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# ON NEVANLINNA DOMAINS WITH FRACTAL BOUNDARIES

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Abstract. A positive answer is given to the question on the existence of a Nevanlinna contour of Hausdorff dimension exceeding 1, posed by K. Yu. Fedorovskiĭ in 2001. In particular, it is shown that this dimension may exceed 3/2.

## §1. Introduction

Recall the definition of a Nevanlinna domain (the details on Nevanlinna domains, their properties, and applications can be found, e.g., in [1, Chapter. 2, §§2.3–2.4]).

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ stand for the unit disk, and let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle on the complex plane $\mathbb{C}$. For an arbitrary open set $E \subset \overline{\mathbb{C}}$, we denote by $H^\infty(E)$ the space of all bounded functions analytic in $E$.

A bounded simply connected domain $\Omega$ in $\mathbb{C}$ with boundary $\Gamma = \partial\Omega$ is called a *Nevanlinna domain* (see [2, Definition 2.1]) if there exist two functions $u, v \in H^\infty(\Omega)$ such that $v \not\equiv 0$ and the identity

$$\overline{\zeta} = \frac{u(\zeta)}{v(\zeta)} \tag{1.1}$$

is fulfilled on $\Gamma$ almost everywhere in the sense of harmonic measure. This means that for a.e. $\xi \in \mathbb{T}$ we have the following identity for boundary values:

$$\overline{h(\xi)} = \frac{u(h(\xi))}{v(h(\xi))}, \tag{1.2}$$

where $h$ is some function analytic and univalent in $\mathbb{D}$ that conformally maps $\mathbb{D}$ onto $\Omega$.

If the domain $\Omega$ is Jordan, then $\Gamma$ is called a *Nevanlinna contour*.

The definition of a Nevanlinna domain is consistent: since the harmonic measure is invariant under conformal mappings, this definition does not depend on the choice of the function $h$; the Fatou theorem shows that the angular boundary values of the functions $u(h(\xi))$ and $v(h(\xi))$ exist almost everywhere on $\mathbb{T}$, and, by the Lusin–Privalov uniqueness theorem, the ratio $u/v$ is determined by those boundary values uniquely. If $\Gamma$ is rectifiable, then formula (1.1) can be understood directly, in the sense of angular boundary values a.e. on $\Gamma$.

For constructing Nevanlinna domains, the following criterion is important, which, in particular, explains the name itself.

**Theorem 1** ([2, Proposition 3.1]). *A domain $\Omega$ is a Nevanlinna one if and only if there exists a function $h$ as in* (1.2) *that admits pseudocontinuation of Nevanlinna type through $\mathbb{T}$, i.e., there exist two function $f_1$, $f_2 \in H^\infty(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ such that $f_2 \not\equiv 0$ and the*

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*angular boundary values of* $h$ *and* $h = f_1/f_2$ (*respectively, from inside and from outside of* $\mathbb{T}$) *are equal a.e. on* $\mathbb{T}$.

The theorem implies that a Nevanlinna domain with analytic boundary is a rational image of the disk; in particular, a circle is a Nevanlinna contour, while an ellipse distinct from a circle, or an arbitrary polygon, are not.

The Nevanlinna domains were introduced by Fedorovskiĭ in [3] in connection with the problem of uniform approximation of continuous functions by polyanalytic functions on rectifiable contours, namely, for such contours he proved Theorem 2 stated below. In the paper [2] the rectifiability restriction was lifted.

We recall that a bounded, simply connected domain $\Omega$ in $\mathbb{C}$ is called a *Caratheodory domain* if $\partial\Omega = \partial\Omega_\infty$, where $\Omega_\infty$ is the unbounded connected component of the set $\overline{\mathbb{C}} \setminus \overline{\Omega}$. As usual, by a polyanalytic function of order $n$ we mean a solution of the equation $\partial^n f/\partial\overline{z}^n = 0$ on an open set in $\mathbb{C}$, where $\partial/\partial\overline{z}$ is the Cauchy–Riemann operator. For $n = 2$ such functions are said to be bianalytic. A polyanalytic polynomial of order $n$ is a corresponding polynomial solution; obviously, it can be written in the form

$$P(z) = \sum_{m=0}^{n-1} P_m(z)\overline{z}^m,$$

where all $P_m$ are polynomials in the complex variable $z$.

**Theorem 2** ([2, Theorem 2.2 (1)]). *Let* $n \geq 2$ *be an integer, and let* $\Omega$ *be a Caratheodory domain with boundary* $\partial\Omega$. *The set of all polyanalytic polynomials of order* $n$ *is dense in the space* $C(\partial\Omega)$ *if and only if* $\Omega$ *is not a Nevanlinna domain.*

Thus, for $n \geq 2$ the situation differs fundamentally from that for $n = 1$, where the corresponding criterion is given by the Mergelyan theorem [4], and where the set of analytic polynomials is not dense in $C(\partial\Omega)$ for whatever $\Omega$.

Nevanlinna domains arise in a series of other problems. Thus, in [2, Theorem 5.5 (1)] it was proved that, for any Nevnlinna domain of Carathéodory type, the homogeneous Dirichlet boundary problem has nontrivial bianalytic solutions (this is not true for "very many" domains). In [5], the relationship was studied between the Nevanlinna domains and the model subspaces of the space $H^2$.

In connection with applications of Nevanlinna domains, a natural question arises as to how wide their class is, in particular, whether the Nevanlinna property of a domain implies any conditions on the dimension of its boundary. In [6] and [2], the question was posed whether any Nevanlinna contour must be rectifiable. The first example of a non-rectifiable Nevanlinna contour was constructed much later, in [7, Example 1]. Earlier, in [8] and [5], various ways for constructing Nevanlinna domains were studied, and examples were given of Nevanlinna domains with rectifiable boundaries possessing various types of irregularity.

It should be noted that the contour presented in [7, Example 1] is analytic outside of an arbitrarily small arc centered at one point, so that it leaves open the fundamental question as to whether or not a Nevanlinna contour can be fractal, i.e., can have Hausdorff dimension exceeding 1. The corresponding question was posed by Fedorovskiĭ in 2001, see [6, Problem 2.10]; in the present paper we answer it in the positive.

We state our main result (Example 1). Recall that, for a bounded set $U \subset \mathbb{C}$, its *Hausdorff content of order* $\nu$ is the quantity

$$M^\nu(U) = \inf \sum_j (r_j)^\nu, \tag{1.3}$$

where $\nu \in (0, 2]$ and the infimum is taken over all coverings of $U$ by at most countable collections of disks $B_j$ of radii $r_j$ (the disks may be closed or open, no matter).

Recall also that the Hausdorff measure of order $\nu$ is defined as

$$H^\nu(U) = \lim_{\delta\to 0} \inf \sum_j (r_j)^\nu,$$

where we require additionally that the radii $r_j$ of the covering disks be not greater than $\delta$. The quantities $M^\nu(U)$ and $H^\nu(U)$ vanish simultaneously; if $0 < H^\nu(U) < \infty$, then the Hausdorff dimension of $U$ is equal to $\nu$, and if $M^\nu(U) > 0$, then the Hausdorff dimension of $U$ is at least $\nu$.

Fixing $\varepsilon > 0$, we consider the set of *admissible functions*

$$h(z) = z + \sum_j \frac{d_j}{z - z_j} \tag{1.4}$$

such that the sum over $j$ is at most countable, $d_j, z_j \in \mathbb{C}$, and the following conditions (1)–(3) are fulfilled:

(1) $|z_j| > 1$, $|d_j| \le |z_j| - 1$ for all $j$;

(2) the "refined" Blaschke condition $\sum_j (|z_j| - 1) < \varepsilon$ is satisfied;

(3) the function $h$ is bounded and continuous in the closed unit disk $\overline{\mathbb{D}}$.

Clearly, any such function $h$ is analytic in $\mathbb{D}$ and admits a Nevanlinna type continuation through $\mathbb{T}$. Indeed, if $B$ is the Blaschke product constructed by the zeros $z_j$, then, in the notations of Theorem 1, in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ we can take $f_2(z) = B(z)/z$ and $f_1 = hf_2$.

**Example 1.** For any $\varepsilon > 0$ and any $\nu < \log_2 3$, there exists a function $h$ satisfying (1.4) and (1)–(3) and such that $h$ maps $\overline{\mathbb{D}}$ conformally and univalently onto a simply connected Jordan domain $\overline{\Omega}$ bounded by a contour $\Gamma = \partial\Omega$ with $M^\nu(\Gamma) > 0$.

The construction of this example is based on the technical Lemma 3 on adding an "almost circle" to the boundary; this lemma can be used for constructing Nevanlinna fractals of various forms.

*Remark.* The dimension $\log_2 3$ is by no means critical; in the present paper simplicity of construction is preferred, rather than the dimension of the contour. The question about the existence of Nevanlinna contours of arbitrary dimension $\nu < 2$ and, possibly, Nevanlinna–Peano contours (i.e., having positive aria) requires further study.

Example 1 can be of interest in the theory of boundary-value problems in domains with fractal boundaries: by [2, Theorem 5.5 (1)], in the domain $\Omega$ as in Example 1 there exists a nonconstant bianalytic function continuous in $\overline{\Omega}$ and equal to zero on $\Gamma$.

## §2. Auxiliary functions and their properties

We introduce some auxiliary functions. Let $\epsilon$ and $\delta$ be positive numbers, and let

$$G(z) = G(z, \delta, \epsilon) = z + \frac{\epsilon}{\delta - z} \tag{2.1}$$

(in fact, this is the Zhukovskiĭ function up to a linear transformation). Clearly, the zeros of the derivative of $G$ are the points $\delta \pm i\sqrt{\epsilon}$, and $G$ is univalent in the half-plane $\Pi_- = \{x \le 0\}$ (where $z = x + iy$). Indeed, for $z_1 \ne z_2$, $z_1, z_2 \in \Pi_-$, we have

$$\frac{G(z_2) - G(z_1)}{z_2 - z_1} = 1 + \frac{\epsilon}{(\delta - z_2)(\delta - z_1)}, \tag{2.2}$$

where the product $(\delta - z_2)(\delta - z_1)$ takes no real negative values.

The closed disk of radius $r$ centered at $z_0$ will be denoted by $B(z_0, r)$, and the corresponding circle by $T(z_0, r)$; for $\lambda > 0$ we put $\lambda B(z_0, r) = B(z_0, \lambda r)$ and $\lambda T(z_0, r) = T(z_0, \lambda r)$.

Clearly, the function $\frac{\epsilon}{\delta - z}$ reshapes the axis $Oy$ to the circle $T(\epsilon/2\delta, \epsilon/2\delta)$.

In the sequel, we shall need the asymptotics of various expressions as $\delta \to 0$ when the ratio $\epsilon/\delta$ is fixed; moreover, it will be assumed that

$$\delta \leq 0.01; \quad \delta^{10/9} \leq \epsilon \leq 2\delta. \tag{2.3}$$

Under the mapping by the function $G$ (see 2.1)), the image of the $y$-axis is, obviously, symmetric with respect to the $x$-axis and is given parametrically by the formulas

$$x = \frac{\epsilon\delta}{t^2 + \delta^2}, \quad y = t + \frac{\epsilon t}{t^2 + \delta^2}, \quad -\infty < t < +\infty.$$

For $t \geq 0$, the function $y(t)$ has 2 points of local extremum, maximum and minimum, respectively, approximately equal to $t = \delta$ and $t = \sqrt{\epsilon}$, if we neglect the quantities of order of $\delta^2$. The coordinates of these points are approximately $(\epsilon/2\delta, \delta + \epsilon/2\delta)$ and $(\delta, 2\sqrt{\epsilon})$.

For $t \geq 0$ we can write

$$y(x) = \delta\sqrt{\frac{\epsilon}{\delta}x^{-1} - 1} + \sqrt{\frac{\epsilon}{\delta}x - x^2}.$$

If we fix $\epsilon/\delta$ and let $\delta$ go to zero, we see that on the interval $(0, \epsilon/\delta)$ the function $y(x)$ tends in $C^\infty$ to the function $\sqrt{\frac{\epsilon}{\delta}x - x^2}$, which gives us an arc of the circle $T(\epsilon/2\delta, \epsilon/2\delta)$.

In Figure 1 we show the mapping of the $y$-axis by the function $G$ as in (2.1) in the Cartesian coordinates for $\delta = \epsilon = 0.01$; note that $2\sqrt{\epsilon} = 0.2$.

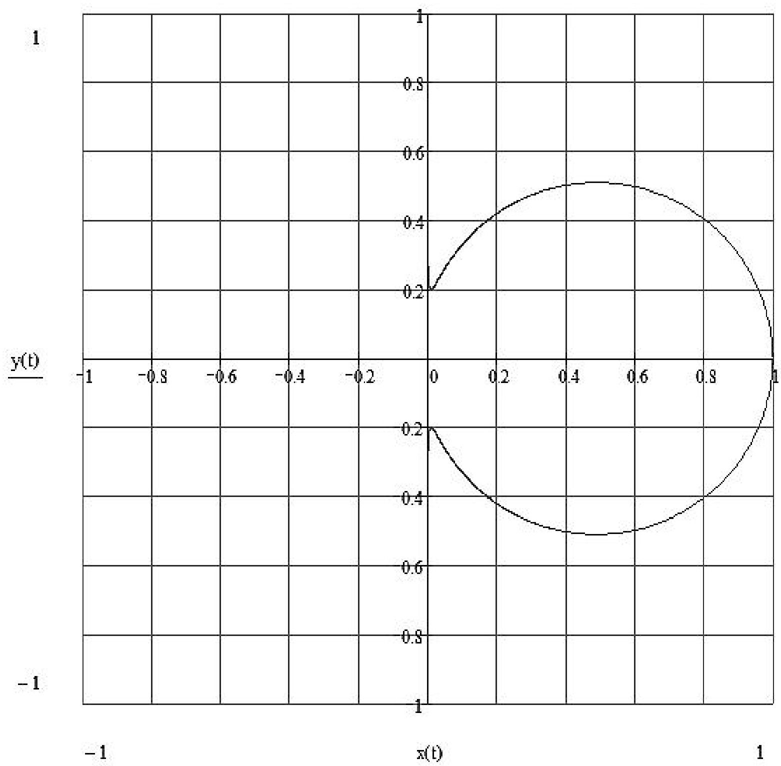


FIGURE 1. An "almost circle".

We state some simple sufficient conditions ensuring the preservation of univalence under variation of the function $G$.

**Lemma 1.** *Put*

$$F(z_1,z_2) = 1 + \frac{\epsilon}{(\delta - z_2)(\delta - z_1)}.$$

*Under conditions* (2.3), *let* $I_+^\delta$ *and* $I_-^\delta$ *be the intervals* $\{y > 2\delta\}$ *and* $\{y < -2\delta\}$ *of the* $y$*-axis. Then*:

(i) *for* $y_1, y_2 \in I_+^\delta$ *we have* $\operatorname{Im} F(iy_1, iy_2) > 0$, *for* $y_1, y_2 \in I_-^\delta$ *we have* $\operatorname{Im} F(iy_1, iy_2) < 0$, *and for the other* $y_1, y_2 \in \mathbb{R}$ *we have* $|F(iy_1, iy_2)| > 1/10$;

(ii)

$$\min_{z_1,z_2\in\Pi_-} |F(z_1,z_2)| > \frac{\delta}{\sqrt{\epsilon}}.$$

*Proof.* Statement (i) follows from considering the arguments of the expressions $(\delta - iy_2)$ and $(\delta - iy_1)$. Namely, for $y_1, y_2 \in I_+^\delta$ we have $\arg F(z_1,z_2) \in (\pi/2, \pi)$, for $y_1, y_2 \in I_-^\delta$ we have $\arg F(z_1,z_2) \in (-\pi, -\pi/2)$, and in the other cases we have $|\arg F(z_1,z_2)| \le \pi/2 + \arctan 2$. As to statement (ii), the maximum modulus principle allows us to assume that $z_1 = iy_1$, $z_2 = iy_2$. Then it suffices to explore the minimum of the expression

$$\left|1 + \frac{(\delta - iy_2)(\delta - iy_1)}{\epsilon}\right|^2 = \left(1 + \frac{\delta^2}{\epsilon} - \frac{y_1 y_2}{\epsilon}\right)^2 + \frac{\delta^2}{\epsilon^2}(y_1 + y_2)^2,$$

which can be done by direct calculation. □

**Corollary to Lemma 1.** *For* $A > 0$ *and* $G_1(z) = Az + \frac{\epsilon}{\delta - z} = A\big(z + \frac{\epsilon/A}{\delta - z}\big)$ *we have* $(z_1 \ne z_2)$:

$$\min_{z_1,z_2\in\Pi_-} \left|\frac{G_1(z_2) - G_1(z_1)}{z_2 - z_1}\right| = A \min_{z_1,z_2\in\Pi_-} \left|1 + \frac{\epsilon/A}{(\delta - z_2)(\delta - z_1)}\right| > \frac{\delta A \sqrt{A}}{\sqrt{\epsilon}}.$$

**Lemma 2.** *Let* $g(z)$ *be a function analytic in a neighborhood of the point* $z_0 = 0$, *with* $g'(0) = A > 0$ *and* $\operatorname{Re} g''(0) = A_1 > 0$, *and let* $G_2(z) = g(z) + \frac{\epsilon}{\delta - z}$, *with* $\epsilon, \delta$ *obeying* (2.3). *Then there exists* $\beta > 0$ *such that for arbitrary* $z_1 = iy_1$ *and* $z_2 = iy_2$ $(z_1 \ne z_2)$ *in the interval* $(-i\beta, i\beta)$ *of the* $y$*-axis we have*

$$\left|\frac{G_2(z_2) - G_2(z_1)}{z_2 - z_1}\right| > \frac{\delta A \sqrt{A}}{\sqrt{\epsilon}}. \tag{2.4}$$

*Proof.* We have the asymptotics

$$\frac{G_2(iy_2) - G_2(iy_1)}{i(y_2 - y_1)} = A + \frac{\epsilon}{(\delta - iy_2)(\delta - iy_1)} + i\frac{g''(0)}{2}(y_1 + y_2) + O(|y_1|^2 + |y_2|^2).$$

We take $\beta > 0$ so small that $O(|y_1|^2 + |y_2|^2) < A_1(|y_1| + |y_2|)/4$ and apply Lemma 1. If $y_1, y_2 \in I_+^\delta$, we use statements (i), (ii) and the fact that $\operatorname{Im}(ig''(0)(y_1 + y_2) > 0$; if $y_1, y_2 \in I_-^\delta$, then our arguments are similar; in the other cases of the location of $y_1$ and $y_2$ we employ the inequality $|F(iy_1, iy_2)| > 1/10$. The lemma is proved. □

*Remark.* Lemma 2 shows that the preservation of local univalence is determined by the terms of the expansion of $g$ of order at most two, and that the conditions $g'(0) > 0$ and $\operatorname{Re} g''(0) > 0$ suffice (in general these conditions are not necessary).

Under shift and rotation, the conditions of Lemma 2 are reshaped in an obvious way, because the invariant sense of the conditions $g'(0) > 0$ and $\operatorname{Re} g''(0) > 0$ is in the fact that the image under $g$ of the one-sided (relative to $\Pi_-$) neighborhood of $z_0 = 0$ is locally convex. Namely, $d^2 g$ makes an acute angle with the inner normal at the point $g(z_0)$, or, in other words,

$$0 < \arg \frac{d^2 g}{dg} < \pi. \tag{2.5}$$

In particular, suppose that $A > 0$, $\epsilon > 0$, $\psi \in (-\pi, \pi]$, $z_0 = iy_0$, $g$ is a function analytic in a neighborhood of $z_0$, and $g'(z_0) = Ae^{i\psi}$. Then, for sufficiently small $\beta > 0$, on the interval $(iy_0 - i\beta, iy_0 + i\beta)$ of the $y$-axis, the function

$$G_2(z) = g(z) + \frac{e^{i\psi}\epsilon}{\delta + iy_0 - z}$$

satisfies estimate (2.4) (recall that $dz = idy$) whenever

$$\operatorname{Re} \frac{g''(z_0)}{g'(z_0)} > 0.$$

Figure 2 illustrates Lemma 2 with shifts and various arguments of the derivative of $g$ taken into account; the $y$-axis is mapped by the function

$$\frac{1}{1-z} + \frac{\delta/2}{\delta - z} + \frac{i\delta/2}{\delta + i - z} - \frac{i\delta/2}{\delta - i - z}, \quad \delta < 0.01.$$

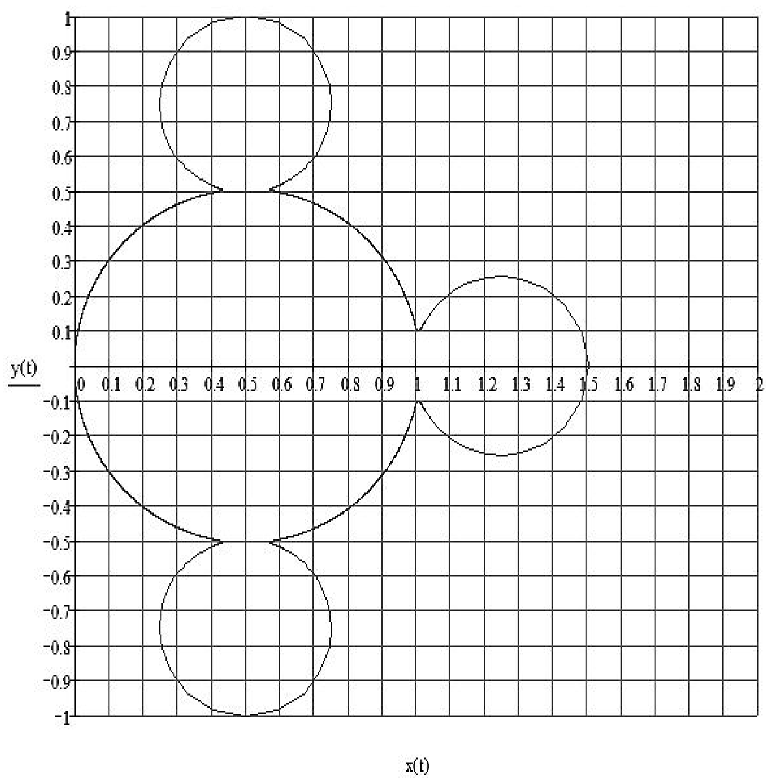


FIGURE 2. The scheme of iteration.

It is not hard to check that the result is a Jordan domain with analytic boundary. The boundary is structured as follows: four “almost circles”, a “base” one and three other circles of half radius (the “next generation”). We have taken into account that for the function $g(z) = \frac{1}{1-z}$ at the points $0, i, -i$ we have, respectively, $\psi = 0,\ \pi/2,\ -\pi/2$.

In the case of rotation, i.e., for the half-plane bounded by the line $z = z_0 + e^{i\alpha}t$ ($t \in \mathbb{R}$; when $t$ grows, the half-plane remains to the left), condition (2.5) becomes

$$0 < \arg \frac{e^{i\alpha} g''(z_0)}{g'(z_0)} < \pi. \tag{2.6}$$

In the construction of Example 1 we shall use functions of the form (1.4) analytic in the unit disk $\mathbb{D}$. Lemmas 1 and 2 are easily carried over from the half-plane $\Pi_-$ to $\overline{\mathbb{D}}$. The function

$$H(z) = z + \frac{\epsilon}{\delta + 1 - z}. \tag{2.7}$$

is univalent in the half-plane $\{x \le 1\}$ and satisfies Lemma 1 (up to shift).

On $\mathbb{T}$, we have the following well-known (see, e.g., [9, Chapter 4, §5]) sufficient condition for local convexity:

$$1 + \operatorname{Re} \frac{z_0 g''(z_0)}{g'(z_0)} > 0. \tag{2.8}$$

Obviously, this condition is weaker than (2.6), which takes the form

$$\operatorname{Re} \frac{z_0 g''(z_0)}{g'(z_0)} > 0 \tag{2.9}$$

for $z_0 = e^{it_0}$; note that in Lemma 3 (iv) we shall be able to obtain (2.9) in the case of "sufficiently massive" subsets of $\mathbb{T}$.

The next lemma will allow us to construct Nevanlinna domains with fractal boundaries via certain local variations of the boundary.

**Lemma 3** (on adding an "almost circle")**.** *Let $g$ be a function analytic in a neighborhood of $\overline{\mathbb{D}}$, and let $g'(z_0) = Ae^{i\psi}$, where $z_0 = e^{it_0} \in \mathbb{T}$ and $A > 0$. Suppose that $g$ maps $\mathbb{D}$ conformally and univalently onto a Jordan domain $E$ bounded by a closed* (*analytic*) *contour $L$, and that estimate* (2.8) *is fulfilled on an are of $\mathbb{T}$ containing $z_0$.*

*We introduce local Cartesian coordinates $(x, y)$ centered at $g(z_0)$, pointing the $x$-axis along the outer normal, i.e., in the direction $e^{i(\psi+t_0)}$, and the $y$-axis in the direction $e^{i(\psi+t_0+\pi/2)}$. By* (2.8)*, there exists $r$, $0 < r < 1/2$, such that for all sufficiently small $\lambda > 0$ the set $B(r, (1 + \lambda)r) \cap L$ is an analytic arc in the half-plane $\{x \le 0\}$ that is the image of some arc $\gamma(\lambda) \subset \mathbb{T}$ containing $z_0$. Fixing $r$ as indicated and taking $\delta \le 0.01$, consider the function*

$$H_\delta(z) = g(z) + \omega_\delta(z), \quad \omega_\delta(z) = \frac{e^{i(\psi+2t_0)}2r\delta}{(1+\delta)e^{it_0} - z}. \tag{2.10}$$

*Then the following is true.*

(i) *As $\delta \to 0$, for all $z \in \overline{\mathbb{D}}$ outside of an arbitrary neighborhood of $e^{it_0}$ the function $\omega_\delta(z)$ tends to zero in $C^\infty$.*

(ii) a) *For any $\lambda > 0$, there exists an arc $\gamma' \subset \mathbb{T}$ containing $z_0$ and such that its image under $H_\delta$ lies in the disk $B(r, (1 + \lambda)r)$ for all sufficiently small $\delta$.*

b) *For any arc $\gamma'' \subset \mathbb{T}$ containing $z_0$, there exists $\lambda > 0$ such that the image of $\mathbb{T} \setminus \gamma''$ under $H_\delta$ does not intersect $B(r, (1 + \lambda)r)$ for all sufficiently small $\delta$.*

(iii) *Fix a sufficiently small $\lambda > 0$ as in the assumptions of the lemma. Then for any $x_0 \in (0, 2r]$, as $\delta \to 0$, the functions* (*given parametrically*)

$$H_\delta(e^{it}) = \operatorname{Re} H_\delta(e^{it}) + i \operatorname{Im} H_\delta(e^{it}) = x(e^{it}, \delta) + iy(e^{it}, \delta)$$

*viewed as implicit functions of $x$ and $y$ converge on the set $\{x \ge x_0\} \cap B(r, (1 + \lambda)r)$ locally in $C^\infty$ to the equation of the circle $T(r, r)$.*

(iv) *Fix an arbitrary $c > 0$; then for all sufficiently small $\delta > 0$ and $|t - t_0| < c\delta$, the function $H_\delta(z) = H_\delta(e^{it})$ satisfies estimate* (2.9): $\operatorname{Re} \frac{zH''_\delta(z)}{H'_\delta(z)} > 0$.

(v) *For all sufficiently small* $\delta > 0$ *the function* $H_\delta$ *is univalent on* $\overline{\mathbb{D}}$, *and for* $z_1, z_2 \in \overline{\mathbb{D}}$ $(z_1 \neq z_2)$ *we have*

$$\left|\frac{H_\delta(z_2) - H_\delta(z_1)}{z_2 - z_1}\right| > \sqrt{\delta}\frac{A\sqrt{A}}{\sqrt{2r}}. \tag{2.11}$$

(vi) *For any* $\lambda$, $0 < \lambda < 1$, *and any two open sets* $E', E''$ *such that* $E' \subset E \subset E''$, *for all sufficiently small* $\delta > 0$ *the image of* $\overline{\mathbb{D}}$ *under* $H_\delta$ *is contained in* $E'' \cup B(r, (1+\lambda)r)$ *and contains* $E' \cup B(r, (1-\lambda)r)$.

*Proof.* In (2.10) we have $g'(z_0)/\omega'(z_0) = A\delta/(2r) > 0$; applying rotation and shift, we may assume in what follows that $z_0 = 1$, $t_0 = \psi = g(1) = 0$, so that

$$\omega_\delta(z) = \frac{2r\delta}{1 + \delta - z}.$$

Statement (i) is obvious, because for $z \neq 1$ and $n \in \mathbb{Z}_+$ we have

$$\lim_{\delta \to 0}\left(\frac{2r\delta}{\delta + 1 - z}\right)^{(n)} = 0.$$

To prove (ii) a), it suffices to observe that the function $\omega_\delta$ reshapes $\mathbb{T}$ to the circle $T_\delta$ that passes through the points $(2r, 0)$ and $(2r/(1+2\delta^{-1}), 0)$ and has center on the $x$-axis, and that $g$ is continuous at $z_0 = 1$; statement (i) reduces (ii) b) to a similar claim for the function $g$, which obviously follows from the assumptions of the lemma.

We prove (iii). Scaling allows us to write

$$H_\delta(e^{i\delta\tau}) = g(e^{i\delta\tau}) + \frac{2r}{1 + (1 - e^{i\delta\tau})\delta^{-1}}, \quad t = \delta\tau, \quad \tau \in (-\pi/\delta, \pi/\delta].$$

Fix an arbitrary $c > 0$; then for all $\tau$ with $|\tau| \le c$ we have

$$\lim_{\delta \to 0}\frac{2r}{1 + (1 - e^{i\delta\tau})\delta^{-1}} = \frac{2r}{1 - i\tau}.$$

This means that, as $\delta \to 0$, the equations of the circles $T_\delta$ converge in the half-plane $x > 0$ to the equation of $T(r, r)$. Since the centers and radii of $T_\delta$ converge in $C^\infty$, so do their equations; observe that the functions $g(e^{i\delta\tau})$ converge in $C^\infty$ to zero.

Since for any sequence $\{\tau_n\}$ with $|\tau_n| \to \infty$ we have

$$\lim_{n \to \infty}\lim_{\delta \to 0}\frac{2r}{1 + (1 - e^{i\delta\tau_n})\delta^{-1}} = 0,$$

to complete the proof of (iii) it suffices to observe that $B(r, (1+\lambda)r) \cap L$ is an arc in the half-plane $\{x \le 0\}$.

We prove statement (iv). Applying rotation, we assume that $t_0 = 0$. Observe that

$$\operatorname{Re}\frac{z\omega_\delta''(z)}{\omega_\delta'(z)} = \operatorname{Re}\frac{2e^{it}}{1 + \delta - e^{it}} = \operatorname{Re}\frac{2}{(1+\delta)e^{-it} - 1} > \delta^{-1}\operatorname{Re}\frac{2}{1 - ic} + O(1),$$

whence (iv) follows via the limit passage.

For the proof of (v), obviously, it suffices to establish estimate (2.11) on $\mathbb{T}$.

We apply Lemma 2. The function $z = \frac{1+w}{1-w}$ maps the half-plane $\Pi_-$ onto $\overline{\mathbb{D}}$. Since $A(z-1) = \frac{2Aw}{1-w}$, whence

$$A(z-1) + A'(z-1)^2 = 2Aw + 2A\left(1 + \frac{A'}{A}\right)w^2 + O(w^3),$$

condition (2.8) shows that, for $A > 0$, Lemma 2 and the identity $\epsilon = 2r\delta$) yield the following: there exists $\beta > 0$ such that on the arc $(e^{-i\beta}, e^{i\beta})$ of the circle $\mathbb{T}$ we have estimate (2.11) for all sufficiently small $\delta$.

By (ii) a), b), there exists $\lambda > 0$ and an arc $\gamma' = (e^{-i\beta'}, e^{i\beta'})$ such that for all sufficiently small $\delta$ the image of $\mathbb{T} \setminus (e^{-i\beta}, e^{i\beta})$ under $H_\delta$ does not intersect the disk $B(r, (1+\lambda)r)$, and the image of $\gamma'$ lies in the disk $B(r, (1+\lambda/2)r)$.

Finally, if at least one of the points $z_1, z_2$, does not belong to the arc $(e^{-i\beta}, e^{i\beta})$, while the other one belongs to $\gamma'$, then statement (v) follows from the location of the disks mentioned above, and if both points do not belong to $\gamma'$, then (v) follows from (i); we use the fact that the right-hand side of (2.11) tends to zero together with $\delta$.

Now we prove (vi). By (v), for all sufficiently small $\delta$ the image of $\mathbb{D}$ under $H_\delta$ is a simply connected Jordan domain. The fact that this image lies in $E'' \cup B(r, (1+\lambda)r)$ follows from (i) and (ii), and the fact that it includes $E' \cup B(r, (1-\lambda)r)$ is an additional consequence of (iii). The lemma is proved. □

## §3. Fractal of circles. Construction of Example 1

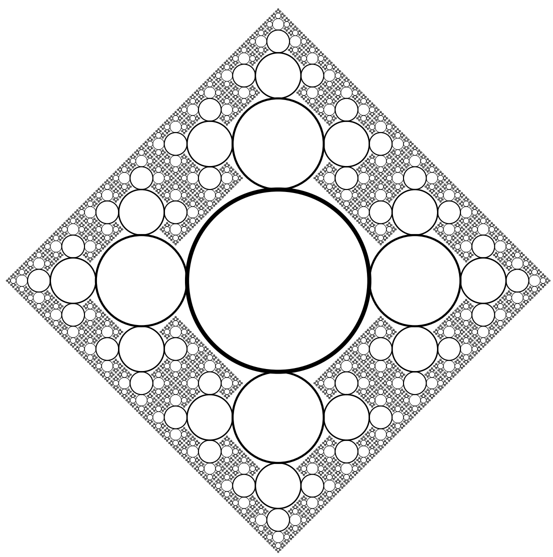

Figure 3. Fractal of circles.

In the sequel, $C$, $C_1$, $C_2, \dots$ will denote positive absolute constants, possibly different in different relations. Let $V$ and $W$ be some positive quantities. We say that $V$ is *not greater in order* than $W$ if there exists an absolute constant $C > 1$ such that $VC > W$, and if, moreover, $VC^{-1} < W < VC$, then $V$ and $W$ are said to be *comparable*.

The role of the model will be played by a fractal of circles the first five generations of which are depicted in Figure 3. The corresponding fractal is known (e.g., in problems on informatics), but we could not get to know its authorship.

We describe the general iteration scheme. Let $k \in (1/3, 1/2]$, and let $\psi$ take one of the four values $0$, $\pi/2$, $\pi$, $-\pi/2$. Let $T = (z_0, r)$ be a circle of generation $n \geq 1$, and let $T' = T(z_0 - e^{i\psi} r(1 + k^{-1}), k^{-1} r)$ be a circle of generation $n-1$ touching $T$. Then the circle $T$ gives rise to three circles of the (next) generation $n+1$:

$$T_j = T(z_0 + e^{i(\psi - \pi + \frac{\pi}{2} j)} r(1+k), rk), \quad j = 1, 2, 3. \tag{3.1}$$

Figure 3 corresponds to the case of $k = 1/2$, where we have a circle of generation 0, $T(0,1)$, and four circles of generation 1, namely, $T(3/2 e^{i\psi}, 1/2)$ with $\psi = 0, \pi/2, \pi, -\pi/2$. With each circle of generation $n \geq 1$, a unique multiindex $(j_1, j_2, \dots, j_n)$ is associated, where $j_1$ takes one of the values $1, 2, 3, 4$, while $j_m$ with the other $m$ is equal to $1, 2$, or $3$. A circle of generation $n$ gives rise to chains of circles of next generations with various $j_{n+1}, j_{n+2}, \dots$. The fractal in question is the closure of the set of points belonging to the circles of all generations.

The case of $k = 1/2$ is limiting in the sense that for $k > 1/2$ the disks $B(z, r)$ corresponding to all possible circles of various generations may have common inner points.

For $k = 1/2$ there are no such common inner points; different chains of circles (and the corresponding disks) may have at most common limit points. For example, the chains of circles generated by $T((3/2)i, 1/2)$ and $T(3/2, 1/2)$ are separated from each other by the line $y = x$; this line contains some limit points of the above chains, e.g., the point $3/2 + i3/2$. By induction, it is easy to show that the distance between the centers of two different circles of generation $n$ (and, consequently, of radius $2^{-n}$) is at least $2^{-n}3\sqrt{2}$.

For $k < 1/2$, the disks $B(z, r)$ that correspond to the circles $T(z, r)$ of different generations have no common inner points, and their different chains have no common limit points. We shall need the following property, which is easily verified by induction.

**Property A.** If $k < 1/2$ and $T_1$ and $T_2$ are different circles of generation $n$ obtained from one circle of generation $n - 1$, then the distances between arbitrary circles that belong to the chains born by $T_1$ and born by $T_2$ are not less than $k^n(1/2 - k)$.

This property may be veiwed as a "good analog" of the Jordan property, because it is not hard to obtain a Jordan curve by modifying our fractal locally in arbitrarily small neighborhoods of the points where circles of neighboring generations touch each other.

The next lemma is standard (see, e.g., [10, Theorem 8.6]).

**Lemma 4.** *Let $k \in (1/3, 1/2]$, let $\Theta$ be the closure of the points belonging to the circles of all generations $n \in \mathbb{Z}_+$, and let the following conditions be satisfied.*

(1) *Starting with some $n$, each circle $T = T(z_0, r)$ of generation $n$ gives rise to precisely three circles of generation $n + 1$, which are at a distance of at most $Cr$ from $T$.*

(2) *The radius of every circle of generation $n$ is comparable to $k^n$.*

(3) *For the circles $T(z, r)$ of an arbitrary generation $n$, the multiplicity of the intersections of the corresponding disks $B(z, r)$ does not exceed an absolute constant $C_1$ (in the case of coincidence, each circle is counted as many times as it occurs in the chains generated by different circles of previous generations).*

*Then the Hausdorff dimension of $\Theta$ is equal to $\mu = -\log_k 3$.*

*Proof.* Let $\Theta(n)$ denote the closure of the set of all points belonging to the circles of generations at least $n$; $\Theta = \Theta(0)$ is the compact set under consideration. Since for any $n$, the union of all circles of generation less than $n$ has finite length, in (1.3) it suffices to deal with coverings by disks corresponding to $\Theta(n)$.

By (1) and (2) for a circle $T = T(z_0, r)$, the disk $B(z_0, C_2 r)$ covers all chains of circles of the next generations born by $T$. Consequently, there exists $C_3 > 0$ such that the collection of all disks $B_j = B(z_j, C_3 k^n)$ corresponding to the circles of generation $n$ covers $\Theta(n)$. This covering will be denoted by $\Pi(n)$; we call $\Pi(n)$ a *regular* covering. By (1) and (3), for a regular covering we have

$$(C_4)^{-1}\big(3k^\nu\big)^n < \sum_j (r(B_j))^\nu < C_4\big(3k^\nu\big)^n$$

with $C_4 > 1$; therefore, if $3k^\nu < 1$, i.e., $\nu > -\log_k 3$, then in the notation of (1.3) we get $M^\nu(\Theta) = 0$.

Now, let $\nu = -\log_k 3$. For the regular coverings $\Pi(n)$ for all $n$ we have

$$(C_4)^{-1} < \sum_j (r(B_j))^\nu < C_4.$$

Consider an arbitrary covering $\Pi$ of the compact set $\Theta(0)$ by an at most countable family of open disks $B'_j$; to complete the proof, it remains to show that the quantity $\sum_j (r(B'_j))^\nu$ for $\Pi$ cannot be less in order than such sums for the regular coverings $\Pi(n)$ for all sufficiently large $n$. Since $\Theta(0)$ is a compact set, the Borel lemma allows us to extract a finite subcovering from $\Pi$. If $n$ is sufficiently large, then the radii of all disks of that subcovering are greater than $k^n$; instead of $\Pi$, it suffices to consider (eliminating some of the disks) a collection $\Pi'$ of disks $B'_j$ belonging to the above subcovering and such that they cover $\Theta(n)$ and each $B'_j$ intersects $\Theta(n)$; we also assume that $r(B'_j) < 1$.

Clearly, for any $B'_j$ there is a circle $T(z', r')$ of generation $m \le n$ such that $B'_j \subset B(z', C_5 r')$, and $r(B'_j)$ and $r'$ are comparable. Thus, for $\Pi'$ we have a covering $\{B''_j\}$ of the compact set $\Theta(n)$ formed by disks $B(z', C_5 r')$ corresponding to circles $T(z', r')$ of generations at most $n$ and such that the quantity $\sum_j (r(B''_j))^\nu$ is not greater in order than that quantity for $\Pi'$.

By (1)–(3) and the relation $3k^\nu = 1$, if we replace a disk $B(z', C_5 r')$ by all disks of class $\Pi(n)$ that intersect $B(z', C_5 r')$, then the sum $(C_3 k^n)^\nu$ over all new disks will not become greater in order compared to $(C_5 r')^\nu$ (indeed, by (1), (2) this is true for the disks corresponding to the circles of generation $n$ born by a circle $T(z', r')$ of generation $m \le n$, and by (3) the number of such circles of generation $m$ is bounded from above by an absolute constant).

Thus, for an arbitrary covering $\Pi$ and all sufficiently large $n$, there is a part of $\Pi(n)$ that covers $\Theta(n)$ and is such that the corresponding sum $(C_3 k^n)^\nu$ is not greater in order than the sum $\sum_j (r(B'_j))^\nu$ for $\Pi$. It remains to observe that, by (3), the number of disks in the part of $\Pi$ mentioned above is comparable to the total number of disks in $\Pi$, and that by (2) their sizes are comparable to one another. The lemma is proved. $\square$

**Corollary to Lemma 4.** *The fractal born by the procedure* (3.1) *with* $k \in (1/3, 1/2]$ *and coinciding with the closure of the set of points of the circles of all generations has Hausdorff dimension* $\mu = -\log_k 3$.

Now we start the construction of Example 1. We fix an arbitrary $k \in (1/3, 1/2)$. The choice of a sufficiently small $\lambda > 0$ and a sufficiently rapidly decaying sequence $\{\delta_n\}$, $\delta_n \searrow 0$, where $n$ denotes the step number, will be specified later. At least, it will be assumed that $k + 2\lambda < 1/2$, $\delta_1 < 0.01$, and $\delta_{n+1} < (\delta_n)^4$. The problem is in approximation of the circles occurring in the iteration scheme (3.1) by "almost circles" of Lemma 3. The number $\lambda$ controls the proximity in question, and the smallness of $\delta_{n+1}$ compared to $\delta_n$ makes it possible to achieve this, by statements (iii) and (vi) of Lemma 3. The passage to the next generation of "almost circles" is performed via the iteration pattern shown in Figure 2 (up to scaling): an "almost circle" of generation $n$ gives rise to three "almost circles" of the next generation. At every step $n \in \mathbb{N}$ we shall construct functions $\omega^{(n)}$ corresponding to the functions $\omega_\delta$ as in (2.10).

Step 1. In accordance with (2.10), consider functions of the form

$$\omega^{(1)}(z) = \frac{2k\delta_1}{\delta_1 + 1 - z} + \frac{2k\delta_1}{-(\delta_1 + 1) - z} + \frac{-2k\delta_1}{i(\delta_1 + 1) - z} + \frac{-2k\delta_1}{-i(\delta_1 + 1) - z};$$
$$h^{(1)}(z) = z + \omega^{(1)}(z).$$

By Lemma 3, the following is true for all sufficiently small $\delta_1 > 0$.

1) The function $h^{(1)}$ maps $\overline{\mathbb{D}}$ univalently onto a closed Jordan domain $\overline{\Omega}^{(1)}$ with analytic boundary. On $\mathbb{T}$, for $z \neq w$ we have

$$\left| \frac{h^{(1)}(z) - h^{(1)}(w)}{z - w} \right| > \sqrt{\delta_1}.$$

2) The domain $\Omega^{(1)}$ includes four disks: $B\big(0,(1-\lambda/2)\big)$ and $B\big((1+k)e^{i\psi},k(1-\lambda/2)\big)$, where $\psi=0,\pi/2,\pi,-\pi/2$, and the domain $\overline{\Omega}^{(1)}$ is contained in the four disks concentric to the previous ones, namely, in the disks $B\big(0,(1+\lambda/2)\big)$ and $B\big((1+k)e^{i\psi},\,k(1+\lambda/2)\big)$.

3) For $z_{2,1}^{(1)}=1$, the outer normal to $\Omega^{(1)}$ at the point $h^{(1)}(z_{2,1}^{(1)})$ is directed along the $x$-axis, and on $\mathbb{T}$ there are two points $z_{1,1}^{(1)}=e^{-i\delta_1+O((\delta_1)^2)}$ and $z_{3,1}^{(1)}=e^{i\delta_1+O((\delta_1)^2)}$ at the images of which under $h^{(1)}$ the outer normal to $\Omega^{(1)}$ goes, respectively, in the direction opposite to that of the $y$-axis, and along the $y$-axis. (Indeed, the claim about the existence of such points $z_{1,1}^{(1)}$ and $z_{3,1}^{(1)}$ is true for the function $\frac{2k\delta_1}{\delta_1+1-z}$, and then we can use statement (iv) of Lemma 3 and the fact that the derivatives of the remaining terms in $h^{(1)}$ are $O(1)$ as $\delta_1\to 0$).

Similar facts (up to rotation by an angle of size $\pi/2$, $\pi$, $-\pi/2$) are valid near the images of the points $i,-1,-i$, and on $\mathbb{T}$ we can find the corresponding triples of points $z_{j,s}^{(1)}$, where $j=1,2,3$, $s=2,3,4$, with the desired directions of the normal to $\Omega^{(1)}$.

The domain $\Omega^{(1)}$ is depicted in Figure 4 ($k=0.49$, $\delta_1<0.01$).

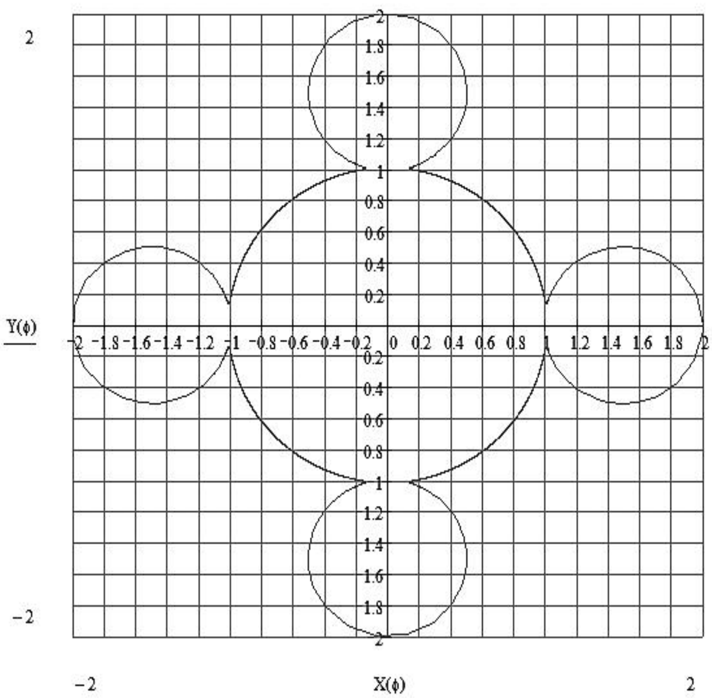


FIGURE 4. The first two generations of "almost circles".

At the next steps we act similarly. At step 2 we use the 12 points $z_{j,s}^{(1)}=e^{it_{j,s}^{(1)}}$ and consider the functions

$$\omega^{(2)}(z)=\sum_{s=1}^{4}\sum_{j=1}^{3}\frac{e^{i\left(\psi_{j,s}^{(1)}+2t_{j,s}^{(1)}\right)}2k^2\delta_2}{(1+\delta_2)e^{it_{j,s}^{(1)}}-z},\quad h^{(2)}(z)=h^{(1)}(z)+\omega^{(2)}(z), \tag{3.2}$$

where in the notation of Lemma 3 we have $g=h^{(1)}$, $t_0=t_{j,s}^{(1)}$, $\psi=\psi_{j,s}^{(1)}$. Recall that $\delta_{n+1}<(\delta_n)^4$ for all $n$; also, $\delta_2$ is chosen sufficiently small compared to $\delta_1$, whence we see that, by Lemma 3, the following properties, similar to 1)–3) for $\Omega^{(1)}$, hold true.

1) The function $h^{(2)}$ maps $\overline{\mathbb{D}}$ univalently onto a closed Jordan domain $\overline{\Omega}^{(2)}$ with analytic boundary; on $\mathbb{T}$ we have

$$\left|\frac{h^{(2)}(z)-h^{(2)}(w)}{z-w}\right| > \sqrt{\delta_2},$$

and for $z$ and $w$ lying outside of the disks of radius $(\delta_2)^{1/3}$ centered at $z^{(1)}_{j,s}$ we have

$$\left|\frac{h^{(2)}(z)-h^{(2)}(w)}{z-w}\right| > \frac{3}{4}\sqrt{\delta_1}$$

(indeed, outside of the above disks the function $\omega^{(2)}$ defined in (3.2) satisfies $\omega^{(2)}(z) = O((\delta_2)^{2/3})$ and $(\omega^{(2)})'(z) = O((\delta_2)^{1/3})$).

2) The domain $\Omega^{(2)}$ includes 17 disks: the "slightly reduced" disks $B(0,(1-3\lambda/4))$ and $B((1+k)e^{i\psi}, k(1-3\lambda/4))$, $\psi = 0,\ \pi/2,\ \pi,\ -\pi/2$, and 12 disks of the form $(1-\lambda/2)B^{(2)}_{j,s}$, where each $B^{(2)}_{j,s}$ is the disk of radius $k^2$ with center on the outer normal to $\Omega^{(1)}$ at the point $h^{(1)}(z^{(1)}_{j,s})$ that touches the boundary of $\Omega^{(1)}$ at that point; moreover, $\overline{\Omega}^{(2)}$ is contained in the union of 17 disks concentric to those mentioned above: $B(0,(1+3\lambda/4))$, $B((1+k)e^{i\psi}$ with $k(1+3\lambda/4))$, and $(1+\lambda/2)B^{(2)}_{j,s}$.

3) On the circle $\mathbb{T}$, we fix 36 points $z^{(2)}_{j,s}$ (where $s = 1,\dots,12$, $j = 1,2,3$) such that $z^{(2)}_{2,s}$ coincides with some $z^{(1)}_{j',s'}$, the points $z^{(2)}_{1,s}$ and $z^{(2)}_{3,s}$ are located on opposite sides of $z^{(2)}_{1,s}$ at a distance of $\delta_2 + O((\delta_2)^2)$, and the normal to $\Omega^{(2)}$ at every point $h^{(2)}(z^{(2)}_{j,s})$ is collinear to one of the coordinate axes.

Arguing similarly, at each step $n > 2$ we construct the functions

$$\omega^{(n)}(z) = \sum_{s=1}^{4\cdot 3^{n-1}} \sum_{j=1}^{3} \frac{e^{i\left(\psi^{(n-1)}_{j,s}+2t^{(n-1)}_{j,s}\right)}2k^n\delta_n}{(1+\delta_n)e^{it^{(n-1)}_{j,s}} - z}, \qquad h^{(n)}(z) = h^{(n-1)}(z) + \omega^{(n)}(z); \tag{3.3}$$

then the function $h$ required in (1.4) looks like this:

$$h(z) = z + \sum_{n=1}^{\infty} \omega^{(n)}(z). \tag{3.4}$$

Arguing much as at steps 1 and 2 considered above in detail, at the expense of a sufficient smallness of $\delta_{n+1}$ compared to $\delta_n$, at every step $n \geq 2$ we can use Lemma 3 and property A to ensure the following conditions A1)–A5).

A1) We have $\delta_n < (\delta_{n-1})^4$, and each point $z^{(n-1)}_{j',s'} = e^{it^{(n-1)}_{j',s'}}$ of generation $n-1$ on $\mathbb{T}$ gives rise to precisely three points of generation $n$: $z^{(n)}_{2,s} = z^{(n-1)}_{j',s'}$ and two points $z^{(n)}_{1,s}$ and $z^{(n)}_{3,s}$ located on $\mathbb{T}$ on opposite sides of $z^{(n)}_{1,s}$ at a distance of $\delta_n + O((\delta_n)^2)$.

A2) The function $h^{(n)}$ maps $\overline{\mathbb{D}}$ univalently onto a closed Jordan domain $\overline{\Omega}^{(n)}$ with analytic boundary, and at each of the points $h^{(n)}(z^{(n)}_{j,s})$ the normal to $\Omega^{(n)}$ is parallel to one of the coordinate axes.

A3) Each point $z^{(n)}_{j,s}$ gives rise to a unique circle $T^{(n)}_{j,s}$ of radius $k^n$, and when we pass from generation $n$ to generation $n+1$, any circle $T(z_0, r) = T^{(n)}_{j,s}$ gives birth to precisely three circles $T_j$ of generation $n+1$ in accordance with the following iteration scheme, which is a "slight variation" of (3.1):

$$T_j = T(z_0 + e^{i(\psi-\pi+\frac{\pi}{2}j)}r(1+k) + O(\delta_{n+1}/\delta_n), rk), \quad j = 1,2,3,$$

where $\psi$ is the angle between the normal to $\overline{\Omega}^{(n)}$ at the point $h^{(n)}(z_{j,s}^{(n)})$ and the $x$-axis.

A4) For each circle $T_{j,s}^{(n)}$, let $B_{j,s}^{(n)}$ be the disk with the same center and radius. Then, for all $m \le n$, $j$, and $s$, the disks $(1-\lambda)B_{j,s}^{(m)}$ are mutually disjoint and are contained in $\Omega^{(n)}$. The closed domain $\overline{\Omega}^{(n)}$ is included in the union of the disks $(1+\lambda)B_{j,s}^{(m)}$ for all $m \le n$, $j$, and $s$, and the disks $B(0,(1+\lambda))$ and $B((1+k)e^{i\psi}, k(1+\lambda))$ with $\psi = 0, \pi/2, \pi, -\pi/2$; the multiplicity of the intersections of the above disks containing $\overline{\Omega}^{(n)}$ is at most 2.

A5) On $\mathbb{T}$, for all $m = 2, \dots, n$, whenever points $z$ and $w$ ($z \ne w$) lie outside of the union of the disks of radius $(\delta_m)^{1/3}$ with centers $z_{j,s}^{(m-1)}$, we have the estimate

$$\left| \frac{h^{(n)}(z) - h^{(n)}(w)}{z - w} \right| > \frac{1}{2}\sqrt{\delta_{m-1}}, \tag{3.5}$$

and each disk of radius $(\delta_{m+1})^{1/3}$ centered at $z_{j,s}^{(m)}$ is included in some disk of radius $(\delta_m)^{1/3}$ centered at $z_{j',s'}^{(m-1)}$.

Compared to steps 1 and 2, explanation is required for the univalence of $h^{(n)}$ in A2), the bounded multiplicity of the disk's intersection in A4), and the estimates in A5). All this is a consequence of property A for $k < 1/2$, combined with a sufficient smallness of the variations (3.1) in A3) and also the condition $\delta_1 < 1/2 - k$.

We shall show that for any $\varepsilon > 0$ and any $\nu < \log_2 3$ there exists a function $h$ of the form (3.4) satisfying all the conditions of Example 1.

The function $h$ is admissible: condition (1) is obvious, condition (2) is satisfied by A1) with sufficiently small $\delta_1$ depending on $\varepsilon$, and condition (3) is implied by the convergence of the series $\sum_{n=1}^{\infty} k^n$ and a sufficient sparseness of the poles of $h$ by A1). We also note that, since $\delta_{n+1} < (\delta_n)^4$, for any $\nu > 0$ the closure of the set of poles of $h$ has Hausdorff content of order $\nu$ equal to zero.

The univalence of the function $h$ follows from (3.5) and Property A. Indeed, let $z, w \in \mathbb{T}$, $z \ne w$. If there exists an index $m$ such that none of the points $z$ and $w$ lies in the union of disks of radius $(\delta_m)^{1/3}$ centered at $z_{j,s}^{(m-1)}$, then the limit passage shows that $h$ satisfies (3.5). Otherwise, at least one of the points $z$ and $w$ lies in a nested sequence of such disks for all $m$, and then Property A and a sufficient smallness of the variations (3.1) in A3) show that the points $h(z)$ and $h(w)$ belong to the disjoint disks $(1+\lambda)B_{j_1,s_1}^{(m_1)}$ and $(1+\lambda)B_{j_2,s_2}^{(m_2)}$.

Thus, the function $h$, analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, is such that $h(z) \ne h(w)$ whenever $z \ne w$, $z$, $w \in \mathbb{T}$; consequently, $h$ maps $\overline{\mathbb{D}}$ conformally and univalently onto a closed and simply connected domain $\overline{\Omega}$ bounded by a contour $\Gamma = \partial\Omega$.

It remains to observe that, by Lemma 4 and conditions A3) and A4), the closure of each of the sets $\bigcup_{j,s,n}(1-\lambda)T_{j,s}^{(n)}$ and $\bigcup_{j,s,n}(1+\lambda)T_{j,s}^{(n)}$ has dimension $\mu = -\log_k 3$. Obviously, the sets of the limit points as $n \to \infty$ of the above closures coincide, lie on $\Gamma$, and have the same dimension.

Since $k$ can be as close to $1/2$ as we wish, the construction of Example 1 is complete.

## References


[1] M. Ya. Mazalov, P. V. Paramonov, and K. Yu. Fedorovskii, *Conditions for the $C^m$-approximability of functions by solutions of elliptic equations*, Uspekhi Mat. Nauk **67** (2012), no. 6, 53–100; English transl., Russian Math. Surveys **67** (2012), no. 6, 1023–1068. MR3075077

[2] Kh. Kh. Karmona, P. V. Paramonov, and K. Yu. Fedorovskii, *Uniform approximation by polyanalytic polynomials and the Dirichlet problem for bianalytic functions*, Mat. Sb. **193** (2002), no. 10, 75–98; English transl., Sb. Math. **193** (2002), no. 9–10, 1469–1492. MR1937036 (2004f:30027)

[3] K. Yu. Fedorovskii, *On uniform approximations of functions by $n$-analytic polynomials on rectifiable contours in* $\mathbb{C}$, Mat. Zametki **59** (1996), no. 4, 604–610; English transl., Math. Notes **59** (1996), no. 3–4, 435–439. MR1445202 (98b:41005)

[4] S. N. Mergelyan, *Uniform approximations of functions of a complex variable*, Uspekhi Mat. Nauk **7** (1952), no. 2, 31–122. (Russian) MR0051921 (14:547e)

[5] A. D. Baranov and K. Yu. Fedorovskii, *Boundary regularity of Nevanlinna domains and univalent functions in model subspaces*, Mat. Sb. **202** (2011), no. 12, 3–22; English transl., Sb. Math. **202** (2011), no. 11–12, 1723–1740. MR2919247

[6] K. Yu. Fedorovskii, *Approximation and boundary properties of polyanalytic functions*, Tr. Mat. Inst. Steklova **235** (2001), 262–271; English transl., Proc. Steklov Inst. Math. **235** (2001), no. 4, 251–260. MR1886586 (2003b:30045)

[7] M. Ya. Mazalov, *An example of nonrectifable Nevanlinna contour*, Algebra i Analiz **27** (2015), no. 4, 50–58; English transl., St. Petersburg Math. J. **27** (2016), no. 4, 625–630. MR3580191

[8] K. Yu. Fedorovskii, *On some properties and examples of Nevanlinna domains*, Tr. Mat. Inst. Steklova **235** (2006), 204–213; English transl., Proc. Steklov Inst. Math. **253** (2006), no. 2, 186–194. MR2338697 (2008g:30030)

[9] G. M. Goluzin, *Geometrical theory of functions of a complex variable*, 2nd ed., Nauka, Moscow, 1966; English transl., Transl. Math. Monogr., vol. 26, Amer. Math. Soc., Providence, RI, 1969. MR0247039 (40308)

[10] K. J. Falconer, *The geometry of fractal sets*, Cambridge Tracts Math., vol. 85, Cambridge Univ., Cambridge, 1986. MR0867284 (88d:28001)

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