sum_{m=1}^{\infty} e^{-\lambda_m t} < \infty \quad \text{for} \quad t > 0,$$

i.e., $e^{-t\boldsymbol{A}_\beta}$ is of trace class and the formula announced in (i) is valid. The estimate for $\operatorname{Tr}(e^{-t\boldsymbol{A}_\beta})$ follows from (6.7), because

$$\int_{\mathbb{Q}_p^n} e^{-Ct\|\xi\|_p^\beta} d^n\xi \leq Dt^{-\frac{n}{\beta}} \quad \text{for} \quad t > 0.$$ □

§7. Analytic continuation of spectral zeta functions

Remark 7.1.
(i) For $a > 0$ we set $a^s := e^{s \ln a}$. Then a^s becomes a holomorphic function on $\mathrm{Re}(s) > 0$.
(ii) We recall the following fact, see, e.g., [18, Lemma 5.3.1]. Let $(X, d\mu)$ denote a measure space, U a nonempty open subset of $\mathbb{C}$, and $f : X \times U \to \mathbb{C}$ a measurable function. Assume that: (1) if $\mathcal{C}$ is a compact subset of U, there exists an integrable function $\phi_C \geq 0$ on X satisfying $|f(\xi, s)| \leq \phi_C(\xi)$ for all $(\xi, s) \in X \times \mathcal{C}$; (2) $f(\xi, \cdot)$ is holomorphic on U for every x in X. Then $\int_X f(\xi, s) d\mu$ is a holomorphic function on U.

Proposition 7.2. *The spectral zeta function for $\boldsymbol{A}_\beta$ is a holomorphic function on* $\mathrm{Re}(s) > \frac{n}{\beta}$ *and*

$$\zeta(s; \boldsymbol{A}_\beta) = \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \frac{d^n\xi}{A_\beta^s(\xi)} \quad \text{for} \quad \mathrm{Re}(s) > \frac{n}{\beta}. \tag{7.1}$$

In particular $\zeta(s; \boldsymbol{A}_\beta)$ does not depend on the basis of $L_0^2(\mathbb{Z}_p^n)$ used in Definition 5.4.

Proof. By Proposition 5.3 and Remark 5.2, the eigenvalues have the form $A_\beta(p^{1-\gamma})$ with $\gamma \leq 0$, and the corresponding multiplicity is the cardinality of $p^\gamma \mathbb{Z}_p^n / \mathbb{Z}_p^n$ times the cardinality of the set of k's, i.e., $p^{-\gamma n}(p^n - 1)$. Therefore,

$$\begin{aligned}\zeta(s; \boldsymbol{A}_\beta) = \sum_{\gamma \leq 0} \frac{p^{-\gamma n}(p^n - 1)}{A_\beta^s(p^{1-\gamma})} &= \sum_{m=1}^{\infty} \frac{p^{mn}(1 - p^{-n})}{A_\beta^s(p^m)} \\ &= \sum_{m=1}^{\infty} \int_{\|\xi\|_p = p^m} \frac{d^n\xi}{A_\beta^s(\|\xi\|_p)} = \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \frac{d^n\xi}{A_\beta^s(\xi)},\end{aligned}$$

and by (4.1) we have

$$|\zeta(s; \boldsymbol{A}_\beta)| \leq \frac{(1 - p^{-n})}{C^{\mathrm{Re}(s)}} \sum_{m=1}^{\infty} p^{m(n - \beta \,\mathrm{Re}(s))} < \infty \quad \text{for} \quad \mathrm{Re}(s) > \frac{n}{\beta}.$$

To establish holomorphy on $\mathrm{Re}(s) > \frac{n}{\beta}$, we use Remark 7.1 (ii). Take

$$X = \mathbb{Q}_p^n \setminus \mathbb{Z}_p^n, \quad d\mu = d^n\xi, \quad U = \Big\{ s \in \mathbb{C};\, \mathrm{Re}(s) > \frac{n}{\beta} \Big\}, \quad f(\xi, s) = A_\beta^{-s}(\|\xi\|_p).$$

Now we verify the two conditions established in Remark 7.1(ii). Let $\mathcal{C}$ be a compact subset of U. By (4.1), we have

$$\left| \frac{1}{A_\beta^s(\|\xi\|_p)} \right| \leq \frac{1}{C^{\mathrm{Re}(s)} \|\xi\|_p^{\beta \,\mathrm{Re}(s)}},$$

where C is a positive constant. Since $\mathrm{Re}(s)$ belongs to a compact subset of

$$\Big\{ s \in \mathbb{R};\, \mathrm{Re}(s) > \frac{n}{\beta} \Big\},$$

we may assume without loss of generality that $\mathrm{Re}(s) \in [\gamma_0, \gamma_1]$ with $\gamma_0 > \frac{n}{\beta}$, whence

$$\frac{1}{C^{\mathrm{Re}(s)} \|\xi\|_p^{\beta \,\mathrm{Re}(s)}} \leq B(\mathcal{C}) \frac{1}{\|\xi\|_p^{\beta \gamma_0}} \in L^1,$$

where $B(\mathcal{C})$ is a positive constant. Condition (2) in Remark 7.1 (ii) follows from Remark 7.1 (i) by observing that $(A_\beta(\|\xi\|_p))^{-s} = \exp(-s \ln A_\beta(\|\xi\|_p))$ with $A_\beta(\|\xi\|_p) > 0$ for $\|\xi\|_p > 1$. □

Remark 7.3. Note that formula (7.1) can be obtained by taking the Mellin transform in (6.4). Indeed,

$$\int_0^\infty \left\{ \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} e^{-tA_\beta(\|\xi\|_p)} t^{s-1} d^n\xi \right\} dt = \int_0^\infty \left\{ \sum_{m=1}^\infty e^{-\lambda_m t} t^{s-1} \right\} dt = \Gamma(s)\zeta(s; \boldsymbol{A}_\beta)$$

for $\mathrm{Re}(s) > 1$, where $\Gamma(s)$ denotes the Archimedean Gamma function. Now, changing variables by the rule $y = A_\beta(\|\xi\|_p)t$ with ξ fixed, we have

$$\zeta(s; \boldsymbol{A}_\beta) = \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \frac{d^n\xi}{A_\beta^s(\|\xi\|_p)} \quad \text{for} \quad \mathrm{Re}(s) > \max\left\{1, \frac{n}{\beta}\right\}.$$

Lemma 7.4. *$\zeta(s; \boldsymbol{A}_\beta)$ has a simple pole at $s = \frac{n}{\boldsymbol{\beta}}$.*

Proof. Set $\sigma \in \mathbb{R}_+$; since

$$\zeta(\sigma; \boldsymbol{A}_\beta) \leq \frac{1}{C_0} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \frac{d^n\xi}{\|\xi\|_p^{\boldsymbol{\beta}\sigma}} = \frac{(1 - p^{-n})p^{-\beta\sigma + n}}{C_0(1 - p^{-\boldsymbol{\beta}\sigma + n})} \quad \text{for} \quad \sigma > \frac{n}{\boldsymbol{\beta}},$$

we have

$$\lim_{\sigma \to \frac{n}{\boldsymbol{\beta}}} (1 - p^{-\boldsymbol{\beta}\sigma + n})\zeta(\sigma; \boldsymbol{A}_\beta) > 0. \tag{7.2}$$

The assertion follows from (7.2), by using the fact that $1 - p^{-\boldsymbol{\beta}\sigma + n}$ has a simple zero at $\frac{n}{\boldsymbol{\beta}}$. Indeed,

$$1 - p^{-\boldsymbol{\beta}\sigma + n} = 1 - \exp\{(-\boldsymbol{\beta}\sigma + n)\ln p\} = \{\boldsymbol{\beta} \ln p\}\left(\sigma - \frac{n}{\boldsymbol{\beta}}\right) + O\left(\left(\sigma - \frac{n}{\boldsymbol{\beta}}\right)^2\right),$$

where O is an analytic function satisfying $O(0) = 0$. □

Theorem 7.5. *The spectral zeta function $\zeta(s; \boldsymbol{A}_\beta)$ possesses the following properties:*
(i) *$\zeta(s; \boldsymbol{A}_\beta)$ is a holomorphic function on* $\mathrm{Re}(s) > \frac{n}{\boldsymbol{\beta}}$, *and on this domain it is given by formula* (7.1);
(ii) *$\zeta(s; \boldsymbol{A}_\beta)$ has a simple pole at $s = \frac{n}{\boldsymbol{\beta}}$, however, this pole is not necessarily unique*;
(iii) *set $N(T) := \sum_{\lambda_m \leq T} 1$ for $T \geq 0$, then $N(T) = O\big(T^{\frac{n}{\beta}}\big)$.*

Proof.
(i) See Proposition 7.2.
(ii) The first part was established in Lemma 7.4. Take $\boldsymbol{A}_\beta$ to be the Taibleson operator $D_T^{\boldsymbol{\beta}}$; then $\zeta(s; \boldsymbol{D}_T^\beta)$ has a meromorphic continuation to the entire complex plane as a rational function of p^{-s} with poles in the set $\frac{n}{\beta} + \frac{2\pi i \mathbb{Z}}{\beta \ln p}$, see Example 5.1.
(iii) The result follows from the formulas

$$\lambda_m = A_\beta(p^m) \quad \text{and} \quad \mathrm{mult}(\lambda_m) = p^{nm}(1 - p^{-n}), \quad m \in \mathbb{N} \setminus \{0\}. \qquad \square$$

Remark 7.6. The fact that $\zeta(s; \boldsymbol{A}_\beta)$ may have several poles on the line $\mathrm{Re}(s) = \frac{n}{\beta}$ prevent us from using the classical Ikehara Tauberian theorem to obtain the asymptotic behavior of $N(T)$, see, e.g., [11, Appendix A] and [31, Chapter 2, §14]. Anyway, we expect that the following is true.

Conjecture 7.7. *$N(T) \sim CT^{\frac{n}{\beta}}$ for some suitable positive constant C.*

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