Алгебра и анализ
Том 29 (2017), № 5

St. Petersburg Math. J.
Vol. 29 (2018), No. 5, Pages 715–747
https://doi.org/10.1090/spmj/1513
Article electronically published on July 26, 2018

MÖBIUS STRUCTURES AND TIMED CAUSAL SPACES ON THE CIRCLE

S. BUYALO

Abstract. A conjectural duality is discussed between hyperbolic spaces on one hand and spacetimes on the other, living on the opposite sides of the common absolute. This duality goes via Möbius structures on the absolute, and it is easily recognized in the classical case of symmetric rank one spaces. In the general case, no trace of such duality is known. As a first step in this direction, it is shown how numerous Möbius structures on the circle, including those that stem from hyperbolic spaces, give rise to 2-dimensional spacetimes, which are axiomatic versions of de Sitter 2-space, and *vice versa*. The paper has two Appendices, one of which is written by V. Schroeder.

Contents

2010 *Mathematics Subject Classification.* Primary 51B10, 53C50.

Key words and phrases. Möbius structures, cross-ratio, harmonic 4-tuple, hyperbolic spaces, spacetimes, de Sitter space.

Supported by RFBR (grant no. 17-01-00128a).

§1. Introduction

It is classical that the quadratic form

$$g(v) = x^2 + y^2 - z^2$$

on $\mathbb{R}^3$, $v = (x, y, z) \in \mathbb{R}^3$, has the following property: on any connected component of the set $g(v) = -1$ it induces a Riemannian metric of the hyperbolic plane H^2, while on the set $g(v) = 1$ it induces a Lorentz metric of the de Sitter 2-space dS^2. The (set of lines in the) cone $g(v) = 0$ serves as the common absolute S^1 of both H^2 and dS^2. A similar picture occurs in any dimension and even for all rank one symmetric spaces of noncompact type.

In other words, we observe a life on the other side of the absolute S^1 of H^2 that is the de Sitter space dS^2. For mathematical aspects of the duality between hyperbolic spaces H^{n+1} and de Sitter spacetimes dS^{n+1}, see, e.g., [Ge, Yu]. The interplay between the geometry of hyperbolic surfaces and the Lorentz (2+1)-spaces was exploited in the famous paper [Mes], see also [A-S]. Duality for quadratic forms of arbitrary signature was discussed in [Ro]. For physical aspects of de Sitter spaces see, e.g., [SSV] and the references therein.

In §2, we describe this duality in intrinsic terms. The basic feature is the canonical Möbius structure M_0 on the absolute S^1, which governs the two sides H^2 and dS^2 of it. In particular, the isometry groups of H^2 and dS^2 coincide with the group of Möbius automorphisms of M_0. We show how to recover the hyperbolic plane H^2 and the de Sitter 2-space dS^2 purely out of M_0.

Moreover, we explain a mechanism of the passage from H^2 to dS^2 and back. In brief, H^2 is the homogeneous space of the M_0-automorphism group $\mathrm{PSL}_2(\mathbb{R})$ over a compact elliptic subgroup isomorphic to S^1, while dS^2 is the homogeneous space of $\mathrm{PSL}_2(\mathbb{R})$ over a (closed) hyperbolic subgroup isomorphic to $\mathbb{R}$.

This rises a bold question: Is there any life (a spacetime) on the other side of the absolute, i.e., the boundary at infinity, of any Gromov hyperbolic space with the same symmetry group? The main result of the paper is the answer "yes" for a large class of hyperbolic spaces with the absolute S^1, see Theorem 1.1.

A Möbius structure on a set X is a class of semimetrics having one and the same cross-ratio on any given ordered 4-tuple of distinct points in X, see §4. Every hyperbolic space Y induces on its boundary at infinity $X = \partial_\infty Y$ a Möbius structure which encodes most essential properties of Y and in a number of cases allows us to recover Y completely, e.g., in the case where Y is a rank one symmetric space of noncompact type, see [BS2, BS3]. In Subsection 4.2, we explain this for the class of *boundary continuous* hyperbolic spaces. In Appendix 1 (§8), we show that every proper Gromov hyperbolic CAT(0) space is boundary continuous.

We axiomatically describe a class $\mathcal{M}$ of *monotone* Möbius structures on the circle S^1, see §5. The class $\mathcal{M}$ includes every Möbius structure M induced on S^1 by a hyperbolic

CAT(0) surface Y without singular points, see Theorem 5.3. In particular, the isometry group of Y is included in the group of Möbius automorphisms of M. Furthermore, the canonical Möbius structure M_0 is the most symmetric representative from $\mathcal{M}$.

On the other hand, the set aY of unordered pairs of distinct points on the circle $X = S^1$ has a natural causal structure, which is independent of anything else, see §2. The points of aY are called *events*. There is a large class $\mathcal{T}$ of 2-dimensional spacetimes compatible with that causal structure, and we characterize it axiomatically in §3. Any spacetime $T \in \mathcal{T}$ is a triple $T = (\mathrm{aY}, \mathcal{H}, t)$, where $\mathcal{H}$ is a class of timelike curves in aY called *timelike lines*, which are actually timelike geodesics, and t is the time between events in the causal relation. The spacetime $T \in \mathcal{T}$ is called the *timed causal space*. We prove the following.

Theorem 1.1. *There are natural mutually inverse maps $\widehat{T}\colon \mathcal{M} \to \mathcal{T}$ and $\widehat{M}\colon \mathcal{T} \to \mathcal{M}$ such that the groups of automorphisms of any $M \in \mathcal{M}$ and of the respective $T = \widehat{T}(M) \in \mathcal{T}$ are canonically isomorphic.*

From constructions of §2 it follows that the canonical Möbius structure M_0 on S^1 determines the de Sitter space dS^2, that is, $\widehat{T}(M_0) = \mathrm{dS}^2$ and $\widehat{M}(\mathrm{dS}^2) = M_0$. In other words, Theorem 1.1 says that a monotone Möbius structure on S^1 on one hand, and the respective timed causal space with the absolute S^1 on the other, are different sides of one and the same phenomenon also in the general case.

The fundamental feature of spacetimes is the *time inequality.* In §7, we discuss a hierarchy of time conditions, in particular, we introduce the *weak time inequality,* and show that every timed causal space $T \in \mathcal{T}$ satisfies the weak time inequality, see Theorem 7.3.

In Subsection 7.5, we introduce the Increment Axiom (I), which implies the time inequality, and show that the subset $\mathcal{I} \subset \mathcal{M}$ of Möbius structures satisfying (I) contains the canonical structure M_0, $M_0 \in \mathcal{I}$ (Proposition 7.10), with a neighborhood of M_0 in the fine topology (Proposition 7.14).

In Subsection 7.7, we introduce the Convexity Axiom (C) for monotone Möbius structures $M \in \mathcal{M}$, which implies the convexity of a functional F_{ab} playing an important role in the hierarchy of time conditions, and show that the subset $\mathcal{C} \subset \mathcal{M}$ of convex Möbius structures contains M_0 (Proposition 7.15).

The spacetimes of class $\mathcal{T}$ are related to the de Sitter 2-space dS^2, at least in a sense, as the hyperbolic CAT(0) surfaces without singular points with the absolute S^1 are related to the hyperbolic plane H^2. Should one extend the results of this paper to more general hyperbolic spaces even with 1-dimensional boundary at infinity, this could potentially produce new interesting classes of spacetimes, e.g., those having a branching time (timelike lines).

§2. On the other side of the absolute

This section serves as a motivation, it contains no new results, and its constructions are widely known. Here, we show how the two sides H^2 and dS^2 of the common absolute S^1 can be recovered by the canonical Möbius structure M_0 on S^1. It is common to define H^2 and dS^2 in the quotient of $\mathbb{R}^3$ out of the antipodal map $x \mapsto -x$. This does not affect H^2, while dS^2 becomes a nontrivial line bundle over S^1, that is, the open Möbius band.

2.1. Recovering the hyperbolic plane H^2. The canonical Möbius structure M_0 on the circle S^1 is determined by the condition that any its representative with infinitely remote point is a standard metric (up to a positive factor) on $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ extended in the sense that the distance between any $x \in \mathbb{R}$ and ∞ is infinite.

To recover H^2 from M_0, we consider the space Y of all Möbius involutions $s\colon S^1 \to S^1$ with respect to M_0 without fixed points. The space Y serves as the underlying space for H^2, and what remains to do is to introduce a respective metric on Y.

A line in Y is determined by a pair x, $x' \in S^1$ of distinct points and consists of all involutions $s \in Y$ that permute x, x', $sx = x'$. Given two distinct points s, $s' \in Y$, the compositions $s's$, $ss'\colon S^1 \to S^1$ have one and the same fixed point set consisting of two distinct points x, $x' \in S^1$. Thus, there is a uniquely determined line in Y through s, s'.

We say that an ordered 4-tuple $q = (x, x', y, z) \in (S^1)^4$ of pairwise distinct points is *harmonic* if

$$|xy| \cdot |x'z| = |xz| \cdot |x'y| \tag{1}$$

for some and hence any metric on S^1 of class M_0. For the canonical Möbius structure M_0, the harmonicity of (x, x', y, z) is equivalent to the fact that the geodesic lines xx', $yz \subset \mathrm{H}^2$ are mutually orthogonal.

A *sphere* S between x, $x' \in S^1$ is a pair $(y, z) \subset S^1$ such that the 4-tuple (x, x', y, z) is harmonic. We take spheres S, $S' \subset S^1$ between x, x' such that S is invariant under s, $s(S) = S$, and S' is invariant under s', $s'(S') = S'$. The spheres S, S' with these properties exist and are determined uniquely. Now, we take $y \in S$, $y' \in S'$ and put

$$|ss'| = |\ln\langle x, y, y', x'\rangle|, \tag{2}$$

where $\langle x, y, y', x'\rangle = \frac{|xy'|\cdot|yx'|}{|xy|\cdot|y'x'|}$ is the cross-ratio of the 4-tuple (x, y, y', x'). This cross-ratio is well defined and independent of the choice of $y \in S$, $y' \in S'$. It is easy to show that $|ss'|$ is the distance in the geometry of H^2, see Subsection 2.3.

Remark 2.1. This construction is easily extended to any rank one symmetric space of noncompact type, see [BS2].

2.2. Recovering the de Sitter space dS^2**.** Let aY be the space of unordered pairs $(x, y) \sim (y, x)$ of distinct points on S^1 with the topology induced from S^1, that is, $\mathrm{aY} = S^1 \times S^1 \setminus \Delta/{\sim}$, where $\Delta = \{(x, x) : x \in S^1\}$ is the diagonal. Then aY is a nontrivial $\mathbb{R}$-bundle over $\mathbb{RP}^1 \approx S^1$, i.e., aY is the open Möbius band. In this case, S^1 is the boundary of aY at infinity, $\partial_\infty \mathrm{aY} = S^1$. The points of aY are called *events.*

We say that two events e, $e' \in \mathrm{aY}$ are in the *causal relation* if and only if e, e' do not separate each other as pairs of points in S^1. This defines the *canonical causality structure* on aY.

A *light line* in aY is determined by any $x \in S^1$ and consists of all events $a = (x, x') \in \mathrm{aY}$, $x' \in S^1 \setminus x$. For this light line p_x, x is a unique point at infinity. Two distinct light lines p_x, p_y have a unique common event $(x, y) \in \mathrm{aY}$, and any two events on a light line are in the causal relation.

The canonical causality structure as well as light lines are inherent in aY, and they do not depend on anything else.

Remark 2.2. In the higher-dimensional case, a causality structure can be defined similarly, but then it depends on the Möbius structure because events are codimension one spheres in S^n.

A *timelike line* in aY is determined by any event $e \in \mathrm{aY}$ and consists of all $a \in \mathrm{aY}$ such that the 4-tuple (e, a) is harmonic. For the timelike line $h_e \subset \mathrm{aY}$ determined by $e = (x, x')$, the points x, $x' \in S^1$ are the ends of h_e at infinity. From the definitions, it follows that $a \in h_e$ if and only if $e \in h_a$.

Any two events on a timelike line are in the causal relation. Conversely, for any two events a, $a' \in \mathrm{aY}$ that are in the causal relation and not on a light line there is a unique timelike line (the common perpendicular) h_e with a, $a' \in h_e$. (This amounts to the existence and uniqueness of a common perpendicular to divergent geodesics in H^2. For a (de Sitter) proof, see Corollary 5.9 and Lemma 5.10.) Let $a = (y, z)$, $a' = (y', z')$, $e = (x, x')$ in this case. Then the *time* $t = t(a, a')$ between the events a, a' is defined by formula (2):

$$t = |\ln\langle x, y, y', x'\rangle|$$

(note that a, a' are spheres between x, x').

It follows that two timelike lines h_e, $h_{e'}$ intersect each other if and only if the events e, $e' \in \mathrm{aY}$ are in the causal relation and not on a light line. In this case, the intersection $h_e \cap h_{e'}$ is a unique event.

An *elliptic line* in aY is determined by any Möbius involution without fixed points $s \in Y$ and consists of all events $a \in \mathrm{aY}$ such that $sa = a$. No two distinct events on an elliptic line are in the causal relation.

Remark 2.3. The last definition makes sense only for the canonical Möbius structure M_0 because in the general case a Möbius structure may fail to admit any Möbius involution without fixed points.

2.3. Automorphisms of M_0. To introduce a metric structure on aY, we consider the Lie algebra $\mathfrak{g}$ of the Lie group $G = \mathrm{SL}_2(\mathbb{R})$. Given α, $\beta \in \mathfrak{g}$, we have the Killing form

$$\langle\alpha, \beta\rangle = \frac{1}{2}\operatorname{Tr}(\alpha\beta) \tag{3}$$

as a scalar product. Note that the matrices $\sigma_1, \sigma_2, \sigma_3 \in \mathfrak{g}$,

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

are mutually orthogonal and $\|\sigma_1\|^2 = \langle\sigma_1, \sigma_1\rangle = 1 = \|\sigma_2\|^2$, $\|\sigma_3\|^2 = -1$.

The group G acts on $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ by the linear-fractional transformations

$$x \mapsto \frac{ax + b}{cx + d} \quad \text{with} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G,$$

which are Möbius with respect to the canonical Möbius structure M_0. The action is not effective with the kernel $\mathbb{Z}_2 = \{\pm\,\mathrm{id}\}$, $G/\mathbb{Z}_2 = \mathrm{PSL}_2(\mathbb{R})$. The group $\mathrm{PSL}_2(\mathbb{R})$ with the left invariant Lorentz metric (3) is the anti de Sitter 3-space AdS^3.

We denote by $K_i = \{\exp(t\sigma_i) : t \in \mathbb{R}\}$, $i = 1, 2, 3$, a 1-parametric subgroup in G, and by $\widehat{K}_i$ its image in $\mathrm{PSL}_2(\mathbb{R})$. Note that $\widehat{K}_i = K_i$ for $i = 1, 2$ and that $\widehat{K}_3 = K_3/\mathbb{Z}_2$.

The space Y of Möbius involutions $s\colon S^1 \to S^1$ without fixed points can be identified with the homogeneous space $G/K_3 = \mathrm{PSL}_2(\mathbb{R})/\widehat{K}_3$, because the group

$$K_3 = \left\{ g_3(t) = \exp(t\sigma_3) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} : t \in \mathbb{R} \right\}$$

stabilizes $s = g_3(\frac{\pi}{2}) = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, which acts on $\widehat{\mathbb{R}}$ as the Möbius involution $s(x) = -\frac{1}{x}$ without fixed points. The space G/K_3 carries a left-invariant Riemannian metric h_3 originated from the subspace $L_3 \subset \mathfrak{g}$ spanned by σ_1, σ_2, and $(G/K_3, h_3)$ is isometric to H^2. To see that, we compute the respective Riemannian distance between two involutions s_1, $s_2 \in Y$. By conjugation we may assume that $s_1 = s$, $s_2 = s'$, where $s' = g_1(t) \cdot s \cdot g_1^{-1}(t)$ for some $t \in \mathbb{R}$,

$$g_1(t) = \exp(t\sigma_1) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t}. \end{bmatrix}$$

Then $s' = \left[\begin{smallmatrix} 0 & e^{2t} \\ -e^{-2t} & 0 \end{smallmatrix}\right]$ and $s'(x) = -\frac{e^{4t}}{x}$. The curve $t \mapsto g_1(t)$ is a unit speed geodesic in G. While projected to $\mathrm{PSL}_2(\mathbb{R})$, the speed is doubled because of linear-fractional action of $\mathrm{PSL}_2(\mathbb{R})$, so we have $|ss'| = 2t$. In the upper half-plane model of H^2 the involution s fixes $i = (0,1)$ with Euclidean distance $|0i|_e = 1$ and s' fixes $ie^{2t} = (0, e^{2t})$ with Euclidean distance $|0ie^{2t}|_e = e^{2t}$, whence $|ss'| = 2t$ equals the H^2-distance $|(0,1)(0,e^{2t})| = \ln \frac{|0ie^{2t}|_e}{|0i|_e} = 2t$.

The action of $\mathrm{PSL}_2(\mathbb{R})$ on $\widehat{\mathbb{R}} \approx S^1$ induces the *standard* action of $\mathrm{PSL}_2(\mathbb{R})$ on aY. Note that $a = \{-1, 1\} \in \mathrm{aY}$ is a fixed point for K_2 because

$$\exp(t\sigma_2) = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix},$$

and

$$\exp(t\sigma_2)a = \left\{ \frac{\cosh t(-1) + \sinh t}{\sinh t(-1) + \cosh t}, \frac{\cosh t + \sinh t}{\sinh t + \cosh t} \right\} = \{-1, 1\} = a.$$

It follows that aY can be identified with the homogeneous space $\mathrm{PSL}_2(\mathbb{R})/K_2$, or similarly with $\mathrm{PSL}_2(\mathbb{R})/K_1$. The space $\mathrm{aY} = \mathrm{PSL}_2(\mathbb{R})/K_2$ carries a left-invariant Lorentz metric h_2 originated from the subspace $L_2 \subset \mathfrak{g}$ spanned by σ_1, σ_3, and $\mathrm{dS}^2 = (\mathrm{aY}, h_2)$.

In the rest of the paper, we explain how a Möbius structure M from a large class of structures on the circle gives rise to a spacetime, and *vice versa*, without any assumption on symmetries of M.

§3. Timed causal spaces on the circle

In this section we list axioms for timed causal spaces on the circle.

3.1. The canonical causality structure. Recall that on the space aY of unordered pairs of distinct point in $X = S^1$, which is homeomorphic to the open Möbius band, we have the canonical causal structure. That is, events e, $e' \in \mathrm{aY}$ are in the causal relation if and only if they do not separate each other as pairs of points in X. Otherwise, we also say that events e, $e' \in \mathrm{aY}$ separate each other. Whenever e, $e' \in \mathrm{aY}$ are in the causal relation and not on a light line, we say that the events e, e' are in the *strong* causal relation.

The canonical causal structure and light lines are inherent to aY, see Subsection 2.2.

For a fixed event $e \in \mathrm{aY}$, the set C_e of all $e' \in \mathrm{aY}$ in the causal relation with e is called the *causal cone*. The pair $e \subset X$ decomposes X into two closed arcs, which we denote by e^+, e^-, with $e^+ \cap e^- = e$. Every $a \in \mathrm{aY}$ with $a \subset e^\pm$ is in the causal relation with e. We let

$$C_e^\pm = \left\{ a \in \mathrm{aY} : a \subset e^\pm \right\}.$$

Therefore, a choice of e^+, e^- induces the decomposition $C_e = C_e^+ \cup C_e^-$ of the causal cone C_e into the *future* cone C_e^+ and the *past* cone C_e^- with $C_e^+ \cap C_e^- = e$, and moreover introduces a partial order on aY in the following way. Every $a \in C_e^\pm$ decomposes X into two closed arcs, and if $a \neq e$, we canonically define $a_e^\pm$ as one of them that does not contain e, otherwise $a_e^\pm = e^\pm$. Now, by definition, $a \leq_e a'$ if and only if one of the following holds true:

- $a \in C_e^-$, $a' \in C_e^+$;
- $a' \subset a_e^+$ if $a, a' \in C_e^+$ and $a \subset (a')_e^-$ if $a, a' \in C_e^-$.

As usual, we say that $a <_e a'$ if $a \leq_e a'$ and $a \neq a'$.

Note that there is no global partial order on aY compatible with the canonical causal structure, and the order defined above only appears if an event $e \in \mathrm{aY}$, the future arc e^+, and the past arc e^- are chosen.

3.2. Timelike lines and a causal space. The notion of a *timelike line* is not inherent to aY, and we define this notion axiomatically.

Axioms for timelike lines

(h1) every event $e \in \mathrm{aY}$ uniquely determines a timelike line $h_e \subset \mathrm{aY}$, and every timelike line in aY is of the form h_e for some $e \in \mathrm{aY}$;
(h2) any event $a \in h_e$ separates e;
(h3) any two events on a timelike line are in the causal relation;
(h4) for any point $x \in X \setminus e$ there is a unique event $x_e = (x, y) \in h_e$;
(h5) if an event $a \in \mathrm{aY}$ is on a timelike line h_e, then $e \in h_a$;
(h6) for any two distinct events a, $a' \in \mathrm{aY}$ there is at most one timelike line h_e with a, $a' \in h_e$.

The space aY with a fixed collection $\mathcal{H}$ of subsets satisfying the axioms of timelike lines is called a *causal space*. We use the notation $(\mathrm{aY}, \mathcal{H})$ for a causal space. In view of Axiom (h1), we say that an event $e \in \mathrm{aY}$ and the timelike line $h_e \subset \mathrm{aY}$ are *dual* to each other.

From (h4) it follows that for every event $e = (z, u) \in \mathrm{aY}$ we have a well-defined map $\rho_e \colon X \to X$ given by $\rho_e(z) = z$, $\rho_e(u) = u$, and $(x, \rho_e(x)) = x_e$ for every $x \in X \setminus e$. The map ρ_e is called *reflection with respect to* e.

Lemma 3.1. *For every $e = (z, u) \in \mathrm{aY}$, the map $\rho_e \colon X \to X$ is an involutive homeomorphism.*

Proof. By definition, $\rho_e^2(z) = z$ and $\rho_e^2(u) = u$. Let $y = \rho_e(x)$ for $x \in X \setminus e$. Then $(x, y), (\rho_e(y), y) \in h_e$, whence $\rho_e(y) = x$ by (h4). Thus $\rho_e^{-1} = \rho_e$, and ρ_e is a bijection.

The event e decomposes X into two closed arcs e^+, e^- with $X = e^+ \cup e^-$, $e^+ \cap e^- = e$. An orientation of X determines linear orders on e^+, e^-. From (h2) and (h3) it follows that $\rho_e \colon e^+ \to e^-$ reverses the orders. Hence, ρ_e is continuous and, therefore, a homeomorphism. □

Proposition 3.2. *Let $(\mathrm{aY}, \mathcal{H})$ be a causal space. Then*:

(a) *for any $e \in \mathrm{aY}$ the line h_e is homeomorphic* (*in the topology induced from* aY) *to $\mathbb{R}$, and the boundary of the closure $\overline{h_e} \subset \mathrm{aY} \cap \partial_\infty \mathrm{aY}$ is e, $\partial \overline{h_e} = e$*;
(b) *for any two distinct events a, $a' \in \mathrm{aY}$ there is a timelike line h including a, a' if and only if a and a' are in the strong causal relation. In this case, h is unique with this property*;
(c) *any two distinct timelike lines h_e, $h_{e'} \in \mathcal{H}$ have a common event a if and only if e, e' are in the strong causal relation. In this case, a is unique*;
(d) *a light line $p_x \subset \mathrm{aY}$ intersects a timelike line $h_e \subset \mathrm{aY}$ if and only if $x \notin e$. In this case, the common event $a \in p_x \cap h_e$ is unique.*

Proof. (a) Let $\rho_e \colon X \to X$ denote reflection with respect to e, and let $e^+ \subset X$ be one of the two closed arcs in which e decomposes X. Then the map $\operatorname{int} e^+ \to h_e$, $x \mapsto (x, \rho_e(x))$, is an order preserving bijection. Extended to e^+, it gives an order preserving bijection to $\overline{h_e} = h_e \cup e$. Thus, h_e is homeomorphic to $\operatorname{int} e^+ \approx \mathbb{R}$, and $\partial \overline{h_e} = e$.

(b) Any distinct a, $a' \in \mathrm{aY}$ on a timelike line h are in the causal relation by (h3), and by (h4) they are not on a light line. Hence, a, a' are in the strong causal relation. Conversely, assume that events a, $a' \in \mathrm{aY}$ are in the strong causal relation. Let $\rho = \rho_a \circ \rho_{a'}$ be the composition of the respective reflections, and let $a^+ \subset X$ be the closed arc determined by a that does not contain a'. Then $\rho(a^+) \subset \operatorname{int} a^+$, and thus there is a fixed point $x \in \operatorname{int} a^+$ of ρ, $\rho(x) = x$. It follows that both reflections ρ_a, $\rho_{a'}$ preserve the event $e = (x, y)$, where $y = \rho_{a'}(x)$. Hence, $e \in h_a \cap h_{a'}$, and by (h5) we have $a, a' \in h_e$. By (h6), h_e is unique with this property.

(c) By the duality (h5), this is a reformulation of (b).

(d) This immediately follows from (h2) and (h4). □

Remark 3.3. Axiom (h6) was not used in Lemma 3.1, and it was only used in Proposition 3.2 to prove the uniqueness in (b) and (c). Thus, all the conclusions of Proposition 3.2 except for the uniqueness in (b) and (c) hold true without Axiom (h6). We shall use this remark in Subsection 5.4.

3.3. Timed causal space. The notion of a *time* is also defined axiomatically.

The time axioms

(t1) A time $t(e,e') \geq 0$ between two events e, $e' \in \mathrm{aY}$ is determined if and only if e, e' are in the causal relation;

(t2) $t(e,e') = 0$ if and only if e, e' are events on a light line;

(t3) $t(e,e') = t(e',e)$ whenever $t(e,e')$ is defined;

(t4) timelike lines are t-geodesics:

(a) if $e, e', e'' \in h_a$ are events on a timelike line such that $e \leq e' \leq e''$, then $t(e,e') + t(e',e'') = t(e,e'')$;

(b) for every $e \in h_a$ and every $s > 0$ there are $e_\pm \in h_a \cap C_e^\pm$ with $t(e,e_\pm) = s$;

(t5) for any events $e = (x,y)$, $d = (z,u)$ in the strong causal relation, we have $t(z_e,u_e) = t(x_d,y_d)$;

(t6) for any $e = (x,y)$ and $d = (z,u) \in h_e$ the 4-tuple (d,e) is *harmonic* in the sense that $t(y_a,u_a) = t(y_b,z_b)$, where $a = (x,z)$, $b = (x,u)$.

A *timed causal space* is defined as $T = (\mathrm{aY}, \mathcal{H}, t)$, where t is a time on the causal space $(\mathrm{aY}, \mathcal{H})$. This is a version of Busemann (locally) timelike spaces, see [Bus], and also an axiomatic version of the de Sitter space dS^2. Since dS^2 is recovered from the canonical Möbius structure M_0 on the circle, see Subsection 2.2, the results of §5 (see Proposition 5.8, Lemma 5.10, and Proposition 5.11) show that dS^2 is a timed causal space.

We denote by $\mathcal{T}$ the set of all timed causal spaces $(\mathrm{aY}, \mathcal{H}, t)$, where the collection $\mathcal{H}$ of timelike lines satisfies Axioms (h1)–(h6), and the time t satisfies Axioms (t1)–(t6). A T-*automorphism*, $T = (\mathrm{aY}, \mathcal{H}, t) \in \mathcal{T}$, is a bijection $g\colon \mathrm{aY} \to \mathrm{aY}$ that preserves the timelike lines $\mathcal{H}$ and the time t, $t(g(e), g(e')) = t(e,e')$ whenever $t(e,e')$ is defined (we do not require that the causality structure be preserved, because this is automatic).

Remark 3.4. I am not satisfied with the terminology in, e.g., [Bus, PY], where the term "timelike (metric) space" is used, because the corresponding object is never a metric space and its basic feature is a causality relation. On the other hand, to talk of "timelike causal spaces" sounds a little bit tautologically. Thus, I use the term "timed causal space" instead.

Remark 3.5. The strange looking Axioms (t5), (t6) are automatically satisfied for the timed causal spaces induced by monotone Möbius structures on the circle, see §5. However, their value and importance are justified by the fact that (t5) and (t6) are indispensable when we recover a Möbius structure on the circle from a timed causal space, see §6, especially Lemma 6.4 and Lemma 6.9. In fact, Axiom (t6) follows from the other axioms, see Appendix 2.

§4. Möbius structures and hyperbolic spaces

On the boundary at infinity of any boundary continuous Gromov hyperbolic space there is an induced Möbius structure. In this section, we recall the details of this fact.

4.1. Semimetrics and topology. Let X be a set. A function $d\colon X^2 \to \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is called a *semimetric* if it is symmetric, $d(x,y) = d(y,x)$ for each x, $y \in X$, is positive outside the diagonal, vanishes on the diagonal and there is at most one infinitely remote point $\omega \in X$ for d, i.e., such that $d(x,\omega) = \infty$ for some $x \in X \setminus \{\omega\}$. Moreover, if $\omega \in X$ is such a point, then $d(x,\omega) = \infty$ for all $x \in X$, $x \neq \omega$. A metric is a semimetric that satisfies the triangle inequality.

A 4-tuple $q = (x_1, x_2, x_3, x_4) \in X^4$ is said to be *nondegenerate* if all its entries are pairwise distinct. We denote by $\operatorname{reg}\mathcal{P}_4 = \operatorname{reg}\mathcal{P}_4(X)$ the set of ordered nondegenerate 4-tuples.

A *Möbius structure* M on X is a class of Möbius equivalent semimetrics on X, where two semimetrics are equivalent if and only if they have the same cross-ratios on every $q \in \operatorname{reg}\mathcal{P}_4$. An *$M$-automorphism* is a bijection $f\colon X \to X$ that preserves cross-ratios.

Given $\omega \in X$, there is a semimetric $d_\omega \in M$ with infinitely remote point ω. It can be obtained from any semimetric $d \in M$ for which ω is not infinitely remote by a *metric inversion*,

$$d_\omega(x,y) = \frac{d(x,y)}{d(x,\omega)d(y,\omega)}.$$

Such a semimetric is unique up to a homothety, see [FS], and we use the notation $|xy|_\omega = d_\omega(x,y)$ for the distance between x, $y \in X$ in that semimetric. We also use the notation $X_\omega = X \setminus \{\omega\}$.

Every Möbius structure M on X determines the *M-topology* whose subbase is given by all open balls centered at the finite points of all semimetrics in M that have infinitely remote points.

For the following fact see [Bu1, Corollary 4.3] in a more general context of sub-Möbius structures. Here we give the proof for the reader's convenience.

Lemma 4.1. *For every $\omega \in X$, every semimetric $d \in M$ with infinitely remote point $\omega \in X$, and every $x \in X_\omega$, the function $f_x\colon X \to \widehat{\mathbb{R}}$, $f_x(y) = d(x,y)$, is continuous in the M-topology.*

Proof. The function f_x takes values in $[0,\infty]$. Given $s, t \in [0,\infty]$, we denote by $B_s(x) = \{y \in X : f_x(y) < s\}$ the open d-ball of radius s centered at x, and we let $C_t(x) = \{y \in X : f_x(y) > t\}$ be the complement of the closed d-ball. The inverse image $f_x^{-1}(I)$ of any open interval $I \subset [0,\infty]$ is either an open ball $B_s(x)$, or a complement $C_t(x)$, or an intersection $B_s(x) \cap C_t(x)$ for some $s > t$.

Let $d_x \in M$ be the metric inversion of d. Then $d_x(y,\omega) = 1/d(x,y)$, whence $C_t(x) = \{y \in X : d_x(y,\omega) < 1/t\}$ is the open d_x-ball of radius $1/t$ centered at ω. It follows that $f_x^{-1}(I)$ is open in the M-topology. $\square$

4.2. Boundary continuous hyperbolic spaces. Let Y be a metric space. Recall that the Gromov product $(x|y)_o$ of x, $y \in Y$ with respect to $o \in Y$ is defined by the formula

$$(x|y)_o = \frac{1}{2}(|xo| + |yo| - |xy|),$$

where $|xy|$ is the distance in Y between x, y. We use the following definition of a hyperbolic space, adapted to the case of geodesic metric spaces.

Definition 4.2. A geodesic metric space Y is *Gromov hyperbolic* if for some $\delta \geq 0$ and any triangle $xyz \subset Y$ the following is true: if $y' \in xy$, $z' \in xz$ are points with

$|xy'| = |xz'| \le (y|z)_x$, then $|y'z'| \le \delta$. In this case, we also say that Y is δ-hyperbolic, and δ is a *hyperbolicity constant* of Y.

A Gromov hyperbolic space Y is *boundary continuous* if the Gromov product extends continuously to the boundary at infinity $\partial_\infty Y = X$ in the following way: given $\xi, \eta \in X$, for any sequences $\{x_i\} \in \xi$, $\{y_i\} \in \eta$ there is a limit $(\xi|\eta)_o = \lim_i (x_i|y_i)_o$ for every $o \in Y$, see [BS1, 3.4.2] for more details. Note that in this case $(\xi|\eta)_o$ is independent of the choice of $\{x_i\} \in \xi$, $\{y_i\} \in \eta$. This allows one to define, for every $o \in Y$, a function $(\xi, \eta) \mapsto d_o(\xi, \eta) = e^{-(\xi|\eta)_o}$, which is a semimetric on X.

Lemma 4.3. *Let Y be a boundary continuous hyperbolic space. Then for any o, $o' \in Y$, the semimetrics d_o, $d_{o'}$ on $X = \partial_\infty Y$ are Möbius equivalent.*

Proof. Given a 4-tuple $(x, y, z, u) \subset Y$, we put

$$\mathrm{cd}_o(x, y, z, u) = (x|u)_o + (y|z)_o - (x|z)_o - (y|u)_o$$

for a fixed $o \in Y$. Then $\mathrm{cd}_o(x, y, z, u) = \mathrm{cd}(x, y, z, u)$ is independent of the choice of o because all entries containing o enter $\mathrm{cd}_o(x, y, z, u)$ twice with the opposite signs.

Now, given a nondegenerate 4-tuple $q = (\alpha, \beta, \delta, \gamma) \in \operatorname{reg} \mathcal{P}_4(X)$, for any $\{x_i\} \in \alpha$, $\{y_i\} \in \beta$, $\{z_i\} \in \gamma$, and $\{u_i\} \in \delta$, the limit

$$\mathrm{cd}(\alpha, \beta, \gamma, \delta) = \lim_i \mathrm{cd}(x_i, y_i, z_i, u_i)$$

exists by the boundary continuity of Y, and this limit coincides with $(\alpha|\delta)_o + (\beta|\gamma)_o - (\alpha|\gamma_o) - (\beta|\delta)_o$. Thus the cross-ratio

$$\frac{d_o(\alpha, \gamma) d_o(\beta, \delta)}{d_o(\alpha, \delta) d_o(\beta, \gamma)} = \exp(-\mathrm{cd}(\alpha, \beta, \gamma, \delta))$$

is independent of o. Hence, the semimetrics d_o, $d_{o'}$ are Möbius equivalent for any o, $o' \in Y$. □

The Möbius structure M on the boundary at infinity $X = \partial_\infty Y$ of a boundary continuous hyperbolic space Y generated by any semimetric $d_o(\xi, \eta) = \exp(-(\xi|\eta)_o)$, $o \in Y$, is said to be *induced* (from Y). For any $\omega \in X$ and $o \in Y$, the metric inversion d_ω of d_o with respect to ω is a semimetric on X of class M with the infinitely remote point ω. Recall that any two semimetrics in M with a common infinitely remote point are proportional to each other. Thus, the metric inversions with respect to ω of the semimetrics d_o, $d_{o'}$ are proportional to each other for any o, $o' \in Y$.

In Appendix 1, we shall show that every proper Gromov hyperbolic CAT(0) space is boundary continuous, see Theorem 8.1.

§5. Monotone Möbius structures on the circle

5.1. Axioms for monotone Möbius structures on the circle. We say that a Möbius structure M on $X = S^1$ is *monotone* if it satisfies the following axioms:

(T) topology: the M-topology on X is that of S^1;

(M) monotonicity: given a 4-tuple $q = (x, y, z, u) \in X^4$ such that the pairs (x, y), (z, u) separate each other, we have

$$|xy| \cdot |zu| > \max\left\{|xz| \cdot |yu|, |xu| \cdot |yz|\right\}$$

for some and hence any semimetric in M.

Remark 5.1. These axioms have arisen in a discussion with V. Schroeder while working on [BS4].

A choice of $\omega \in X$ uniquely determines the interval $xy \subset X_\omega$ for any distinct $x, y \in X$ different from ω as the arc in X with the endpoints x, y that does not contain ω. As an useful reformulation of Axiom (M), we have the following statement.

Corollary 5.2. *For a nondegenerate 4-tuple $q = (x, y, z, u) \in \operatorname{reg}\mathcal{P}_4$, assume that the interval $xz \subset X_u$ is contained in xy, $xz \subset xy \subset X_u$. Then $|xz|_u < |xy|_u$.*

Proof. By assumption, the pairs (x, y), (z, u) separate each other. Hence, by Axiom (M) we have $|xz|\,|yu| < |xy|\,|zu|$ for any semimetric in M. In particular, $|xz|_u < |xy|_u$. □

We denote by $\mathcal{M}$ the class of monotone Möbius structures on S^1.

5.2. Examples of monotone Möbius structures on the circle. By Theorem 8.1, every proper Gromov hyperbolic CAT(0) space Y is boundary continuous, and, thus, $\partial_\infty Y$ possesses an induced Möbius structure.

Recall that in any CAT(0) space Y, the angle $\angle_o(x, x')$ between geodesic segments ox, ox' with a common vertex o is well defined and, by definition, it is at most π, $\angle_o(x, x') \le \pi$.

A point o in a CAT(0) space Y with $\partial_\infty Y$ homeomorphic to the circle S^1 is said to be *singular* if there are two geodesics $\xi\xi'$, $\eta\eta' \subset Y$ through o such that the pairs of points (ξ, ξ') and (η, η') in $\partial_\infty Y$ separate each other and $\angle_o(\xi, \eta) + \angle_o(\xi', \eta') \ge 2\pi$.

Theorem 5.3. *Let Y be a Gromov hyperbolic* CAT(0) *surface with $\partial_\infty Y = S^1$ and without singular points. Then the induced Möbius structure M on $X = \partial_\infty Y$ is monotone.*

Proof. For the induced Möbius structure, the M-topology on X coincides with the standard Gromov topology, see [BS1, 2.2.3] or [Bu1, Lemma 5.1]. Thus M satisfies Axiom (T) by assumption. To check Axiom (M), consider a 4-tuple $q = (\xi, \xi', \eta, \eta') \in X^4$ such that the pairs (ξ, ξ') and (η, η') separate each other. Since Y is Gromov hyperbolic, there are geodesics $\xi\xi'$, $\eta\eta' \subset Y$ with the endpoints at infinity ξ, ξ' and η, η', respectively. The assumption on separation and the fact that Y is a CAT(0) surface imply that these geodesics intersect at some point o. We have $(\xi|\xi')_o = 0 = (\eta|\eta')_o$. Thus, $|\xi\xi'| = 1 = |\eta\eta'|$ for the semimetric $|xy| = \exp(-(x|y)_o)$ on X. Recall that this semimetric is a semimetric of M.

The angles at o between the rays ox, $x = \xi, \eta, \xi', \eta'$ in this cyclic order, form two opposite pairs. Since o is not singular in Y, at least one of the angles $\angle_o(x, z)$ is less than π for each opposite pair. By Corollary 8.4, $(x|z)_o > 0$, so that $|xz| < 1$. It follows that

$$|\xi\xi'| \cdot |\eta\eta| > \max\left\{|\xi\eta| \cdot |\xi'\eta'|, |\xi'\eta| \cdot |\xi\eta'|\right\},$$

i.e., M satisfies Axiom (M). □

Examples 5.4. 1. The canonical Möbius structure M_0 on the circle is monotone.

2. Let S be a closed surface of negative Euler characteristic with an Euclidean metric having cone type singularities with complete angles strictly greater than 2π about every singular point, and let Y be the universal covering of S with the lifted metric. Then Y is a Gromov hyperbolic CAT(0) surface with $\partial_\infty Y = S^1$. From Theorem 8.1 it follows that Y induces a Möbius structure M on $\partial_\infty Y$. However, M is not monotone.

3. The absence of singular points on a Gromov hyperbolic CAT(0) surface Y does not mean that Y has no metric singularities. Remove an open equidistant neighborhood of a geodesic line in H^2 and glue the remaining pieces by an isometry between their boundaries. Then the resulting Y is a proper Gromov hyperbolic CAT(-1) surface with $\partial_\infty Y = S^1$ without singular points, and Theorem 5.3 can be applied to Y. At the same time, Y has metric singularities along the gluing line.

4. Every (topological) embedding $f\colon S^1 \to \widehat{\mathbb{R}}^2$ induces on S^1 some metric and therefore a Möbius structure M_f. For $f = \mathrm{id}$, the Möbius structure M_f coincides with the canonical one, and thus it is monotone. However, if $f(S^1) \subset \mathbb{R}^2$ is an ellipse with principal semiaxes a, b such that $4ab \le a^2 + b^2$, then M_f is not monotone. That is, in general, the Möbius structure induced on a convex curve in $\widehat{\mathbb{R}}^2$ is not monotone.

In what follows, we assume that a monotone Möbius structure M on $X = S^1$ is fixed.

5.3. Harmonic pairs. A pair $a = (x, y)$, $b = (z, u) \in \mathrm{aY}$ of events is said to be *M-harmonic*, or to form an *M-harmonic 4-tuple*, if

$$|xz| \cdot |yu| = |xu| \cdot |yz| \tag{4}$$

for some and hence any semimetric of the Möbius structure (this is the same as (1); however, for a general Möbius structure M on S^1, the interpretation of (1) for the canonical M_0 as orthogonality of the corresponding geodesic lines in H^2 makes no sense). Nevertheless, in this case we also say that a, b are mutually *orthogonal,* $a \perp b$. Note that any harmonic 4-tuple $q = (a, b)$ is nondegenerate.

Lemma 5.5. *If events $a = (x, y)$ and $b = (z, u) \in \mathrm{aY}$ are mutually orthogonal, $a \perp b$, then they separate each other and $z \in xy \subset X_u$ is a unique midpoint.*

Proof. We have $|xz|_u = |zy|_u$. By Corollary 5.2, $z \in xy \subset X_u$ is a unique midpoint, and thus a, b separate each other. □

Lemma 5.6. *For every $e \in \mathrm{aY}$ and every $x \in X \setminus e$, there is a unique $y \in X$ such that the pair $a = (x, y)$, $e \in \mathrm{aY}$ of events is harmonic.*

Proof. Let $e = (z, u)$. By Axiom (T) and Lemma 4.1, the functions $f_z, f_u\colon X \to \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, $f_z(t) = |zt|_x$, $f_u(t) = |ut|_x$ are continuous on $X = S^1$, and they take values between $0 = f_z(z) = f_u(u)$ and $f_z(u) = f_u(z) > 0$ on the segment $zu \subset X_x$. Thus, there is a midpoint $y \in zu \subset X_x$ between z and u, $|zy|_x = |yu|_x$. By Corollary 5.2, such a point y is unique. □

Proposition 5.7. *For every monotone Möbius structure $M \in \mathcal{M}$ there is a uniquely determined timed causal space $T = \widehat{T}(M) \in \mathcal{T}$ such that the automorphism group G_M of M injects into the automorphism group G_T of T: If $g\colon X \to X$ is an M-Möbius automorphism, then the induced $\widehat{g}\colon \mathrm{aY} \to \mathrm{aY}$ is an automorphism of T.*

We prove Proposition 5.7 in Subsections 5.4 and 5.5.

5.4. Timelike lines. Any timelike line h in aY is associated with an event $e \in \mathrm{aY}$, $h = h_e$, and is defined as the set of events $a \in \mathrm{aY}$ such that the pair (a, e) is harmonic, $h_e = \{a \in \mathrm{aY} : a \perp e\}$. We denote by $\mathcal{H} = \mathcal{H}_M$ the collection of timelike lines in aY.

Proposition 5.8. *The collection $\mathcal{H}$ satisfies Axioms* (h1)–(h5).

Proof. Axioms (h1), (h5) are fulfilled by definition, (h2) follows from Lemma 5.5 and (h4) follows from Lemma 5.6.

To check (h3), assume that $e = (z, u) \in \mathrm{aY}$, $a = (x, y)$, $a' = (x', y') \in h_e$. Then z is the midpoint of the segments xy, $x'y' \subset X_u$. Therefore, by Axiom (M), the pairs (x, y), (x', y') do not separate each other, that is, the events a and a' are in the causal relation. Hence, (h3) is valid. □

We say that an event $a \in \mathrm{aY}$ is a *common perpendicular* to events e, $e' \in \mathrm{aY}$ if e, $e' \in h_a$.

Corollary 5.9. *Given e, $e' \in \mathrm{aY}$ in the strong causal relation, there is a common perpendicular $a \in \mathrm{aY}$ to e, e'.*

Proof. By Proposition 5.8, the collection $\mathcal{H} = \mathcal{H}_M$ of timelike lines in aY determined by the Möbius structure M satisfies Axioms (h1)–(h5). Thus, the assertion follows from Proposition 3.2(b), see Remark 3.3. $\square$

We postpone the proof of (h6) to Subsection 5.5, see Lemma 5.10.

5.5. Time between events. The time between events a, $a' \in \mathrm{aY}$ is defined if and only they are in causal relation. We do this essentially as in Subsection 2.2, using formula (2). First of all, by definition, the time between events on a light line is zero, $t(a, a') = 0$ for a, $a' \in p_x$, $x \in X$.

Next, assume that a, $a' \in \mathrm{aY}$ are in strong causal relation. Then by Corollary 5.9 there is a common perpendicular $e \in \mathrm{aY}$ to a, a', that is, a, $a' \in h_e$. Let $e = (x, y)$, $a = (z, u)$, $a' = (z', u')$. Then, by definition,

$$t_e(a, a') = \left|\ln \frac{|xz'| \cdot |yz|}{|xz| \cdot |yz'|}\right| \tag{5}$$

for some and hence any semimetric on X from M. The harmonicity of (a, e) and (a', e) implies that

$$t_e(a, a') = \left|\ln \frac{|xu'| \cdot |yu|}{|xu| \cdot |yu'|}\right| = \left|\ln \frac{|xu'| \cdot |yz|}{|xz| \cdot |yu'|}\right| = \left|\ln \frac{|xz'| \cdot |yu|}{|xu| \cdot |yz'|}\right|, \tag{6}$$

and we often use these different representations of $t_e(a, a')$.

Lemma 5.10. *Given distinct a, $a' \in \mathrm{aY}$ in causal relation, there is at most one common perpendicular $b \in \mathrm{aY}$ to a, a'. In particular, the time $t(a, a') = t_e(a, a')$ is well defined, and Axiom (h6) is fulfilled for the collection $\mathcal{H}$ of timelike lines.*

Proof. The idea is taken from [BS4]. If the events a, a' are on a light line, then they cannot lie on a timelike line, say h_e, because by Axiom (h1) they both must separate e, which would contradict (h4). Thus, we assume that a, a' are not on a light line.

Assume there are common perpendiculars $b = (z, u)$, $b' = (z', u') \in \mathrm{aY}$ to a, a', that is $b \perp a, a'$ and $b' \perp a, a'$, or which is the same, b, $b' \in h_a \cap h_{a'}$. By the already established Axiom (h3), see Proposition 5.8, b and b' do not separate each other. Let $a = (x, y)$, $a' = (x', y')$. Without loss of generality, we assume that on X_x we have the following order of points: $zz'yy'u'ux'$.

By Axiom (h5), a, $a' \in h_b \cap h_{b'}$. The times $t = t_b(a, a')$, $t' = t_{b'}(a, a')$ were already defined by (5). Computing them in a semimetric of the Möbius structure with infinitely remote point x, we obtain

$$e^t = \frac{|zx'|}{|x'u|}, \quad e^{t'} = \frac{|z'x'|}{|x'u'|}.$$

Using the order of points $zz'yy'u'ux'$ on X_x, we see, in particular, that the interval $z'x'$ is included in the interval zx'. By Corollary 5.2, $|zx'| \geq |z'x'|$. Similarly, $x'u \subset x'u'$, whence $|x'u| \leq |x'u'|$. Thus, $t \geq t'$, and if $b' \neq b$, then the inequality is strong. Applying this argument with the infinitely remote point y, we obtain $t \leq t'$. Therefore, $t = t'$ and $b = b'$. $\square$

Proposition 5.11. *The time between events in* aY *defined above satisfies Axioms* (t1)–(t6).

Proof. Axiom (t1) is satisfied by the definition of the time t.

Axiom (t2): If events a, a' are on a light line, then $t(a,a')=0$ by definition. Conversely, assume that $t(a,a')=0$ for events a, $a'\in \mathrm{aY}$ in causal relation that are not on a light line, in particular, $a\neq a'$. Then by Lemmas 5.9 and 5.10, there is a unique $e\in\mathrm{aY}$ with a, $a'\in h_e$. Let $e=(x,y)$, $a=(z,u)$, $a'=(z',u')$. Since $a\neq a'$, we have $z'\neq z,u$. On the other hand, from (5) it follows that $|xz'|\cdot|yz|=|xz|\cdot|yz'|$ for any semimetric on X from M. In particular, $|xz'|_y=|xz|_y$, and by monotonicity (M), x is the midpoint between z, z' in X_y. Hence, $\rho_e(z)=z'=u$, a contradiction.

Axiom (t3) follows from the definition of the time t and (5).

Axiom (t4a): Let e, e', $e''\in h_a$ with $e\le e'\le e''$. If e' coincides with e or e'', then the required identity is trivial. Thus, we assume that $e<e'<e''$. Without loss of generality, we may assume that for $a=(x,y)$, $e=(z,u)$, $e'=(z',u')$, $e''=(z'',u'')$, the points z, z', z'' lie on one and the same arc determined by a in the order $xzz'z''y$. Then

$$\exp(t(e,e'))=\frac{|xz'|\cdot|yz|}{|xz|\cdot|yz'|},\quad \exp(t(e',e''))=\frac{|xz''|\cdot|yz'|}{|xz'|\cdot|yz''|},$$

and we obtain

$$\exp(t(e,e')+t(e',e''))=\frac{|xz''|\cdot|yz|}{|xz|\cdot|yz''|}=\exp(t(e,e'')).$$

Axiom (t4b): given an event $e=(z,u)$ on a timelike line $h_a\subset \mathrm{aY}$ with $a=(x,y)$, and $s>0$, we take a semimetric of class M with the infinitely remote point y. Then $|xz|_y=|xu|_y:=t>0$, and there is no loss of generality in assuming that $t=1$. Note that the function $f_x\colon X_y\to\mathbb{R}$, $f_x(x')=|xx'|_y$, is continuous, see Lemma 4.1, and monotone, see Corollary 5.2. It varies from $0=f_x(x)$ to $\infty=f_x(y)$. Thus, there is $z_-\in xz$ and $z_+\in zy$ with $f_x(z_\pm)=e^{\pm s}$. For the events $e_\pm=(z_\pm,u_\pm)\in h_a$, where $u_\pm=\rho_a(z_\pm)$, we have

$$t(e,e_\pm)=\left|\ln\frac{|xz_\pm|_y}{|xz|_y}\right|=|\ln|xz_\pm|_y|=s.$$

Choosing a decomposition $X=e^+\cup e^-$ determined by e so that $y\in e^+$, we have $e_\pm\in h_a\cap C_e^\pm$ and $t(e,e_\pm)=s$.

Axiom (t5): Let $e=(x,y)$, $d=(z,u)$ be events in strong causal relation. Then by (5), (6) we have

$$t(z_e,u_e)=\left|\ln\frac{|xu|\cdot|yz|}{|xz|\cdot|yu|}\right|=t(x_d,y_d).$$

Axiom (t6): given $e=(x,y)\in\mathrm{aY}$, $d=(z,u)\in h_e$, we put $a=(x,z)$, $b=(x,u)$. Then by (5), (6) we have

$$t(y_a,u_a)=\left|\ln\frac{|xy|\cdot|zu|}{|xu|\cdot|zy|}\right|,\quad t(y_b,z_b)=\left|\ln\frac{|xy|\cdot|zu|}{|xz|\cdot|yu|}\right|.$$

On the other hand, the pair (a,e) is harmonic, so that $|xu|\cdot|zy|=|xz|\cdot|yu|$. Hence, $t(y_a,u_a)=t(y_b,z_b)$. □

Proof of Proposition 5.7. Given a monotone Möbius structure $M\in\mathcal{M}$, above we have defined a class $\mathcal{H}=\mathcal{H}_M$ of timelike lines in aY that satisfies Axioms (h1)–(h6), and a time t on $(\mathrm{aY},\mathcal{H})$ that satisfies Axioms (t1)–(t6). Therefore, a timed causal space $T=(\mathrm{aY},\mathcal{H},t)$, $T=\widehat{T}(M)$, is defined.

Let $g\colon X\to X$ be an M-Möbius automorphism. Then the induced map $\widehat{g}\colon \mathrm{aY}\to\mathrm{aY}$ preserves the causality structure, the class of timelike lines $\mathcal{H}$, and the time t, because the last two are defined via cross-ratios. Thus, $\widehat{g}$ is an automorphism of T. If $\widehat{g} = \mathrm{id}_T$, then, in particular, it preserves every light line. Hence, $g = \mathrm{id}$, and the group G_M of M-automorphisms injects into the group G_T of T-automorphisms. □

§6. Timed causal spaces and Möbius structures

In this section we adopt the following more advanced point of view on Möbius structures, see [Bu1].

6.1. Möbius and sub-Möbius structures. Let X be a set, $\operatorname{reg}\mathcal{P}_4 = \operatorname{reg}\mathcal{P}_4(X)$, see Subsection 4.1. For any semimetric d on X we have three cross-ratios

$$q \mapsto \mathrm{cr}_1(q) = \frac{|x_1x_3|\,|x_2x_4|}{|x_1x_4|\,|x_2x_3|};\quad \mathrm{cr}_2(q) = \frac{|x_1x_4|\,|x_2x_3|}{|x_1x_2|\,|x_3x_4|};\quad \mathrm{cr}_3(q) = \frac{|x_1x_2|\,|x_3x_4|}{|x_2x_4|\,|x_1x_3|}$$

for $q = (x_1, x_2, x_3, x_4) \in \operatorname{reg}\mathcal{P}_4$, the product of which equals 1, where $|x_ix_j| = d(x_i, x_j)$. With d we associate a map $M_d\colon \operatorname{reg}\mathcal{P}_4 \to L_4$ defined by

$$M_d(q) = \big(\ln \mathrm{cr}_1(q), \ln \mathrm{cr}_2(q), \ln \mathrm{cr}_3(q)\big), \tag{7}$$

where $L_4 \subset \mathbb{R}^3$ is the 2-plane given by the equation $a + b + c = 0$.

Two semimetrics d, d' on X are Möbius equivalent if and only $M_d = M_{d'}$. Thus, a Möbius structure on X is completely determined by a map $M = M_d$ for any semimetric d of the Möbius structure, and we often identify a Möbius structure with the corresponding map M. A bijection $f\colon X \to X$ is an M-automorphism if and only if $M \circ \overline{f(q)} = M(q)$ for every ordered 4-tuple $q \in \operatorname{reg}\mathcal{P}_4$ and the induced $\overline{f}\colon \operatorname{reg}\mathcal{P}_4 \to \operatorname{reg}\mathcal{P}_4$.

Let S_n be the symmetry group of n elements. The group S_4 acts on $\operatorname{reg}\mathcal{P}_4$ by permutation of the entries of any $q \in \operatorname{reg}\mathcal{P}_4$. The group S_3 acts on L_4 by signed permutations of coordinates, where a permutation $\sigma\colon L_4 \to L_4$ has the sign "-1" if and only if σ is odd.

The *cross-ratio* homomorphism $\phi\colon S_4 \to S_3$ can be described as follows: a permutation of ordered vertices $(1, 2, 3, 4)$ of a tetrahedron gives rise to a permutation of pairs of opposite edges $((12)(34), (13)(24), (14)(23))$. Thus, the kernel K of ϕ consists of four elements 1234, 2143, 4321, 3412, and is isomorphic to the dihedral group D_4 of automorphisms of a square. We denote by $\mathrm{sgn}\colon S_4 \to \{\pm 1\}$ the homomorphism that associates the sign "-1" with every odd permutation.

It is easy to check that any Möbius structure $M\colon \operatorname{reg}\mathcal{P}_4 \to L_4$ is equivariant with respect to the signed cross-ratio homomorphism,

$$M(\pi(q)) = \mathrm{sgn}(\pi)\phi(\pi)M(q) \tag{8}$$

for every $q \in \operatorname{reg}\mathcal{P}_4$, $\pi \in S_4$, where $\phi\colon S_4 \to S_3$ is the cross-ratio homomorphism.

A *sub-Möbius structure* on X is a map $M\colon \mathcal{P}_4 \to L_4$ with the basic property (8) (we drop the details related to degenerate 4-tuples, which can be found in [Bu1]). Now, we describe a criterion for a sub-Möbius structure to be Möbius. Given an ordered collection $q = (x_1, \dots, x_k) \in X^k$, we use the notation

$$q_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k),$$

$i = 1, \dots, k$.

For a sub-Möbius structure M on X we define its *codifferential* $\delta M\colon \operatorname{reg}\mathcal{P}_5 \to L_5 = L_4^5$ by

$$(\delta M(q))_i = M(q_i),\ i = 1, \dots, 5.$$

Furthermore, we denote $M(q_i) = (a(q_i), b(q_i), c(q_i))$, $i = 1, \dots, 5$, $q \in \operatorname{reg}\mathcal{P}_5$. The following theorem was proved in [Bu1, Theorem 3.4].

Theorem 6.1. *A sub-Möbius structure M on X is a Möbius structure if and only if for every nondegenerate* 5*-tuple $q \in X^5$ the following conditions* (A) *and* (B) *are satisfied*:

(A) $b(q_1) + b(q_4) = b(q_3) - a(q_1)$;
(B) $b(q_2) = -a(q_4) + b(q_1)$.

Remark 6.2. Conditions (A) and (B) are in fact equivalent to each other. This follows from the S_5-symmetry of the codifferential δM and was explained in detail in [Bu2].

6.2. Timed causal space and a sub-Möbius structure. We begin with the following remark.

Remark 6.3. For the monotone Möbius structures on the circle, Axiom (M) is equivalent to the fact that $a(q) < 0$ and $b(q) > 0$, where $M(q) = (a(q), b(q), c(q)) \in L_4$ whenever $q = (x, y, z, u) \in \operatorname{reg}\mathcal{P}_4$ is such that the pairs (x, u) and (y, z) separate each other. This follows from (7).

With every timed causal space $T = (\mathrm{aY}, \mathcal{H}, t)$ we associate a sub-Möbius structure M on $X = S^1$ as follows.

We fix an orientation of S^1. Then for any 4-tuple $q \in \operatorname{reg}\mathcal{P}_4$ we have a well-defined *cyclic order* $\operatorname{co}(q)$. Let $A \subset \mathcal{P}_4$ be the set $\{\pi q : \pi \in S_4\}$, $A = A(q)$, for a given $q \in \mathcal{P}_4$. Note that the cyclic order $\operatorname{co}(\pi q) = \operatorname{co}(q)$ is independent of $\pi \in S_4$, and we denote it by $\operatorname{co}(A)$.

Let $\operatorname{co}(A) = 1234$. With any pair $i, i+1$ of consecutive points in $\operatorname{co}(A)$, we associate the timelike line $h_{i,i+1}$ dual to the event $(i, i+1)$. Two other points of $\operatorname{co}(A)$ determine the events $a = (i+2)_{(i,i+1)}$, $a' = (i+3)_{(i,i+1)} \in h_{i,i+1}$, and with $(i, i+1)$ we associate the time $t_{i(i+1)} > 0$ between a and a'. In that way, every pair $(i, i+1)$ of consecutive points in $\operatorname{co}(A)$ is labeled by a positive time $t_{i(i+1)}$.

Adjacent pairs are labeled in general by distinct numbers $t_{i(i+1)}$, $t_{(i+1)(i+2)}$; however, by Axiom (t5), the opposite pairs are labeled by one and the same number, that is, $t_{i(i+1)} = t_{(i+2)(i+3)}$, where indices are taken modulo 4.

Assume that a 4-tuple $q = (x, y, z, u) \in \operatorname{reg}\mathcal{P}_4$ is obtained from $\operatorname{co}(A)$ by fixing the initial point and by the transposition of two last entries of $\operatorname{co}(A)$, that is, x is the chosen initial point and $\operatorname{co}(A) = xyuz$. Then we put

$$M(q) = (-t_{xy}, t_{yu}, t_{xy} - t_{yu}) \in L_4. \tag{9}$$

We denote by $B \subset A$ the subset consisting of all 4-tuples obtained from $\operatorname{co}(A)$ by fixing an initial entry and transposing two last entries of $\operatorname{co}(A)$. The set B consists of 4 elements, $|B| = 4$, and it is the orbit of a cyclic subgroup $\Gamma_\pi \subset S_4$ generated by the permutation $\pi = 2413 \in S_4$, that is, $\Gamma_\pi = \{\mathrm{id}, \pi, \pi^2, \pi^3\}$ and $B = \{\sigma q : \sigma \in \Gamma_\pi\}$ for any $q \in B$. For example, if $\operatorname{co}(A) = xyuz$, then $q = (x, y, z, u)$, $\pi q = (y, u, x, z)$, $\pi^2 q = (u, z, y, x)$, $\pi^3 q = (z, x, u, y) \in B$.

Lemma 6.4. *For any q, $q' = \sigma q \in B$ with $\sigma \in \Gamma_\pi$, we have*

$$M(q') = \operatorname{sgn}(\sigma)\phi(\sigma)M(q).$$

Proof. It suffices to prove this for the generator $\sigma = 2413$ of Γ_π. Assume without loss of generality that $q = (x, y, z, u)$ and, hence, $\operatorname{co}(A) = xyuz$. We also write $\operatorname{co}(A) = 1234$. Then $q' = \sigma q = (y, u, x, z)$. By definition, $M(q) = (-t_{12}, t_{23}, t_{12} - t_{23})$, $M(q') = (-t_{23}, t_{34}, t_{23} - t_{34})$. On the other hand, $\operatorname{sgn}(\sigma) = -1$ because σ is odd, and $\phi(\sigma) = 213$. Therefore,

$$\operatorname{sgn}(\sigma)\phi(\sigma)M(q) = -(t_{23}, -t_{12}, t_{12} - t_{23}) = (-t_{23}, t_{12}, t_{23} - t_{34}) = M(q'),$$

because $t_{12} = t_{34}$ by Axiom (t5). $\square$

Furthermore, for every $p \in A$ there is $\sigma \in S_4$ and $q \in B$ such that $p = \sigma q$. We put

$$M(p) = \mathrm{sgn}(\sigma)\phi(\sigma)M(q). \tag{10}$$

Proposition 6.5. *Formula* (10) *defines unambiguously a map* $M\colon \operatorname{reg}\mathcal{P}_4 \to L_4$, *which is a sub-Möbius structure on* X.

Proof. We show that for another representation $p = \sigma' q'$ with $\sigma' \in S_4$, $q' \in B$, formula (10) gives the same value $M(p)$. We have $\sigma' q' = \sigma q$, so that $q' = \rho q$ with $\sigma' \rho = \sigma$. Since q, $q' \in B$ and the group S_4 acts on A effectively, we have $\rho \in \Gamma_\pi$. Then

$$M(q') = \mathrm{sgn}(\rho)\phi(\rho)M(q)$$

by Lemma 6.4, and we obtain

$$\mathrm{sgn}(\sigma')\phi(\sigma')M(q') = \mathrm{sgn}(\sigma'\rho)\phi(\sigma'\rho)M(q) = M(p).$$

Thus identity (10) defines unambiguously a map $M\colon \operatorname{reg}\mathcal{P}_4 \to L_4$, which now satisfies (10) for any $p = \sigma q$ with $q \in \operatorname{reg}\mathcal{P}_4$, $\sigma \in S_4$. Hence, M is a sub-Möbius structure on X. □

Note that to define the sub-Möbius structure M we do not use Axiom (t6).

6.3. The sub-Möbius structure M is a Möbius one. Given a nondegenerate 5-tuple $q \in \operatorname{reg}\mathcal{P}_5$, we label its cyclic order by $\mathrm{co}(q) = 12345$. Assuming that the order of $q = xyzuv$ is cyclic, we have $\mathrm{co}(q_i) = \mathrm{co}(q)_i$ for $i = 1, \dots, 5$. For every $i \in \mathrm{co}(q)$, we consider three variables $t^i_{(i+1)(i+2)}$, $t^i_{(i+2)(i+3)}$, $t^i_{(i+3)(i+4)}$, associated with the 4-tuple $\mathrm{co}(q_i) = \mathrm{co}(q)_i$ as in Subsection 6.2, where indices are taken modulo 5. These 15 variables satisfy 10 equations

$$t^i_{(i+1)(i+2)} = t^i_{(i+3)(i+4)}, \tag{11}$$

$$t^i_{(i+2)(i+3)} = t^{i+1}_{(i+2)(i+3)} + t^{i+4}_{(i+2))i+3)}, \tag{12}$$

which follow from Axioms (t5) and (t4a), respectively. We compute $\delta M(q) = v$ for $q \in \operatorname{reg}\mathcal{P}_5$ with $\mathrm{co}(q) = 12345$ as follows. The time-labeling of $\mathrm{co}(q_i) = \mathrm{co}(q)_i$ is given by $t^i_{(i+1)(i+2)}$, $t^i_{(i+2)(i+3)}$, $t^i_{(i+3)(i+4)}$, $t^i_{(i+4)(i+6)}$. Thus, in accordance with our definition of the sub-Möbius structure M, we have

$$\big(-t^i_{(i+1)(i+2)}, t^i_{(i+2)(i+3)}, t^i_{(i+1)(i+2)} - t^i_{(i+2)(i+3)}\big) = M(\pi\sigma^{i-1}q_i),$$

where $\pi = 1243$, $\sigma = 4123$. Therefore,

$$M(\sigma^{i-1}q_i) = \mathrm{sgn}(\pi)\phi(\pi)M(\pi\sigma^{i-1}q_i) = \big(t^i_{(i+1)(i+2)}, t^i_{(i+2)(i+3)} - t^i_{(i+1)(i+2)}, -t^i_{(i+2)(i+3)}\big),$$

and we obtain

$$\delta M(q) = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \\ a_5 & b_5 & c_5 \end{pmatrix} = \begin{pmatrix} t^1_{23} & t^1_{34} - t^1_{23} & -t^1_{34} \\ t^2_{45} & t^2_{34} - t^2_{45} & -t^2_{34} \\ t^3_{12} & t^3_{15} - t^3_{12} & -t^3_{15} \\ t^4_{12} & t^4_{23} - t^4_{12} & -t^4_{23} \\ t^5_{12} & t^5_{23} - t^5_{12} & -t^5_{23} \end{pmatrix}.$$

Theorem 6.6. *The sub-Möbius structure M associated with any timed causal space* $(\mathrm{aY}, \mathcal{H}, t)$ *is Möbius.*

Proof. We show that M satisfies equations (A) and (B) of Theorem 6.1. It suffices to check that for every unordered 5-tuple $x, y, z, u, v \subset X$ of pairwise distinct points, equations (A) and (B) are satisfied for some ordering $q \in \operatorname{reg}\mathcal{P}_5$ of the 5-tuple, because in this case $\delta M(q)$ lies in an irreducible invariant subspace R of the corresponding representation of S_5, describing the Möbius structures, see [Bu2]. Hence, $\delta M(q) \in R$ for any ordering of the 5-tuple. Or, applying the procedure above, we may check (A) and (B) directly. Thus, we assume that $q = (x, y, z, u, v) \in \operatorname{reg}\mathcal{P}_5$ has the cyclic order $xyzuv$.

Equation (A) can be rewritten as

$$0 = b_1 + b_4 - b_3 + a_1 = -c_1 - b_3 + b_4 =: A(v)$$

because $a_1 + b_1 + c_1 = 0$, and using (11), (12) we get

$$\begin{aligned} A(v) = -c_1 - b_3 + b_4 &= t^1_{34} + t^3_{12} - t^3_{15} + t^4_{23} - t^4_{12} \\ &= t^1_{34} - t^3_{15} + t^4_{23} - t^5_{12} \\ &= t^5_{34} + t^2_{34} - t^3_{15} + t^4_{23} - t^5_{12} \\ &= t^2_{34} - t^3_{15} + t^4_{23} \\ &= t^2_{34} + t^4_{23} - t^2_{15} - t^4_{15} \\ &= t^2_{34} - t^2_{15} = 0. \end{aligned}$$

Similarly, equation (B) can be rewritten as $0 = b_1 - b_2 - a_4 =: B(v)$, and using (11), (12) we see that

$$\begin{aligned} B(v) = b_1 - b_2 - a_4 &= t^1_{34} - t^1_{23} - t^2_{34} + t^2_{45} - t^4_{12} \\ &= t^5_{34} - t^1_{23} + t^2_{45} - t^4_{12} \\ &= t^5_{34} - t^1_{23} + t^1_{45} + t^3_{45} - t^4_{12} \\ &= t^5_{34} + t^3_{45} - t^4_{12} \\ &= t^5_{34} + t^3_{45} - t^3_{12} - t^5_{12} = 0. \end{aligned}$$

Therefore, M is a Möbius structure by Theorem 6.1. □

Proposition 6.7. *The Möbius structure $M = \widehat{M}(T)$ associated with a timed causal space $T \in \mathcal{T}$ is monotone, $M \in \mathcal{M}$, and the timed causal space $\widehat{T}(M)$ associated with M coincides with T, $\widehat{T}(M) = T$.*

The proof proceeds in three steps, Lemmas 6.8 – 6.10.

Lemma 6.8. *The Möbius structure $M = \widehat{M}(T)$ satisfies Axiom* (M)*, and the time of the timed causal space $T = (\mathrm{aY}, \mathcal{H}, t)$ is computed in the usual way via M-cross-ratios.*

Proof. We check Axiom (M) and simultaneously compute the time $t(e, e')$ between events e, $e' \in \mathrm{aY}$, assuming without loss of generality that $e = (y, y')$, $e' = (u, u') \in h_a$ for $a = (x, z)$ such that the 4-tuple $q = (x, y, z, u) \in \operatorname{reg}\mathcal{P}_4$ is obtained from $\operatorname{co}(q) = xyuz$ by fixing the initial point x and by transposing the last two entries of $\operatorname{co}(q)$. Note that the pairs (x, u) and (y, z) separate each other. Then, by definition, $M(q) = (a(q), b(q), c(q)) = (-t_{xy}, t_{yu}, t_{xy} - t_{yu})$ with the negative first entry $a(q) = -t_{xy}$ and the positive second entry $b(q) = t_{yu}$. By Theorem 6.6, we have $M(q) = M_d(q)$ for any semimetric $d \in M$. Thus,

$$\operatorname{cr}_1(q) = e^{a(q)} = \frac{d(x, z)d(y, u)}{d(x, u)d(y, z)} < 1, \quad \operatorname{cr}_2(q) = e^{b(q)} = \frac{d(x, u)d(y, z)}{d(x, y)d(z, u)} > 1.$$

This shows that M satisfies Axiom (M), see Remark 6.3, and that $t(e, e') = t_{xz} = t_{yu} = \ln \operatorname{cr}_2(q)$. □

Lemma 6.9. *Let $h = h_e$ be a timelike line in a timed causal space T. An event d belongs to h_e if and only if the 4-tuple (d, e) is M-harmonic, that is, harmonic with respect to the Möbius structure $M = \widehat{M}(T)$.*

Proof. Let $e = (x, y)$ and $d = (z, u)$.

If $d \in h_e$, then by Axiom (h2), d separates e, and by Axiom (t6), we have $t(y_a, u_a) = t(y_b, u_b)$, where $a = (x, z)$, $b = (x, u)$. Thus, we may assume without loss of generality that $\mathrm{co}(q) = xzyu$ for a nondegenerate 4-tuple $q = (e, d)$. Note that $t(y_a, u_a) = t_{xz}$ and $t(y_b, z_b) = t_{xu}$. Therefore, $t_{xz} = t_{xu}$. The 4-tuple $\widetilde{q} = (u, x, y, z)$ is obtained from $\mathrm{co}(q) = uxzy$ by fixing the first entry u and permuting the last two entries z, y. Therefore, by definition, $M(\widetilde{q}) = (-t_{xu}, t_{xz}, 0)$. Since $M(\widetilde{q}) = M_d(\widetilde{q})$ for any semimetric $d \in M$, we obtain

$$1 = \mathrm{cr}_3(\widetilde{q}) = \frac{d(x, u) \cdot d(y, z)}{d(y, u) \cdot d(x, z)}.$$

Hence (d, e) is M-harmonic.

Conversely, if (d, e) is M-harmonic, then, by Lemma 5.5, d and e separate each other. By Axiom (h3), there is a unique $u' \in X \setminus e$ such that $d' = (z, u') \in h_e$. By the first part of the proof, the 4-tuple (d', e) is M-harmonic. Taking a semimetric $d \in M$ with the infinitely remote point z, we observe that

$$d(x, u) = d(u, y) \quad \text{and} \quad d(x, u') = d(u', y), \tag{13}$$

because the 4-tuples (d, e), (d', e) are M-harmonic. Suppose $u \neq u'$. Then the 4-tuple (x, y, u, u') is nondegenerate, and u, u' are on the arc determined by e that does not contain z. Without loss of generality, we assume that (x, u) separates (y, u'). By Lemma 6.8, M satisfies Axiom (M). Thus,

$$d(x, u) \cdot (y, u') > d(x, u') \cdot d(y, u)$$

in contradiction with (13). Hence $u = u'$ and $d = d' \in h_e$. □

Lemma 6.10. *The set $\mathcal{A}$ of open arcs in X coincides with the set $\mathcal{B}$ of open balls with respect to the semimetrics $d \in M = \widehat{M}(T)$ with infinitely remote point that are centered at finite points of d, $\mathcal{A} = \mathcal{B}$.*

Proof. Let $\alpha \in \mathcal{A}$ be an open arc in X and let x, $y \in X$ be the endpoints of α. We put $e = (x, y) \in \mathrm{aY}$ and take $z \in \alpha$. Then for $u = \rho_e(z)$ the event $d = (z, u)$ lies on the timelike line h_e. By Lemma 6.9, the 4-tuple (d, e) is M-harmonic. Thus, z is the midpoint between x, y with respect to any semimetric $d \in M$ with infinitely remote point u. By Axiom (M), $v \in \alpha$ if and only if $d(z, v) < r := d(x, z) = d(y, z)$. Therefore, α coincides with the open ball $B_r(z)$ with respect to d of radius r centered at z. This means that $\mathcal{A} \subset \mathcal{B}$.

Let $\beta = B_r(o) \in \mathcal{B}$ be the open ball with respect to a semimetric $\delta \in M$ with the infinitely remote point ω, of radius $r > 0$ and centered at $o \in X_\omega$. We show that $\beta \in \mathcal{A}$.

Let $d = (o, \omega) \in \mathrm{aY}$. By (h4), for any $y \in X_\omega$, $y \neq o$, there is a unique event $e = y_d = (y, y') \in h_d$. We fix such y, denote by $e^+ \subset X$ the closed arc determined by e that contains ω, and consider the corresponding linear order $<=<_e$ on h_d with the future arc e^+.

First, we show that for $r = \delta(y, o)$ the open ball $B_r(o)$ coincides with the open arc $\mathrm{int}\, e^- \in \mathcal{A}$ determined by e that contains o. We denote by $d^+ \subset X$ the closed arc determined by d that contains y, and by d^- the opposite closed arc. Then $\mathrm{int}\, e^- = (d^+ \cap \mathrm{int}\, e^-) \cup (d^- \cap \mathrm{int}\, e^-)$.

We have $u \in d^+ \cap \mathrm{int}\, e^-$ if and only if the pairs (y, o), (u, ω) separate each other. By Axiom (M), this is equivalent to $\delta(u, o) < \delta(y, o) = r$. On the other hand, $u \in d^- \cap \mathrm{int}\, e^-$ if and only if $u' = \rho_d(u) \in d^+ \cap \mathrm{int}\, e^-$. By the above, this is equivalent to $\delta(u', o) < r$. By Lemma 6.9, the 4-tuple (d, u_d) is harmonic, where $u_d = (u, u')$. Thus, $\delta(u, o) = \delta(u', o) < r$. Therefore, $\mathrm{int}\, e^- = B_r(o)$ for $r = \delta(y, o)$.

It remains to show that for any $r > 0$ there is $y \in X_\omega$ with $\delta(y,o) = r$. We fix some $y \in X_\omega$, $y \neq o$, and use the notation introduced above. By (t4b), for any $s > 0$ there is $e_\pm = (u_\pm, u'_\pm) \in h_d \cap C_e^\pm$ with $t(e, e_\pm) = s$. By Lemma 6.9, the 4-tuples (d,e), $(d,e_\pm)$ are M-harmonic. Hence, $\delta(o,y) = \delta(o,y')$, $\delta(o,u_\pm) = \delta(o,u'_\pm)$. As above, Axiom (M) implies

$$\delta(u_-,o) < \delta(y,o) < \delta(u_+,o).$$

By Lemma 6.8, the time $t(e,e_\pm)$ is computed via M-cross-ratios,

$$t(e,e_\pm) = \left|\ln \frac{\delta(\omega,y)\delta(u_\pm,o)}{\delta(\omega,u_\pm)\delta(y,o)}\right| = \left|\ln \frac{\delta(u_\pm,o)}{\delta(y,o)}\right|,$$

whence $s = \pm \ln \frac{\delta(u_\pm,o)}{\delta(y,o)}$. This shows that for any $\lambda > 0$ there is $u \in X_\omega$, $u \neq o$, with $\delta(u,o) = \lambda\delta(y,o)$. Hence, for any $r > 0$ there is $y \in X_\omega$ with $\delta(y,o) = r$. □

Proof of Proposition 6.7. By Lemma 6.8, the Möbius structure $M = \widehat{M}(T)$ satisfies Axiom (M) for any timed causal space $T = (\mathrm{aY}, \mathcal{H}, t) \in \mathcal{T}$. Lemma 6.10 shows that M satisfies Axiom (T). Thus, M is monotone, $M \in \mathcal{M}$.

Let $T' = (\mathrm{aY}, \mathcal{H}', t') = \widehat{T}(M) \in \mathcal{T}$ be the timed causal space determined by M. By Lemma 6.9 we have $\mathcal{H}' = \mathcal{H}$, and by Lemma 6.8, $t' = t$. Thus, $T' = T$. □

Proof of Theorem 1.1. Given $M \in \mathcal{M}$, we show that $M' = M$, where $M' = \widehat{M} \circ \widehat{T}(M)$, that is, $M'(q) = M(q)$ for every $q \in \operatorname{reg} \mathcal{P}_4$. Using (8), we may assume without loss of generality that the cyclic order of $q = (x,y,z,u)$ is $\mathrm{co}(q) = xyuz$, so that q is obtained from $\mathrm{co}(q)$ by choosing the first entry x and permuting the last two entries. In particular, (x,u) and (y,z) separate each other. Then, by the definition (9), we have

$$M'(q) = (-t_{xy}, t_{yu}, t_{xy} - t_{yu}),$$

where $t_{xy} = t(z_a, u_a)$, $t_{yu} = t(x_b, z_b)$ for $a = (x,y)$, $b = (y,u) \in \mathrm{aY}$, see Axiom (h4), for $T = \widehat{T}(M) = (\mathrm{aY}, \mathcal{H}, t) \in \mathcal{T}$. By the definition (5) of the time t, we have $t(z_a,u_a) = \left|\ln \frac{|xz|\cdot|yu|}{|xu|\cdot|yz|}\right| = -\ln \mathrm{cr}_1(q)$, $t(x_b,z_b) = \left|\ln \frac{|xu|\cdot|yz|}{|xy|\cdot|uz|}\right| = \ln \mathrm{cr}_2(q)$ (to choose the signs, we have used the fact that (x,u), (y,z) separate each other and the monotonicity of M). Therefore, $M'(q) = M(q)$. Together with Proposition 6.7, this shows that $\widehat{T}\colon \mathcal{M} \to \mathcal{T}$ and $\widehat{M}\colon \mathcal{T} \to \mathcal{M}$ are mutually inverse maps.

Let $\widehat{g}\colon \mathrm{aY} \to \mathrm{aY}$ be an automorphism of some $T = (\mathrm{aY}, \mathcal{H}, t) \in \mathcal{T}$. Since $t(e,e') = 0$ if and only if the events e, $e' \in \mathrm{aY}$ lie on a light line, and $\widehat{g}$ preserves the time t, we see that $\widehat{g}$ maps every light line to a light line. Thus, $\widehat{g}$ determines a map $g\colon X \to X$ with $\widehat{g}(p_x) = p_{g(x)}$, see Subsection 2.2. For any event $e = (x,y) \in \mathrm{aY}$ we have $e = p_x \cap p_y$. Therefore, $\widehat{g}(e) = \widehat{g}(p_x) \cap \widehat{g}(p_y) = p_{g(x)} \cap p_{g(y)} = (g(x), g(y))$. Hence, $\widehat{g}$ is induced by g.

Since $T = \widehat{T}(M)$ for some $M \in \mathcal{M}$, the timelike lines and the time of T are determined by cross-ratios of M, see Proposition 5.7. Therefore, g is an M-automorphism. If $g = \mathrm{id}_X$, then $\widehat{g} = \mathrm{id}_{\mathrm{aY}}$. Thus, the group G_T of T-automorphisms injects into the group G_M of M-automorphisms. Together with Proposition 5.7, this shows that the groups G_M and G_T are canonically isomorphic. □

§7. Time inequalities

The *time inequality* for the de Sitter 2-space dS^2 says that

$$t(a,b) + t(b,c) \le t(a,c)$$

for any events $a < b < c$, with equality in the case where $t(a,c) > 0$ if and only if a, b, c are events on a timelike line. First, in Subsection 7.1 we show that this inequality follows from the properties of Lambert quadrilaterals. Then in Subsection 7.2, we discuss a hierarchy of time conditions, which includes the time inequality, and show that every timed causal space $T \in \mathcal{T}$ satisfies the *weak time inequality,* see Theorem 7.3. In Subsection 7.5 we describe the monotone Möbius structures that satisfy the *variational principle*, (VP), the strongest time condition on the list, and in Subsection 7.7 we also describe the *convex* Möbius structures. We show that these two classes include the canonical Möbius structure, and that the first of them contains a neighborhood of the canonical structure in a fine topology.

7.1. The time inequality for dS^2 via H^2. The time inequality for de Sitter 2-space dS^2 follows from the properties of Lambert quadrilaterals in H^2. This goes of course via the canonical Möbius structure M_0 on the common absolute S^1. More precisely, we use the fact that the harmonicity of a 4-tuple $((x,y),(z,u)) \subset S^1$ with respect to M_0 is equivalent to the orthogonality of the geodesics xy, $zu \subset \mathrm{H}^2$.

Recall that a *Lambert quadrilateral* $\alpha\beta\gamma o$ in the hyperbolic plane H^2 has three right angles at α, β, and γ. The fourth angle at o is acute, and $|\alpha\beta| < |o\gamma|$, $|\beta\gamma| < |\alpha o|$. Now, we explain how these properties imply the time inequality for dS^2.

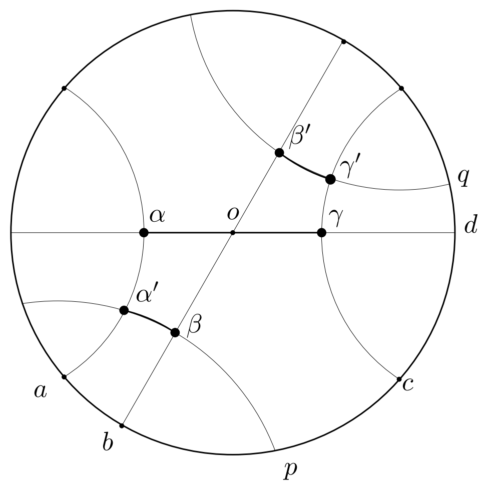

FIGURE 1. The time inequality in dS^2.

Let a, b, c be events in aY such that $a < b < c$ for the order $< := <_b$. We consider a generic case when no pair of events (a,b), (b,c) lies on a light line, and the events are not on a common timelike line. Then there are events d, p, $q \in$ aY with $a, b \in h_p$, $a, c \in h_d$, $b, c \in h_q$. We pass to the H^2-picture, and draw the respective timelike lines as geodesics in H^2 with the same ends on the absolute S^1. Since the time in dS^2 and the distance in H^2 are computed via cross-ratios with respect to M_0, we have $t(a,c) = |\alpha\gamma|$, $t(a,b) = |\alpha'\beta|$, $t(b,c) = |\beta'\gamma'|$, see Figure 1.

But $|\alpha\gamma| = |\alpha o| + |o\gamma|$, and the quadrilaterals $\alpha\alpha'\beta o$ and $\gamma\gamma'\beta' o$ have right angles at α, α', β and, respectively, at γ, γ', β', i.e., they are Lambert quadrilaterals. Thus, $|\alpha o| > |\alpha'\beta|$, $|o\gamma| > |\beta'\gamma'|$, and we obtain

$$t(a,c) > t(a,b) + t(b,c).$$

7.2. Hierarchy of time conditions. We assume that a timed causal space $T = \{\mathrm{aY}, \mathcal{H}, t\} \in \mathcal{T}$ is fixed together with the respective monotone Möbius structure $M = \widehat{M}(T) \in \mathcal{M}$.

We say that an event $b \in \mathrm{aY}$ is *strictly between* events a and $c \in \mathrm{aY}$ if a and c lie on different open arcs in X determined by b. Note that in this case, a, b, c are pairwise in strong causal relation, in particular, $a < b < c$ for appropriately chosen $<:=<_b$.

Let $a = (o, o')$, $b = (\omega, \omega') \in \mathrm{aY}$ be events in strong causal relation such that the pairs (o, ω') and (o', ω) separate each other. Then $(o, \omega), (o', \omega')$ are also in strong causal relation. Let $d = (x, x') \in \mathrm{aY}$ be an event strictly between (o, ω) and (o', ω'). We denote

$$t_d^+(a, b) = t(o_d, \omega_d), \quad t_d^-(a, b) = t(o'_d, \omega'_d).$$

In general, $t_d^+(a, b) \neq t_d^-(a, b)$. However, if $a, b \in h_d$, then $t_d^+(a, b) = t_d^-(a, b) = t(a, b)$ by the definition of $t(a, b)$, see (5), (6), and Lemma 5.10.

We consider the function

$$F_{ab}(d) = \frac{1}{2}(t_d^+(a, b) + t_d^-(a, b))$$

on the set D_{ab} of events $d \in \mathrm{aY}$ that are strictly between (o, ω) and (o', ω'), and introduce the following list of time conditions for T and therefore simultaneously for M.

(VP) Variational principle: the infimum of F_{ab} is attained at a unique $d_0 \in D_{ab}$ for which $a, b \in h_{d_0}$.

(LQI) Lambert quadrilateral inequality:

$$F_{ab}(d) > F_{ab}(d_0)$$

for every $d \in D_{ab} \setminus d_0$ such that $a \in h_d$.

(TI) Time inequality:

$$t(a, b) + t(b, c) \leq t(a, c)$$

for any $a < b < c$ with equality in the case where $t(a, c) > 0$ if and only if a, b, c are events on a timelike line.

(WTI) Weak time inequality:

$$t(a, b) + t(b, c) < t(a, c)$$

for any $a < b < c$ such that b lies on a light line either with a or with c, and a, c are not on a light line.

We have the following implications

$$(\mathrm{VP}) \Rightarrow (\mathrm{LQI}) \Rightarrow (\mathrm{TI}) \Rightarrow (\mathrm{WTI}).$$

The first and the last implications are obvious, and the second implication is explained in Proposition 7.6. For the canonical Möbius structure M_0, the geometric meaning of the function $F_{ab} \colon D_{ab} \to \mathbb{R}$ is especially clear.

Proposition 7.1. *Let $a = (o, o')$, $b = (\omega, \omega') \in \mathrm{aY}$ be events in the strong causal relation such that the pairs (o, ω') and (o', ω) separate each other, $d = (x, x') \in D_{ab}$. Then for the canonical Möbius structure M_0, the value $F_{ab}(d)$ is the distance in H^2 between the points $p = oo' \cap xx'$ and $q = \omega\omega' \cap xx'$ at which the geodesic $xx' \subset \mathrm{H}^2$ intersects the geodesics oo' and $\omega\omega'$.*

Proof. Since the time between events in a timed causal space and the distance in H^2 are computed via the respective cross-ratios, we see that $t_d^+(a, b) = t(o_d, \omega_d)$ is the distance in H^2 between the projections $\widehat{o}$, $\widehat{\omega}$ of o, ω to the geodesic $xx' \subset \mathrm{H}^2$, and similarly, $t_d^-(a, b) = t(o'_d, \omega'_d)$ is the distance between the projections $\widehat{o}'$, $\widehat{\omega}'$ of o', ω' to the same geodesic xx'. By the angle parallelism formula, $\widehat{o}p = p\widehat{o}'$ and $\widehat{\omega}q = q\widehat{\omega}'$. Therefore, $F_{ab}(d) = \frac{1}{2}(|\widehat{o}\widehat{\omega}| + |\widehat{o}'\widehat{\omega}'|) = |pq|$. □

Corollary 7.2. *The canonical Möbius structure satisfies* (VP).

Proof. This immediately follows from the properties of the distance in H^2 between points on geodesics. $\square$

7.3. The weak time inequality. Here, we prove the following claim.

Theorem 7.3. *Any timed causal space $T = \{\mathrm{aY}, \mathcal{H}, t\} \in \mathcal{T}$ satisfies* (WTI).

Lemma 7.4. *Assume that distinct events a, b lie on a common timelike line, $a, b \in h_c$, where $a = (x, y)$, $b = (z, u)$, $c = (v, w) \in \mathrm{aY}$, and suppose that v lies on the open arc γ between x, y that does not contain b. Then for every $s \in \gamma$, $s \neq v$, and for $d = (s, t) \in h_a$, $d' = (s, t') \in h_b$, the following is true: t' lies on the open arc σ between w, t that does not contain s.*

Proof. Moving s along γ, observe that for $s = v$ we have $t = t'$, while for s approaching x or y the point t is not on the arc between z, u that contains w. Therefore, t' lies on σ for these extremal cases. By the continuity of reflections ρ_a, $\rho_b \colon X \to X$ and Lemma 5.10, we have $t' \in \sigma$ for every $s \in \gamma$. $\square$

Lemma 7.5. *Let $a = (o, o')$ and $b = (\omega, \omega') \in \mathrm{aY}$ be events in the strong causal relation such that the pairs (o, ω') and (o', ω) separate each other. Then the function $F^+_{ab}(d) = t(o_d, \omega_d)$ is monotone on the set D_{ab} of events $d \in \mathrm{aY}$ that are strictly between $e = (o, \omega)$ and $e' = (o', \omega')$, $F^+_{ab}(d) < F^+_{ab}(d')$ for any d, $d' \in D_{ab}$ with $d < d' < e$.*

Proof. Let $d = (x, y)$. By Axiom (t5) we have $t(o_d, \omega_d) = t(x_e, y_e)$. Therefore, $F^+_{ab}(d) = t(x_e, y_e)$. For $d' = (x', y')$ between d and e, the segment $x_e y_e \subset h_e$ is contained in the segment $x'_e y'_e \subset h_e$ and does not coincide with it (though, we do not exclude that these segments may have a common end). Thus, $t(x_e, y_e) < t(x'_e, y'_e)$. $\square$

Proof of Theorem 7.3. Let $a, b, c \in \mathrm{aY}$ be events in the causal relation, $a < b < c$, and assume without loss of generality that b, c are on a light line. Then $t(b, c) = 0$, and the required inequality reduces to $t(a, b) < t(a, c)$. Furthermore, there is no loss of generality in assuming that $a = (o, o')$, $b = (\omega, \omega')$ and the pairs (o, ω'), (o', ω) separate each other. Since b, c are on a light line, we may assume that $c = (\omega, \omega'')$. Then the assumption $a < b < c$ implies that ω'' is on the (open) arc α between ω, ω' that does not contain a.

There is $d = (x, y) \in \mathrm{aY}$ with $a, b \in h_d$. We assume that x is on the arc β between o, o' that does not contain b. Then $y \in \alpha$. Similarly, there is $d' = (x', y') \in \mathrm{aY}$ with $a, c \in h_{d'}$. We also assume that x' is on the arc β. Then y' is on the arc $\alpha' \subset \alpha$ between ω, ω''. Note that d, $d' \in D_{ab}$ and that $d \neq d'$ because $b \neq c$, whence $x' \neq x$ because d, $d' \in h_a$.

We claim that x' lies on the arc β between x and o. Indeed, otherwise, since d, $d' \in h_a$, substituting o for v, o' for w, ω for s, ω' for t, and ω'' for t' and using Lemma 7.4, we would see that $\omega'' \notin \alpha$ in contradiction with the previously established fact that $\omega'' \in \alpha$.

It follows that $d < d' < e = (o, \omega)$. By Lemma 7.5, $F^+_{ab}(d) < F^+_{ab}(d')$. On the other hand, $F^+_{ab}(d) = t(o_d, \omega_d) = t(a, b)$ and $F^+_{ab}(d') = t(o_{d'}, \omega_{d'}) = t(a, c)$. $\square$

7.4. The implication (LQI)$\Longrightarrow$(TI). Here, we show that the Lambert quadrilateral inequality implies the time inequality.

Proposition 7.6. (LQI) $\Rightarrow$ (TI).

Assume that $b \in \mathrm{aY}$ is strictly between $a, c \in \mathrm{aY}$. Then by Corollary 5.9, there are common perpendiculars p to a, b and q to b, c.

Lemma 7.7. *Suppose that a, $c \in h_d$ and $b \in \mathrm{aY} \setminus h_d$ is strictly between a and c. Then d is strictly between the common perpendiculars p to a, b and q to b, c.*

Proof. By assumption, b is not on the timelike line $h_d \subset \mathrm{aY}$. Hence, the common perpendicular $p = (p', p'') \in \mathrm{aY}$ to a, b is not equal to d, $p \neq d$, and the common perpendicular $q = (q', q'') \in \mathrm{aY}$ to b, c is not equal to d, $q \neq d$.

Since p, $d \in h_a$, the events p, d are not on a light line, and some closed arc in X determined by d does not include p. We denote that arc by $d^+ \subset X$. Hence, $p <_d d$ for the respective partial order $<_d$.

We also denote by $b^+ \subset X$ the closed arc determined by b that includes c. Without loss of generality, we assume that p'', w, $q' \in b^+$, where $d = (v, w)$. Then by Lemma 7.4 applied to a, $b \in h_p$ and $\gamma = b^+$, we see that v lies on the arc σ determined by (p', t) that does not contain w, where $r = (t, w)$ is orthogonal to b. Therefore, $d <_d r$ and $t \in d^+$.

We denote by r_d^+ the closed arc in X determined by r that does not include d, see Subsection 3.1. Since d is also orthogonal to c, applying Lemma 7.4 to b, $c \in h_q$, we see that t lies on the arc σ' determined by (v, q'') that does not contain $q' \in d^+$. Since r, $q \in h_b$, this means that $q \subset r_d^+$, whence $r <_d q$.

Therefore, $p <_d d <_d r <_d q$. Since by construction, p, r, $q \in h_b$ and p, d are not on a light line, we conclude that d is strictly between p and q. $\square$

Corollary 7.8. *Suppose a, $c \in h_d$ are as in Lemma* 7.7. *If p, $q \in \mathrm{aY}$ with $p \perp a$, $q \perp c$ are such that d is strictly between p and q, then the common perpendicular to p, q is strictly between a and c.*

Proof. Since d is strictly between p and q, the events p, q are in strong causal relation. Therefore, their common perpendicular $b \in \mathrm{aY}$ exists and is uniquely determined by Corollary 5.9 and Lemma 5.10. Since a is the common perpendicular to d, p and c is the common perpendicular to d, q, Lemma 7.7 implies that b is strictly between a and c. $\square$

Proof of Proposition 7.6. Assume that $a < b < c$ for events in aY. If $t(a, c) = 0$, then by Axiom (t2), a, c are on a light line, $a, c \in p_x$ for some $x \in X$. Then $b \in p_x$, and we have $t(a, b) = t(b, c) = t(a, c) = 0$.

Therefore, we may assume that $t(a, c) > 0$, and, hence, $a, c \in h_d$ for some timelike line $h_d \subset \mathrm{aY}$. Using Theorem 7.3, we may also assume that b does not lie on a light line either with a or with b. If b is also on h_d, then by Axiom (t4a), $t(a, b) + t(b, c) = t(a, c)$. To complete the proof, we show that the assumption $b \notin h_d$ implies the strict inequality in the time inequality. In this case, b is strictly between a, c by our assumption, and there are p, $q \in \mathrm{aY}$ with $a, b \in h_p$, $b, c \in h_q$. By Lemma 7.7, d is strictly between p and q.

Since $a \in h_p$ and p, $d \in D_{ab}$, (LQI) applied to a, b gives $F_{ab}(d) > F_{ab}(p)$. Since $c \in h_q$ and q, $d \in D_{bc}$, (LQI) applied to b, c gives $F_{bc}(d) > F_{bc}(q)$. On the other hand, $F_{ab}(p) = t(a, b)$, $F_{bc}(q) = t(b, c)$, and it remains to show that $F_{ab}(d) + F_{bc}(d) = t(a, c)$.

We fix the decomposition $X = d^+ \cup d^-$, $d^+ \cap d^- = d$, induced by d, and write $a = (a^+, a^-)$, $b = (b^+, b^-)$, $c = (c^+, c^-)$, where $a^\pm, b^\pm, c^\pm \in d^\pm$. By (t4a), we have $t(a_d^\pm, b_d^\pm) + t(b_d^\pm, c_d^\pm) = t(a_d^\pm, c_d^\pm)$. Therefore, $F_{ab}(d) + F_{bc}(d) = \frac{1}{2}(t(a_d^+, c_d^+) + t(a_d^-, c_d^-)) = t(a, c)$ because $a, c \in h_d$. $\square$

Corollary 7.9. *The variational principle implies the time inequality,* (VP)$\Longrightarrow$(TI), *cf.* [PY].

7.5. Monotone Möbius structures with (VP). Some important properties of Möbius structures $\mathcal{M}$ that do not follow from the monotonicity Axiom (M) can be expressed as an inequality $\mathrm{cr}(q) > \mathrm{cr}(q')$ between cross-ratios of 4-tuples q, q' with two common entries, $|q \cap q'| = 2$, under the assumption that a symmetry between q, q' is broken down in a certain way.

We use the notation $\mathrm{reg}\,\mathcal{P}_n$ for the set of ordered nondegenerate n-tuples of points in $X = S^1$, $n \in \mathbb{N}$. For $q \in \mathrm{reg}\,\mathcal{P}_n$ and a proper subset $I \subset \{1, \dots, n\}$, we denote by

$q_I \in \operatorname{reg}\mathcal{P}_k$, $k = n-|I|$, the k-tuple obtained from q (with the induced order) by crossing out all entries that correspond to the elements of I.

We introduce the following axiom for a Möbius structure $M \in \mathcal{M}$, which implies the variational principle (VP).

(I) Increment: for any $q \in \operatorname{reg}\mathcal{P}_7$ with cyclic order $\operatorname{co}(q) = 1234567$ such that q_{247} and q_{157} are harmonic, we have

$$\operatorname{cr}_1(q_{345}) > \operatorname{cr}_1(q_{123}).$$

This means the following. Assume we are given two events $e = (o, \omega)$, $e' = (o', \omega') \in \mathrm{aY}$ in strong causal relation such that (o, ω') and (o', ω) separate each other. Let $oo' \subset X$ be the arc between o, o' that does not contain ω, ω', and let $(u, v) \in \mathrm{aY}$, $u \in oo'$, be the common perpendicular to $a = (o, o')$, $b = (\omega, \omega')$, i.e., $(u, v) \in h_a \cap h_b$. Given $x \in oo'$ such that (o, u) and (o', x) separate each other, we put $g_+(u, x) = \exp t_e(u_e, x_e)$, $g_-(u, x) = \exp(-t_{e'}(u_{e'}, x_{e'}))$, $\Delta(u, x) = g_+(u, x)g_-(u, x)$. Then Axiom (I) says that $\Delta(u, x) > 1$.

Indeed, consider $q = (o, \omega, v, \omega', o', u, x) \in \operatorname{reg}\mathcal{P}_7$ written in the cyclic order $\operatorname{co}(q) = 1234567$. The assumption that the 4-tuples q_{247} and q_{157} are harmonic means that the 4-tuples (u, o, v, o') and (u, ω, v, ω') are harmonic with the common axis (u, v), i.e., $(u, v) \in h_a \cap h_b$. Since $q_{345} = (o, \omega, u, x)$ and $q_{123} = (\omega', o', u, x)$, we have $g_+(u, x) = \operatorname{cr}_1(q_{345})$, $g_-(u, x) = 1/\operatorname{cr}_1(q_{123})$. Thus, the condition $\operatorname{cr}_1(q_{345}) > \operatorname{cr}_1(q_{123})$ means that $\Delta(u, x) > 1$.

Proposition 7.10. *The canonical Möbius structure M_0 on X satisfies Axiom* (I).

Proof. Let $q = (o, \omega, v, \omega', o', u, x) \in \operatorname{reg}\mathcal{P}_7$ be as above. In the metric on X from M_0 with infinitely remote point u, we have $|vo| = |vo'|$, $|v\omega| = |v\omega'|$. Since M_0 is canonical, $|vo| = |v\omega| + |o\omega|$, so that $|o\omega| = |o'\omega'|$. Furthermore, $\operatorname{cr}_1(q_{345}) = \operatorname{cr}_1(o, \omega, u, x) = |x\omega|/|ox|$ and $\operatorname{cr}_1(q_{123}) = \operatorname{cr}_1(\omega', o', u, x) = |xo'|/|x\omega'|$.

Note that $xo \subset x\omega' \subset X_u$. Therefore, $|xo| < |x\omega'|$. Using $|x\omega| = |xo| + |o\omega|$ and $|xo'| = |x\omega'| + |o'\omega'| = |x\omega'| + |o\omega|$, we obtain $|x\omega|/|ox| > |xo'|/|x\omega'|$. Hence, $\operatorname{cr}_1(q_{345}) > \operatorname{cr}_1(q_{123})$, and M_0 satisfies (I). □

Proposition 7.11. *The increment Axiom* (I) *implies the Variational Principle* (VP).

Proof. Let $a = (o, o')$, $b = (\omega, \omega') \in \mathrm{aY}$ be events in strong causal relation such that the pairs (o, ω') and (o', ω) separate each other. Then the events $e = (o, \omega)$, $e' = (o', \omega')$ are also in strong causal relation.

Let $d_0 = (u, v) \in D_{ab}$ be a unique event with a, $b \in h_{d_0}$. We show that $F_{ab}(d) > F_{ab}(d_0)$ for any $d = (x, x') \in D_{ab}$, $d \neq d_0$. Let $oo' \subset X$ be the arc between o, o' that does not include b. Without loss of generality we may assume that $u, x \in oo'$ and $x \neq u$. It suffices to show that $F_{ab}(d) > F_{ab}(d')$ for $d' = (u, x')$.

Let $\sigma \subset h_e$ be the segment between u_e, $x'_e \in h_e$, and let $\sigma' \subset h_{e'}$ be the segment between $u_{e'}$, $x'_{e'} \in h_{e'}$. Since $x \neq u$, one of the events $x_e \in h_e$, $x_{e'} \in h_{e'}$ lies in the respective segment σ, σ', while the other does not. We assume without loss of generality that $x_{e'} \in \sigma'$. Then $x_e \notin \sigma$, and moreover u_e separates the events x_e and x'_e on the timelike line h_e. Therefore, $t(x_e, x'_e) > t(u_e, x'_e)$, while $t(x_{e'}, x'_{e'}) < t(u_{e'}, x'_{e'})$. By Axiom (I), $t(x_e, u_e) > t(x_{e'}, u_{e'})$, and, consequently $t(x_e, x'_e) - t(u_e, x'_e) > t(u_{e'}, x'_{e'}) - t(x_{e'}, x'_{e'})$.

Recall that

$$F_{ab}(d) = \frac{1}{2}(t_d^+(a, b) + t_d^-(a, b)),$$

where $t_d^+(a,b) = t(o_d, \omega_d)$, $t_d^-(a,b) = t(o'_d, \omega'_d)$. By (t5) we have $t(o_d, \omega_d) = t(x_e, x'_e)$, $t(o'_d, \omega'_d) = t(x_{e'}, x'_{e'})$. Hence,

$$F_{ab}(d) - F_{ab}(d') = \frac{1}{2}(t(x_e, x'_e) - t(u_e, x'_e) + t(x_{e'}, x'_{e'}) - t(u_{e'}, x'_{e'})) > 0,$$

which completes the proof. □

Using Corollary 7.9, we immediately obtain the following statement.

Corollary 7.12. *The Increment Axiom* (I) *implies the time inequality* (TI).

7.6. The fine topology and axiom (I). We denote by $\mathcal{I}$ the class of monotone Möbius structures on the circle that satisfy Axiom (I). This paper does not provide any tools to answer natural questions like to characterize the hyperbolic spaces Y with $\partial_\infty Y = S^1$ for which the respective Möbius structure is in the class $\mathcal{I}$. Here we only show that a neighborhood of the canonical Möbius structure M_0 on $X = S^1$ in an appropriate topology lies in $\mathcal{I}$.

Recall that a Möbius structure M on a set X determines the M-topology on X (see Subsection 4.1) and hence the induced topology on the set $\operatorname{reg}\mathcal{P}_n(X) \subset X^n$. A Möbius structure can be viewed as a map defined on $\operatorname{reg}\mathcal{P}_4$ with values in a vector space (see Subsection 6.1). Thus, it not clear how to define a topology on the set of Möbius structures on X, because the topology of X may change together when a Möbius structure changes.

However, for monotone Möbius structures on $X = S^1$ such a problem does not exist in view of Axiom (T): on X all Möbius structures $M \in \mathcal{M}$ induce one and the same topology of the circle. We define a fine topology on $\mathcal{M}$ as follows.

Let $\operatorname{reg}^+\mathcal{P}_7 \subset X^7$ be the subset of $\operatorname{reg}\mathcal{P}_7$ that consists of all $q \in \operatorname{reg}\mathcal{P}_7$ with the cyclic order. That is, for $q \in \operatorname{reg}^+\mathcal{P}_7$ we have $\operatorname{co}(q) = q$. On $\operatorname{reg}^+\mathcal{P}_7$ we take the topology induced from the standard topology of the 7-torus X^7. With a Möbius structure $M \in \mathcal{M}$ we associate a section of the trivial bundle $\operatorname{reg}^+\mathcal{P}_7 \times \mathbb{R}^4 \to \operatorname{reg}^+\mathcal{P}_7$ given by

$$M(q) = \big(q, \operatorname{cr}_2(q_{247}), \operatorname{cr}_2(q_{157}), \operatorname{cr}_1(q_{345}), \operatorname{cr}_1(q_{123})\big)$$

for $q = 1234567 \in \operatorname{reg}^+\mathcal{P}_7$. Taking the product topology on $\operatorname{reg}^+\mathcal{P}_7 \times \mathbb{R}^4$, we define the *fine* topology on $\mathcal{M}$ with base given by the sets

$$U_V = \left\{M \in \mathcal{M} : \ M(\operatorname{reg}^+\mathcal{P}_7) \subset V\right\},$$

where V runs over the open subsets of $\operatorname{reg}^+\mathcal{P}_7 \times \mathbb{R}^4$.

We show that the canonical Möbius structure M_0 on X possesses a neighborhood U_V in the fine topology that lies in $\mathcal{I}$, that is, every Möbius structure $M \in U_V$ satisfies Axiom (I). For this, consider a function $\varepsilon\colon \operatorname{reg}^+\mathcal{P}_7 \to \mathbb{R}$ given by

$$\varepsilon(q) = \frac{|o\omega|_0^2}{4|x\omega'|_0^2}$$

for $q = (o, \omega, v, \omega', o', u, x) \in \operatorname{reg}^+\mathcal{P}_7$, where $|\cdot\cdot|_0$ is a standard metric on $X_u = \mathbb{R}$ from the canonical Möbius structure M_0 with infinitely remote point u. Such a metric is determined up to a homothety, but clearly ε does not depend on that.

Lemma 7.13. *The function* $\varepsilon\colon \operatorname{reg}^+\mathcal{P}_7 \to \mathbb{R}$ *is continuous.*

Proof. Obviously, it suffices to check that ε varies continuously in the variable $u \in q$. We switch to the notation $d_u(x,y) = |xy|_0$ for a metric from M_0 with infinitely remote point u. Applying a metric inversion to $u' \in X$, $u' \neq u$, we have

$$d_{u'}(x,y) = \frac{d_u(x,y)}{d_u(u',x)d_u(u',y)}.$$

The point $u' \in X$ is infinitely remote for $d_{u'}$. Thus, for $q' = (o, \omega, v, \omega', o', u', x)$ and $q = (o, \omega, v, \omega', o', u, x)$, we obtain

$$\varepsilon(q') = \frac{d^2_{u'}(o, \omega)}{4d^2_{u'}(x, \omega')} = \varepsilon(q)\frac{d^2_u(u', \omega')d^2_u(u', x)}{d^2_u(u', o)d^2_u(u', \omega)}.$$

The factor after $\varepsilon(q)$ on the right-hand side tends to 1 as $u' \to u$. Therefore, $\varepsilon(q') \to \varepsilon(q)$ as $u' \to u$, that is, as $q' \to q$. □

The set

$$V = \left\{(q, r) \in \operatorname{reg}^+ \mathcal{P}_7 \times \mathbb{R}^4 : |r - \operatorname{pr}_2 \circ M_0(q)| < \varepsilon(q)\right\},$$

where $\operatorname{pr}_2 \colon \operatorname{reg}^+ \mathcal{P}_7 \times \mathbb{R}^4 \to \mathbb{R}^4$ is the projection to the second factor, is the ε-neighborhood of $M_0(\operatorname{reg}^+ \mathcal{P}_7)$ with variable $\varepsilon = \varepsilon(q)$ in $\operatorname{reg}^+ \mathcal{P}_7 \times \mathbb{R}^4$. Lemma 7.13 shows that V is open in $\operatorname{reg}^+ \mathcal{P}_7 \times \mathbb{R}^4$. Thus, the set

$$U_V = \left\{M \in \mathcal{M} : M(\operatorname{reg}^+ \mathcal{P}_7) \subset V\right\}$$

of Möbius structures is open in the fine topology. The following is a pertubed version of Proposition 7.10.

Proposition 7.14. *Every Möbius structure $M \in U_V$ satisfies the Increment Axiom* (I), *i.e., $U_V \subset \mathcal{I}$.*

Proof. Given $M \in U_V$, for any $q \in \operatorname{reg}^+ \mathcal{P}_7$, $q = 1234567$, such that the 4-tuples q_{247}, q_{157} are M-harmonic, i.e., $\operatorname{cr}_2(q_{247}) = 1 = \operatorname{cr}_2(q_{157})$, we must show that $\operatorname{cr}_1(q_{345}) > \operatorname{cr}_1(q_{123})$ for M-cross-ratios.

We assume that $q = (o, \omega, v, \omega', o', u, x)$, and for (semi)metrics $d_u \in M$, $d^0_u \in M_0$ with infinitely remote point u we use the notations $d_u(a, b) = |ab|$, $d^0_u(a, b) = |ab|_0$. The assumption $M \in U_V$ implies that $|\operatorname{cr}^0_2(q_{247}) - 1| < \varepsilon$, $|\operatorname{cr}^0_2(q_{157}) - 1| < \varepsilon$ for M_0-cross-ratios, where $\varepsilon = \varepsilon(q)$. Since $q_{247} = (o, v, o', u)$, $q_{157} = (\omega, v, \omega', u)$, we have $1 = \operatorname{cr}_2(q_{247}) = \frac{|vo'|\cdot|ou|}{|ov|\cdot|o'u|} = \frac{|vo'|}{|ov|}$, $1 = \operatorname{cr}_2(q_{157}) = \frac{|v\omega'|\cdot|\omega u|}{|\omega v|\cdot|\omega' u|} = \frac{|v\omega'|}{|\omega v|}$. Hence,

$$\left|\frac{|vo'|_0}{|ov|_0} - 1\right| < \varepsilon, \quad \left|\frac{|v\omega'|_0}{|\omega v|_0} - 1\right| < \varepsilon. \tag{14}$$

Since $|o\omega|_0 = |ov|_0 - |\omega v|_0$, $|\omega' o'|_0 = |vo'|_0 - |v\omega'|_0$ because M_0 is canonical, we have

$$|o\omega|_0 - |\omega' o'|_0 = |ov|_0 - |vo'|_0 + |v\omega'|_0 - |\omega v|_0$$

and using (14), we obtain

$$-\varepsilon(|ov|_0 + |\omega v|_0) \le |o\omega|_0 - |\omega' o'|_0 \le \varepsilon(|ov|_0 + |\omega v|_0). \tag{15}$$

Similarly, since $|x\omega|_0 = |xo|_0 + |o\omega|_0$ and $|xo'|_0 = |x\omega'|_0 + |\omega' o'|_0$, we have

$$\operatorname{cr}^0_1(q_{345}) - \operatorname{cr}^0_1(q_{123}) = \frac{|x\omega|_0}{|xo|_0} - \frac{|xo'|_0}{|x\omega'|_0} = \frac{|o\omega|_0}{|xo|_0} - \frac{|\omega' o'|_0}{|x\omega'|_0}.$$

Using (15) and the identity $|x\omega'|_0 - |xo|_0 = |o\omega'|_0$, we obtain

$$\operatorname{cr}^0_1(q_{345}) - \operatorname{cr}^0_1(q_{123}) \ge \frac{|o\omega|_0 \cdot |o\omega'|_0}{|xo|_0 \cdot |x\omega'|_0} - \varepsilon\frac{|ov|_0 + |\omega v|_0}{|x\omega'|_0}. \tag{16}$$

By the assumption $M \in U_V$, we have $|\operatorname{cr}_1(p) - \operatorname{cr}^0_1(p)| < \varepsilon$ for $p = q_{345}$ and $p = q_{123}$. Hence, $\operatorname{cr}_1(q_{345}) - \operatorname{cr}_1(q_{123}) \ge \operatorname{cr}^0_1(q_{345}) - \operatorname{cr}^0_1(q_{123}) - 2\varepsilon$. Therefore, using (16), we obtain

$$\operatorname{cr}_1(q_{345}) - \operatorname{cr}_1(q_{123}) \ge \frac{|o\omega|_0 \cdot |o\omega'|_0}{|xo|_0 \cdot |x\omega'|_0} - \varepsilon\left(2 + \frac{|ov|_0 + |\omega v|_0}{|x\omega'|_0}\right). \tag{17}$$

We have $o\omega \subset o\omega'$, $xo \subset x\omega'$, $\omega v \subset ov \subset x\omega'$ in X_u. Thus, $|o\omega|_0 < |o\omega'|_0$, $|xo|_0 < |x\omega'|_0$, and $|\omega v|_0 < |ov|_0 < |x\omega'|_0$, whence

$$\frac{|o\omega|_0 \cdot |o\omega'|_0}{|xo|_0 \cdot |x\omega'|_0} > \frac{|o\omega|_0^2}{|x\omega'|_0^2}, \quad \frac{|ov|_0 + |\omega v|_0}{|x\omega'|_0} < \frac{2|ov|_0}{|x\omega'|_0} < 2.$$

It follows that $\mathrm{cr}_1(q_{345}) - \mathrm{cr}_1(q_{123}) > \frac{|o\omega|_0^2}{|x\omega'|_0^2} - 4\varepsilon = 0$. □

7.7. Convex Möbius structures. We introduce the following axiom for a Möbius structure $M \in \mathcal{M}$, which implies the convexity of the function F_{ab}.

(C) Convexity: for any $q \in \operatorname{reg}\mathcal{P}_6$ with cyclic order $\mathrm{co}(q) = 123456$ such that $\mathrm{cr}_3(q_{46}) = \mathrm{cr}_3(q_{26})$, we have

$$\mathrm{cr}_1(q_{12}) > \mathrm{cr}_1(q_{14}).$$

A Möbius structure $M \in \mathcal{M}$ is *convex* if it satisfies Axiom (C).

Axiom (C) can be rewritten in the following way. Assume we have $q = (o', x, y, z, o, \omega) \in \operatorname{reg}\mathcal{P}_6$ written in the cyclic order, $\mathrm{co}(q) = 123456$. Then $q_{46} = (o', x, y, o)$, $q_{26} = (o', y, z, o)$, and the assumption $\mathrm{cr}_3(q_{46}) = \mathrm{cr}_3(q_{26})$ is equivalent to $\delta_{x,y,z}(o) = \delta_{x,y,z}(o')$, where

$$\delta_{x,y,z}(o) = \frac{|yo|^2}{|xo| \cdot |zo|}.$$

Next, we have $q_{12} = (y, z, o, \omega)$, $q_{14} = (x, y, o, \omega)$. Thus, the condition $\mathrm{cr}_1(q_{12}) > \mathrm{cr}_1(q_{14})$ is equivalent to $\delta_{x,y,z}(o) > \delta_{x,y,z}(\omega)$.

Proposition 7.15. *The canonical Möbius structure M_0 on X is convex.*

Proof. In the metric from M_0 with infinitely remote point o', we have $\delta_{x,y,z}(o') = 1$. Thus, $\delta_{x,y,z}(o) = 1$, whence $|yo|^2 = |xo| \cdot |zo|$. Let $\sigma = |o\omega|$. Since M_0 is canonical, we have $|y\omega| = |yo| + \sigma$, $|x\omega| = |xo| + \sigma$, and $|z\omega| = |zo| + \sigma$. Therefore,

$$\delta_{x,y,z}(\omega) = \frac{(|yo| + \sigma)^2}{(|xo| + \sigma)(|zo| + \sigma)} = \frac{1 + \alpha\sigma + \beta\sigma^2}{1 + \gamma\sigma + \beta'\sigma^2},$$

where $\alpha = 2/|yo|$, $\beta = 1/|yo|^2$, $\gamma = \frac{|xo|+|zo|}{|xo|\cdot|zo|}$, $\beta' = 1/(|xo| \cdot |zo|)$. Since $|yo|^2 = |xo| \cdot |zo|$, we have $\beta = \beta'$, and, thus, the inequality $\delta_{x,y,z}(\omega) < 1$ is equivalent to $\sqrt{|xo|/|zo|} + \sqrt{|zo|/|xo|} > 2$, which is always true because $x \neq z$. □

Let $a = (o, o')$ and $b = (\omega, \omega') \in \mathrm{aY}$ be events in the strong causal relation such that the pairs (o, ω') and (o', ω) separate each other. Using the parametrization $x \leftrightarrow x_a$ of the arc oo' between o, o' that does not contain b by the timelike line h_a, $x \in oo'$, $x_a \in h_a$, and the parametrization $x' \leftrightarrow x'_b$ of the arc $\omega\omega'$ between ω, ω' that does not contain a by the timelike line h_b, we view the function $F_{ab}\colon D_{ab} \to \mathbb{R}$, see Subsection 7.2, as a function defined on $h_a \times h_b$, $F_{ab}\colon h_a \times h_b \to \mathbb{R}$.

Proposition 7.16. *The Convexity Axiom* (C) *implies that the function $F_{ab}\colon h_a \times h_b \to \mathbb{R}$ is strictly convex for any events a, $b \in \mathrm{aY}$ in strong causal relation.*

Remark 7.17. 1. The convexity of the function F_{ab} is a precise analog of the convexity of the distance function in $\mathrm{CAT}(-1)$ spaces, cf. Proposition 7.1.

2. The convexity property depends on a parametrization up to an affine equivalence. Here, the parametrization of D_{ab} by $h_a \times h_b$ is chosen, because $h_a \times h_b$ is an affine space isomorphic to $\mathbb{R} \times \mathbb{R}$.

Proof of Proposition 7.16. As usual, we assume that $a = (o, o')$ and $b = (\omega, \omega') \in \mathrm{aY}$ are events in strong causal relation such that the pairs (o, ω') and (o', ω) separate each other, and $e = (o, \omega)$, $e' = (o', \omega')$. We show that the increment of the function F_{ab}

strictly monotone increases along any line in $h_a \times h_b = \mathbb{R}^2$. For this, it suffices to show that for any x_a, y_a, $z_a \in h_a$ with $x_a < y_a < z_a$ such that $t(x_a, y_a) = t(y_a, z_a)$, we have $\Delta F_{a,b}(z_a, y_a) > \Delta F_{a,b}(y_a, x_a)$, where the increment

$$\Delta F_{a,b}(y_a, x_a) = \frac{1}{2}(t(y_e, x'_e) + t(y_{e'}, x'_{e'}) - t(x_e, x'_e) - t(x_{e'}, x'_{e'})),$$

calculated for some $x'_b \in h_b$, is in fact independent of x'_b (recall that here we use parametrizations $x \leftrightarrow x_a$ and $x' \leftrightarrow x'_b$). Indeed, without loss of generality we may assume that $q = (o', x, y, z, o, \omega) \in \operatorname{reg} \mathcal{P}_6$ is written in the cyclic order. Then $t(y_e, x'_e) - t(x_e, x'_e) = t(y_e, x_e)$ and $t(y_{e'}, x'_{e'}) - t(x_{e'}, x'_{e'}) = -t(y_{e'}, x_{e'})$, so that

$$\Delta F_{a,b}(y_a, x_a) = \frac{1}{2}(t(y_e, x_e) - t(y_{e'}, x_{e'})).$$

The condition $t(x_a, y_a) = t(y_a, z_a)$ is equivalent to $\frac{|yo'|\cdot|xo|}{|yo|\cdot|xo'|} = \frac{|zo'|\cdot|yo|}{|zo|\cdot|yo'|}$ for any semimetric from M, or, which is the same, to $\delta_{x,y,z}(o) = \delta_{x,y,z}(o')$. Axiom (C) implies that $\delta_{x,y,z}(o) > \delta_{x,y,z}(\omega)$. Since

$$t(z_e, y_e) = \frac{|yo|\cdot|z\omega|}{|zo|\cdot|y\omega|} \quad \text{and} \quad t(y_e, x_e) = \frac{|xo|\cdot|y\omega|}{|yo|\cdot|x\omega|},$$

this is equivalent to $t(z_e, y_e) > t(y_e, x_e)$.

Applying the same argument to $q' = (o, z, y, x, o', \omega') \in \operatorname{reg} \mathcal{P}_6$, we see that Axiom (C) implies $\delta_{x,y,z}(o') > \delta_{x,y,z}(\omega')$, which is equivalent to $t(z_{e'}, y_{e'}) < t(y_{e'}, x_{e'})$. Therefore, $\Delta F_{a,b}(z_a, y_a) > \Delta F_{a,b}(y_a, x_a)$, and the strict convexity of the function F_{ab} follows. $\square$

Remark 7.18. By Proposition 7.16, Axiom C implies that the function $F_{ab}\colon D_{ab} \to \mathbb{R}$ attains its infimum at a unique point $d'_0 \in D_{ab}$ for any a, $b \in \mathrm{aY}$ in the strong causal relation, because $F_{ab}(d) \to \infty$ as d approaches the boundary ∂D_{ab} of D_{ab}. However, in general there is no reason to think that $d'_0 = h_a \cap h_b$. It seems that Axioms (I) and (C) are independent of each other.

§8. Appendix 1

We show that Gromov hyperbolic spaces from a large class are boundary continuous, see Subsection 4.2.

Theorem 8.1. *Every proper Gromov hyperbolic* CAT(0) *space* Y *is boundary continuous.*

For CAT(-1) spaces this was established in [BS1, Proposition 3.4.2]. Here, we extend this result to CAT(0) spaces. The distinction between CAT(-1) and CAT(0) cases relevant to our arguments is that $\operatorname{dist}(\gamma, \gamma') = \inf\{d(s, s') : s \in \gamma, s' \in \gamma'\} = 0$ for asymptotic geodesic rays γ, γ' in the former case, while that distance is only finite in the latter. This distinction is compensated for by the following lemma.

We use the notation $o_t(1)$ for a quantity with $o_t(1) \to 0$ as $t \to \infty$.

Lemma 8.2. *Let* $xyz \subset \mathbb{R}^2$ *be a triangle with* $|yz| \le d$ *for some fixed* $d > 0$ *and* $|xy|, |xz| \ge t$. *Assume that* $\angle_z(x, y)$, $\angle_y(x, z) \ge \pi/2 - o_t(1)$. *Then* $\big||xy| - |xz|\big| = o_t(1)$.

Proof. The required estimate follows from the convexity of the distance function on $\mathbb{R}^2$ and the first variation formula. We leave the details to the reader. $\square$

Recall that in a geodesic metric space, the Gromov product is *monotone* in the following sense, see, e.g., [BS1, Lemma 2.1.1].

Lemma 8.3. *Let* Y *be a geodesic metric space,* $xyz \subset Y$ *a geodesic triangle. Then for any* $y' \in xy$, $u \in yz$ *we have*

$$(y'|z)_x \le (y|z)_x \le \min\{(y|u)_x, (u|z)_x\}.$$

Proof. The left-hand side inequality is equivalent to $|y'x| - |y'z| \le |yx| - |yz|$, which follows from the triangle inequality $|yz| \le |yy'| + |y'z|$ because $|yx| - |y'x| = |yy'|$. A similar argument using $|yz| = |yu| + |uz|$ proves the right-hand side inequality. $\square$

All necessary information about CAT(0) spaces like the definition of the angles, the triangle inequality for angles, the comparison of angles, the first variation formula, etc. used in the proof below can be found in [BH].

Proof of Theorem 8.1. Given $o \in Y$, $\xi, \xi' \in \partial_\infty Y$, we need to show that for any sequences $\{x_i\} \in \xi$, $\{x_i'\} \in \xi'$ the limit $\lim_i (x_i|x_i')_o$ exists. We may assume that $\xi \ne \xi'$ because otherwise there is nothing to prove.

We use the notation $\xi = \xi(t)$ for the unit speed parametrization of the geodesic ray $o\xi$ with $\xi(0) = o$. By the monotonicity of the Gromov product, see Lemma 8.3, the limit

$$a = \lim_{t\to\infty} (\xi(t)|\xi'(t))_o$$

exists. We have $a < \infty$ because Y is hyperbolic and $\xi \ne \xi'$, which implies that the geodesic segment $\xi(t)\xi'(t)$ stays at a bounded distance from o uniformly in t. Since Y is proper, the segments $\xi(t)\xi'(t)$ subconverge in the compact-open topology as $t \to \infty$ to a geodesic $\gamma \subset Y$ with the endpoints ξ, ξ' at infinity.

(1) We fix $p \in \gamma$ and show that $|x_ip| + |px_i'| = |x_ix_i'| + o_i(1)$. The geodesic segments px_i, px_i' converge to subrays $p\xi$, $p\xi' \subset \gamma$ (respectively) in the compact-open topology as $i \to \infty$. It follows that $\angle_p(x_i, x_i') \ge \pi - o_i(1)$.

Let $q_i \in x_ix_i'$ be the point closest to p. By the hyperbolicity of Y we have $|pq_i| = \operatorname{dist}(p, x_ix_i') \le d$ for some $d > 0$ and all i. For the triangles $\Delta_i = pq_ix_i$, $\Delta_i' = pq_ix_i'$ we have $\angle_{q_i}(p, x_i), \angle_{q_i}(p, x_i') \ge \pi/2$, and $\angle_p(x_i, q_i) + \angle_p(q_i, x_i') \ge \angle_p(x_i, x_i') \ge \pi - o_i(1)$.

Using the comparison of angles for CAT(0) spaces, we see that the comparison triangles $\widetilde{\Delta}_i = \widetilde{p}\widetilde{q}_i\widetilde{x}_i$, $\widetilde{\Delta}_i' = \widetilde{p}\widetilde{q}_i\widetilde{x}_i' \subset \mathbb{R}^2$ have angles at least $\pi/2$ at $\widetilde{q}_i$, and $\angle_{\widetilde{p}}(\widetilde{x}_i, \widetilde{q}_i) \ge \angle_p(x_i, q_i)$, $\angle_{\widetilde{p}}(\widetilde{q}_i, \widetilde{x}_i') \ge \angle_p(q_i, x_i')$. Thus, $\angle_{\widetilde{p}}(\widetilde{x}_i, \widetilde{q}_i), \angle_{\widetilde{p}}(\widetilde{q}_i, \widetilde{x}_i') < \pi/2$, and we see that

$$\pi - o_i(1) \le \angle_{\widetilde{p}}(\widetilde{x}_i, \widetilde{q}_i) + \angle_{\widetilde{p}}(\widetilde{q}_i, \widetilde{x}_i') < \pi.$$

Hence, $\angle_{\widetilde{p}}(\widetilde{x}_i, \widetilde{q}_i), \angle_{\widetilde{p}}(\widetilde{q}_i, \widetilde{x}_i') \ge \pi/2 - o_i(1)$. Since $|\widetilde{p}\widetilde{q}_i| \le d$, we can apply Lemma 8.2 and conclude that $|\widetilde{x}_i\widetilde{p}| = |\widetilde{x}_i\widetilde{q}_i| + o_i(1)$, $|\widetilde{p}\widetilde{x}_i'| = |\widetilde{q}_i\widetilde{x}_i'| + o_i(1)$. Therefore $|x_ip| + |px_i'| = |x_ix_i'| + o_i(1)$.

(2) By the hyperbolicity of Y, there are points $u \in o\xi$, $u' \in o\xi'$, $v_t \in \xi(t)\xi'(t)$ with mutual distances bounded above independently of t. Thus,

$$\angle_{\xi(t)}(o, \xi'(t)) = \angle_{\xi(t)}(o, v_t) = o_t(1), \quad \angle_{\xi'(t)}(o, \xi(t)) = \angle_{\xi'(t)}(o, v_t) = o_t(1),$$

that is, the segment ov_t is observed from $\xi(t)$ and $\xi'(t)$ under arbitrarily small angles as $t \to \infty$.

(3) Let $\eta(t)$, $\eta'(t) \in \gamma$ be points closest to $\xi(t)$ and $\xi'(t)$, respectively. Since the geodesic γ is convex as a set in Y, we have $|\eta(t)\eta'(t)| \le |\xi(t)\xi'(t)|$. Our next goal is to show that $|\xi(t)\xi'(t)| \le |\eta(t)\eta'(t)| + o_t(1)$.

Since the geodesic rays $o\xi$, $p\xi$ are asymptotic, the distance $\operatorname{dist}(\xi(t), \gamma)$ is uniformly bounded above. Using the convexity of the distance function on Y, we conclude that $g(t) = \operatorname{dist}(\xi(t), \gamma)$ and similarly $g'(t) = \operatorname{dist}(\xi'(t), \gamma)$ are monotone decreasing as $t \to \infty$. Then for $t' > t$ we have $g(t') \le g(t) \le |\xi(t)\eta(t')|$ and similarly $g'(t') \le g'(t) \le |\xi'(t)\eta'(t')|$. The first variation formula for CAT(0) spaces, see [BH, Corollary 3.6], implies that $\angle_{\xi(t)}(\eta(t), o), \angle_{\xi'(t)}(\eta'(t), o) \ge \pi/2$ for all $t > 0$. Combining this with the estimates in (2) for the angles $\angle_{\xi(t)}(o, \xi'(t)), \angle_{\xi'(t)}(o, \xi(t)) = o_t(1)$, we conclude that $\angle_{\xi(t)}(\eta(t), \xi'(t)), \angle_{\xi'(t)}(\eta'(t), \xi(t)) \ge \pi/2 - o_t(1)$. Therefore, all the angles of the quadrilateral $\eta(t)\xi(t)\xi'(t)\eta'(t)$ are at least $\pi/2 - o_t(1)$. We also note that $g(t) = |\xi(t)\eta(t)|$ and $g'(t) = |\xi(t)\eta(t)| \le c$ for all $t \ge 0$ and some $c > 0$ independent of t.

Let $x(t)y(t)u(t)$, $y(t)z(t)u(t)$ be comparison triangles in $\mathbb{R}^2$ with vertices $x(t)$, $z(t)$ separated by the common side $y(t)u(t)$ for the triangles $\eta(t)\xi(t)\eta'(t)$, $\xi(t)\xi'(t)\eta'(t)$ in Y, respectively. Using the comparison of angles in CAT(0) spaces and the triangle inequality for angles, we see that all the angles of the quadrilateral $x(t)y(t)z(t)u(t) \subset \mathbb{R}^2$ are at least $\pi/2 - o_t(1)$. Since $|x(t)y(t)|$, $|z(t)u(t)| \le c$, we have $\angle_{y(t)}(z(t), u(t))$, $\angle_{u(t)}(x(t), y(t)) = o_t(1)$. Thus, $\angle_{y(t)}(x(t), u(t))$, $\angle_{u(t)}(z(t), y(t)) \ge \pi/2 - o_t(1)$. By Lemma 8.2, $|y(t)z(t)|$, $|x(t)u(t)| = |y(t)u(t)| + o_t(1)$, whence $|\xi(t)\xi'(t)| \le |\eta(t)\eta'(t)| + o_t(1)$.

(4) Now, we show that $\alpha(t)$, $\alpha'(t) \ge \pi/2 - o_t(1)$, where $\alpha(t) = \angle_{\xi(t)}(\eta(t), \xi)$, $\alpha'(t) = \angle_{\xi'(t)}(\eta'(t), \xi')$. For brevity, we only prove this estimate for the angles $\alpha(t)$.

By the first variation formula, we have $|\xi(t+s)\eta(t)| = |\xi(t)\eta(t)| - s\cos\alpha(t) + o(s)$ for all sufficiently small $s \ge 0$. On the other hand, the function $g = g(t)$ is convex. Therefore, at every point it has the right derivative d_+g/dt, which is monotone nondecreasing. It is monotone nonpositive because $g(t)$ decreases. Thus $-d_+g(t)/dt = o_t(1)$. Since $g(t+s) \le |\xi(t+s)\eta(t)|$ for every $s \ge 0$, we obtain the inequality

$$g(t) - s\cos\alpha(t) + o(s) = |\xi(t+s)\eta(t)| \ge g(t+s) \ge g(t) + s \cdot d_+g(t)/dt$$

for all sufficiently small $s > 0$, whence $\cos\alpha(t) \le -d_+g(t)/dt = o_t(1)$, and, therefore, $\alpha(t) \ge \pi/2 - o_t(1)$.

(5) We show that $|\xi(t)x_i| = |\eta(t)x_i| + o_{t,i}(1)$ for every sufficiently large fixed t, and similarly $|\xi'(t)x_i'| = |\eta'(t)x_i'| + o_{t,i}(1)$. The geodesic segments $\xi(t)x_i$, $\eta(t)x_i$ converge in the compact-open topology to subrays $\xi(t)\xi$, $\eta(t)\xi$, respectively, as $i \to \infty$. Thus, $\angle_{\xi(t)}(\eta(t), x_i) \ge \alpha(t) - o_i(1)$ and $\angle_{\eta(t)}(\xi(t), x_i) \ge \beta(t) - o_i(1)$, where $\beta(t) = \angle_{\eta(t)}(\xi(t), \xi) \ge \pi/2$. Using (4) and the comparison of angles, we see that the angles at x, y of the comparison triangle $xyz \subset \mathbb{R}^2$ for $\xi(t)\eta(t)x_i$ are at least $\pi/2 - o_{t,i}(1)$. By Lemma 8.2, $|\xi(t)x_i| = |\eta(t)x_i| + o_{t,i}(1)$.

(6) Since the geodesic segments ox_i converge to the ray $o\xi$, we have $|ox_i| = |o\xi(t)| + |\xi(t)x_i| - o_{t,i}(1)$ for every fixed $t > 0$ and all sufficiently large i. Similarly, $|px_i| = |p\eta(t)| + |\eta(t)x_i| - o_{t,i}(1)$. By (5), $|ox_i| - |px_i| = |o\xi(t)| - |p\eta(t)| + o_{t,i}(1)$. Using (1), (3) and the identity $|\eta(t)p| + |p\eta'(t)| = |\eta(t)\eta'(t)|$, we finally obtain $(x_i|x_i')_o = (\xi(t)|\xi'(t))_o + o_{t,i}(1)$. Hence, $\lim_i (x_i|x_i')_o = a$. □

Corollary 8.4. *In a proper Gromov hyperbolic* CAT(0) *space Y, we have $(\xi|\xi')_o = 0$ if and only if $\angle_o(\xi, \xi') = \pi$ for $o \in Y$, ξ, $\xi' \in \partial_\infty Y$.*

Proof. If $\angle_o(\xi, \xi') = \pi$, then $|xo| + |ox'| = |xx'|$, and $(x|x')_o = 0$ for every $x \in o\xi$, $x' \in o\xi'$. By Theorem 8.1, $(\xi|\xi')_o = 0$.

Conversely, assume that $\angle_o(\xi, \xi') < \pi$. Then for $x \in o\xi$, $x' \in o\xi'$ sufficiently close to o, we have $|xo| + |ox'| > |xx'|$, and, thus, $(x|x')_o > 0$. By the monotonicity of the Gromov product and Theorem 8.1, $(\xi|\xi')_o \ge (x|x')_o > 0$. □

§9. Appendix 2

Viktor Schroeder

Here, it will be shown that Axiom (t6) follows from the other axioms of timed causal spaces. That is, we assume Axioms (h1)–(h6) and (t1)–(t5) but not (t6) and show that (t6) follows. Given an event $e = (\alpha, \beta)$ we have a reflection $\rho = \rho_e \colon S^1 \to S^1$ fixing α, β. The Möbius structure M was obtained in Theorem 6.6 without using (t6). This gives another timelike line structure $\mathcal{H}_M$ and hence, for e, another reflection $\tau = \tau_e \colon S^1 \to S^1$. Choose x, y in the same component of $S^1 \setminus \{\alpha, \beta\}$ in the order $\alpha xy\beta$. We use the notation $[\ ,\ ,\ ,\ ]$ for the cross-ratio cr_3,

$$[x, y, z, u] := \frac{|xy|\,|zu|}{|xz|\,|yu|}.$$

Then

$$[\alpha, x, \tau(x), \beta] = [\alpha, y, \tau(y), \beta] = 1. \tag{18}$$

This cross-ratio satisfies the *cocycle property*

$$[\alpha, x, y, \beta][\alpha, y, z, \beta] = [\alpha, x, z, \beta]$$

for any x, y, z. Axiom (t6) was not used in the proof of Lemma 6.8. By that lemma, the time of the timed causal space is computed in the usual way via M-cross-ratios. Therefore,

$$\ln[\alpha, x, y, \beta] = -t((x, \rho(x)), (y, \rho(y))) = \ln[\alpha, \rho(x), \rho(y), \beta],$$

and by the cocycle property and (18) we have $[\alpha, x, y, \beta] = [\alpha, \tau(x), \tau(y), \beta]$. Thus,

$$[\alpha, \rho(x), \tau(x), \beta][\alpha, \tau(x), \rho(y), \beta] = [\alpha, \rho(x), \rho(y), \beta]$$

equals

$$[\alpha, \tau(x), \rho(y), \beta][\alpha, \rho(y), \tau(y), \beta] = [\alpha, \tau(x), \tau(y), \beta].$$

Therefore, $[\alpha, \rho(x), \tau(x), \beta]$ is constant for x in a connected component of $S^1 \setminus \{\alpha, \beta\}$. In order to prove the result, we need to show that $[\alpha, \rho(x), \tau(x), \beta] = 1$. Then $\rho(x) = \tau(x)$ by monotonicity, and we arrive at (t6).

Now $[\alpha, \rho(x), \tau(x), \beta] = [\alpha, \rho(x), x, \beta]$ because $[\alpha, \tau(x), x, \beta] = 1$ and, hence, also

$$[\alpha, x, \rho(x), \beta] \quad \text{is constant in } x. \tag{19}$$

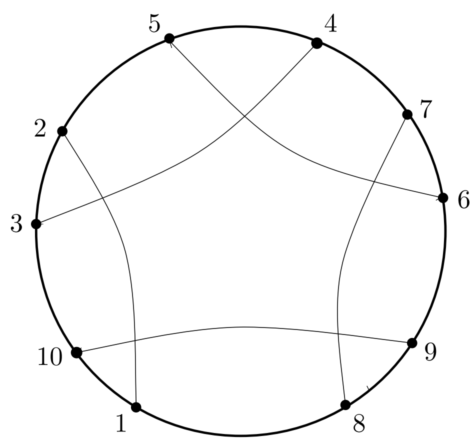

FIGURE 2. The pentagon P.

Now, we construct a pentagon $P = x_1x_2, x_2x_3, \dots, x_9x_{10}$ of consecutively "orthogonal" timelike lines, i.e., $\rho_{x_i,x_{i+1}}(x_{i+2}) = x_{i+3}$ for $i = 1, \dots, 9$, where the indices are taken modulo 10 (the existence of P easily follows from Proposition 3.2(b)). Then (19) implies (we write $[i, j, k, l] = [x_i, x_j, x_k, x_l]$)

$$\begin{aligned}[1, 3, 4, 2] &= [6, 3, 4, 5]\\ &= [6, 8, 7, 5]\\ &= [9, 8, 7, 10]\\ &= [9, 1, 2, 10]\\ &= [4, 1, 2, 3] = [1, 4, 3, 2] = 1/[1, 3, 4, 2],\end{aligned}$$

whence $[1, 3, 4, 2] = 1$.

Acknowledgments

The author is very much thankful to Prof. Dr. Viktor Schroeder for attention to this paper and valuable remarks. Especially, I am grateful to him for pointing out to me that Axiom (t6) in Subsection 3.3 follows from the other axioms of timed causal spaces. This is explained in detail in Appendix 2 written by Viktor Schroeder.

References

[A-S] L. Andersson, T. Barbot, R. Benedetti, F. Bonsante, W. Goldman, F. Labourie, K. Scannell, and J.-M. Schlenker, *Notes on "Lorents spacetimes of constant curvature"* [*Geom. Dedicata* 126 (2007), 3–45] *by G. Mess*, Geom. Dedicata **126** (2007), 47–70. MR2328922 (2010a:53155)

[BH] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren Math. Wiss., Bd. 319, Springer-Verlag, Berlin, 1999. MR1744486 (2000k:53038)

[Bus] H. Busemann, *Timelike spaces*, Dissertationes Math. Rozprawy Mat. **53** (1967), 1–52. MR0220238 (36:3304)

[Bu1] S. Buyalo, *Möbius sub-Möbius structures*, Algebra i Analiz **28** (2016), no. 5, 1–20; English transl., St. Petersburg Math. J. **28** (2017), no. 5, 555–568. MR3637585

[Bu2] ______, *Deformation of Möbius structures*, preprint.

[BS1] S. Buyalo and V. Schroeder, *Elements of asymptotic geometry*, EMS Monogr. Math., Eur. Math. Soc. (EMS), Zürich, 2007. MR2327160 (2009a:53068)

[BS2] ______, *Möbius characterization of the boundary at infinity of rank one symmetric spaces*, Geom. Dedicata **172** (2014), no. 1, 1–45. MR3253769

[BS3] ______, *Incidence axioms for the boundary at infinity of complex hyperbolic spaces*, Anal. Geom. Metr. Spaces **3** (2015), 244–277. MR3399374

[BS4] ______, *Möbius structures on the circle.* (in preparation)

[FS] T. Foertsch and V. Schroeder, *Metric Möbius geometry and a characterization of spheres*, Manuscripta Math. **140** (2013), no. 3–4, 613–620. MR3019142

[Ge] C. Gerhardt, Pinching estimates for dual flows in hyperbolic and de Sitter space, `arXiv:1510.03747 [math.DG]` (2015)

[Mes] G. Mess, *Lorentz spacetimes of constant curvature*, Geom. Dedicata **126** (2007), no. 1, 3–45. MR2010a:53154

[PY] A. Papadopoulos and S. Yamada, *Timelike Hilbert and Funk geometries*, `arXiv:1602.07072 [math.DG]` (2016).

[Ro] B. A. Rozenfel′d, *Noneuclidean spaces*, Nauka, Moscow, 1969. (Russian) MR0253136 (40:6351)

[SSV] M. Spradlin, A. Strominger, and A. Volovich, *Les Houches lectures on de Sitter space*, `arXiv:hep-th/0110007` (2001).

[Yu] H. Yu, *Dual flows in hyperbolic space and de Sitter space*, `arXiv:1604.02369 [math.DG]` (2016).

St. Petersburg Branch, Steklov Institute of Mathematics, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191023, Russia

Email address: `sbuyalo@pdmi.ras.ru`

Received 5/MAY/2016

Translated by THE AUTHOR