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MATHEMATICAL PROBLEMS OF THE THEORY OF PHASE TRANSITIONS IN CONTINUUM MECHANICS

V. G. OSMOLOVSKIĬ

Abstract. The paper is a survey of the author's results related to variational problems for phase transitions in continuum mechanics. The main emphasis is on the study of the relationship between the solutions and the parameters of the problem, which allows one to trace the process of phase transitions when these parameters vary.

§1. Introduction

The stationary problem of phase transitions in continuum mechanics can be classified as a nonstandard problem of calculus of variatious. To motivate the mathematical setting (given in the next sections), in the introduction we describe the physical setting of the problem.

In quadratic approximation, the free energy density of a single-phase nonhomogeneous anisotropic medium that fills a domain $\Omega \subset \mathbb{R}^m$, $m = 1, 2, 3$, is written in the form

$$\begin{aligned} F(\nabla u, t', x) = a_{ijkl}(e_{ij}(\nabla u) - \zeta_{ij})(e_{kl}(\nabla u) - \zeta_{kl}) \\ - t'\kappa_{kl} a_{ijkl}(e_{ij}(\nabla u) - \zeta_{ij}) + F_0(t'), \end{aligned} \tag{1.1}$$

where $u = u(x)$, $x \in \Omega$, is the field of displacements,

$$(\nabla u)_{ij} = u^i_{x_j}, \quad e_{ij}(\nabla u) = 1/2(u^i_{x_j} + u^j_{x_i})$$

is the strain tensor, $\zeta_{ij} = \zeta_{ij}(x)$ is the residual strain tensor, $t' = t'(x)$ is the temperature deviation from a fixed value, $F_0(t')$ is a second order polynomial in t'. The functions a_{ijkl}, κ_{ij}, which are the coefficients of the polynomial F_0, depend on $x \in \Omega$. They are determined by the elastic and thermodynamic characteristics of a medium and obey traditional restrictions; summation from 1 to m is taken over the repeating indices.

Let g and f be the fields of volume and surface forces that act on the elastic medium. Then, for the density (1.1), the strain energy functional is defined by the formula

$$I[u, t'] = \int_\Omega F(\nabla u, t', x)\, dx - \int_\Omega g \cdot u\, dx - \int_{\partial\Omega} f \cdot u\, dS. \tag{1.2}$$

For fixed temperature distribution, the equilibrium displacement field $\widehat{u}$ is a solution of the variational problem

$$I[\widehat{u}, t'] = \inf_{u \in \mathbb{H}} I[u, t'], \quad \widehat{u} \in \mathbb{H}, \tag{1.3}$$

where $\mathbb{H}$ is the set of admissible fields of displacements determined by the boundary value of the function u on a part of the boundary (possibly empty or coinciding with $\partial\Omega$) of the domain Ω.

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For multiphase elastic media, in the process of deformation phase transitions occur, associated with a change in the crystal structure. For two-phase media, it is assumed that only two possibilities are realized, distinguished by the symbols $+$ and $-$, with collection of values $a^{\pm}_{ijkl}$, $\kappa^{\pm}_{ij}$, $\zeta^{\pm}_{ijkl}$, $F^{\pm}_0$ in the representation (1.1) of the energy density $F^{\pm}(\nabla u, t', x)$.

Denote by $\chi(x)$ the characteristic function of the subset of Ω corresponding to the phase with index $+$. Given the field of displacement u, the phase distribution χ, and the temperature t', for the strain energy of a two-phase elastic medium we have

$$\begin{aligned} I_0[u,\chi,t'] &= \int_\Omega \{\chi F^+(\nabla u, t', x) + (1-\chi)F^-(\nabla u, t', x)\}\,dx \\ &\qquad - \int_\Omega g\cdot u\,dx - \int_{\partial\Omega} f\cdot u\,dS, \\ F^{\pm}(\nabla u, t', x) &= a^{\pm}_{ijkl}(e_{ij}(\nabla u) - \zeta^{\pm}_{ij})(e_{kl}(\nabla u) - \zeta^{\pm}_{kl}) \\ &\qquad - t'\kappa^{\pm}_{kl} a^{\pm}_{ijkl}(e_{ij}(\nabla u) - \zeta^{\pm}_{ij}) + F^{\pm}_0(t'). \end{aligned} \tag{1.4}$$

By the equilibrium field of displacements $\widehat{u}$ and equilibrium phase distribution $\widehat{\chi}$ we understand the pair $\widehat{u}$, $\widehat{\chi}$ that, for given t', minimizes the energy functional

$$I_0[\widehat{u},\widehat{\chi},t'] = \inf_{\substack{u\in\mathbb{H},\\ \chi\in\mathbb{Z}'}} I_0[u,\chi,t'], \qquad \widehat{u}\in\mathbb{H}, \quad \widehat{\chi}\in\mathbb{Z}', \tag{1.5}$$

where $\mathbb{Z}'$ is the set of all characteristic functions. We emphasize that an unknown quantity in the variational problem (1.5) is not only the equilibrium field of displacements $\widehat{u}$, but also the phase distribution $\widehat{\chi}$. If in (1.4) we fix the function χ by setting $\chi = \widetilde{\chi}$, then problem (1.3) with $I_0[u,\widetilde{\chi},t']$ describes the equilibrium state of a composite material with the phase distribution fixed by the function $\widetilde{\chi}$.

The functional (1.4) consists of the strain energy of each phase and does not involve the surface energy of the boundary of their separation. The latter is traditionally assumed to be proportional to the area of the phase interface. We denote this area by $S[\chi]$ and the coefficient of proportionality (surface tension coefficient) by σ. Then the energy functional, taking the surface energy of the phase interface into account, has the form

$$I[u,\chi,t',\sigma] = I_0[u,\chi,t'] + \sigma S[\chi]. \tag{1.6}$$

By the state of equilibrium of the two-phase elastic medium with energy functional (1.6) for fixed t' and σ we understand the solution $\widehat{u}$, $\widehat{\chi}$ of the following variational problem:

$$I[\widehat{u},\widehat{\chi},t',\sigma] = \inf_{\substack{u\in\mathbb{H},\\ \chi\in\mathbb{Z}}} I[u,\chi,t',\sigma], \qquad \widehat{u}\in\mathbb{H}, \quad \widehat{\chi}\in\mathbb{Z}, \tag{1.7}$$

where $\mathbb{Z}$ is the set of all characteristic functions with finite phase interface area.

We investigate the properties of solutions of problems (1.5), (1.7) under the following additional restrictions: the two-phase elastic medium is homogeneous (the numbers $a^{\pm}_{ijkl}$, $\kappa^{\pm}_{ij}$, $\zeta^{\pm}_{ij}$, the polynomial coefficients $F^{\pm}_0$, and the temperature deviation t' do not depend on x), the volume-expansion coefficients $\kappa^{\pm}_{ij}$ are 0, the fields of force g and f are absent, and a fixing condition is satisfied on the boundary of the domain (for fields of displacement belonging to the set $\mathbb{H}$ the boundary condition $u|_{\partial\Omega} = 0$ is fulfilled).

Put $t = F^+_0(t') - F^-_0(t')$. Then, up to a term that does not affect the solutions of (1.5) and (1.7), the functionals $I_0[u,\chi,t']$ and $I[u,\chi,t',\sigma]$ can be replaced with

$$\begin{aligned} I_0[u,\chi,t] &= \int_\Omega \{\chi(F^+(\nabla u) + t) + (1-\chi)F^-(\nabla u)\}\,dx, \\ I[u,\chi,t,\sigma] &= I_0[u,\chi,t] + \sigma S[\chi], \\ F^{\pm}(\nabla u) &= a^{\pm}_{ijkl}(e_{ij}(\nabla u) - \zeta^{\pm}_{ij})(e_{kl}(\nabla u) - \zeta^{\pm}_{kl}), \end{aligned} \tag{1.8}$$

where a number t (assumed to be arbitrary) and a positive number σ play the role of parameters. In what follows, for simplicity, we call the number t temperature. The variation problems (1.5), (1.7) for the functionals (1.8) are replaced by the first and second problems

$$\begin{aligned} I_0[\widehat{u},\widehat{\chi},t] &= \inf_{\substack{u\in\mathbb{H},\\ \chi\in\mathbb{Z}'}} I_0[u,\chi,t], \quad \widehat{u}\in\mathbb{H}, \quad \widehat{\chi}\in\mathbb{Z}', \\ I[\widehat{u},\widehat{\chi},t,\sigma] &= \inf_{\substack{u\in\mathbb{H},\\ \chi\in\mathbb{Z}}} I[u,\chi,t,\sigma], \quad \widehat{u}\in\mathbb{H}, \quad \widehat{\chi}\in\mathbb{Z}, \end{aligned} \tag{1.9}$$

respectively.

Our goal is not only to establish the solvability or absence of solutions of these problems, but also investigate the dependence of solutions on t and σ in all possible details, and, thus, to demonstrate the possibility of using mathematical methods to characterize phase transformations when these parameters vary. The statement of the variational problem formulated above is traditional in the theory of elasticity, see [9, 13, 48]. The first step to multiphase media is the composite theory [10]. The suggested statement of the problem on phase transformations was formulated in [3]. Note that the macroscopic approach, according to which the energy of a two-phase object consists of the sum of the energies of both phases and the energy of their phase interface, is applicable to various problems. However, even in the theory of elasticity the above setting is only one among many possible settings [63, 64].

From a mathematical point of view, problems (1.9) belong to the class of nonconvex variational problems with free surface. To comment on these statements, we apply the minimization procedure to (1.9) in the variable $\chi\in\mathbb{Z}'$, reducing it to the variation problem

$$\begin{aligned} I_0^{\min}[\widehat{u},t] &= \inf_{u\in\mathbb{H}} I_0^{\min}[u,t], \quad \widehat{u}\in\mathbb{H}, \\ I_0^{\min}[u,t] &= \int_\Omega F^{\min}(\nabla u,t)\,dx, \\ F^{\min}(\nabla u,t) &= \min\bigl\{F^+(\nabla u)+t, F^-(\nabla u)\bigr\}. \end{aligned} \tag{1.10}$$

Obviously, the function $F^{\min}(\,.\,,t)$ fails to be convex for all t. The absence of convexity of the functional $I_0^{\min}[\,.\,,t]$ leads in some cases to the unsolvability of problem (1.10). Note that the addition of surface energy significantly improves the mathematical properties of the strain energy functional. In our case the term “free surface” means that the phase interface defined by the function $\widehat{\chi}$ is not fixed initially and is to be determined in the process of solving the problem.

Let us briefly describe the content of the paper. In §2, the model one-dimensional case is considered. In that section, no general theorems are used. All results are obtained “by hand”. This is done intentionally for the readers who specialize in mechames and do not want to go deeply into detail of mathematical proofs and are capable to believe that all (to be true, almost all) results obtained for the one-dimensional case extend to the multidimensional case. In the one-dimensional case, the states of equilibrium for a two-phase elastic medium exist in both models, when the surface energy of the phase interface boundary is taken into account, and when it is not. All states of equilibrium are found explicitly, which allows us to study their dependence on temperature, the surface-tension coefficient, and the size of a two-phase rod. Explicit formulas for phase transition temperatures and the volume of each phase of a two-phase equilibrium state are constructed, and the limit points of the equilibrium states are found as the surface tension coefficient approaches zero. The critical rod size is calculated such that two-phase equilibrium states do not exist for the smaller sizes. The relationship between the critical rod size and the coefficient of surface tension is found. The notion of a critical point for the energy functional and the chemical potential is introduced. A description of the set of all critical points is given, and the character of the critical points that are not states of equilibrium is established.

In §3, a multidimensional problem is investigated without taking into account the surface energy of the phase interface. For this problem, the energy functional is not very good from the mathematical viewpoint, because for arbitrary coefficients $a^{\pm}_{ijkl}$ in the energy density formula (1.8), there are tensors of residual deformation $\zeta^{\pm}$ such that the first problem in (1.9) is not solvable for some values of the temperature t. The situation improves significantly in the case of an isotropic two-phase medium. In this case (under some addition conditions) the first problem in (1.9) is solvable for all temperatures, and its solutions (possibly not all) have a fractal character and are represented explicitly for any domain Ω. The existence of explicit formulas allows us to repeat the results of the one-dimensional case. For anisotropic media, the results are more modest. In particular, we have managed to prove the existence of phase transitions temperatures independent of a domain and obtain two-sided estimates for them. These estimates allow us to formulate a criterion for the coincidence of the lower and upper temperatures of phase transitions. We give examples of anisotropic energy densities, for which, as in the isotropic case, explicit formulas for the phase-transition temperatures are obtained. In conclusion of that section, we derive equilibrium equations for a two-phase elastic medium with zero coefficient of surface tension in the case of a smooth phase interface. The stability of some classes of critical points is investigated.

In §4, we study a multidimensional problem for the functional $I[u, \chi, t, \sigma]$. The component $\sigma S[\chi]$, $\sigma > 0$, essentially improves the mathematical properties of the energy functional, which leads to the solvability of the second problem in (1.9) for each value of temperature. Using direct methods of calculus of variations, we study the dependence of the equilibrium states on the parameters t and σ at a qualitative level. In particular, we establish the character of the temperature dependence of the phase transitions on the coefficient σ, study the jump-like process of appearance of a new phase, and estimate the volume of its embryo. The contribution of the surface tension in the equilibrium equations is found and the role of single-phase critical points is determined. In that section, at a qualitative level, we repeat the results of §2 for the one-dimensional problem.

In §5, we study the behavior of equilibrium states of the functional $I[u, \chi, t, \sigma_n]$ as $\sigma_n \to 0$. It is established that this sequence is minimizing for the functional $I_0[u, \chi, t]$, and, consequently, converges in some sense to some minimizer of the relaxed variation problem. For a series of media, the quasiconvex hull of the density (1.10) is calculated, and a characterization of all minimizers of the relaxed problem is given. In the general case, we obtain a two-sided estimate for the quasiconvex hull. On the basis of our study of the behavior of the phase interface area as $\sigma \to 0$, we determine which of the minimizers of the relaxed problem is a limit point for equilibrium states.

A minor part of the above results was justified in [16]. Complete proofs of all statements are contained in the preprint [47].

Alternative methods of the investigation of similar problems (both close and not so close to those presented in this work) were given in the book [54]. There, one can also find an extensive bibliography. We shall not touch upon those methods with rare exceptions (see the bibliographical notes to §2), when the subject of study (but not the results) is almost analogous.

§2. The one-dimensional problem on phase transitions

Our goal in this section is, by the example of a one-dimensional variational problem, to understand what we may expect and what we should achieve in our study of the multi-dimensional case. We start our investigation with the one-dimensional problem because its solution admits an explicit characterization.

2.1. The setting of the problem. In the model one-dimensional case, we assume that the domain Ω is a segment $(0,l)$. The scalar function $u(x)$, $x\in(0,l)$, is a field of displacements, and the strain energy densities of each phase are defined by the formulas

$$F^{\pm}(M)=a_{\pm}(M-c_{\pm})^2,\qquad a_{\pm},c_{\pm},M\in\mathbb{R},\quad a_{\pm}>0. \tag{2.1}$$

If the surface tension coefficient is equal to zero, we define the strain energy of a two-phase elastic medium by the formula

$$I_0[u,\chi,t]=\int_0^l\{\chi(x)(F^+(u'(x))+t)+(1-\chi(x))F^-(u'(x))\}\,dx, \qquad u\in\mathbb{H},\ \ \chi\in\mathbb{Z}',\ \ t\in\mathbb{R}, \tag{2.2}$$

where $\mathbb{H}=\overset{\circ}{W}{}_2^1(\Omega)$, and $\mathbb{Z}'$ is the set of all measurable characteristic functions.

In the case of the zero surface tension coefficient, by the state of equilibrium of a two-phase elastic medium we mean the equilibrium field of displacements $\widehat{u}_t$ and the equilibrium phase distribution $\widehat{\chi}_t$ that, for fixed t, solve the following variational problem:

$$I_0[\widehat{u}_t,\widehat{\chi}_t,t]=\inf_{\substack{u\in\mathbb{H},\\ \chi\in\mathbb{Z}'}}I_0[u,\chi,t],\qquad \widehat{u}_t\in\mathbb{H},\quad \widehat{\chi}_t\in\mathbb{Z}'. \tag{2.3}$$

In the case where the surface-tension coefficient is positive, we need to change the set of all admissible phases by replacing $\mathbb{Z}'$ with $\mathbb{Z}$:

$$\begin{aligned}&\chi\in\mathbb{Z}\ \text{ provided that there exists a finite collection of open intervals}\\ &l_j\subset(0,l),\quad j=1,\dots,N[\chi],\qquad \overline{l}_i\cap\overline{l}_k=\varnothing\quad\text{for } i\neq k\text{, such that}\\ &\chi(x)=\begin{cases}1 & \text{if } x\in\bigcup_j l_j,\\ 0 & \text{if } x\notin\bigcup_j l_j,\end{cases}\qquad x\in(0,l).\end{aligned} \tag{2.4}$$

For each function $\chi\in\mathbb{Z}$ corresponding to the set of intervals l_j, $j=1,\dots,N[\chi]$, we denote by $S[\chi]$ the number of endpoints of l_j belonging to the interval $(0,l)$. The quantity $S[\chi]$ is viewed as the phase interface area with distribution function χ.

For a positive coefficient σ of the surface tension, the strain energy functional of a two-phase elastic medium is defined by the relation

$$I[u,\chi,t,\sigma]=I_0[u,\chi,t]+\sigma S[\chi],\qquad u\in\mathbb{H},\quad \chi\in\mathbb{Z},\quad t,\sigma\in\mathbb{R},\quad \sigma>0. \tag{2.5}$$

By the state of equilibrium we mean the equilibrium field of displacements and the equilibrium phase distribution $\widehat{u}_{t,\sigma}$ that for given t and σ, solve of the following variational problem:

$$I[\widehat{u}_{t,\sigma},\widehat{\chi}_{t,\sigma},t,\sigma]=\inf_{\substack{u\in\mathbb{H},\\ \chi\in\mathbb{Z}}}I[u,\chi,t,\sigma],\qquad \widehat{u}_{t,\sigma}\in\mathbb{H},\quad \widehat{\chi}_{t,\sigma}\in\mathbb{Z}. \tag{2.6}$$

We say that the state of equilibrium for problem (2.3) is *single-phase* if $\widehat{\chi}_t(x)=0$ or $\widehat{\chi}_t(x)=1$ almost everywhere on the interval $(0,l)$, and *two-phase* otherwise. Obviously, for single-phase states of equilibrium, $\widehat{u}_t\equiv0$.

The state of equilibrium for problem (2.6) is said to be single-phase if $\widehat{\chi}_{t,\sigma}(x)=0$ or $\widehat{\chi}_{t,\sigma}(x)=1$ on the interval $(0,l)$, and two-phase otherwise. Obviously, for the single-phase states of equilibrium we have $\widehat{u}_{t,\sigma}\equiv0$.

2.2. The problem with the zero surface tension coefficient. The following statement is the basis for solution of the variational problem (2.3).

Lemma 2.1. *For all $u \in \mathbb{H}$, $\chi \in \mathbb{Z}'$, and $t \in \mathbb{R}$, the functional* (2.2) *is representable in the form*

$$
\begin{gathered}
I_0[u,\chi,t] = \int_0^l \big(a_+\chi + a_-(1-\chi)\big)\big(u' - \alpha(Q)(\chi - Q)\big)^2\,dx + lG(Q,t),\\
Q = \frac{1}{l}\int_0^l \chi(x)\,dx, \quad \alpha(Q) = \frac{[ac]}{a_-Q + a_+(1-Q)},\\
G(Q,t) = tQ + a_+c_+^2 Q + a_-c_-^2(1-Q) - [ac]\alpha(Q)Q(1-Q),
\end{gathered}
\tag{2.7}
$$

where $[\alpha] = \alpha_+ - \alpha_-$ is the jump of the quantity α that takes two values α_+ and α_-.

Representation (2.7) leads to a characterization of the set of all solutions of problem (2.3).

Theorem 2.1. *The variational problem* (2.3) *is solvable. For each t, the set of all its solutions admits the following characterization*:

$$
\begin{gathered}
\widehat{\chi}_t \text{ is an arbitrary element of the set } \mathbb{Z}' \text{ for which } \frac{1}{l}\int_0^l \widehat{\chi}_t(x)\,dx = \widehat{Q}(t),\\
\text{and } \widehat{u}_t(x) \text{ is defined by the formula } \widehat{u}_t(x) = \alpha(\widehat{Q}(t))\int_0^x \big(\widehat{\chi}_t(y) - \widehat{Q}(t)\big)\,dy,
\end{gathered}
\tag{2.8}
$$

where $\widehat{Q}(t)$ is given by

$$
G(\widehat{Q}(t),t) = \min_{Q\in[0,1]} G(Q,t), \quad \widehat{Q}(t) \in [0,1]. \tag{2.9}
$$

To explore problem (2.9), we need the following notation:

$$
t_+ = t^* + \frac{[ac]^2}{a_+}, \quad t_- = t^* - \frac{[ac]^2}{a_-}, \qquad t^* = -[ac^2]. \tag{2.10}
$$

The numbers $t_\pm$ are called the temperatures (t_+ is the upper one, t_- is the lower one; obviously, $t_+ \geq t_-$) of phase transitions. The following lemma motivates these names.

Lemma 2.2. *Let $t_+ = t_-$. Then for $t < t^*$ the only solution of problem* (2.9) *is the number $\widehat{Q}(t) = 1$, for $t > t^*$ the only solution of this problem is the number $\widehat{Q}(t) = 0$, while if $t = t^*$, then any number in the interval $[0,1]$ serves as a solution $\widehat{Q}(t)$.*

Let $t_- < t_+$. Then for $t \leq t_-$ the only solution of problem (2.9) *is the number $\widehat{Q}(t) = 1$, for $t \geq t_+$ the only solution of this problem is the number $\widehat{Q}(t) = 0$, while if $t \in (t_-, t_+)$, then the solution $\widehat{Q}(t)$ of problem* (2.9) *is given by*

$$
\begin{gathered}
\widehat{Q}(t) = \begin{cases} h(t), & [a] = 0,\\ \frac{a_+ + a_-}{2[a]} + \frac{1}{2} - \frac{1}{[a]g^{1/2}(t)}, & [a] \neq 0, \end{cases}\\
h(t) = \frac{t_+ - t}{t_+ - t_-}, \quad g(t) = \frac{1}{a_-^2}h(t) + \frac{1}{a_+^2}(1 - h(t)).
\end{gathered}
\tag{2.11}
$$

The function $\widehat{Q}(t)$ is the volume fraction of the phase with index $+$ in the state of equilibrium. It is easily seen that for $t_- < t_+$, the function $\widehat{Q}(t)$ is continuous in the variable $t \in \mathbb{R}$ and strictly monotone decreasing on the interval $[t_-, t_+]$. On this interval, the function is concave for $[a] > 0$, convex for $[a] < 0$, and linear for $[a] = 0$.

Theorem 2.1 and Lemma 2.2 allows us to describe the phase transition process for a two-phase elastic medium with the energy functional (2.2) when the temperature t varies from very low to very high values.

The result of this characterization looks like this.

Let $t_- < t_+$. Then:

(1) for $t \in (-\infty, t_-]$, only the single-phase state of equilibrium $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 1$ is realized;
(2) for each $t \in (t_-, t_+)$, infinitely many different states of equilibrium are realized, all of them being two-phase with common volume fraction of the phase with index $+$, which is a single-valued function of temperature;
(3) for $t \in [t_+, \infty)$, only the single-phase state of equilibrium $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 0$ is realized.

Let $t_- = t_+ = t^*$. Then:

(4) for $t \in (-\infty, t^*)$, only the single-phase state of equilibrium $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 1$ is realized;
(5) for $t \in (t^*, +\infty)$, only the single-phase state of equilibrium $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 0$ is realized;
(6) for $t = t^*$, the set of all states of equilibrium is exhausted by the pairs $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 1$ and $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 0$ as well as by an infinite family of different two-phase states of equilibrium with an arbitrary volume fraction of the phase with index $+$ and the zero field of displacements $\widehat{u}_t$.

Therefore, for sufficiently low and sufficiently high values of temperature, a two-phase elastic medium consists of a substance of only one phase (for low temperature $\widehat{\chi}_t \equiv 1$, and for high temperature $\widehat{\chi}_t \equiv 0$).

The question about the stability of these pairs arises in the case of temperatures t for which they are not states of equilibrium of the two-phase elastic medium under study. The following theorem answers this question. To formulate the theorem, we need a definition.

We say that, for some t, the pair $\widetilde{u} \in \mathbb{H}$, $\widetilde{\chi} \in \mathbb{Z}'$ is a saddle point for the energy functional (2.2) if for any $\delta > 0$ there exist functions $v_\pm \in \mathbb{H}$, $\psi_\pm \in \mathbb{Z}'$ such that

$$\begin{gathered} \|v_\pm\|_{W_2^1} < \delta, \quad \|\widetilde{\chi} - \psi_\pm\|_{L_1} < \delta, \\ I_0[\widetilde{u} + v_+, \psi_+, t] > I_0[\widetilde{u}, \widetilde{\chi}, t], \quad I_0[\widetilde{u} + v_-, \psi_-, t] < I_0[\widetilde{u}, \widetilde{\chi}, t]. \end{gathered} \tag{2.12}$$

When we study the stability of the pair $\widetilde{u}$, $\widetilde{\chi}$ with respect to perturbations of $\widetilde{u}$ weaker than energy ones (we need a perturbation in the class $W_\infty^1(0, l) \cap \mathbb{H}$), in the definition of a saddle point (2.12) we replace the norm W_2^1 with the norm W_∞^1.

Theorem 2.2. (1). *If for given t, one of the pairs $u \equiv 0$, $\chi \equiv 1$ or $u \equiv 0$, $\chi \equiv 0$ is not a solution of problem* (2.3)*, then this pair is a saddle point for the energy functional* (2.2).

(2). *For $t_- < t_+$ and $t \in (t_-, t^*)$, the pair $u \equiv 0$, $\chi \equiv 1$ is a local minimum with respect to $W_\infty^1(0, l)$-small perturbations of the function u and arbitrary perturbations of the function χ in the space $L_1(0, l)$. For $t > t^*$, this pair is a saddle point for the functional* (2.2) *with respect to the perturbations mentioned above.*

(3). *For $t_- < t_+$ and $t \in (t^*, t_+)$, the pair $u \equiv 0$, $\chi \equiv 0$ is a local minimum with respect to $W_\infty^1(0, l)$-small perturbations of the function u and arbitrary perturbations of the function χ in the space $L_1(0, l)$. For $t < t^*$, this pair is a saddle point for the functional* (2.2) *with respect to the perturbations mentioned above.*

2.3. The problem with a positive surface tension coefficient. As in the previous subsection, solution of problem (2.6) is based on the representation (2.7) of the functional (2.2). For the functional (2.5), it has the following form:

$$\begin{aligned} I[u,\chi,t,\sigma] &= \int_0^l \big(a_+\chi + a_-(1-\chi)\big)\big(u' - \alpha(Q)(\chi - Q)\big)^2\,dx + J[\chi,t,\sigma],\\ J[\chi,t,\sigma] &= lG(Q,t) + \sigma S[\chi], \quad u \in \mathbb{H}, \quad \chi \in \mathbb{Z},\\ Q &= \frac{1}{l}\int_0^l \chi(x)\,dx, \quad t,\sigma \in \mathbb{R}, \quad \sigma > 0. \end{aligned} \tag{2.13}$$

In this case, an analog of (2.9) is the problem

$$J[\widehat{\chi}_{t,\sigma}, t, \sigma] = \inf_{\chi\in\mathbb{Z}} J[\chi,t,\sigma], \quad \widehat{\chi}_{t,\sigma} \in \mathbb{Z}. \tag{2.14}$$

Lemma 2.3. *The problem* (2.14) *is solvable. If for some t and σ, among all its solutions there is a solution $\widehat{\chi}_{t,\sigma}$ with the quantity*

$$\widehat{Q}(t,\sigma) = \frac{1}{l}\int_0^l \widehat{\chi}_{t,\sigma}(x)\,dx \tag{2.15}$$

different from zero and one, then $\widehat{\chi}_{t,\sigma}$ is the characteristic function of any of the segments

$$(0, l\widehat{Q}(t,\sigma)), \quad (l(1-\widehat{Q}(t,\sigma)), l). \tag{2.16}$$

The representation (2.13) and Lemma 2.3 allow us to justify the solvability of the variational problem (2.6).

Theorem 2.3. *The variational problem* (2.6) *is solvable. For each t and σ, the set of all its solutions admits the following characterization*:

$$\begin{aligned} &\widehat{\chi}_{t,\sigma} \text{ is an arbitrary solution of the problem } (2.14),\\ &\widehat{u}_{t,\sigma}(x) = \alpha(\widehat{Q}(t,\sigma))\int_0^x \{\widehat{\chi}_{t,\sigma}(y) - \widehat{Q}(t,\sigma)\}\,dy, \end{aligned} \tag{2.17}$$

where $\widehat{Q}(t,\sigma)$ is calculated by the function $\widehat{\chi}_{t,\sigma}$ in accordance with (2.15).

Theorem 2.3 shows that, in order to describe the set of all solutions of problem (2.6), we need a detailed characterization of all solutions of problem (2.14). This requires a series of additional considerations.

We introduce the function $\sigma(t)$, $t \in \mathbb{R}$, that acts by the following rule:

$$\begin{aligned} &\sigma(t) \equiv 0 \quad \text{if} \quad t_- = t_+,\\ &\sigma(t) = 0 \quad \text{for} \quad t \in (-\infty, t_-] \cup [t_+, \infty) \quad \text{if} \quad t_- < t_+. \end{aligned} \tag{2.18}$$

For $t_- < t_+$ and $t \in [t_-, t_+]$, we set (the function $g(t)$ was defined in (2.11))

$$\begin{aligned} &\text{for } [a] = 0:\\ &\sigma(t) = \frac{l}{2}\frac{(t-t_-)^2}{t_+ - t_-} \quad \text{if } t \in [t_-, t^*],\\ &\sigma(t) = \frac{l}{2}\frac{(t-t_+)^2}{t_+ - t_-} \quad \text{if } t \in [t^*, t_+];\\ &\text{for } [a] \neq 0:\\ &\sigma(t) = \frac{l}{2[a]}([a] - (a_+ + a_-))(t - t_-) - 2l\frac{a_+a_-[ac]^2}{[a]^2}(g^{1/2}(t) - g^{1/2}(t_-)),\\ &\qquad t \in [t_-, t^*],\\ &\sigma(t) = \frac{l}{2[a]}([a] + (a_+ + a_-))(t_+ - t) + 2l\frac{a_+a_-[ac]^2}{[a]^2}(g^{1/2}(t_+) - g^{1/2}(t)),\\ &\qquad t \in [t^*, t_+]. \end{aligned} \tag{2.19}$$

It can be checked that for $t_- < t_+$, the function $\sigma \in C(\mathbb{R}) \cap C^1(-\infty, t^*] \cap C^1[t^*, \infty)$ is strictly convex and infinitely differentiable on the intervals $[t_-, t^*]$, $[t^*, t_+]$, is positive for $t \in (t_-, t_+)$, and takes its maximum value σ^* at the point t^*:

$$\sigma^* = \sigma(t^*) = \frac{l[ac]^2}{(\sqrt{a_+} + \sqrt{a_-})^2}. \tag{2.20}$$

Relation (2.20) holds true in both cases, provided that $[a] = 0$ or $[a] \neq 0$. Note that for $t_+ = t_-$, this also gives the true result $\sigma^* = 0$.

The graph of the nonnegative function $\sigma(t)$ splits the half-plane of parameters $t, \sigma \in \mathbb{R}$, $\sigma > 0$, in the following regions:

$$\begin{aligned} V_< &= \{t, \sigma \,:\, \sigma \in (0, \sigma(t))\}, \\ V_>^- &= \{t, \sigma \,:\, t < t^*, \sigma > \sigma(t)\}, \\ V_>^+ &= \{t, \sigma \,:\, t > t^*, \sigma > \sigma(t)\}, \\ V_>^* &= \{t, \sigma \,:\, t = t^*, \sigma > \sigma(t^*)\}, \\ V_=^- &= \{t, \sigma \,:\, t \in (t_-, t^*), \sigma = \sigma(t)\}, \\ V_=^+ &= \{t, \sigma \,:\, t \in (t^*, t_+), \sigma = \sigma(t)\}, \\ V_=^* &= \{t, \sigma \,:\, t = t^*, \sigma = \sigma(t^*)\}. \end{aligned} \tag{2.21}$$

Obviously, for $t_- = t_+$, the sets $V_<$, $V_=^{\pm}$, $V_=^*$ are empty. It is important to note that by Lemma 2.2,

$$\text{if } t_- < t_+, \text{ then } \widehat{Q}(t) \in (0,1) \text{ in the regions } V_<, V_=^{\pm}, V_=^*. \tag{2.22}$$

The following lemma gives a complete characterization of the set of all solutions of problem (2.14) for each of the regions (2.21).

Lemma 2.4. *For all solutions of problem* (2.14), *the quantity* (2.15) *admits the following characterization*:

$$\widehat{Q}(t,\sigma) = \begin{cases} \widehat{Q}(t) & \textit{if } t,\sigma \in V_<, \\ 1 & \textit{if } t,\sigma \in V_>^-, \\ 0 & \textit{if } t,\sigma \in V_>^+, \end{cases} \qquad \widehat{Q}(t,\sigma) = \begin{cases} 0 \textit{ and } 1 & \textit{if } t,\sigma \in V_>^*, \\ 1 \textit{ and } \widehat{Q}(t) & \textit{if } t,\sigma \in V_=^-, \\ 0 \textit{ and } \widehat{Q}(t) & \textit{if } t,\sigma \in V_=^+, \end{cases}$$
$$\widehat{Q}(t,\sigma) = \{0, 1, \widehat{Q}(t)\} \quad \textit{if } t,\sigma \in V_=^*, \tag{2.23}$$

where $\widehat{Q}(t)$ is the solution of problem (2.9).

For the problem with positive surface tension coefficient we also introduce the temperatures of phase transitions $t_{\pm}(\sigma)$, $\sigma > 0$, keeping the notation (2.10) for $t_{\pm}$ and t^* used in the case where the surface tension coefficient is zero. We put

$$\begin{aligned} t_+(\sigma) = t_-(\sigma) = t^* \text{ for } t_+ = t_- = t^*, &\quad \sigma > 0, \\ \text{and for } t_- < t_+, &\quad \sigma \geq \sigma^*, \end{aligned} \tag{2.24}$$

and let $t_{\pm}(\sigma)$ be the smallest and the greatest of the two solutions of the equation

$$\sigma = \sigma(t) \text{ for } t_- < t_+, \;\; 0 < \sigma < \sigma^*. \tag{2.25}$$

We call the numbers $t_{\pm}(\sigma)$ the upper and lower temperatures of phase transitions for a two-phase elastic medium with the positive surface-tension coefficient σ.

In accordance with the definition, only in the case of (2.25) the numbers $t_{\pm}(\sigma)$ are different and are not defined explicitly. Their explicit form is given by the following lemma. We use the notation (2.11) for the function $g(t)$.

Lemma 2.5. *Suppose $t_- < t_+$, $\sigma \in [0, \sigma^*]$. Then for $[a] = 0$ we have*

$$
\begin{aligned}
&t_-(\sigma) = (t^* - t_-)\sqrt{\frac{\sigma}{\sigma^*}} + t_-, \quad t_+(\sigma) = (t^* - t_+)\sqrt{\frac{\sigma}{\sigma^*}} + t_+,\\
&\widehat{Q}(t_-(\sigma)) = 1 - \frac{t^* - t_-}{t_+ - t_-}\sqrt{\frac{\sigma}{\sigma^*}}, \quad \widehat{Q}(t_+(\sigma)) = \frac{t_+ - t^*}{t_+ - t_-}\sqrt{\frac{\sigma}{\sigma^*}};
\end{aligned}
\tag{2.26}
$$

for $[a] \neq 0$ we have

$$
\begin{aligned}
t_\pm(\sigma) &= \frac{t^* - t_\pm}{g(t^*) - g(t_\pm)}\Big((g^{1/2}(t^*) - g^{1/2}(t_\pm))\sqrt{\frac{\sigma}{\sigma^*}} + g^{1/2}(t_\pm)\Big)^2\\
&\qquad + \frac{t_\pm g(t^*) - t^* g(t_\pm)}{g(t^*) - g(t_\pm)},\\
\widehat{Q}(t_\pm(\sigma)) &= \widehat{Q}(t^*) + \frac{\widehat{Q}(t^*) - \widehat{Q}(t_\pm)}{g^{1/2}(t^*) - g^{1/2}(t_\pm)} g^{1/2}(t_\pm)\\
&\qquad \times \left(1 - \frac{g^{1/2}(t^*)}{(g^{1/2}(t^*) - g^{1/2}(t_\pm))\sqrt{\frac{\sigma}{\sigma^*}} + g^{1/2}(t_\pm)}\right),\\
\widehat{Q}(t_+) &= 0, \quad \widehat{Q}(t_-) = 1, \quad \widehat{Q}(t^*) = \frac{1}{2}\left(1 + \frac{[a]}{(\sqrt{a_+} + \sqrt{a_-})^2}\right).
\end{aligned}
\tag{2.27}
$$

Theorem 2.3 and Lemmas 2.4, 2.5 allows us to describe the phase transition process for a two-phase elastic medium with the energy functional (2.5) when the temperature t varies from very low to very high values. In the statements below, we use the pairs $\widehat{u}_t^\pm$, $\widehat{\chi}_t^\pm$, where

$\widehat{\chi}_t^+$ is the characteristic function of the interval $(0, l\widehat{Q}(t))$,

$\widehat{\chi}_t^-$ is the characteristic function of the interval $(l(1 - \widehat{Q}(t)), l)$,

and the functions $\widehat{u}_t^\pm$ is obtained from $\chi_t^\pm$ via formula (2.8).

The result of this description looks like this. Let $t_- < t_+$ and $\sigma \in (0, \sigma^*)$. Then:

(1) for $t \in (-\infty, t_-(\sigma))$, only the single-phase state of equilibrium $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 1$ is realized;
(2) for $t = t_-(\sigma)$, the set of all states of equilibrium consists of the single-phase state $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 1$ and the two two-phase states $\widehat{u}_t = \widehat{u}^\pm_{t_-(\sigma)}$, $\widehat{\chi}_t = \widehat{\chi}^\pm_{t_-(\sigma)}$;
(3) for $t \in (t_-(\sigma), t_+(\sigma))$, only the two two-phase states of equilibrium $\widehat{u}_t = \widehat{u}_t^\pm$, $\widehat{\chi}_t = \widehat{\chi}_t^\pm$ are realized;
(4) for $t = t_+(\sigma)$, the set of all states of equilibrium consists of the single-phase state $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 0$ and the two two-phase states $\widehat{u}_t = \widehat{u}^\pm_{t_+(\sigma)}$, $\widehat{\chi}_t = \widehat{\chi}^\pm_{t_+(\sigma)}$;
(5) for $t \in (t_+(\sigma), \infty)$, only the single-phase state of equilibrium $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 0$ is realized;
(6) the volume fraction of the phase with index + is the function $\widehat{Q}(t, \sigma)$ equal to one when $t < t_-(\sigma)$, and to zero when $t > t_+(\sigma)$. For $t \in (t_-(\sigma), t_+(\sigma))$, it coincides with $\widehat{Q}(t)$. It takes the two values 1, $\widehat{Q}(t_-(\sigma))$ for $t = t_-(\sigma)$ and 0, $\widehat{Q}(t_+(\sigma))$ for $t = t_+(\sigma)$.

Let $t_- < t_+$, $\sigma = \sigma^*$. Then:

(7) for $t \in (-\infty, t^*)$, only the single-phase state of equilibrium $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 1$ is realized;
(8) for $t = t^*$, the set of all states of equilibrium consists of the two single-phase states $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 1$ and $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 0$ and the two two-phase states $\widehat{u}_t = \widehat{u}^\pm_{t^*}$, $\widehat{\chi}_t = \widehat{\chi}^\pm_{t^*}$;

(9) for $t \in (t^*, \infty)$, only the single-phase state of equilibrium $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 0$ is realized;

(10) the volume fraction of the phase with index $+$ is the function $\widehat{Q}(t,\sigma)$ equal to one for $t < t^*$ and to zero for $t > t^*$ and taking the three values 1, $\widehat{Q}(t^*)$, 0 for $t = t^*$.

Let $t_- < t_+$ and $\sigma > \sigma^*$ or $t_- = t_+$ and $\sigma > 0$. Then:

(11) for $t \in (-\infty, t^*)$, only the single-phase state of equilibrium $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 1$ is realized;

(12) for $t = t^*$, only the two states of equilibrium $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 1$ and $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 0$ are realized;

(13) for $t \in (t^*, \infty)$, only the single-phase state of equilibrium $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 0$ is realized.

It turns out that, in contrast to the case where $\sigma = 0$ (Theorem 2.2), for $\sigma > 0$ each of the pairs $u \equiv 0$, $\chi \equiv 1$ and $u \equiv 0$, $\chi \equiv 0$ is a local minimum of the energy functional for all t.

Theorem 2.4. *For each t, the energy functional* (2.5) *has local minima at the pairs $u \equiv 0$, $\chi \equiv 1$ and $u \equiv 0$, $\chi \equiv 0$ with respect to any perturbations u of class $\mathbb{H}$ and sufficiently $L_1(0,l)$-small perturbations χ class $\mathbb{Z}$.*

We mention a number of distinctions in the process of phase transformations between the cases of zero and positive surface tension coefficients.

The first distinction is a jump-like beginning of a new phase when t varies in the case where $\sigma > 0$, in contrast to a continuous behavior in the case of $\sigma = 0$. In the first case, the existence of jumps of the function $\widehat{Q}(\,.\,,\sigma)$ at the points $t_\pm(\sigma)$ follows from the characterization (1)–(13). The exact values of the volume fraction of the embryo of a new phase when the parameter t crosses the temperature of phase transitions for $t_- < t_+$ and $\sigma \in (0, \sigma^*]$ are given in formulas (2.26), (2.27) for the quantities $\widehat{Q}(t_\pm(\sigma))$. For $t_- = t_+$ or $t_- < t_+$ and $\sigma > \sigma^*$, during the phase transition (when $t = t^*$) a jump-like change of the phase with index $+$ to the phase with index $-$ occurs on the whole interval $(0,l)$. In the second case, the continuity of formation of a new phase follows from the continuity of the function $\widehat{Q}(t)$ for $t_- < t_+$ and the fact that its values fill the interval $[0,1]$ for $t_- = t_+ = t^*$. Note that, for $\sigma > 0$, a new phase starts at one of the endpoints of the interval $(0,l)$, while for $\sigma = 0$ it can appear in any part of the interval.

The second distinction is the dependence of the temperatures $t_\pm(\sigma)$ of phase transitions on the length l of the segment in the case where $t_- < t_+$. The parameter l is involved in formulas (2.26), (2.27) for $t_\pm(\sigma)$ via the quantity σ^* given by (2.20). To take this relationship into account, we add the variable l to the list of arguments of the phase transition temperatures, using the notation $t_\pm(\sigma, l)$. It is not difficult to check that the functions $t_\pm(\sigma, .)$ are continuous in the variable $l \in (0,\infty)$,

$$
\begin{aligned}
&t_+(\sigma,l) = t_-(\sigma,l) \quad \text{for} \quad 0 < l \le l^* = \frac{(\sqrt{a_+} + \sqrt{a_-})^2}{[ac]^2}\sigma, \\
&t_-(\sigma,l) < t_+(\sigma,l) \quad \text{for} \quad l > l^*,
\end{aligned}
\tag{2.28}
$$

the function $t_-(\sigma,\,.\,)$ is strictly monotone decreasing, the function $t_+(\sigma,\,.\,)$ is strictly monotone increasing on the interval $[l^*, \infty)$, and

$$
t_-(\sigma,l) \to t_-, \quad t_+(\sigma,l) \to t_+ \qquad \text{as} \quad l \to \infty. \tag{2.29}
$$

From (2.28) it follows that the critical length l^* of a rod is characterized by the fact that for a rod of length $l \le l^*$, the upper and the lower temperatures of phase transitions coincide. However, they are different for $l > l^*$.

Formulas (2.28) allow us to make a hypothetical experiment to determine the parameter σ that characterizes a two-phase elastic medium with $[ac] \neq 0$. We need to warm up a rod of length l filled with this medium from the temperature t_- up to the temperature t_+ and determine the temperatures $t_\pm(\sigma, l)$ of phase transitions. Changing the length l, we define experimentally the number l^*. This allows us to calculate the surface tension coefficient σ by formula (2.28).

In conclusion of the section, we consider a problem related to the behavior of the state of equilibrium $\widehat{u}_{t,\sigma}$, $\widehat{\chi}_{t,\sigma}$ for the functional $I[u, \chi, t, \sigma]$ as $\sigma \to 0$. From the characterization of the phase transitions (1)–(13) it follows that the pair $\widehat{u}_{t,\sigma}$, $\widehat{\chi}_{t,\sigma}$ does not depend on σ when $t \notin (t_-, t_+)$, and for $t \in (t_-, t_+)$ this pair does not depend on σ when $0 < \sigma < \sigma(t)$. Moreover, the same characterization shows that this pair represents a state of equilibrium for the functional $I_0[u, \chi, t]$ for all σ when $t \notin (t_-, t_+)$ and for $0 < \sigma < \sigma(t)$ when $t \in (t_-, t_+)$. Note that, in the second case, for the states of equilibrium (for each $t \in (t_-, t_+)$, there are exactly two such states) $\widehat{u}_{t,\sigma}$, $\widehat{\chi}_{t,\sigma}$ of the functional I_0 the phase interface area is minimal among all states of equilibrium $\widehat{u}_t$, $\widehat{\chi}_t$ of this functional with $\widehat{\chi}_t \in \mathbb{Z}$. Therefore, "the method of vanishing surface tension", which employs the passage to the limit as $\sigma \to 0$ in the states of equilibrium of the functional I, gives all single-phase states of equilibrium for the functional I_0. Among the two-phase states of equilibrium for this functional for $t \in (t_-, t_+) \neq \varnothing$, only those with the minimal area of a phase interface boundary are preserved. For $t = t_+ = t_- = t^*$, this method gives only the single-phase states of equilibrium.

2.4. Critical points of the energy functional. Let us calculate the first variation of the energy functional in a two-phase elastic medium. The vanishing of the first variation gives a necessary condition for extremum. Since the sets $\mathbb{Z}'$ and $\mathbb{Z}$ are not linear spaces, we need to use the inner variation technique.

Consider the diffeomorphisms $y = y(x)$ of class $C^1[0, l]$ taking the interval $[0, l]$ onto itself and such that the inverse maps $x = x(y)$ have the form

$$x(y) = y + h(y), \quad h \in C^1_0[0, l], \quad |h'(y)| \leq \frac{1}{2} \tag{2.30}$$

with any function h as in (2.30).

We fix functions $\widetilde{u} \in \mathbb{H}$, $\widetilde{\chi} \in \mathbb{Z}'$ and construct their perturbations u, χ by the rule

$$u(x) = \widetilde{u}(y(x)) + v(y(x)), \quad v \in \mathbb{H}, \qquad \chi(x) = \widetilde{\chi}(y(x)). \tag{2.31}$$

Obviously, $u \in \mathbb{H}$, $\chi \in \mathbb{Z}'$, and if $\widetilde{\chi} \in \mathbb{Z}$, then the perturbation (2.31) also belongs to $\mathbb{Z}$. Moreover, $S[\chi] = S[\widetilde{\chi}]$.

Lemma 2.6. *We have*

$$\begin{aligned} I_0[u, \chi, t] - I_0[\widetilde{u}, \widetilde{\chi}, t] &= \int_0^l \{\widetilde{\chi} F_M^+(\widetilde{u}') + (1 - \widetilde{\chi}) F_M^-(\widetilde{u}')\} v' \, dx \\ &+ \int_0^l \Big\{\widetilde{\chi}\big(F^+(\widetilde{u}') + t - \widetilde{u}' F_M^+(\widetilde{u}')\big) + (1 - \widetilde{\chi})\big(F^-(\widetilde{u}') - \widetilde{u}' F_M^-(\widetilde{u}')\big)\Big\} h' \, dx + R, \\ &\widetilde{u} \in \mathbb{H}, \quad \widetilde{\chi} \in \mathbb{Z}', \quad v \in \mathbb{H}, \quad h \in C^1_0(0, l), \quad |h'(x)| \leq \tfrac{1}{2}; \end{aligned}$$

$$\begin{aligned} I[u, \chi, t, \sigma] - I[\widetilde{u}, \widetilde{\chi}, t, \sigma] &= \int_0^l \{\widetilde{\chi} F_M^+(\widetilde{u}') + (1 - \widetilde{\chi}) F_M^-(\widetilde{u}')\} v' \, dx \\ &+ \int_0^l \Big\{\widetilde{\chi}\big(F^+(\widetilde{u}') + t - \widetilde{u}' F_M^+(\widetilde{u}')\big) + (1 - \widetilde{\chi})\big(F^-(\widetilde{u}') - \widetilde{u}' F_M^-(\widetilde{u}')\big)\Big\} h' dx + R, \\ &\widetilde{u} \in \mathbb{H}, \quad \widetilde{\chi} \in \mathbb{Z}, \quad v \in \mathbb{H}, \quad h \in C^1_0(0, l), \quad |h'(x)| \leq \tfrac{1}{2}, \\ &|R| \leq C \int_0^l (|v'|^2 + |h'|^2 + |h' \widetilde{u}'|^2) \, dx. \end{aligned} \tag{2.32}$$

Given t (or t,σ), we say that the pair $\widetilde{u} \equiv \widetilde{u}_t \in \mathbb{H}$, $\widetilde{\chi} \equiv \widetilde{\chi}_t \in \mathbb{Z}'$ (or the pair $\widetilde{u} \equiv \widetilde{u}_{t,\sigma} \in \mathbb{H}$, $\widetilde{\chi} \equiv \widetilde{\chi}_{t,\sigma} \in \mathbb{Z}$) is a critical point of the functional I_0 (or I) if

$$\begin{aligned}\int_0^l \{\widetilde{\chi}(F^+(\widetilde{u}') + t - \widetilde{u}' F_M^+(\widetilde{u}')) + (1-\widetilde{\chi})(F_M^-(\widetilde{u}') - \widetilde{u}' F_M^-(\widetilde{u}'))\} h'\,dx \\ + \int_0^l \{\widetilde{\chi} F_M^+(\widetilde{u}') + (1-\widetilde{\chi}) F_M^-(\widetilde{u}')\} v'\,dx = 0 \quad \text{for all } v,h \in C_0^\infty(0,l).\end{aligned} \tag{2.33}$$

We introduce the functions

$$\varTheta^\pm(M) = F_M^\pm(M), \quad \varPhi^\pm(M) = F^\pm(M) - M F_M^\pm(M), \qquad M \in \mathbb{R}. \tag{2.34}$$

The quantities $\varTheta^\pm(M)$ determine electric potentials, while $\varPhi^\pm(M)$ gives chemical potentials for each energy density $F^\pm(M)$. Then the functions

$$\begin{gathered}\varTheta[u,\chi](x) = \chi(x)\varTheta^+(u'(x)) + (1-\chi(x))\varTheta^-(u'(x)), \\ \varPhi[u,\chi](x,t) = \chi(x)(\varPhi^+(u'(x)) + t) + (1-\chi(x))\varPhi^-(u'(x)), \\ u \in \mathbb{H}, \quad \chi \in \mathbb{Z}', \quad x \in (0,l), \quad t \in \mathbb{R}\end{gathered} \tag{2.35}$$

determine the distributions of electric and chemical potentials for the field of displacements u and phase distribution χ. Evidently, identity (2.33) is equivalent to the fact that the electric and chemical potentials are constant on the interval $(0,l)$:

$$\begin{gathered}\varTheta[\widetilde{u},\widetilde{\chi}](x) = C_\varTheta, \quad \varPhi[\widetilde{u},\widetilde{\chi}](x,t) = C_\varPhi(t), \quad \text{for almost all } x \in (0,l) \\ \text{and some constant } C_\varTheta \text{ and some function } C_\varPhi(t).\end{gathered} \tag{2.36}$$

In accordance with (2.32), the left-hand side of (2.33) gives the linear part of the increment of the energy functionals I_0 and I under the perturbation (2.31). Hence, the states of equilibrium for these functionals are their critical points.

The following theorem answers the question about the existence of critical points for energy functionals besides the states of equilibrium, and specifies all critical points.

Theorem 2.5. (a) *For each fixed t, the set of all critical points $\widetilde{u}_t$, $\widetilde{\chi}_t$ of the functional I_0 consists of its states of equilibrium $\widehat{u}_t$, $\widehat{\chi}_t$ and the single-phase states $u \equiv 0$, $\chi \equiv 1$ and $u \equiv 0$, $\chi \equiv 0$.*
(b) *For any fixed t and σ, the set of all critical points $\widetilde{u}_{t,\sigma}$, $\widetilde{\chi}_{t,\sigma}$ of the functional I coincides with the set of all critical points $\widetilde{u}_t$, $\widetilde{\chi}_t$ of the functional I_0 for which $\widetilde{\chi}_t \in \mathbb{Z}$.*
(c) *The critical points $\widetilde{u}_t$, $\widetilde{\chi}_t$ of the functional I_0 that do not coincide with states of equilibrium are saddle points of this functional.*
(d) *The critical points $\widetilde{u}_{t,\sigma}$, $\widetilde{\chi}_{t,\sigma}$ of the functional I that do not coincide with states of equilibrium are local minima of this functional relative to any perturbations of the function $\widetilde{u}_{t,\sigma}$ in the space $\mathbb{H}$ and sufficiently $L_1(0,l)$-small norm deviations of class $\mathbb{Z}$ from the function $\widetilde{\chi}_{t,\sigma}$.*

Suppose that, for some point $x_0 \in (0,l)$ and sufficiently small $\delta > 0$, the function $\widetilde{\chi}$ is constant for $x \in (x_0-\delta, x_0)$ and for $x \in (x_0, x_0+\delta)$, but is nonconstant on the interval $(x_0-\delta, x_0+\delta)$. Then x_0 is a boundary point of the phase interface. By (2.36), we have

$$\begin{gathered}[F_M(\widetilde{u}')] = 0, \quad [F(\widetilde{u}') - \widetilde{u}' F_M(\widetilde{u}')] + t = 0, \\ \text{where } \ [\alpha(\widetilde{u}')] = \alpha^+(M_+) - \alpha^-(M_-).\end{gathered} \tag{2.37}$$

Here, the $M_\pm$ are the limit values of the function $\widetilde{u}'(x)$ at the point x_0 from the direction of the positive and the zero function $\widetilde{\chi}$, respectively.

Conditions (2.37) are the classical necessary conditions for the Weierstrass–Erdmann extremum of the integral functional

$$I_0^{\min}[u,t] = \int_0^l F^{\min}(u',t)\,dx, \tag{2.38}$$
$$F^{\min}(M,t) = \min\{F^+(M)+t, F^-(M)\}, \quad u \in \mathbb{H},$$

on the set of functions with admissible jump of the derivative at an arbitrary point x_0 unknown in advance.

Formulas (2.37) admit the following geometric interpretation showing that the two points $(M_-, F^-(M_-))$ and $(M_+, F^+(M_+)+t)$ are jointed by the common tangent line to the graphs of the functions $F^-(M)$ and $F^+(M)+t$ at these points. Depending on $a_\pm$, $c_\pm$, and t, the following situations are possible: the pair $M_\pm$ does not exist, there exists one such pair, or two such pairs.

2.5. Bibliographical notes. Statements of this section are based on the papers [17–20]. These papers also contain different approaches, which allow one to obtain similar results for other boundary conditions and nonzero fields of force.

In the book [2], one can find applications of inner variations to one-dimensional variation problems. The canonical proof of the Weierstrass–Erdmann conditions is given, e.g., in [4]. It is based on the common formula for the first variation of an integral functional and its geometric interpretation. A multidimensional analog of the geometric interpretation of the Weierstrass–Erdmann conditions was used in [60] for the classification of the stable critical points of the energy functional of a two-phase medium.

The equilibrium conditions (2.36) for a one-dimensional two-phase medium coincide with those for the two-component gases: the constancy of the chemical potential and pressure. For the multidimensional two-phase medium, this coincidence fails.

§3. The multidimensional problem with zero coefficient of surface tension

An attempt is made to partially extend results of Subsection 2.2 *related to the one-dimensional problem with zero surface tension coefficient to the multidimensional* ($m \geq 2$) *case. The essential distinction between the multidimensional and the one-dimensional settings is the possible absence of equilibrium.*

3.1. The setting of the problem. To formulate the multidimensional problem on phase transitions with zero surface tension coefficient, for every phase $\pm$ we introduce the elastic modulus tensors $a^\pm_{ijkl}$, $i,j,k,l = 1,\dots,m$, $m \geq 2$, satisfying the symmetry and positive definiteness conditions:

$$a^\pm_{ijkl} = a^\pm_{klij} = a^\pm_{jikl} = a^\pm_{ijlk}, \tag{3.1}$$
$$\nu^{-1}\xi_{ij}\xi_{ij} \geq a^\pm_{ijkl}\xi_{ij}\xi_{kl} \geq \nu\xi_{ij}\xi_{ij} \qquad \text{for all matrices} \quad \xi \in \mathbb{R}^{m\times m}_s,$$

where $\mathbb{R}^{m\times m}_s$ is the space of symmetric matrices of size $m\times m$, $\nu \in (0,1)$; in (3.1) and in what follows, summation from 1 to m is assumed over the repeating indices.

In the space $\mathbb{R}^{m\times m}$ of $(m\times m)$-matrices, we define the Hilbert–Schmidt scalar product

$$\langle \alpha, \beta\rangle = \operatorname{tr}\alpha\beta^*, \qquad \alpha,\beta \in \mathbb{R}^{m\times m}. \tag{3.2}$$

For each sign $\pm$, the coefficients $a^\pm_{ijkl}$ generate a linear map

$$A^\pm\colon \mathbb{R}^{m\times m}_s \to \mathbb{R}^{m\times m}_s, \quad (A^\pm\xi)_{ij} = a^\pm_{ijkl}\xi_{kl}, \tag{3.3}$$

which is symmetric and positive definite with respect to the scalar product (3.2):

$$\langle A^{\pm}\xi,\zeta\rangle = \langle \xi, A^{\pm}\zeta\rangle, \quad \nu^{-1}|\xi|^2 \geq \langle A^{\pm}\xi,\xi\rangle \geq \nu|\xi|^2, \quad |\xi|^2 = \langle \xi,\xi\rangle, \quad \text{for all} \quad \xi,\zeta \in \mathbb{R}^{m\times m}_s. \tag{3.4}$$

Besides the tensors of elastic moduli, we need the residual strain tensors $\zeta^{\pm}_{ij}$ and the strain tensors $e_{ij}(M)$:

$$\zeta^{\pm} \in \mathbb{R}^{m\times m}_s, \quad e(M) = \frac{M+M^*}{2} \in \mathbb{R}^{m\times m}_s, \quad M \in \mathbb{R}^{m\times m}. \tag{3.5}$$

Let functions $F^{\pm}(M)$ be defined by the formula

$$F^{\pm}(M) = \langle A^{\pm}(e(M)-\zeta^{\pm}), e(M)-\zeta^{\pm}\rangle. \tag{3.6}$$

Clearly, the quadratic functions $F^{\pm}(M)$ are convex:

$$F^{\pm}_{M_{ij}M_{kl}} C_{ij}C_{kl} = 2\langle A^{\pm}e(C), e(C)\rangle \geq 0 \quad \text{for all} \quad C \in \mathbb{R}^{m\times m} \tag{3.7}$$

and satisfy the Legendre–Hadamard condition

$$F^{\pm}_{M_{ij}M_{kl}} \xi_i\xi_k\lambda_j\lambda_l \geq \nu|\lambda|^2|\xi|^2 \quad \text{for all vectors} \quad \lambda,\xi \in \mathbb{R}^m. \tag{3.8}$$

Using (3.6), we define the deformation energy functional of a two-phase elastic medium in the bounded domain $\Omega \subset \mathbb{R}^m$, $m \geq 2$, by the identity

$$I_0[u,\chi,t] = \int_\Omega \{\chi(F^+(\nabla u)+t) + (1-\chi)F^-(\nabla u)\}\,dx, \quad u \in \mathbb{H}, \quad \chi \in \mathbb{Z}', \quad t \in \mathbb{R}, \tag{3.9}$$

where $\mathbb{H} = \overset{\circ}{W}{}^1_2(\Omega,\mathbb{R}^m)$, $\mathbb{Z}'$ is the set of all measurable characteristic functions, and $(\nabla u)_{ij} = u^i_{x_j}$.

By an equilibrium state of a two-phase elastic medium with zero surface tension coefficient for fixed temperature t we mean an equilibrium displacement field $\widehat{u}_t$ and an equilibrium phase distribution $\widehat{\chi}_t$ minimizing the energy functional for given t:

$$I_0[\widehat{u}_t,\widehat{\chi}_t,t] = \inf_{\substack{u\in\mathbb{H},\\ \chi\in\mathbb{Z}'}} I_0[u,\chi,t]. \qquad \widehat{u}_t \in \mathbb{H}, \quad \widehat{\chi}_t \in \mathbb{Z}'. \tag{3.10}$$

An equilibrium state $\widehat{u}_t$, $\widehat{\chi}_t$ is said to be single-phase if $\widehat{\chi}_t = 1$ or $\widehat{\chi}_t = 0$ almost everywhere on Ω, and two-phase otherwise. Obviously, for a single-phase equilibrium state we have $\widehat{u}_t = 0$.

The main difficulty in studying problem (3.10) is that it may be unsolvable.

Lemma 3.1. *Let $\lambda \in \mathbb{R}^m$, $|\lambda| = 1$, and let*

$$\zeta^{\pm} = \pm\lambda\otimes\lambda, \quad (\lambda\otimes\lambda)_{ij} = \lambda_i\lambda_j.$$

Then problem (3.10) *for $t = 0$ has no solutions.*

There are cases when the energy densities (3.6) can be simplified by making residual strain tensors $\zeta^{\pm}$ equal to each other.

Lemma 3.2. *Suppose that there exists a matrix $\xi \in \mathbb{R}^{m\times m}_s$ such that*

$$[A]\xi = [A\zeta]. \tag{3.11}$$

Then the energy functional (3.9) *coincides with the energy functional for the densities* (3.6) *with the equal residual strain tensors $\zeta^{\pm} = \xi$.*

Thus, problem (3.10) may (or may not) be unsolvable. In the next section, not only shall we establish the existence of its solutions for an important class of two-phase elastic media, but also obtain explicit formulas for them.

3.2. Phase transitions for isotropic media. A two-phase medium is said to be *isotropic* if

$$
\begin{gathered}
a^{\pm}_{ijkl} = \frac{a_{\pm}}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + b_{\pm}\delta_{ij}\delta_{kl}, \quad \zeta^{\pm}_{ij} = c_{\pm}\delta_{ij}, \\
a_{\pm}, b_{\pm}, c_{\pm} \in \mathbb{R}, \quad a_{\pm} > 0, \;\; b_{\pm} \geq 0.
\end{gathered} \tag{3.12}
$$

In this case,

$$
\begin{gathered}
F^{\pm}(M) = a_{\pm}\operatorname{tr}(e(M) - c_{\pm}i)^2 + b_{\pm}\operatorname{tr}^2(e(M) - c_{\pm}i), \\
i \text{ is the unit matrix in the space } \mathbb{R}^m.
\end{gathered} \tag{3.13}
$$

Our goal is to prove the solvability of the problem (3.10) for the energy densities (3.12) and describe the dependence of equilibrium states $\widehat{u}_t$, $\widehat{\chi}_t$ on the temperature t. Both results and a sketch of their proofs are similar to those described in Subsection 2.2 for the one-dimensional case. Unfortunately, the realization of these plans is successful only under the additional restriction

$$
a_+ = a_- \equiv a. \tag{3.14}
$$

We start with the evaluation of the energy functional (3.9) for the densities (3.13). The following lemma is a multidimensional analog of Lemma 2.1.

Lemma 3.3. *For* $u \in \mathbb{H}$ *and* $\chi \in \mathbb{Z}'$, *the energy functional* (3.9) *with the densities* (3.13) *can be written in the form*

$$
\begin{aligned}
I_0[u,\chi,t] = {}& \int_{\Omega} (a_+\chi + a_-(1-\chi))(u^i_{x_j}u^j_{x_i} - u^i_{x_i}u^j_{x_j})\,dx \\
& + \int_{\Omega} \biggl\{ \frac{a_+\chi + a_-(1-\chi)}{4}|\operatorname{curl} u|^2 \\
& \qquad + ((a_+ + b_+)\chi + (a_- + b_-)(1-\chi))(\operatorname{div} u - \alpha(Q)(\chi - Q))^2 \biggr\}\,dx \\
& + |\Omega| G(Q,t), \\
& (\operatorname{curl} u)_{ij} = u^i_{x_j} - u^j_{x_i}, |\operatorname{curl} u|^2 = (\operatorname{curl} u)_{ij}(\operatorname{curl} u)_{ij}, \\
& Q = \frac{1}{|\Omega|}\int_{\Omega}\chi\,dx, \quad \alpha(Q) = \frac{[c(a+bm)]}{(a_- + b_-)Q + (a_+ + b_+)(1-Q)}, \\
G(Q,t) = {}& Qt + mc_+^2(a_+ + b_+m)Q \\
& + mc_-^2(a_- + b_-m)(1-Q) - [c(a+bm)]\alpha(Q)Q(1-Q).
\end{aligned} \tag{3.15}
$$

By the assumptions (3.14), the first integral on the right-hand side of (3.15) is equal to zero for all $u \in \mathbb{H}$. Therefore, as in the one-dimensional case, the representation (3.15) allows us to split the variational problem (3.10) for the energy densities (3.13), (3.14) into two problems. One of them is a system of equations for the functions u and χ:

$$
\operatorname{div} u = \alpha(Q)(\chi - Q), \quad \operatorname{curl} u = 0, \quad u \in \mathbb{H}, \quad \chi \in \mathbb{Z}', \quad Q = \frac{1}{|\Omega|}\int_{\Omega}\chi\,dx. \tag{3.16}
$$

The second is the problem on the extremum of the function $G(\,.\,,t)$ for a fixed t:

$$
G(\widehat{Q}(t),t) = \min_{Q\in[0,1]} G(Q,t), \quad \widehat{Q}(t) \in [0,1]. \tag{3.17}
$$

The solvability of problem (3.17) is obvious in view of the continuity of the function $G(\,.\,,t)$. Since the solvability of system (3.16) is not yet established, the next lemma still has a conditional character.

Lemma 3.4. *Suppose that system* (3.16) *is solvable for any* $Q \in [0,1]$. *Then problem* (3.10) *with the energy densities* (3.13) *is also solvable provided that condition* (3.14) *is fulfilled and the set of all solutions* $\widehat{u}_t$, $\widehat{\chi}_t$ *of* (3.10) *coincides with the set all solutions of the system*

$$\begin{aligned} &\operatorname{div} \widehat{u}_t = \alpha(\widehat{Q}(t))(\widehat{\chi}_t - \widehat{Q}(t)), \quad \operatorname{curl} \widehat{u}_t = 0, \quad \widehat{u}_t \in \mathbb{H}, \quad \widehat{\chi}_t \in \mathbb{Z}', \\ &\frac{1}{|\Omega|} \int_\Omega \widehat{\chi}_t \, dx = \widehat{Q}(t), \quad G(\widehat{Q}(t), t) = \min_{Q \in [0,1]} G(Q,t). \end{aligned} \tag{3.18}$$

We turn to the study of system (3.16). In the following lemma, we prove that this system is solvable for an arbitrary constant α and every number $Q \in [0,1]$. The merit of the lemma is not only in establishing solvability, but also in constructing explicit formulas for some class of its solutions.

Lemma 3.5. *The system*

$$\operatorname{div} v = \alpha(\chi - Q), \quad \operatorname{curl} v = 0, \quad v \in \mathbb{H}, \quad \chi \in \mathbb{Z}', \quad Q = \frac{1}{|\Omega|} \int_\Omega \chi \, dx \tag{3.19}$$

for the unknown functions v *and* χ *is solvable in an arbitrary bounded domain* $\Omega \subset \mathbb{R}^m$ *for any number* α *and every* $Q \in [0,1]$.

We briefly describe the method of obtaining (possibly not all) solutions (3.19) in an explicit form. First, we consider this system in the ball $\Omega = B_R$ of radius R centered at the origin. We are looking for spherically symmetric solutions

$$\chi(x) = \chi(|x|), \quad v(x) = \frac{x}{|x|} w(|x|), \quad Q = \frac{1}{|B_R|} \int_{B_R} \chi(|x|) \, dx \tag{3.20}$$

with some scalar function $w(.)$. For the representation (3.20), the second equation of system (3.19) is satisfied automatically. Solving the first equation, we obtain:

$$v(x) = \alpha \frac{x}{|x|} \frac{|x|}{m} \frac{1}{|B_{|x|}|} \int_{B_{|x|}} (\chi(|z|) - Q) \, dz. \tag{3.21}$$

Using a shift, we carry the solution (3.20), (3.21) over to a ball centered at a point $y \in \mathbb{R}^m$.

To construct a solution in an arbitrary domain Ω, we need the construction called the Vitali covering of an open set Ω.

> *Let* $\Omega \subset \mathbb{R}^m$ *be an open set, and let* $\delta > 0$. *Then there exists a countable family of disjoint closed balls* $\bar{B}_{R_j}(x_j) \subset \Omega$, $j = 1, \dots,$ *such that* $\operatorname{diam} B_{R_j}(x_j) \le \delta$ *for all* j *and* $|\Omega \setminus \bigcup_j \bar{B}_{R_j}(x_j)| = 0$.

Obviously, for each domain Ω one can find infinitely many different Vitali coverings.

We fix a Vitali covering in the domain Ω. In each ball $B_{R_j}(x_j)$ of this covering, we take an arbitrary spherically symmetric solution $v^{(j)}$, $\chi^{(j)}$. The solution in the domain Ω is obtained if we sum all these solutions overs balls of the covering. It is easily seen that for each such solution v, χ the function v belongs to the space $\mathbb{H} \cap W^1_\infty(\Omega, \mathbb{R}^m)$, and

$$\|v\|_{C(\bar{\Omega})} \le C\delta, \quad \|\nabla v\|_{L_\infty(\Omega)} \le C \tag{3.22}$$

for some positive constant $C = C(m, \alpha)$ and the fixed parameter δ of the Vitali covering.

The above constructions show that for $Q \in (0,1)$ and $\alpha \neq 0$, a solution of system (3.19) is certainly not unique. This nonuniqueness is provided by the nonuniqueness of a solution of the system in the ball $B_{R_j}(x^j)$ and the nonuniqueness of the Vitali covering.

For $Q = 0$, $Q = 1$, or $\alpha = 0$, system (3.19) has the form

$$\operatorname{curl} v = 0, \quad \operatorname{div} v = 0, \qquad v \in \mathbb{H}. \tag{3.23}$$

Therefore, its only solution is the function $v = 0$.

To investigate problem (3.17), it is convenient to introduce the following notation:

$$t_- = t^* - \frac{[c(a+bm)]^2}{a_- + b_-}, \quad t_+ = t^* + \frac{[c(a+bm)]^2}{a_+ + b_+}, \quad t^* = -[mc^2(a+bm)]. \tag{3.24}$$

The numbers $t_\pm$ will be called the temperatures of the phase transitions.

Theorem 3.1. *Problem* (3.10) *with the energy densities* (3.13) *is solvable for each value of the parameter* t *under condition* (3.14)*. The set of all solutions is characterized by the properties* (1)–(6) *described in Subsection* 2.2*. In* (3.24)*, the phase transition temperatures* $t_\pm$ *are fixed.*

As in the one-dimensional case, the function $\widehat{Q}(t)$, $t \in (t_-, t_+)$ can be found explicitly.

Lemma 3.6. *Let* $[c(a+bm)] \neq 0$*. For* $t \in (t_-, t_+)$*, put*

$$h(t) = \frac{t_+ - t}{t_+ - t_-}, \quad g(t) = \frac{1}{(a_- + b_-)^2} h(t) + \frac{1}{(a_+ + b_+)^2}(1 - h(t)). \tag{3.25}$$

Then

$$\widehat{Q}(t) = \begin{cases} h(t) & \textit{if } [a+b] = 0, \\ \frac{(a_+ + b_+) + (a_- + b_-)}{2[a+b]} + \frac{1}{2} - \frac{1}{[a+b] g^{1/2}(t)} & \textit{if } [a+b] \neq 0. \end{cases} \tag{3.26}$$

Note that the assumption (3.14) is essential for the validity of Theorem 3.1, while formulas (3.26) are true even of (3.14) fails.

The next theorem, an analog of Theorem 2.2, answers the question about the role of the pairs $u \equiv 0$, $\chi \equiv 1$ and $u \equiv 0$, $\chi \equiv 0$ in the case where they do not minimize the energy functional of a two-phase elastic medium.

Theorem 3.2. *For the functional* (3.9) *with the energy densities* (3.13) *satisfying condition* (3.14)*, statements* (1)–(3) *of Theorem* 2.2 *hold true.*

The equilibrium states $\widehat{u}_t$, $\widehat{\chi}_t$ constructed in this section have a relatively complicated structure, and the phase distributions with indices "+" (i.e., the supports of $\widehat{\chi}_t$) can have fractal features characterized by self-similarity.

For the convenience of the following presentation, we introduce the following set of pairs of functions:

$$\mathbb{Y}'_t = \left\{ u \in \mathbb{H}, \chi \in \mathbb{Z}' : \operatorname{curl} u = 0,\ \operatorname{div} u = \alpha(\widehat{Q}(t))(\chi - \widehat{Q}(t)),\ \frac{1}{|\Omega|}\int_\Omega \chi\, dx = \widehat{Q}(t) \right\}. \tag{3.27}$$

This is the set of all solutions of the variational problem (3.10) with the energy density (3.13) under condition (3.14). If $[c(a+bm)] \neq 0$, then the function $\widehat{Q}(t)$ is single-valued and $\alpha(\widehat{Q}(t)) \neq 0$. Therefore, in this case, each component of the pair $u, \chi \in \mathbb{Y}'_t$ is uniquely determined by the other one. If $[c(a+bm)] = 0$ and $t \neq t^* (= t_\pm)$, then $\widehat{Q}(t)$ is still single-valued, but $\alpha(\widehat{Q}(t)) = 0$. Hence, in this case, the set $\mathbb{Y}'_t$ is exhausted by the pairs $u = 0$, $\chi = 1$ for $t < t^*$ and $u = 0$, $\chi = 0$ for $t > t^*$. If $[c(a+bm)] = 0$ and $t = t^*$, the set $\mathbb{Y}'_t$ consists of the pairs for which $u = 0$ and χ is an arbitrary element of the set $\mathbb{Z}'$, because $\widehat{Q}(t)$ is nonhomogeneous and the number $\alpha(\widehat{Q}(t))$ is equal to zero.

3.3. Temperature of phase transitions. In view of formula (3.9), it seems to be plausible that for the energy functional of a two-phase medium there exist phase transition temperatures $t_\pm$, $t_- \le t_+$ such that

$$\begin{aligned} &\text{for } t < t_-, \text{ only the state of equilibrium } \widehat{u}_t \equiv 0,\ \widehat{\chi}_t \equiv 1 \text{ is realized;} \\ &\text{for } t > t_+, \text{ only the state of equilibrium } \widehat{u}_t \equiv 0,\ \widehat{\chi}_t \equiv 0 \text{ is realized;} \\ &\text{for } t \in (t_-, t_+), \text{ no single-phase equilibrium states exist.} \end{aligned} \tag{3.28}$$

This conjecture can be justified in the one-dimensional case with $t_\pm$ defined by (2.10), as well as for multidimensional media under condition (3.14), where the temperatures $t_\pm$ are defined by (3.24). In both cases the temperatures $t_\pm$ do not depend on the domain filled by the medium, and for each $t \in (t_-, t_+)$ there exists a (two-phase) state of equilibrium. Taking Lemma 3.1 into account, for the general form of the energy functional (3.9) we can hope only to prove the existence of the phase transition temperatures $t_\pm$ that do not depend on the domain Ω and are determined only by the characteristics of the two-phase medium in question. Of course, the constructions below are also valid in the one-dimensional case, but we did not need them, thanks to the explicit solvability of the one-dimensional problem.

We shall need the following functions:

$$\begin{gathered} i^+(t,\Omega) = \inf_{u\in\mathbb{H}} I_0[u,1,t], \quad i^-(t,\Omega) = \inf_{u\in\mathbb{H}} I_0[u,0,t], \\ i_{\min}(t,\Omega) = \min\{i^(t,\Omega), t^-(t,\Omega)\}, \quad i(t,\Omega) = \inf_{u\in\mathbb{H},\chi\in\mathbb{Z}'} I_0[u,\chi,t] \end{gathered} \tag{3.29}$$

where the energy functional $I_0[u,\chi,t]$ is as defined in (3.9).

By these definitions, we have

$$\begin{gathered} i^+(t,\Omega) = |\Omega|(\langle A^+\zeta^+,\zeta^+\rangle + t), \quad i^-(t,\Omega) = |\Omega|\langle A^-\zeta^-,\zeta^-\rangle, \\ i_{\min}(t,\Omega) = |\Omega| \begin{cases} \langle A^+\zeta^+,\zeta^+\rangle + t & \text{if } t \le t^*, \\ \langle A^-\zeta^-,\zeta^-\rangle & \text{if } t \ge t^*, \end{cases} \quad t^* = -[\langle A\zeta,\zeta\rangle]. \end{gathered} \tag{3.30}$$

Since

$$i_{\min}(t,\Omega) = \inf_{u\in\mathbb{H},\chi=\chi^\pm} I_0[u,\chi,t], \quad \chi^+ \equiv 1, \quad \chi^- \equiv 0, \tag{3.31}$$

we have

$$i(t,\Omega) \le i_{\min}(t,\Omega). \tag{3.32}$$

Clearly, the existence of single-phase states of equilibrium for the functional (3.9) is determined by the function $i_{\min}(t,\Omega) - i(t,\Omega)$:

(3.33)

for given t, if $i_{\min}(t,\Omega) - i(t,\Omega) > 0$,
then for such t, the functional (3.9) has no single-phase equilibrium states;

for given $t < t^*$, if $i_{\min}(t,\Omega) - i(t,\Omega) = 0$,
then for such t, the functional (3.9) admits only one single-phase equilibrium state $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 1$;

for given $t > t^*$, if $i_{\min}(t,\Omega) - i(t,\Omega) = 0$,
then for such t, the functional (3.9) admits only one single-phase equilibrium state $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 0$;

if $i_{\min}(t^*,\Omega) - i(t^*,\Omega) = 0$, then for the functional (3.9) with $t = t^*$ both single-phase equilibrium states are realized:
$\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 1$ and $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 0$.

In order to apply (3.33) to prove the existence of phase transition temperatures, we need the following lemma.

Lemma 3.7. *For every fixed domain $\Omega \neq \varnothing$, the function $i(\,.\,,\Omega)$ is concave and satisfies the local Lipschitz condition.*

If for some t'_- such that $t'_- \le t^*$ we have $i(t'_-, \Omega) = i_{\min}(t'_-, \Omega)$, then $i(t,\Omega) = i_{\min}(t,\Omega)$ for all $t < t'_-$, and for such t, the only state of equilibrium of the functional (1.9) *is the pair $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 1$.*

If for some t'_+ such that $t'_+ \ge t^*$ we have $i(t'_+, \Omega) = i_{\min}(t'_+, \Omega)$, then $i(t,\Omega) = i_{\min}(t,\Omega)$ for all $t > t'_+$, and for such t, the only state of equilibrium of the functional (1.9) *is the pair $\widehat{u}_t \equiv 0$, $\widehat{\chi}_t \equiv 0$.*

This lemma shows that each set $\{t'_\pm\}$ is closed and the following statements are true:

$$
\begin{aligned}
&\text{if the numbers } t_\pm \text{ exist, then } t_- \le t^* \le t_+, \text{ where equality}\\
&\text{occurs or does not occur in the two inequalities at the same time;}\\
&\text{if the numbers } t^\pm \text{ exist, then for } t_- = t_+ = t^* \text{ we have}\\
&i(t,\Omega) = t_{\min}(t,\Omega) \text{ for all } t \in \mathbb{R},\\
&\text{if the numbers } t_\pm \text{ exist, then for } t_- < t_+ \text{ we have}\\
&i(t,\Omega) < i_{\min}(t,\Omega) \text{ for } t \in (t_-, t_+), \text{ and } i(t,\Omega) = i_{\min}(t,\Omega) \text{ for } t \notin (t_-, t_+).
\end{aligned}
\tag{3.34}
$$

As the next step, we discuss the existence of phase transition temperatures in the case where $\Omega = B$, the unit ball centered at the origin.

Denote

$$
\begin{aligned}
\mu_1^+ &= t^* - \frac{\nu\,\mathrm{tr}^2[A\zeta]}{m^2}, \quad \mu_1^- = t^* - \frac{|[A\zeta]|^2}{\nu},\\
\mu_2^+ &= t^* + \frac{\nu\,\mathrm{tr}^2[A\zeta]}{m^2}, \quad \mu_2^- = t^* + \frac{|[A\zeta]|^2}{\nu}.
\end{aligned}
\tag{3.35}
$$

The inequality

$$
\begin{gathered}
|\,\mathrm{tr}\,S| = |\langle S, i\rangle| \le \langle S, S\rangle^{1/2}\langle i, i\rangle^{1/2} = |S|\sqrt{m},\\
S \in \mathbb{R}_s^{m\times m}, \quad i \text{ is the unit matrix in } R^m,
\end{gathered}
\tag{3.36}
$$

implies the following estimates for the numbers (3.35):

$$
\mu_1^- \le \mu_1^+ \le t^*, \quad t^* \le \mu_2^+ \le \mu_2^-. \tag{3.37}
$$

Lemma 3.8. *In the case where $\Omega = B$, the phase transition temperatures exist and satisfy the two-sided estimates*

$$
t_- \in [\mu_1^-, \mu_1^+], \quad t_+ \in [\mu_2^+, \mu_2^-]. \tag{3.38}
$$

The proof is based on the construction of functions $g_\pm(t, B)$ such that

$$
\begin{aligned}
&g_-(t,B) \le i(t,B) \le g_+(t,B) \le i_{\min}(t,B),\\
&g_+(t,B) = i_{\min}(t,B) \ \text{ if and only if } \ t \notin (\mu_1^+, \mu_2^+),\\
&g_-(t,B) = i_{\min}(t,B) \ \text{ if and only if } \ t \notin (\mu_1^-, \mu_2^-);
\end{aligned}
\tag{3.39}
$$

combining this with (3.34), we get the existence of the temperatures $t_\pm$ and the validity of (3.38).

When constructing the functions $g_\pm(t,B)$, we have to "clamp" the functional (3.9) between two energy functionals of isotropic two-phase media, the information on the temperatures of the phase transitions for which is obtained in Subsection 3.2 of this section. If inequalities (3.4) are applied for the two-sided estimate of the functional (3.9), additional terms arise besides the energy functionals of isotropic media, because the tensors of residual deformation $\zeta^\pm$ may fail to coincide with the tensors $c_\pm i$ for isotropic media. The evaluation of these terms is the main difficulty in the proof of the lemma. We set

$$
L(\Omega) = \{t \in R^1 \,:\, i_{\min}(t,\Omega) - i(t,\Omega) > 0\}. \tag{3.40}
$$

By Lemma 3.7 and statements (3.33) and (3.34), the set $L(\Omega)$ is an open (in particular, empty) interval. If $L(\Omega) \neq \varnothing$, then $t^* \in L(\Omega)$ and the boundedness of the interval $L(\Omega)$ (from the left or form the right) means the existence of phase transition temperatures (t_- or t_+, respectively) coinciding with its endpoints. If $L(\Omega) = \varnothing$, then the phase transition temperatures exist and $t_\pm = t^*$.

The next lemma contains a series of statements about the dependence of the set $L(\Omega)$ on the domain Ω.

Lemma 3.9. *We have*

$$\begin{aligned} &L(\Omega_e) = L(\Omega), \ \textit{where} \\ &\qquad \Omega_e = \{x + e : x \in \Omega,\ e \textit{ is a fixed vector in the space } \mathbb{R}^m\}; \\ &L(\Omega^\lambda) = L(\Omega), \ \textit{where} \\ &\qquad \Omega^\lambda = \{\lambda x : x \in \Omega,\ \lambda \textit{ is a fixed number belonging the interval } (0, \infty)\}; \\ &L(\Omega') \supset L(\Omega) \textit{ for arbitrary bounded domain } \Omega' \subset R^m,\ \Omega' \supset \Omega. \end{aligned} \tag{3.41}$$

The lemmas obtained above enable us to prove the basic statement of this section about the existence and estimates of the phase transition temperatures for the functional (3.9) and their independence of the domain Ω.

Theorem 3.3. *For the functional* (3.9)*, the temperatures of phase transitions exist, they do not depend on the domain Ω, and obey* (3.38).

Theorem 3.3 not only guarantees the existence of the phase transition temperatures $t_\pm$, but also shows that

$$t_- < t^* < t_+ \quad \text{for} \quad \operatorname{tr}[A\zeta] \neq 0, \quad t_- = t^* = t_+ \quad \text{for} \quad [A\zeta] = 0. \tag{3.42}$$

Note that in the isotropic case (3.12) we have

$$A^\pm \zeta^\pm = c_\pm (a_\pm + b_\pm m) i. \tag{3.43}$$

It follows that, in the isotropic case,

$$\operatorname{tr}^2[A\zeta] = m^2 [c(a + bm)]^2, \quad |[A\zeta]|^2 = m[c(a + bm)]^2. \tag{3.44}$$

Therefore, even without the assumption (3.14), the condition $[c(a+bm)] = 0$ is a criterion for the coincidence of the temperatures of phase transitions for the energy densities (3.13).

Estimates (3.42) show that for the general form of the energy densities (3.6) the identity $[A\zeta] = 0$ implies the coincidence of the phase transition temperatures, but their coincidence only implies that $\operatorname{tr}[A\zeta] = 0$. However, for isotropic media (see (3.12)), there is a criterion (the relation $[c(a + bm)] = 0$) for the coincidence of the phase transition temperatures. It turns out that the criterion for $t_- = t_+$ can be obtained for the general form of the densities (3.6), and it coincides with the sufficient condition ($[A\zeta] = 0$) for the fact that $t_- = t_+$.

Lemma 3.10. *For the general form of the energy densities* (3.6)*, the criterion for the coincidence of the temperatures of phase transitions is the relation $[A\zeta] = 0$. In case it is true, the set of all equilibrium states of the functional $I_0[u, \chi, t^*]$ is exhausted by the pairs for which $\widehat{u}_{t^*} = 0$ and $\widehat{\chi}_{t^*}$ is an arbitrary element of $\mathbb{Z}'$.*

Let us investigate the role of the single-phase states $u \equiv 0$, $\chi \equiv 1$ and $u \equiv 0$, $\chi \equiv 0$ for a two-phase medium with the energy functional (3.9). In the one-dimensional case (Theorem 2.2) and for multidimensional isotropic media (Theorem 3.2) it was established that for the values of t for which any one of these pairs is not a state of equilibrium, it is a saddle point of the energy functional. For the general form of the energy densities (3.6), we succeeded to obtain only a more modest result.

Theorem 3.4. *For the functional* (3.9) *with arbitrary densities of energy* (3.6), *statements* (2) *and* (3) *of Theorem* 2.2 *are valid. Moreover, for* $t > \mu_1^+$ *the pair* $u \equiv 0$, $\chi \equiv 1$ *and for* $t < \mu_2^+$ *the pair* $u \equiv 0$, $\chi \equiv 0$ *are saddle points of this functional.*

The definition of the temperatures of phase transitions shows that for $t \in (t_-, \mu_1^+]$, the single-phase state $u \equiv 0$, $\chi \equiv 1$, and for $t \in [\mu_2^+, t_+)$ the single-phase state $u \equiv 0$, $\chi \equiv 0$ do not minimize the energy functional of a two-phase elastic medium. However, Theorem 3.4 does not guarantee the instability of these states at the indicated values of t.

Theorem 3.3 allows us to extend the set of two-phase elastic media for which it is possible to calculate the temperatures of phase transitions explicitly. The method of finding the temperatures $t_\pm$ consists of comparison of the interval (3.40) of the problem under study with the interval for a problem with known phase transition temperatures in a domain convenient for calculations. The coincidence of the intervals of the two problems guarantees that of the phase transition temperatures.

Consider the following energy densities of two-phase elastic media:

$$F^\pm(M) = a_\pm \operatorname{tr}(e(M) - c_\pm P^{(k)})^2, \quad M \in \mathbb{R}^{m\times m}, \quad a_\pm, c_\pm \in \mathbb{R}, \quad a_\pm > 0, \tag{3.45}$$

where $P^{(k)}$, $1 \le k \le m$, is an orthogonal projector in $\mathbb{R}^m$, $m \ge 2$, onto a k-dimensional subspace.

In the case where $k = m$, the densities (3.45) are particular cases of the densities (3.13), for which, provided $a_+ = a_- = a$, the phase transitions temperatures $t_\pm$ are given by formulas (3.24), and Theorem 3.1 also guarantees the existence of the equilibrium states for $t \in (t_-, t_+)$. In the case where $k = 1$, it is known (Lemma 3.1) that for $t = 0$, there are no equilibrium states.

Theorem 3.5. *Let* $a_\pm$ *be arbitrary for* $k = 1$ *and* $a_+ = a_-$ *for* $1 < k \le m$. *Then the temperatures of phase transitions for the densities* (3.45) *are given by the relations*

$$t_- = t^* - \frac{[ac]^2}{a_-}, \quad t_+ = t^* + \frac{[ac]^2}{a_+}, \qquad t^* = -k[ac^2]. \tag{3.46}$$

3.4. Critical points of the energy functional. We consider diffeomorphisms $y = y(x)$ of class $C^1(\bar{\Omega}, R^m)$ of the domain Ω onto itself such that the inverse mappings $x = x(y)$ have the form

$$x(y) = y + h(y), \quad h \in C_0^1(\Omega, \mathbb{R}^m), \quad \|h\|_{C^1} \le \frac{1}{2}, \tag{3.47}$$

where h is an arbitrary function as in (3.47). We fix functions $\widetilde{u} \in \mathbb{H}$ and $\widetilde{\chi} \in \mathbb{Z}'$ and construct their perturbations u, χ by a rule similar to (2.31):

$$u(x) = \widetilde{u}(y(x)) + v(y(x)), \quad v \in \mathbb{H}, \qquad \chi(x) = \widetilde{\chi}(y(x)). \tag{3.48}$$

Lemma 3.11. *We have*

$$\begin{aligned} I_0[u,\chi,t] - I_0[\widetilde{u},\widetilde{\chi},t] &= \int_\Omega \{\widetilde{\chi} F^+_{M_{ij}}(\nabla\widetilde{u}) + (1-\widetilde{\chi}) F^-_{M_{ij}}(\nabla\widetilde{u})\} v^i_{x_j}\, dx \\ &\quad + \int_\Omega \big\{\widetilde{\chi}((F^+(\nabla\widetilde{u}) + t)\delta_{kj} - \widetilde{u}^i_{x_k} F^+_{M_{ij}}(\nabla\widetilde{u})) \\ &\qquad + (1-\widetilde{\chi})(F^-(\nabla\widetilde{u})\delta_{kj} - \widetilde{u}^i_{x_k} F^-_{M_{ij}}(\nabla\widetilde{u}))\big\} h^k_{x_j}\, dx + R, \\ &\widetilde{\chi} = \widetilde{\chi}(x), \quad \widetilde{u} = \widetilde{u}(x), \quad v = v(x), \quad h = h(x), \\ &|R| \le C \int_\Omega (|\nabla v|^2 + \|h\|^2_{C^1}(1 + |\nabla\widetilde{u}|^2))\, dx. \end{aligned} \tag{3.49}$$

We say that a pair $\widetilde{u} \equiv \widetilde{u}_t \in \mathbb{H}$ and $\widetilde{\chi} \equiv \widetilde{\chi}_t \in \mathbb{Z}'$ is a critical point of the functional I_0 for given t if

$$\begin{aligned}
&\int_\Omega \{\widetilde{\chi} F^+_{M_{ij}}(\nabla\widetilde{u}) + (1-\widetilde{\chi}) F^-_{M_{ij}}(\nabla\widetilde{u})\} v^i_{x_j}\, dx \\
&\quad + \int_\Omega \Big\{\widetilde{\chi}((F^+(\nabla\widetilde{u}) + t)\delta_{kj} - \widetilde{u}^i_{x_k} F^+_{M_{ij}}(\nabla\widetilde{u})) \\
&\qquad + (1-\widetilde{\chi})(F^-(\nabla\widetilde{u})\delta_{kj} - \widetilde{u}^i_{x_k} F^-_{M_{ij}}(\nabla\widetilde{u}))\Big\} h^k_{x_j}\, dx = 0 \\
&\qquad\qquad \text{for all } v \in \mathbb{H},\ \ h \in C^1_0(\Omega, \mathbb{R}^m).
\end{aligned} \tag{3.50}$$

In view of (3.49), the left-hand side of (3.50) is the linear part of the increment of the functional I_0 under the perturbation (3.48). Hence, the equilibrium states $\widehat{u}_t$, $\widehat{\chi}_t$ must be critical points. However, not every critical point is an equilibrium state. For example, the pairs $\widetilde{u} \equiv 0$, $\widetilde{\chi} \equiv 1$ and $\widetilde{u} \equiv 0$, $\widetilde{\chi} \equiv 0$ are critical points of the energy functional for any t, while they are the equilibrium states only for $t \le t_-$ and $t \ge t_+$, respectevely.

It is natural to call condition (3.50) imposed on the pair $\widetilde{u}$, $\widetilde{\chi}$ a generalized form of the equilibrium equations for a two-phase elastic medium. To describe the classical form of these equations, we introduce the following notation:

$$\begin{aligned}
\Theta_{kj}[u,\chi](x) &= \chi(x)\Theta^+_{kj}(\nabla u(x)) + (1-\chi(x))\Theta^-_{kj}(\nabla u(x)), \\
\Theta^\pm_{kj}(M) &= F^\pm_{M_{kj}}(M), \\
\Phi_{kj}[u,\chi](x,t) &= \chi(x)(\Phi^+_{kj}(\nabla u(x)) + t\delta_{kj}) + (1-\chi(x))\Phi^-_{kj}(\nabla u(x)), \\
\Phi^\pm_{kj}(M) &= F^\pm(M)\delta_{kj} - M_{ik}F^\pm_{M_{ij}}(M), \\
u \in \mathbb{H},\ \chi &\in \mathbb{Z}',\ \ x \in \Omega,\ t \in \mathbb{R},\ \ k,j = 1,\dots,m.
\end{aligned} \tag{3.51}$$

For the displacement field u and phase distribution χ, Θ is the stress tensor, and Φ is called the chemical potential tensor.

Theorem 3.6. *Let a ball $B_r(x_0) \subset \Omega$ and a critical point $\widetilde{u}$, $\widetilde{\chi}$ of the energy functional I_0 be fixed.*
(a) *If the function $\widetilde{\chi}$ is constant in the ball $B_r(x_0)$, then the function $\widetilde{u} \in C^\infty(B_r(x_0), \mathbb{R}^m)$ satisfies the system of equations*

$$\begin{aligned}
&-\frac{d}{dx_j} F^+_{M_{ij}}(\nabla\widetilde{u}(x)) = 0,\ \ x \in B_r(x_0),\ \ i = 1,\dots,m,\ \ \textit{whenever } \widetilde{\chi} \equiv 1 \textit{ in the ball } B_r(x_0), \\
&-\frac{d}{dx_j} F^-_{M_{ij}}(\nabla\widetilde{u}(x)) = 0,\ \ x \in B_r(x_0),\ \ i = 1,\dots,m,\ \ \textit{when } \widetilde{\chi} \equiv 0 \textit{ in the ball } B_r(x_0).
\end{aligned} \tag{3.52}$$

(b) *Suppose that the ball $B_r(x_0)$ is divided into two parts by an $(m-1)$-dimensional surface Γ of class $C^{k,\epsilon}$, $k \ge 2$, $\epsilon \in (0,1)$. Suppose that, $\widetilde{\chi} \equiv 1$ in $B^+_r(x_0)$ and $\widetilde{\chi} \equiv 0$ in $B^-_r(x_0)$. Then the function $\widetilde{u}$ belongs to the class $C^{k,\epsilon}$ in each of the domains $B^\pm_r(x_0)$ up to the boundary of their separation Γ, and*

$$\begin{aligned}
&-\frac{d}{dx_j} F^+_{M_{ij}}(\nabla\widetilde{u}(x)) = 0, \quad x \in B^+_r(x_0), \\
&-\frac{d}{dx_j} F^-_{M_{ij}}(\nabla\widetilde{u}(x)) = 0, \quad x \in B^-_r(x_0), \quad i = 1,\dots,m, \\
&\qquad [\Theta_{kj}[\widetilde{u},\widetilde{\chi}]]|_\Gamma n_j = 0, \quad k = 1,\dots,m, \qquad [\Phi_{kj}[\widetilde{u},\widetilde{\chi}]]|_\Gamma n_j n_k = 0,
\end{aligned} \tag{3.53}$$

where $[Z]|_\Gamma$ is the jump of Z when crossing the interface boundary Γ, and n is the unit normal vector to this boundary.

Equations (3.52) are the classical equilibrium equations for single-phase media with energy densities $F^{\pm}$. Equations (3.53) and the condition on the stress jump when crossing the surface Γ correspond to the standard necessary conditions for equilibrium states in composite media. The *a priori* uncertainty of the phase interface boundary Γ makes the problem of phase transitions different from the equilibrium problem in composite media. This fact leads to the arising of an additional condition in the equations of equilibrium, namely, to conditions for a jump of the chemical potential.

Note that the assumptions on the smoothness of the phase interface boundary for equilibrium states (and consequently, for critical points) remain only suppositions. Indeed, for equilibrium states of isotropic two-phase media, the restriction of the function $\widehat{\chi}_t$ to any ball of a Vitali covering can be an arbitrary measurable spherically symmetric characteristic function. Therefore, for equilibrium states, no smoothness of the phase interface boundary can be talked of. The situation becomes better when we take the surface energy into account. In this case, for sufficiently close values of the coefficients $a^{\pm}_{ijkl}$, it is possible to prove a certain smoothness of the phase interface.

Theorem 2.5 describes the set of all critical points for the one-dimensional problem. In the multidimensional case, a similar result can be obtained for the isotropic problem (3.12), (3.14).

A critical point $\widetilde{u}$, $\widetilde{\chi}$ of the functional $I_0[u,\chi,t]$ is said to be *regular* if there exists an open set $\omega \subset \Omega$, $\varnothing \neq \omega \neq \Omega$, for which $\partial\omega \cap \Omega$ consists of a finite collection of $(m-1)$-dimensional surfaces $\Gamma_l \in C^2$, $l = 1,\dots,k$, such that

$$\begin{gathered} \widetilde{\chi}(x) = 1 \quad \text{for} \quad x \in \overline{\omega}, \qquad \widetilde{\chi}(x) = 0 \;\; \text{for} \;\; x \in \Omega \setminus \overline{\omega}, \\ \widetilde{u} \in C^2(\overline{\omega}, \mathbb{R}^m) \cap C^2(\overline{\Omega} \setminus \omega, \mathbb{R}^m) \cap C(\overline{\Omega}, \mathbb{R}^m). \end{gathered} \tag{3.54}$$

Obviously, for a regular critical point, the statements (3.53) are valid.

We say that a critical point $\widetilde{u}$, $\widetilde{\chi}$ of the functional $I_0[u,\chi,t]$ is *potential* if

$$\operatorname{curl} \widetilde{u} = 0 \quad \text{in} \quad \Omega. \tag{3.55}$$

In the statement of the following theorem, the phase-transition temperatures (3.24) for isotropic two-phase media are employed.

Theorem 3.7. *Let $\widetilde{u}$, $\widetilde{\chi}$ be a regular potential critical point of the energy functional $I_0[u,\chi,t]$ with the densities* (3.13), (3.14). *Then either $t_- < t_+$ and $t \in (t_-, t_+)$, or $t_- = t_+ = t^*$ and $t = t^*$, and at this critical point the functional $I_0[\,.\,,.\,,t]$ takes its minimum value.*

If $\Omega = B_R$ and a regular critical point is spherically symmetric (in the sense described by (3.20)), then it is automatically potential. Hence, for the densities (3.13), (3.14), a spherically symmetric regular critical point in $\Omega = B_R$ minimizes the energy functional.

3.5. Bibliographical notes. The results of this section are based on the papers [21–23, 25–34, 36, 40, 65]. They also contain a number of statements related to other boundary conditions, nonzero force fields, and quasistationary problems on the evolution of the phase interface boundary. A construction close to the Vitali covering was used by specialists in mechanics in the study of composite media, see [62]. The method of constructing the Vitali covering was described in [51]. In [3], equilibrium equations for two-phase media were obtained by other means. The method of inner variation makes it possible to write out relatively short expressions for the second variation of the energy functional of a two-phase medium, see [12]. The assertions stated above on the smoothness of generalized solutions of the elliptic (see (3.8)) system of equilibrium equations in the theory of elasticity were given in [49]. The self-similarity of fractal sets was discussed in [11]. A problem for more than two-phase media was considered in [14].

It should be noted that the equilibrium energy $i(t,\Omega)$ is expressed in terms of a quasiconvex hull $\mathcal{F}(M,t)$ of the function $F^{\min}(M,t)=\min\{F^{+}(M)+t,F^{-}(M)\}$ by the formula $i(t,\Omega)=|\Omega|\mathcal{F}(0,t)$. The last formula specifies the dependence of $i(t,\Omega)$ on Ω. The quasiconvex hulls of the energy densities of a two-phase elastic medium will be investigated in §5.

A study of stability of the spherical phase interfaces in a ball under the condition $a_{+}\neq a_{-}$ was given in [7]. In that paper, it was shown that the critical points of the energy functional of isotropic two-phase media can be nonstable.

An approach closest to ours in the study of isotropic media is contained in [53]. In that paper, in the case where $a_{-}<a_{+}$, $b_{\pm}=0$, $t\geq 0$, for the energy densities (3.12), an analog of Lemma 3.2 was used to obtain a relatively explicit criterion for the existence of equilibrium states. The monograph [54] is very useful, together with the bibliography therein.

§4. Multidimensional problem with positive surface tension coefficient

The results of Subsection 2.3 *for the problem with a positive coefficient of surface tension are partially caried over to the multidimensional* ($m\geq 2$) *case. Construction of equilibrium states in explicit form being impossible, the main emphasis is made on their qualitative analysis.*

4.1. Statement of the problem and preliminary constructions. In order to determine the area of the phase interface boundary, to which the surface energy is proportional, for each function $\chi\in\mathbb{Z}'$ we introduce the quantity

$$S[\chi]=\sup_{\substack{h\in C_0^1(\Omega,\mathbb{R}^m),\\ |h|\leq 1}}\int_{\Omega}\chi\operatorname{div}h\,dx. \tag{4.1}$$

We set

$$\mathbb{Z}=\{\chi\in\mathbb{Z}'\,:\,S[\chi]<\infty\}. \tag{4.2}$$

In the case where the support of the function χ is separated from its complement in the domain Ω by a continuously differentiable $(m-1)$-dimensional surface Γ, the Stokes formula yields

$$S[\chi]=\sup_{\substack{h\in C_0^1(\Omega,R^m),\\ |h|\leq 1}}\int_{\Gamma}h\cdot n\,dS, \tag{4.3}$$

where n is the unit normal vector to Γ, looking outward with respect to $\operatorname{supp}\chi$. Calculating the supremum on the right-hand side of (4.3), we see that, for smooth phase interfaces, $S[\chi]$ coincides with the area of the interface boundary. For an arbitrary function $\chi\in\mathbb{Z}'$, by the interface boundary area of $\operatorname{supp}\chi$ and its complement we mean the quantity (4.1). By definition, this quantity is finite if and only if $\chi\in\mathbb{Z}$.

Using the definition (4.2), we write the energy functional of a two-phase elastic medium that takes into account the surface energy of the phase interface boundary in the form

$$I[u,\chi,t,\sigma]=I_0[u,\chi,t]+\sigma S[\chi],\quad u\in\mathbb{H},\quad \chi\in\mathbb{Z},\quad \sigma>0, \tag{4.4}$$

where the functional $I_0[u,\chi,t]$ and the space $\mathbb{H}$ are as defined in §3.

By an equilibrium state of a two-phase elastic medium with the energy functional (4.4), we understand a pair $\widehat{u}_{t,\sigma}$, $\widehat{\chi}_{t,\sigma}$ that minimizes this functional:

$$I[\widehat{u}_{t,\sigma},\widehat{\chi}_{t,\sigma},t,\sigma]=\inf_{u\in\mathbb{H},\chi\in\mathbb{Z}}I[u,\chi,t,\sigma],\qquad \widehat{u}_{t,\sigma}\in\mathbb{H},\quad \widehat{\chi}_{t,\sigma}\in\mathbb{Z}. \tag{4.5}$$

As above, an equilibrium state is said to be single-phase if $\widehat{\chi}_{t,\sigma} = 0$ or $\widehat{\chi}_{t,\sigma} = 1$ almost everywhere in Ω, and two-phase otherwise.

Now we list some properties of $S[\chi]$ needed in what follows. For the validity of (some among) them, certain restrictions on the smoothness of the boundary of the domain Ω are required; such restructions are always assumed to be fulfilled in what follows.

Let

$$\Omega \subset \mathbb{R}^m,\ m \geq 2,\ \text{be a bounded domain with Lipschitz boundary.} \tag{4.6}$$

Then:

(1) if a sequence of functions $\chi_n \in \mathbb{Z}$ converges almost everywhere to a function χ, then

$$S[\chi] \leq \liminf_{n\to\infty} S[\chi_n], \tag{4.7}$$

provided that the right-hand side of (4.7) is finite, and $\chi \in \mathbb{Z}$;

(2) any sequence $\chi_n \in \mathbb{Z}$ with $S[\chi_n] \leq R \neq R(n)$ admits selection of a subsequence $\chi_{n'}$ such that

$$\chi_{n'}(x) \to \chi(x) \quad \text{almost everywhere on} \quad \Omega \quad \text{and} \quad \chi \in \mathbb{Z}; \tag{4.8}$$

(3) if $\chi \in \mathbb{Z}$ is such that for fixed $\gamma \in (0, 1)$ we have

$$\frac{1}{|\Omega|} \int_\Omega \chi\, dx \leq \gamma, \tag{4.9}$$

then

$$\left(\int_\Omega \chi\, dx \right)^{\frac{m-1}{m}} \leq \kappa_\gamma S[\chi], \tag{4.10}$$

where $\kappa_\gamma = \kappa_\gamma(\Omega)$ is a positive constant.

Statement (1) means the lower semicontinuity of the function $S[\chi]$ with respect to convergence almost everywhere, Statement (2) is related to the fact that the embedding of the space $BV(\Omega)$ in $L_1(\Omega)$ is compact, and statement (3) is called the isoperimetric inequality. Obviously, $S[\chi] = 0$ if and only if $\chi \equiv 0$ or $\chi \equiv 1$.

The following example of a function $\chi \in \mathbb{Z}' \setminus \mathbb{Z}$ will be useful.

Lemma 4.1. *Let v, χ be an arbitrary solution of system* (3.19) *with $\alpha \neq 0$ and $Q \in (0, 1)$ constructed in Lemma* 3.5 *for some Vitali covering of the domain Ω. Then $\chi \in \mathbb{Z}' \setminus \mathbb{Z}$.*

To establish the relationship between the sets $\mathbb{Z}'$ and $\mathbb{Z}$, we introduce the additional set

$$\mathbb{Z}'' = \big\{ \chi \in L_\infty(\Omega) : 0 \leq \chi(x) \leq 1 \ \text{ almost everywhere on } \ \Omega \big\}. \tag{4.11}$$

Clearly,

$$\mathbb{Z} \subset \mathbb{Z}' \subset \mathbb{Z}''. \tag{4.12}$$

We recall the definition of $*$-weak convergence. We say that $\chi_n \xrightarrow{*} \chi$, $\chi_n, \chi \in L_\infty(\Omega)$, if for all $f \in L_1(\Omega)$ we have

$$\int_\Omega f \chi_n\, dx \to \int_\Omega f \chi\, dx. \tag{4.13}$$

It is known that the set $\mathbb{Z}''$ is compact with respect to $*$-weak convergence:

from any sequence $\chi_n \in \mathbb{Z}''$ one can select a subsequence $\chi_{n'}$ $*$-weak convergent to a function $\chi \in \mathbb{Z}''$.

Let $\mathbb{Z}_*$, $\mathbb{Z}'_*$, $\mathbb{Z}''_*$ denote the $*$-weak closure of sets $\mathbb{Z}$, $\mathbb{Z}'$, $\mathbb{Z}''$, respectively. It follows that

$$\mathbb{Z}''_* = \mathbb{Z}''. \tag{4.14}$$

Lemma 4.2. (a) *We have*

$$\mathbb{Z}_* = \mathbb{Z}'_* = \mathbb{Z}''_*. \tag{4.15}$$

(b) *If a sequence $\chi_n \in \mathbb{Z}'$ is $*$-weak convergent to $\chi \in \mathbb{Z}'$, then $\chi_n \to \chi$ in the space $L_p(\Omega)$ for any $p \in [1, \infty)$.*
(c) *For any $\chi \in \mathbb{Z}'$ there exists a sequence $\chi_n \in \mathbb{Z}$ such that $\chi_n \to \chi$ in the space $L_p(\Omega)$ for any $p \in [1, \infty)$. The sequence χ_n can be chosen so that, for each n,*

$$\operatorname{supp} \chi_n = \bigcup_{j=1}^{N(n)} \bar{\omega}_j, \tag{4.16}$$

where the ω_j are strictly inner subdomains of the domain Ω with smooth boundaries, $\bar{\omega}_j \cap \bar{\omega}_k = \varnothing$ for $j \neq k$.

We return for a while to the functional $I_0[u, \chi, t]$ with zero coefficient of surface tension. As before, we assume that the admissible displacement fields are functions of class $\mathbb{H}$, and as admissible phase distributions we choose one of the three sets $\mathbb{Z}$, $\mathbb{Z}'$, or $\mathbb{Z}''$. If we assume that the set $\mathbb{Z}'$ is a standard admissible set of phase distributions, then the set $\mathbb{Z}$ narrows, and the set $\mathbb{Z}''$ extends the domain of the functional I_0.

For each t, we set

$$\begin{aligned} \mu(t) &= \inf_{u \in \mathbb{H}, \chi \in \mathbb{Z}} I_0[u, \chi, t], \\ \mu'(t) &= \inf_{u \in \mathbb{H}, \chi \in \mathbb{Z}'} I_0[u, \chi, t], \\ \mu''(t) &= \inf_{u \in \mathbb{H}, \chi \in \mathbb{Z}''} I_0[u, \chi, t]. \end{aligned} \tag{4.17}$$

For these quantities and each t, the following inequalities are obvious:

$$-\infty < \mu''(t) \leq \mu'(t) \leq \mu(t) < \infty. \tag{4.18}$$

Lemma 4.3. *We have*

$$\mu''(t) = \mu'(t) = \mu(t). \tag{4.19}$$

The importance of the second identity in (4.19) is that in the definition (3.29) of the function $i(t, \Omega)$, the replacement of the set $\mathbb{Z}'$ by the set $\mathbb{Z}$ does not affect the magnitude of this function.

The replacement of the admissible set of phase distributions $\mathbb{Z}'$ by the set $\mathbb{Z}''$ has a mechanical interpretation. The function $\chi \in \mathbb{Z}'$ describes the situation when at each point $x \in \Omega$ a substance can be realized only in one of the phases, while the function $\chi \in \mathbb{Z}''$ admits the existence of a mixture of phases at the point x with indices $+$ and $-$ in fractions $\chi(x)$ and $1 - \chi(x)$, respectively.

In the next lemma, we discuss the relationship between the initial and extended problems for the functional $I_0[u, \chi, t]$.

Lemma 4.4. *Suppose that, for some t, the variational problem*

$$I_0[\tilde{u}_t, \tilde{\chi}_t, t] = \inf_{\substack{u \in \mathbb{H}, \\ \chi \in \mathbb{Z}''}} I_0[u, \chi, t], \qquad \tilde{u}_t \in \mathbb{H}, \quad \tilde{\chi}_t \in \mathbb{Z}'' \tag{4.20}$$

is solvable. Then the following is true.

(a) *For any its solution* $\widetilde{u}_t$, $\widetilde{\chi}_t$ *we have*

$$
\begin{aligned}
&\widetilde{\chi}_t(x) = 1 \quad \textit{for almost all } x \in \Omega \textit{ such that } \phi(x,t) < 0,\\
&\widetilde{\chi}_t(x) = 0 \quad \textit{for almost all } x \in \Omega \textit{ such that } \phi(x,t) > 0,\\
&\widetilde{\chi}_t(x) \in [0,1] \quad \textit{for almost all } x \in \Omega \textit{ such that } \phi(x,t) = 0,\\
&\phi(x,t) = \big(F^+(\nabla \widetilde{u}_t(x)) - F^-(\nabla \widetilde{u}_t(x))\big) + t;
\end{aligned}
\tag{4.21}
$$

(b) *for such* t, *problem* (3.10) *is also solvable*; *every its solution* $\widehat{u}_t$, $\widehat{\chi}_t$ *is a solution of problem* (4.20), *and any solution* $\widetilde{u}_t$, $\widetilde{\chi}_t$ *of problem* (4.20) *gives rise to a solution* $\widehat{u}_t$, $\widehat{\chi}_t$ *of problem* (3.10) *by the following rule*:

$$
\begin{aligned}
&\widehat{u}_t(x) = \widetilde{u}_t(x) \textit{ everywhere in } \Omega,\\
&\widehat{\chi}_t(x) = \widetilde{\chi}_t(x) \quad \textit{for } x \in \Omega \textit{ such that } \phi(x,t) \neq 0,\\
&\widehat{\chi}_t(x) \textit{ is an arbitrary characteristic function}\\
&\qquad\qquad \textit{on the set of points } x \in \Omega \textit{ such that } \phi(x,t) = 0.
\end{aligned}
\tag{4.22}
$$

Let us find out where the assertions of Lemma 4.4 lead to in the case of isotropic two-phase media.

Lemma 4.5. *Suppose that identities* (3.12) *and* (3.14) *are true. Then*:
(a) *for* $t_- < t_+$, *each solution of problem* (4.20) *is a solution of problem* (3.10);
(b) *for* $t_- = t_+ = t^*$ *and* $t \neq t^*$, *each solution of problem* (4.20) *is a solution of problem* (3.10);
(c) *for* $t_- = t_+ = t^*$ *and* $t = t^*$, *the set of all solutions of problem* (4.20) *is exhausted by the pairs where* $\widetilde{u}_t = 0$ *and* $\widetilde{\chi}_t$ *is an arbitrary function in* $\mathbb{Z}''$, *while the set of all solutions of problem* (3.10) *has the form* $\widehat{u}_t = 0$, $\widehat{\chi}_t$ *is an arbitrary element of* $\mathbb{Z}'$.

Thus, in the case of isotropic media, in general, transition to a mixture of phases proposed in Lemma 4.4 does not lead to equilibrium states in which these mixtures are realized. An exception is only the degenerate case treated in part (*c*) of the lemma.

4.2. The existence of equilibrium states. Taking the surface energy of the interface boundary into account significantly improves the mathematical properties of the energy functional of a two-phase elastic medium. In particular, this leads to the solvability of the variational problem (4.5) for any temperatures t and any positive surface tension coefficients σ. The proof of the existence of equilibrium states is based on traditional methods of calculus of variations, based on the coerciveness and lower semicontinuity of the functional under study. The existence of these properties for the functional (4.4) is established in the next lemma.

Lemma 4.6. *The energy functional* (4.4) *is coercive*:

$$
\begin{aligned}
&I[u_n, \chi_n, t, \sigma] \to \infty \quad \textit{for fixed } t, \sigma \in \mathbb{R},\ \sigma > 0,\\
&\textit{and any sequences } u_n \in \mathbb{H},\ \chi_n \in \mathbb{Z},\ n = 1, 2, \dots,\ \textit{with}\\
&\qquad\qquad \|u_n\|_{W_2^1} + S[\chi_n] \to \infty \quad \textit{as} \quad n \to \infty,
\end{aligned}
\tag{4.23}
$$

and lower semicontinuous: *for any functions* $u_n, u \in W_2^1(\Omega, \mathbb{R}^m)$ *and* $\chi_n, \chi \in \mathbb{Z}$, $n = 1, 2, \dots$, *such that*

$$
u_n \rightharpoonup u \quad \textit{in the space } W_2^1(\Omega, \mathbb{R}^m) \quad \textit{and} \quad \chi_n \to \chi \textit{ almost everywhere on } \Omega \tag{4.24}
$$

for all $t, \sigma \in \mathbb{R}$, $\sigma > 0$, *we have*

$$
I[u, \chi, t, \sigma] \le \liminf_{n \to \infty} I[u_n, \chi_n, t, \sigma]. \tag{4.25}
$$

Lemma 4.6 ensures the following statement.

Theorem 4.1. *The variational problem* (4.5) *for the functional* (4.4) *is solvable.*

We shall try to answer the question as to why the above technique does not yield the solvability of problem (3.10).

Let $u_n \in \mathbb{H}$, $\chi_n \in \mathbb{Z}'$ be a minimizing sequence of the functional $I_0[u,\chi,t]$ for some t. Then inequality (3.4) implies the uniform boundedness of the sequence $\|u_n\|_{W_2^1}$. Hence, from this minimizing sequence we can select a subsequence (we keep the same notation) such that

$$u_n \rightharpoonup u \text{ in the space } W_2^1(\Omega,\mathbb{R}^m), \quad \chi_n \overset{*}{\rightharpoonup} \chi, \quad u \in W_2^1(\Omega,\mathbb{R}^m), \quad \chi \in \mathbb{Z}''. \tag{4.26}$$

Since the limit function χ in (4.26) may fail to belong to the set $\mathbb{Z}'$, we need to replace problem (3.10) by problem (4.20). By Lemma 4.4, the solvability of problem (4.20) implies the solvability of (3.10). For problem (4.20), an analog of Theorem 4.1 will be true if we are able to prove that for any sequences $u_n \in W_2^1(\Omega,\mathbb{R}^m)$, $\chi_n \in \mathbb{Z}''$ satisfying (4.26), we have

$$I_0[u,\chi,t] \le \liminf_{n\to\infty} I_0[u_n,\chi_n,t]. \tag{4.27}$$

Theorem 4.2. *Inequality* (4.27) *is fulfilled for some t and all sequences $u_n \in W_2^1(\Omega,R^m)$, $\chi_n \in \mathbb{Z}''$ satisfying* (4.26) *if and only if*

$$A^+ = A^-, \quad \zeta^+ = \zeta^-. \tag{4.28}$$

This theorem removes a dissonance between the example of the nonexistence of equilibrium states for $\sigma = 0$, as constructed in Lemma 3.1, and the method of the proof of the existence of equilibrium states for $\sigma > 0$ based on Lemma 4.6.

The positivity of the surface tension coefficient not only leads to the existence of equilibrium states, but also has a certain impact on their properties. In the following theorem we use the temperatures of phase transitions (3.28) and the constant occurring in the isoperimetric inequality (4.10).

Theorem 4.3. (a) *The two-phase states of equilibrium may exist only under the following restrictions on the parameters t and σ*:

$$t_- < t < t_+, \quad 0 < \sigma \le \frac{3\kappa_{1/2}(\Omega)}{\nu}|[A\zeta]|^2\Big(\frac{|\Omega|}{2}\Big)^{\frac{1}{m}}. \tag{4.29}$$

(b) *For all two-phase states of equilibrium $\widehat{u}_{t,\sigma}$, $\widehat{\chi}_{t,\sigma}$, the volume fraction of the phase with index $+$ satisfies the inequality*

$$\begin{gathered} \gamma(\sigma) \le \frac{1}{|\Omega|}\int_\Omega \widehat{\chi}_{t,\sigma}\,dx \le 1-\gamma(\sigma), \quad \sigma > 0, \\ (0,1/2] \ni \gamma(\sigma) = \frac{1}{|\Omega|}\Big(\frac{\nu\sigma}{3\kappa_{1/2}(\Omega)|[A\zeta]|^2}\Big)^m. \end{gathered} \tag{4.30}$$

(c) *For every t and any positive σ, the single-phase states $u \equiv 0$, $\chi \equiv 0$ and $u \equiv 0$, $\chi \equiv 1$ are local minima of the functional* (4.4) *with respect to any perturbation in the space $\mathbb{H}$ of functions u and sufficiently $L_1(\Omega)$-small perturbations of class $\mathbb{Z}$ of functions χ.*

Statement (a) of Theorem 4.3 provides localization of the values of t,σ for which two-phase equilibrium states can exist. The fact of localization itself is quite plausible: the existence of two-phase equilibrium states is energetically disadvantageous for large values of $|t|$ and σ. Estimates (4.29) and the coordination of the first of them with the case of $\sigma = 0$ deserve our attention. Note that for $t_- = t_+$, the functional (4.4) has no two-phase equilibrium states for any σ. Assertions (b) and (c) of the theorem indicate from different viewpoints that for $\sigma > 0$, the arising of new phase embryos with an arbitrarily small volume is energetically unfavorable.

4.3. Temperatures of phase transitions. In this subsection, we give a partial generalization of the one-dimensional description of the set of all equilibrium states for positive coefficient of surface tension to the multidimensional case. Let us study the dependence of the phase transition temperatures on the coefficient of surface tension. First, we specify the set of values of the parameters t and σ for which two-phase equilibrium states obtained in Theorem 4.3(a) are possible.

By analogy with (3.29), we introduce the functions

$$
\begin{gathered}
j^+(t,\sigma,\Omega)=\inf_{u\in\mathbb{H}} I[u,\chi^+,t,\sigma], \quad j^-(t,\sigma,\Omega)=\inf_{u\in\mathbb{H}} I[u,\chi^-,t,\sigma], \\
\text{where} \quad \chi^+\equiv 1, \quad \chi^-\equiv 0, \\
j_{\min}(t,\sigma,\Omega)=\min\{j^-(t,\sigma,\Omega), j^+(t,\sigma,\Omega)\}, \\
j(t,\sigma,\Omega)=\inf_{\substack{u\in\mathbb{H},\\ \chi\in\mathbb{Z}}} I[u,\chi,t,\sigma], \quad t,\sigma\in R^1, \quad \sigma\geq 0.
\end{gathered} \tag{4.31}
$$

Obviously,

$$
j^+(t,\sigma,\Omega)=i^+(t,\Omega), \quad j^-(t,\sigma,\Omega)=i^-(t,\Omega), \quad j_{\min}(t,\sigma,\Omega)=i_{\min}(t,\Omega),
$$

the function $j(\,.\,,\,.\,,\Omega)$ is locally bounded from below, and by Lemma 4.3 we have $j(t,0,\Omega)=i(t,\Omega)$. The definitions (4.31) show that

$$
\begin{gathered}
j_{\min}(t,\sigma,\Omega)-j(t,\sigma,\Omega)\geq 0 \quad \text{for all} \quad t,\sigma\in\mathbb{R}, \quad \sigma\geq 0, \\
\text{and the strict inequality is a criterion for the} \\
\text{absence of single-phase equilibrium states.}
\end{gathered} \tag{4.32}
$$

In the following lemmas, we give the description of the set where the left-hand side of inequality (4.32) is positive.

Lemma 4.7. *The function $j(\,.\,,\,.\,,\Omega)$ is concave, continuous for $t,\sigma\in\mathbb{R}$, $\sigma\geq 0$, and satisfies the local Lipschitz condition for $\sigma>0$. For fixed t, the function $j(t,\,.\,,\Omega)$ increases monotonically.*

Lemma 4.8. *There exists a nonnegative bounded function $\sigma(t,\Omega)$ such that*

$$
\begin{aligned}
&j_{\min}(t,\sigma,\Omega)-j(t,\sigma,\Omega)>0 \quad \text{if} \quad \sigma\in(0,\sigma(t,\Omega)),\\
&j_{\min}(t,\sigma,\Omega)-j(t,\sigma,\Omega)=0 \quad \text{if} \quad \sigma\geq\sigma(t,\Omega).
\end{aligned} \tag{4.33}
$$

Moreover,

$$
\sigma(t,\Omega)=0 \quad \text{if} \quad t\notin(t_-,t_+), \quad \sigma(t,\Omega)>0 \quad \text{if} \quad t\in(t_-,t_+). \tag{4.34}
$$

Lemma 4.9. *The function $\sigma(\,.\,,\Omega)$ is convex on each of the semiaxes $(-\infty,t^*]$, $[t^*,\infty)$. This function is strictly monotone increasing in the variable $t\in[t_-,t^*]$, strictly monotone decreasing in the variable $t\in[t^*,t_+]$, and Lipschitz on the real line.*

The graph of the function $\sigma(\,.\,,\Omega)$ splits the half-plane of parameters t, σ, $\sigma>0$, into the zones (2.21). Our next goal is to describe the sets of all equilibrium states in each of these zones. We start with the simplest case where the parameters t,σ belong to one of the zones $V_<$, $V_>^-$, $V_>^+$, $V_>^*$.

Theorem 4.4. *The following statements are true*:

$$
\widehat{u}_{t,\sigma},\widehat{\chi}_{t,\sigma}=\begin{cases}\widehat{u}^+,\chi^+ & \text{if} \quad t,\sigma\in V_>^-,\\ \widehat{u}^-,\chi^- & \text{if} \quad t,\sigma\in V_>^+,\\ \widehat{u}^\pm,\chi^\pm & \text{if} \quad t,\sigma\in V_>^*,\end{cases} \qquad u^\pm\equiv 0,\ \chi^+\equiv 1,\ \chi^-\equiv 0, \tag{4.35}
$$

and for $t,\sigma\in V_<$ only two-phase equilibrium states exist.

To describe the set of all equilibrium states in the remaining zones $V_{=}^{-}$, $V_{=}^{+}$, $V_{=}^{*}$, we need additional facts related to the properties of solutions of problem (4.5). These properties are given in Lemmas 4.10–4.13; they are not only used to describe the equilibrium states in the remaining zones, but are also of independent interest. The first of them generalizes the property of lower semicontinuity to the case of variable parameters t and σ.

Lemma 4.10. (a) *Suppose that* $t_n, t \in \mathbb{R}$, $u_n, u \in \mathbb{H}$, $\chi_n, \chi \in \mathbb{Z}'$, *and*

$$\begin{aligned} &t_n \to t, \quad u_n \rightharpoondown u \ \text{ in the space } \ W_2^1(\Omega, R^m), \\ &\chi_n(x) \to \chi(x) \ \text{ almost everywhere on } \ \Omega \ \text{ as } \ n \to \infty. \end{aligned} \tag{4.36}$$

Then

$$I_0[u, \chi, t] \le \liminf_{n\to\infty} I_0[u_n, \chi_n, t_n]. \tag{4.37}$$

(b) *Suppose that* $t_n, t, \sigma_n, \sigma \in \mathbb{R}$, $\sigma_n, \sigma > 0$, $u_n, u \in \mathbb{H}$, $\chi_n, \chi \in \mathbb{Z}$, *and*

$$\begin{aligned} &t_n \to t, \quad \sigma_n \to \sigma, \quad u_n \rightharpoondown u \text{ in the space } W_2^1(\Omega, R^m), \\ &\chi_n(x) \to \chi(x) \text{ almost everywhere in } \Omega \text{ as } n \to \infty. \end{aligned} \tag{4.38}$$

Then under the condition $S[\chi_n] \le R \ne R(n)$, *we have*

$$I[u, \chi, t, \sigma] \le \liminf_{n\to\infty} I[u_n, \chi_n, t_n, \sigma_n]. \tag{4.39}$$

The next lemma asserts that convergence of energy functionals improves the convergence of their arguments.

Lemma 4.11. (a) *Under condition* (4.36), *suppose that*

$$I_0[u_n, \chi_n, t_n] \to I_0[u, \chi, t]. \tag{4.40}$$

Then $u_n \to u$ *in the space* $W_2^1(\Omega, \mathbb{R}^m)$.
(b) *Under condition* (4.38), *suppose that*

$$I[u_n, \chi_n, t_n, \sigma_n] \to I[u, \chi, t, \sigma], \quad S[\chi_n] \le R \ne R(n). \tag{4.41}$$

Then

$$u_n \to u \ \text{ in the space } \ W_2^1(\Omega, \mathbb{R}^m), \quad S[\chi_n] \to S[\chi]. \tag{4.42}$$

Now we establish the continuous dependence of equilibrium states on the parameters t and σ. Since for fixed t and σ, even in the one-dimensional case, in general, an equilibrium state is not unique, the concept of continuous dependence itself needs to be specified.

Lemma 4.12. *For any sequences* t_n, σ_n *such that*

$$t_n \to t, \qquad \sigma_n \to \sigma, \quad \sigma, \sigma_n > 0, \tag{4.43}$$

and any sequence of equilibrium states $\widehat{u}_{t_n,\sigma_n}$, $\widehat{\chi}_{t_n,\sigma_n}$ *there exists a sequence* $t_{n'}$, $\sigma_{n'}$ *and a state of equilibrium* $\widehat{u}_{t,\sigma}$, $\widehat{\chi}_{t,\sigma}$ *such that*

$$\begin{aligned} &\widehat{u}_{t_{n'},\sigma_{n'}} \to \widehat{u}_{t,\sigma} \text{ in the space } W_2^1(\Omega, \mathbb{R}^m), \\ &\widehat{\chi}_{t_{n'},\sigma_{n'}} \to \widehat{\chi}_{t,\sigma} \text{ almost everywhere on } \Omega, \quad S[\widehat{\chi}_{t_{n'},\sigma_{n'}}] \to S[\widehat{\chi}_{t,\sigma}]. \end{aligned} \tag{4.44}$$

We denote by $\mathfrak{B}_{t,\sigma}$, $t, \sigma \in \mathbb{R}$, $\sigma > 0$, the set of all equilibrium states of the functional $I[u, \chi, t, \sigma]$. By Theorem 4.1, $\mathfrak{B}_{t,\sigma} \ne \varnothing$ for any given t and σ.

Lemma 4.13. *Any sequence* $\widehat{u}_n, \widehat{\chi}_n \in \mathfrak{B}_{t,\sigma}$ *has a subsequence* $\widehat{u}_{n'}$, $\widehat{\chi}_{n'}$ *such that*

$$\begin{aligned} &\widehat{u}_{n'} \to \widehat{u} \in \mathbb{H} \ \text{ in the space } \ W_2^1(\Omega, R^m), \\ &\widehat{\chi}_{n'} \to \widehat{\chi} \in \mathbb{Z} \ \text{ almost everywhere on } \ \Omega, \quad S[\widehat{\chi}_{n'}] \to S[\widehat{\chi}], \end{aligned} \tag{4.45}$$

and $\widehat{u}, \widehat{\chi} \in \mathfrak{B}_{t,\sigma}$.

Lemmas 4.10–4.13 are used partially in the proof of the following theorem.

Theorem 4.5.

$$\mathfrak{B}_{t,\sigma} = \begin{cases} \widehat{u}^+, \chi^+ \text{ and at least one} \\ \qquad \text{two-phase equilibrium state if } t,\sigma \in V_=^-, \\ \widehat{u}^-, \chi^- \text{ and at least one} \\ \qquad \text{two-phase equilibrium state if } t,\sigma \in V_=^+, \\ \widehat{u}^\pm, \chi^\pm \text{ and at least one} \\ \qquad \text{two-phase equilibrium state if } t,\sigma \in V_=^*. \end{cases} \tag{4.46}$$

$$\widehat{u}^\pm \equiv 0, \qquad \text{here } \chi^+ \equiv 1, \quad \chi^- \equiv 0.$$

By Lemma 4.9, for $t_- < t_+$, the function $\sigma(\,.\,,\Omega)$ takes its maximal value σ^* at a single point $t = t^*$. As in the one-dimensional case, we define the temperatures of the phase transitions $t_\pm(\sigma,\Omega)$ by formulas similar to (2.24), (2.25). Lemma 4.9 shows that the functions $t_\pm(\,.\,,\Omega)$ are continuous. In case $t_- < t_+$, the function $t_-(\,.\,,\Omega)$ is strictly monotone increasing on the interval $[0,\sigma^*]$, the function $t_+(\,.\,,\Omega)$ is strictly monotone decreasing on the same interval, and

$$t_\pm(0,\Omega) = t_\pm, \quad t_\pm(\sigma^*,\Omega) = t^*.$$

Since the functions $t_\pm(\,.\,,\Omega)$ are inverses to the function $\sigma(\,.\,,\Omega)$ on the intervals $(-\infty,t^*]$ and $[t^*,\infty)$, respectively, the function $t_+(\,.\,,\Omega)$ is convex, and the function $t_-(\,.\,,\Omega)$ is concave. Therefore, the functions $t_\pm(\,.\,,\Omega)$ satisfy the Lipschitz condition on the half-axis $(0,\infty)$. The one-dimensional case demonstrates (see formulas (2.26), (2.27)) that the Lipschitz constant can grow unboundedly as $\sigma \to 0$.

Theorems 4.4 and 4.5 imply a partial validity of the characterization 1–13 in Subsection 2.3 of the phase transition process with the temperature t varying from $-\infty$ to $+\infty$. Statements 1–5 and 11–13 are valid, in which information on two-phase equilibrium states should be replaced by the words "and at least one two-phase equilibrium state".

To reformulate statements 6 and 10 of the characterization 1–13 for the multidimensional case, we need to study the multivalued function

$$\widehat{Q}[t,\sigma] = \frac{1}{|\Omega|}\int_\Omega \widehat{\chi}_{t,\sigma}(x)\,dx, \qquad \widehat{u}_{t,\sigma}, \widehat{\chi}_{t,\sigma} \in \mathfrak{B}_{t,\sigma}, \tag{4.47}$$

the value of which for every t and σ represents a subset of the interval $[0,1]$ (which may also be a singleton).

Lemma 4.14. (a) *The set $\widehat{Q}[t,\sigma]$ is closed. There exist equilibrium states*

$$\widehat{u}_{t,\sigma,\min}, \quad \widehat{\chi}_{t,\sigma,\min} \quad \text{and} \quad \widehat{u}_{t,\sigma,\max}, \quad \widehat{\chi}_{t,\sigma,\max}, \tag{4.48}$$

such that the numbers

$$\widehat{Q}_{\min}(t,\sigma) = \frac{1}{|\Omega|}\int_\Omega \widehat{\chi}_{t,\sigma,\min}(x)\,dx, \quad \widehat{Q}_{\max}(t,\sigma) = \frac{1}{|\Omega|}\int_\Omega \widehat{\chi}_{t,\sigma,\max}(x)\,dx \tag{4.49}$$

are the minimal and maximal elements of the set $\widehat{Q}[t,\sigma]$.
(b) *The functions $\widehat{Q}_{\min}(\,.\,,\sigma)$, $\widehat{Q}_{\max}(\,.\,,\sigma)$ decrease monotonically. For $t_2 > t_1$ we have*

$$\widehat{Q}_{\min}(t_1,\sigma) \geq \widehat{Q}_{\max}(t_2,\sigma). \tag{4.50}$$

(c) *The limits*

$$\begin{gathered} \lim_{t\uparrow t_0}\widehat{Q}_{\min}(t,\sigma) \equiv \widehat{Q}_{\min}(t_0-0,\sigma), \quad \lim_{t\downarrow t_0}\widehat{Q}_{\min}(t,\sigma) \equiv \widehat{Q}_{\min}(t_0+0,\sigma), \\ \lim_{t\uparrow t_0}\widehat{Q}_{\max}(t,\sigma) \equiv \widehat{Q}_{\max}(t_0-0,\sigma), \quad \lim_{t\downarrow t_0}\widehat{Q}_{\max}(t,\sigma) \equiv \widehat{Q}_{\max}(t_0+0,\sigma) \end{gathered} \tag{4.51}$$

exist and satisfy

$$\begin{aligned} &\widehat{Q}_{\min}(t_0-0,\sigma)=\widehat{Q}_{\max}(t_0,\sigma), \quad \widehat{Q}_{\min}(t_0+0,\sigma)=\widehat{Q}_{\min}(t_0,\sigma),\\ &\widehat{Q}_{\max}(t_0-0,\sigma)=\widehat{Q}_{\max}(t_0,\sigma), \quad \widehat{Q}_{\max}(t_0+0,\sigma)=\widehat{Q}_{\min}(t_0,\sigma). \end{aligned} \tag{4.52}$$

This implies that the function $\widehat{Q}_{\min}(\,.\,,\sigma)$ is continuous from the right, and the function $\widehat{Q}_{\max}(\,.\,,\sigma)$ is continuous from the left.

Theorems 4.4, 4.5 and Lemma 4.14 allow us to formulate a multidimensional analog of statements 6 and 10 of the characterization 1–13 of phase transitions in the one-dimensional case, proved in Subsection 2.3.

6′. Let $t_- < t_+$, $\sigma\in(0,\sigma^*)$. Then

$$\widehat{Q}_{\min}(t,\sigma)=\widehat{Q}_{\max}(t,\sigma)=\begin{cases}1 \text{ if } \ t<t_-(\sigma,\Omega),\\ 0 \text{ if } \ t>t_+(\sigma,\Omega),\end{cases}$$

$$0<\widehat{Q}_{\min}(t,\sigma)\le\widehat{Q}_{\max}(t,\sigma)<1 \text{ for } t\in(t_-(\sigma,\Omega),t_+(\sigma,\Omega)),$$

$$\widehat{Q}_{\max}(t_-(\sigma,\Omega),\sigma)=1, \quad \widehat{Q}_{\min}(t_-(\sigma,\Omega),\sigma)\in(0,1),$$

$$\widehat{Q}_{\max}(t_+(\sigma,\Omega),\sigma)\in(0,1), \quad \widehat{Q}_{\min}(t_+(\sigma,\Omega),\sigma)=0.$$

10′. Let $t_- < t_+$, $\sigma=\sigma^*$. Then

$$\widehat{Q}_{\min}(t,\sigma^*)=\widehat{Q}_{\max}(t,\sigma^*)=\begin{cases}1 \text{ if } \ t<t^*,\\ 0 \text{ if } \ t>t^*,\end{cases}$$

$$\widehat{Q}_{\max}(t^*,\sigma^*)=1, \quad \widehat{Q}_{\min}(t^*,\sigma^*)=0,$$

$$\widehat{Q}[t^*,\sigma^*] \text{ contains a nonempty subset of the interval } (0,1).$$

We proceed to the study of the dependence of the phase transition temperatures $t_\pm(\sigma,\Omega)$ on the second argument. A complete result is obtained only for the families of domains Ω_e and Ω^λ defined in (3.41).

In the notation (4.1) of the area of the phase interface boundary, we incorporate temporarily the argument Ω indicating its dependence on the domain of integration. Clearly, for $\widetilde{\chi}\in\mathbb{Z}\equiv\mathbb{Z}(\Omega)$ we have

$$\begin{aligned} &S[\chi,\Omega_e]=S[\widetilde{\chi},\Omega], \ x\in\Omega_e, \ \widetilde{x}\in\Omega, \ x=\widetilde{x}+e, \ \chi(x)=\widetilde{\chi}(\widetilde{x}), \ \chi\in\mathbb{Z}(\Omega_e),\\ &S[\chi,\Omega^\lambda]=\lambda^{m-1}S[\widetilde{\chi},\Omega], \ x\in\Omega^\lambda, \ \widetilde{x}\in\Omega, \ x=\lambda\widetilde{x}, \ \chi(x)=\widetilde{\chi}(\widetilde{x}), \ \chi\in\mathbb{Z}(\Omega^\lambda). \end{aligned} \tag{4.53}$$

In the notation (4.4) of the energy functional of a two-phase medium, we also temporarily incorporate the argument Ω indicating its dependence on the domain of integration. Using (4.53), we obtain

$$\begin{aligned} &I[u,\chi,t,\sigma,\Omega_e]=I[\widetilde{u},\widetilde{\chi},t,\sigma,\Omega], \quad u(x)=\widetilde{u}(\widetilde{x}),\\ &I[u,\chi,t,\sigma,\Omega^\lambda]=\lambda^m I\Big[\widetilde{u},\widetilde{\chi},t,\frac{\sigma}{\lambda},\Omega\Big], \quad u(x)=\lambda\widetilde{u}(\widetilde{x}). \end{aligned} \tag{4.54}$$

Then for equilibrium energies (4.31) we get

$$j(t,\sigma,\Omega_e)=j(t,\sigma,\Omega), \quad j(t,\sigma,\Omega^\lambda)=\lambda^m j\Big(t,\frac{\sigma}{\lambda},\Omega\Big). \tag{4.55}$$

Moreover,

$$\begin{aligned} &j_{\min}(t,\sigma,\Omega_e)=j_{\min}(t,\sigma,\Omega)=i_{\min}(t,\Omega),\\ &j_{\min}(t,\sigma,\Omega^\lambda)=\lambda^m j_{\min}(t,\frac{\sigma}{\lambda},\Omega)=\lambda^m i_{\min}(t,\Omega). \end{aligned} \tag{4.56}$$

Formulas (4.55), (4.56) lead to a physically obvious conclusion that even with a positive surface tension coefficient the process of phase transitions does not depend on the shift of the domain Ω by a fixed vector e. These formulas also prove that the process of phase transitions in the domain Ω^λ for a medium with a coefficient of surface tension $\sigma > 0$ is the same as in the domain Ω for a medium with a coefficient of surface tension σ/λ. The last result is a direct analog of the one-dimensional situation and is caused by different scales of the volume and surface components of the energy functional when the domain of integration is stretched.

For $t_- < t_+$, as in the one-dimensional case, for a medium with a surface tension coefficient $\sigma > 0$, we introduce the critical value of the stretching parameter λ,

$$\lambda^* = \frac{\sigma}{\sigma^*}, \tag{4.57}$$

characterized by the properties

$$\begin{aligned} t_-(\sigma, \Omega^\lambda) &< t_+(\sigma, \Omega^\lambda) \quad \text{if} \quad \lambda > \lambda^*, \\ t_-(\sigma, \Omega^\lambda) &= t_+(\sigma, \Omega^\lambda) \quad \text{if} \quad \lambda \leq \lambda^*. \end{aligned} \tag{4.58}$$

Determining the quantity λ^* experimentally, we can use formula (4.57) to calculate the coefficient of surface tension σ characterizing the given two-phase medium.

Unfortunately, in the multidimensional case we have no explicit formula for σ^* similar to relation (2.20) in the one-dimensional case. This defect does not allow us to replace (4.57) by a multidimensional analog of (2.28). We are forced to be only restricted to a two-sided estimate of the quantity σ^* in the unit ball.

Lemma 4.15. *Let $\Omega = B$ be the unit ball in the space $\mathbb{R}^3$ centered at the origin. Then*

$$\begin{gathered} \sigma^*_{\min} \leq \sigma^* \leq \sigma^*_{\max}, \\ \sigma^*_{\min} = \frac{1}{54\sqrt[3]{2}} \nu \operatorname{tr}^2[A\zeta], \quad \sigma^*_{\max} = \frac{2}{\nu} |[A\zeta]|^2. \end{gathered} \tag{4.59}$$

Taking estimates (4.59) and formula (4.57) into account, we obtain an interval for the surface tension coefficient σ as a function of the parameter λ^*, determined experimentally for the unit ball:

$$\sigma \in [\lambda^* \sigma^*_{\min}, \lambda^* \sigma^*_{\max}]. \tag{4.60}$$

4.4. Critical points of the energy functional. For functions $\tilde{u} \in \mathbb{H}$ and $\tilde{\chi} \in \mathbb{Z}$, consider their perturbations u, χ constructed by formula (3.48).

(1) The function χ belongs to $\mathbb{Z}$, and the linear in h part of the increment $S[\chi] - S[\tilde{\chi}]$ coincides with the quantity

$$\int_\Omega \big(\operatorname{div} h(x) - (\nabla h(x) \tilde{\nu}(x), \tilde{\nu}(x))\big)\, d\tilde{\mu}(x), \tag{4.61}$$

where the Borel measure $\tilde{\mu}$ and the vector-valued function $\tilde{\nu}$ are determined by the function $\tilde{\chi}$, the support of the measure $\tilde{\mu}$ lies in

$$\Gamma = \Omega \cap \partial(\operatorname{supp} \tilde{\chi}),$$

and $|\tilde{\nu}(x)| = 1$ for $\tilde{\mu}$-almost all points $x \in \Omega$.

(2) If the set Γ is a continuously differentiable surface on the support of the function h, then in (4.61) we have

$$\tilde{\nu}(x) = n(x), \quad x \in \Gamma, \qquad \tilde{\mu}(x) = dS_x, \tag{4.62}$$

where $n(x)$ is the unit normal vector to Γ outward with respect to $\operatorname{supp} \tilde{\chi}$.

We say that a pair $\widetilde{u} \equiv \widetilde{u}_{t,\sigma} \in \mathbb{H}$, $\widetilde{\chi} \equiv \widetilde{\chi}_{t,\sigma} \in \mathbb{Z}$ is a critical point of the energy functional I if

$$
\begin{aligned}
&\int_\Omega \Big\{\widetilde{\chi} F^+_{M_{ij}}(\nabla\widetilde{u}) + (1-\widetilde{\chi}) F^-_{M_{ij}}(\nabla\widetilde{u})\Big\} v^i_{x_j}\, dx \\
&\quad + \int_\Omega \Big\{\widetilde{\chi}((F^+(\nabla\widetilde{u}) + t)\delta_{kj} - \widetilde{u}^i_{x_k} F^+_{M_{ij}}(\nabla\widetilde{u})) \\
&\qquad\qquad + (1-\widetilde{\chi})(F^-(\nabla\widetilde{u})\delta_{kj} - \widetilde{u}^i_{x_k} F^-_{M_{ij}}(\nabla\widetilde{u}))\Big\} h^k_{x_j}\, dx \\
&\quad + \sigma \int_\Omega \big(\operatorname{div} h(x) - (\nabla h(x)\widetilde{\nu}(x), \widetilde{\nu}(x))\big)\, d\widetilde{\mu}(x) = 0 \\
&\qquad\qquad\qquad\qquad \text{for all } v \in \mathbb{H},\ h \in C^1_0(\Omega, \mathbb{R}^m).
\end{aligned}
\tag{4.63}
$$

Since the left-hand side in (4.63) is the linear part of the increment of I under the perturbation (3.48), the equilibrium states $\widehat{u}_{t,\sigma}$, $\widehat{\chi}_{t,\sigma}$ must be critical points. Nevertheless, not every critical point is an equilibrium state. For example, the pairs $\widetilde{u} \equiv 0$, $\widetilde{\chi} \equiv 0$ and $\widetilde{u} \equiv 0$, $\widetilde{\chi} \equiv 1$ are critical points of the energy functional I for all values of t and σ, while they are equilibrium states only in the cases indicated in (4.35), (4.46).

Theorem 4.6. *Let a ball $B_r(x_0) \subset \Omega$ and critical point $\widetilde{u}$, $\widetilde{\chi}$ of the energy functional I be fixed.*
(a) *If the function $\widetilde{\chi}$ is constant in $B_r(x_0)$, then $\widetilde{u}$ is of class $C^\infty(B_r(x_0), R^m)$ and satisfies equations* (3.52)*.*
(b) *Let the ball $B_r(x_0)$ be divided into two parts by an $(m-1)$-dimensional surface Γ of class $C^{k,\epsilon}$, $k \geq 2$, $\epsilon \in (0,1)$. Suppose that $\widetilde{\chi} = 1$ in $B^+_r(x_0)$, and $\widetilde{\chi} = 0$ in $B^-_r(x_0)$. The function $\widetilde{u}$ is of class $C^{k,\epsilon}$ in each of the domains $B^\pm_r(x_0)$ up to the phase interface boundary Γ and satisfies equations* (3.53)*. Condition* (3.53) *on the jump of the stress tensor Θ is fulfilled on Γ, and the condition on the jump of the tensor of the chemical potential Φ is replaced by the relation*

$$
[\Phi_{kl}[\widetilde{u}, \widetilde{\chi}]]|_\Gamma n_k n_l + \sigma H = 0,
\tag{4.64}
$$

where H is the mean curvature of the surface Γ in the direction n.

In particular, identity (4.63) for the critical point $\widetilde{u}$, $\widetilde{\chi}$ means that, by (3.8)), the function $\widetilde{u}$ represents a generalized solution of an elliptic system, which we rewrite in the form

$$
\begin{gathered}
\mathcal{L}_{\widetilde{\chi}}\widetilde{u} = L\widetilde{u} + L_{\widetilde{\chi}}\widetilde{u} = F, \\
(L\widetilde{u})_i = -(a^-_{ijkl}\widetilde{u}^k_{x_l})_{x_j}, \quad (L_{\widetilde{\chi}}\widetilde{u})_i = -(\widetilde{\chi}[a_{ijkl}]\widetilde{u}^k_{x_l})_{x_j}, \quad F_i = -(\widetilde{\chi}[a_{ijkl}\zeta_{kl}])_{x_j}, \\
i = 1, \dots, m.
\end{gathered}
\tag{4.65}
$$

It is known that

$$
\begin{aligned}
&\text{the operator } L \text{ is an isomorphism} \\
&\text{of the spaces } \mathring{W}^1_p(\Omega, R^m) \text{ and } W^{-1}_p(\Omega, R^m) \text{ for any } p \in [2, \infty).
\end{aligned}
\tag{4.66}
$$

Obviously, the operator $L_{\widetilde{\chi}}$ maps the space $\mathring{W}^1_p(\Omega, R^m)$ into $W^{-1}_p(\Omega, R^m)$ and for each $p \in [2, \infty)$ we have

$$
\|L_{\widetilde{\chi}}\|_{\mathring{W}^1_p \to W^{-1}_p} \leq C_p |[A]|
\tag{4.67}
$$

uniformly with respect to the functions $\widetilde{\chi}$, and $F \in W^{-1}_p(\Omega, R^m)$ for the same p.

Therefore, there exists a number $\alpha_p > 0$ such that whenever

$$
|[A]| < \alpha_p,
\tag{4.68}
$$

the operator $\mathcal{L}_{\widetilde{\chi}}$ is an isomorphism between $\overset{\circ}{W}{}^1_p(\Omega, R^m)$ and $W_p^{-1}(\Omega, R^m)$ for all $\widetilde{\chi} \in \mathbb{Z}$. Hence, the function $\widetilde{u}$, belonging originally to the space $\mathbb{H}$, falls into the space $\overset{\circ}{W}{}^1_p(\Omega, R^m)$, $p > 2$, for sufficiently close matrices A^+ and A^-.

In particular, if for given $p > 2$, estimate (4.68) holds true, then the equilibrium displacement field $\widehat{u}_{t,\sigma}$ has higher smoothness:

$$\widehat{u}_{t,\sigma} \in \overset{\circ}{W}{}^1_p(\Omega, R^m).$$

This allows us to judge whether the interface boundary is smooth.

Theorem 4.7. *Suppose that $m \le 7$ and estimate (4.68) is fulfilled for $p > 2m$. Then for a two-phase equilibrium state, the phase interface boundary is equivalent to a continuously differentiable surface.*

4.5. Bibliographical notes. The results of this section are based on the papers [24, 37–39, 41–43, 55]. There, also the case of various boundary conditions and the possible presence of strong fields was treated. From a mathematical point of view, the surface energy serves as a regularization of the functional with zero coefficient of the surface tension. Among all possible definitions of the phase interface boundary area [1], we chose the perimeter of the set, see [6]. It is most convenient for problems of calculus of variations and, moreover, it does not change when the support of the function χ changes by a set of measure zero. The properties (4.7)–(4.10) of the area (4.1) were proved in [51]. In that paper, it was also mentioned that in the one-dimensional case, the definition (4.1) coincides with (2.4). The proof of Lemma 4.1 was given in [15].

As a different regularization, we can propose the term of the form $\sigma\|u\|^q_{W_2^2}$ of degree q with a positive coefficient σ. This regularization occurs in the moment theory of elasticity. If the number q is chosen properly, then the dependence of equilibrium states on the parameters t and σ at a qualitative level is similar to the dependence on these parameters for the regularization studied in this section, see [56]. Variational calculus statements related to the lower semicontinuity of functionals and improving the convergence of their arguments as energies converge are traditional, see [52]. Formula (4.61) for the perimeter variation is contained in [6].

The question about the smoothness of free surfaces is very complicated. The available techniques of its investigation (see [50]) usually provide answers only in model cases. The smoothness of the phase boundary was also established in the model case (4.68). It is based on the result on the smoothness of the generalized minimal surface, see [66], and the statement (4.66) presented in [61].

We briefly dwell on the terminology. In the mechanics of continuous media, a two-phase elastic medium for which only one of the phases can occur at each point of the domain Ω in a state of equilibrium, is called a heterogeneous medium. The passage to phase mixtures realized in Lemmas 4.4 and 4.5 is an attempt of lifting the heterogeneity requirement.

§5. Passage to the limit as the surface tension coefficient tends to zero

Our goal in this section is an attempt to construct limit points of equilibrium states as the surface tension coefficient tends to zero. The following results only partially coincide with analogous statements in the one-dimensional case. This happens because equilibrium states with finite area of the phase interface boundary may fail to exist if the coefficient of surface tension is zero and, moreover, equilibrium states may even be totally absent.

5.1. Equilibrium states and minimizing sequences. At the first step, we establish the role of the equilibrium states $\widehat{u}_{t,\sigma}$, $\widehat{\chi}_{t,\sigma}$ of the energy functional $I[u, \chi, t, \sigma]$ with positive σ and fixed t as $\sigma \to 0$.

Lemma 5.1. *Any sequence of equilibrium states*

$$\widehat{u}_{t,\sigma_n}, \quad \widehat{\chi}_{t,\sigma_n}, \quad \sigma_n > 0, \quad \sigma_n \to 0, \tag{5.1}$$

is minimizing for the functional $I_0[u, \chi, t]$.

At the second step, we construct an auxiliary functional $I^{\min}[u, t]$ such that the set of accumulation points of all minimizing sequences contains the set of accumulation points of the first components of all sequences (5.1). The accumulation points are understood in the sense of weak convergence in $W_2^1(\Omega, \mathbb{R}^m)$.

Inequality (3.1) implies the existence of a number $R = R(t) > 0$ such that $\|\widehat{u}_{t,\sigma_n}\|_{W_2^1} \leq R$ for all terms of the sequence (5.1). Therefore, for the sequences (5.1) the set we are interested in is nonempty.

Using the energy densities (3.6), we introduce the function

$$F^{\min}(M, t) = \min\{F^+(M) + t, F^-(M)\}, \qquad M \in \mathbb{R}^{m\times m}, \quad t \in \mathbb{R}, \tag{5.2}$$

and associate with it the functional

$$I_0^{\min}[u, t] = \int_\Omega F^{\min}(\nabla u, t)\, dx, \quad u \in \mathbb{H}, \tag{5.3}$$

for which we set the following variational problem:

$$I_0^{\min}[\widehat{u}_t, t] = \inf_{u\in\mathbb{H}} I_0^{\min}[u, t], \quad \widehat{u}_t \in \mathbb{H}. \tag{5.4}$$

For convenience of presentation, for each function $u \in \mathbb{H}$ we construct a function χ_u by the rule

$$\mathbb{Z}' \ni \chi_u(x) = \begin{cases} 1 & \text{if } F^+(\nabla u(x)) + t < F^-(\nabla u(x)), \\ 0 & \text{if } F^+(\nabla u(x)) + t > F^-(\nabla u(x)), \\ \text{any number} & \text{if } F^+(\nabla u(x)) + t = F^-(\nabla u(x)) \end{cases} \tag{5.5}$$
$$\text{almost everywhere on } \Omega.$$

In (5.5), for compatibility of inclusion and the third condition it is required that for almost all $x \in \Omega$ satisfying $F^+(\nabla u(x)) + t = F^-(\nabla u(x))$, the function $\chi_u(x)$ take only two values: 0 and 1.

Lemma 5.2. (a) *Each solution* $\widehat{u}_t$ *of problem* (5.4) *gives rise to a solution* $\widehat{u}_t$, $\widehat{\chi}_t$ *of problem* (3.10) *with the same function* $\widehat{u}_t$ *and* $\widehat{\chi}_t = \chi_{\widehat{u}_t}$. *For each solution* $\widehat{u}_t$, $\widehat{\chi}_t$ *of problem* (3.10), *the function* $\widehat{u}_t$ *solves problem* (5.4) *and* $\widehat{\chi}_t = \chi_{\widehat{u}_t}$.
(b) *For each minimizing sequence* u_n, χ_n, $n = 1, \dots,$ *of the functional* $I_0[u, \chi, t]$, *the sequence* u_n *is minimizing for the functional* $I_0^{\min}[u, t]$. *If* u_n *is a minimizing sequence for* $I_0^{\min}[u, t]$, *then the sequence* u_n, χ_{u_n} *is minimizing for* $I_0[u, \chi, t]$.

At the third step, we identify the accumulation points of the minimizing sequences of the functional (5.3) with solutions of the relaxed variational problem. We consider the following quasiconvex hull of the function (5.2):

$$\mathcal{F}(M, t, \Omega) = |\Omega|^{-1} \inf_{u\in\mathbb{H}} \int_\Omega F^{\min}(M + \nabla u, t)\, dx = |\Omega|^{-1} \inf_{\substack{u\in\mathbb{H},\\ \chi\in\mathbb{Z}'}} I_0[u + v_M, \chi, t], \tag{5.6}$$
$$M \in \mathbb{R}^{m\times m}, \ t \in \mathbb{R}, \ v_M(x) = Mx.$$

It is known that the function (5.6) does not depend on the domain Ω:

$$\mathcal{F}(M, t, \Omega) \equiv \mathcal{F}(M, t).$$

Lemma 5.3. *The function $\mathcal{F}(\,.\,,\,.\,)$ is locally Lipschitz and satisfies the inequalities*

$$C_1(|e(M)|^2 - |t| - 1) \le \mathcal{F}(M,t) \le C_2(|e(M)|^2 + |t| + 1), \qquad C_i \ne C_i(M,t,\Omega) > 0, \;\; i = 1,2. \tag{5.7}$$

With the function (5.6) we associate the functional

$$\mathfrak{I}[u,t] = \int_\Omega \mathcal{F}(\nabla u, t)\,dx, \quad u \in \mathbb{H}, \tag{5.8}$$

and consider the following variational problem:

$$\mathfrak{I}[\breve{u}_t, t] = \inf_{u\in\mathbb{H}} \mathfrak{I}[u,t], \quad \breve{u}_t \in \mathbb{H}. \tag{5.9}$$

The following facts are known.

(1) By the properties of the quasiconvex hull (5.6) and Lemma 5.3, the functional (5.8) is weakly lower semicontinuous and coercive in the space $\mathbb{H}$. Therefore, problem (5.9) is solvable. Moreover,

$$\inf_{u\in\mathbb{H}} I_0^{\min}[u,t] = \min_{u\in\mathbb{H}} \mathfrak{I}[u,t].$$

(2) Each solution of problem (5.9) is a weak limit in the space $\mathbb{H}$ of some minimizing sequence of the functional (5.3).

(3) Every weakly convergent sequence in $\mathbb{H}$ minimizing the functional (5.3), weakly converges in $\mathbb{H}$ to a solution of problem (5.9). In particular, every solution of problem (5.4) solves problem (5.9).

Since problem (5.4) can be unsolvable (Lemma 3.1), properties (1)–(3) allow us to interpret the solution of (5.9) as a generalized solution of (5.4). Problem (5.9) is usually said to be relaxed relative to problem (5.4).

The following statement is a result of the above three steps.

Theorem 5.1. *For any weakly convergent subsequence $\widehat{u}_{t,\sigma_{n'}}$, $\widehat{\chi}_{t,\sigma_{n'}}$*

$$\widehat{u}_{t,\sigma_{n'}} \rightharpoonup \breve{u}_t \;\; \textit{in the space } \mathbb{H}, \quad \widehat{\chi}_{t,\sigma_{n'}} \overset{*}{\rightharpoonup} \breve{\chi}_t \in \mathbb{Z}'' \tag{5.10}$$

of a sequence of equilibrium states $\widehat{u}_{t,\sigma_n}$, $\widehat{\chi}_{t,\sigma_n}$, $\sigma_n > 0$, $\sigma_n \to 0$, the function $\breve{u}_t$ is a solution of problem (5.9).

Each such sequence of equilibrium states contains a subsequence weakly convergent in the sense of (5.10).

At the fourth step, we are going to study the behavior of the interface boundary area for equilibrium states $\widehat{u}_{t,\sigma}$, $\widehat{\chi}_{t,\sigma}$, $\sigma > 0$ as $\sigma \to 0$. The resulting information will give a negative answer to the question as to whether any solution of problem (5.9) can be obtained by the method presented in Theorem 5.1. Recall the notation $\mathfrak{B}_{t,\sigma}$, $t,\sigma \in \mathbb{R}$, $\sigma > 0$, for the set of all equilibrium states of the functional $I[u,\chi,t,\sigma]$. Let $\mathfrak{B}_t$ denote the set (possibly empty) of all equilibrium states of the functional $I_0[u,\chi,t]$.

Lemma 5.4. (a) *For any $t,\sigma \in \mathbb{R}$, $\sigma > 0$, there exist pairs*

$$\widehat{u}_{t,\sigma}^{\min}, \widehat{\chi}_{t,\sigma}^{\min} \in \mathfrak{B}_{t,\sigma}, \quad \widehat{u}_{t,\sigma}^{\max}, \widehat{\chi}_{t,\sigma}^{\max} \in \mathfrak{B}_{t,\sigma} \tag{5.11}$$

such that

$$S[\widehat{\chi}_{t,\sigma}^{\min}] = \inf_{u,\chi\in\mathfrak{B}_{t,\sigma}} S[\chi], \quad S[\widehat{\chi}_{t,\sigma}^{\max}] = \sup_{u,\chi\in\mathfrak{B}_{t,\sigma}} S[\chi]. \tag{5.12}$$

(b) *For given t, suppose that the set $\mathfrak{B}_t$ is nonempty and contains equilibrium states $\widehat{u}_t$, $\widehat{\chi}_t$ with $\widehat{\chi}_t \in \mathbb{Z}$. Then there exists a pair*

$$\widehat{u}_t^{\min}, \widehat{\chi}_t^{\min} \in \mathfrak{B}_t \tag{5.13}$$

such that

$$S[\widehat{\chi}_t^{\min}] = \inf_{u,\chi \in \mathfrak{B}_t} S[\chi]. \tag{5.14}$$

We note that if condition (*b*) is fulfilled, then an analog of the second pair in (5.11) with a finite area of phase interface boundary may fail to exist. This can be easily verified by using the solution of the homogeneous isotropic problem on phase transitions in a ball. The lemma does not guarantee the uniqueness of solutions of problems (5.12), (5.14).

We put

$$S^{\max}(t,\sigma) = S[\widehat{\chi}_{t,\sigma}^{\max}], \quad S^{\min}(t,\sigma) = S[\widehat{\chi}_{t,\sigma}^{\min}], \qquad t,\sigma \in \mathbb{R}, \quad \sigma > 0. \tag{5.15}$$

Using Theorems 4.4 and 4.5, we conclude that

$$\begin{aligned} &S^{\max}(t,\sigma) = S^{\min}(t,\sigma) = 0 \ \text{ for } \ \sigma > \sigma(t), \\ &S^{\max}(t,\sigma) > S^{\min}(t,\sigma) = 0 \ \text{ for } \ \sigma = \sigma(t) > 0, \\ &S^{\max}(t,\sigma) \geq S^{\min}(t,\sigma) > 0 \ \text{ for } \ \sigma \in (0,\sigma(t)). \end{aligned} \tag{5.16}$$

The following statement is a basis for our next results.

Lemma 5.5. *For any t and any $\sigma_2 > \sigma_1 > 0$, we have*

$$S^{\max}(t,\sigma_2) \leq S^{\min}(t,\sigma_1). \tag{5.17}$$

By (5.17) and the obvious inequality $S^{\min}(t,\sigma) \leq S^{\max}(t,\sigma)$, we have

$$\begin{gathered} S^{\max}(t,\sigma_2) \leq S^{\max}(t,\sigma_1), \quad S^{\min}(t,\sigma_2) \leq S^{\min}(t,\sigma_1) \\ \text{for all } t,\sigma_1,\sigma_2, \quad \text{where} \quad \sigma_2 > \sigma_1 > 0. \end{gathered} \tag{5.18}$$

The monotonicity (5.18) implies the existence of the limits

$$\begin{aligned} S^{\max}(t,\sigma \pm 0) &= \lim_{\epsilon\to 0} S^{\max}(t,\sigma \pm \epsilon), \\ S^{\min}(t,\sigma \pm 0) &= \lim_{\epsilon\to 0} S^{\min}(t,\sigma \pm \epsilon), \quad \sigma > 0, \quad \epsilon > 0. \end{aligned} \tag{5.19}$$

Lemma 5.6. *For the limits* (5.19), *the following identities are true*:

$$\begin{gathered} S^{\min}(t,\sigma - 0) = S^{\max}(t,\sigma), \quad S^{\min}(t,\sigma + 0) = S^{\min}(t,\sigma), \\ S^{\max}(t,\sigma - 0) = S^{\max}(t,\sigma), \quad S^{\max}(t,\sigma + 0) = S^{\min}(t,\sigma) \\ \textit{for all} \ \ t,\sigma \ \textit{where} \ \ \sigma > 0. \end{gathered} \tag{5.20}$$

In particular, relations (5.20) mean the continuity of the function $S^{\min}(t,\,.\,)$ from the right, and the continuity of the function $S^{\max}(t,\,.\,)$ from the left.

The following lemma is devoted to the behavior of the phase interface boundary area as $\sigma \to 0$.

Lemma 5.7. *We have*

$$\lim_{\sigma\downarrow 0} S^{\max}(t,\sigma) = \lim_{\sigma\downarrow 0} S^{\min}(t,\sigma), \quad \lim_{\sigma\downarrow 0} \sigma S^{\max}(t,\sigma) = 0, \tag{5.21}$$

where the limits in the first identity can be infinite.

Based on the behavior of the phase interface boundary area as $\sigma \to 0$, we formulate a criterion for the existence of a solution $\widehat{u}_t, \widehat{\chi}_t \in \mathfrak{B}_t$ with $\widehat{\chi}_t \in \mathbb{Z}$.

Lemma 5.8. (a) *The set $\mathfrak{B}_t$ of all solutions of the variational problem* (3.10) *contains a solution $\widehat{u}_t$, $\widehat{\chi}_t$ with $\widehat{\chi}_t \in \mathbb{Z}$ if and only if*

$$\lim_{\sigma\downarrow 0} S^{\min}(t,\sigma) < \infty. \tag{5.22}$$

(b) *Under condition* (5.22), *any sequence of equilibrium states* $\widehat{u}_{t,\sigma_n}$, $\widehat{\chi}_{t,\sigma_n} \in \mathfrak{B}_{t,\sigma_n}$, $\sigma_n > 0$, $\sigma_n \to 0$, *has a subsequence* $\widehat{u}_{t,\sigma_{n'}}$, $\widehat{\chi}_{t,\sigma_{n'}}$ *such that for some pair* $\widehat{u}_t^{\min}, \widehat{\chi}_t^{\min} \in \mathfrak{B}_t$ *we have*

$$(5.23)\qquad \begin{aligned} &\widehat{u}_{t,\sigma_{n'}} \to \widehat{u}_t^{\min} \quad \text{in the space} \quad \mathbb{H}, \\ &\widehat{\chi}_{t,\sigma_{n'}} \to \widehat{\chi}_t^{\min} \text{ almost everywhere on } \Omega, \quad S[\widehat{\chi}_{t,\sigma_{n'}}] \uparrow S[\widehat{\chi}_t^{\min}]. \end{aligned}$$

The last of the proposed lemmas gives a sufficient condition for the solvability of problem (3.10).

Lemma 5.9. *Suppose that in* (5.10) *we have* $\check{\chi}_t \in \mathbb{Z}'$. *Then* $\mathfrak{B}_t \neq \varnothing$.

From the assertions of the fourth step, the next theorem follows.

Theorem 5.2. *Suppose that the set* $\mathfrak{B}_t$ *is nonempty and contains a pair* $\widehat{u}_t$, $\widehat{\chi}_t$ *with* $\widehat{\chi}_t \in \mathbb{Z}$. *Then in* (5.10), *only one of the pairs* $\widehat{u}_t^{\min}$, $\widehat{\chi}_t^{\min}$ *can play the role of the pair* $\check{u}_t$, $\check{\chi}_t$.

5.2. Quasiconvex hull in the isotropic case. In the case of an isotropic two-phase medium, we need to describe the set of all solutions of problem (5.9). For this, we calculate explicitly the quasiconvex hull (5.6) for the energy densities (3.12) under the condition (3.14).

Theorem 5.3. *For the energy densities* (3.12), (3.14), *the quasiconvex hull* (5.6) *is defined by the formula*

$$(5.24)\qquad \mathcal{F}(\nabla u, t) = a\frac{|\operatorname{curl} u|^2}{4} + a(u^i_{x_j}u^j_{x_i} - |\operatorname{div} u|^2) + R_c^{\min}(t,z), \quad z = \frac{\operatorname{div} u}{m},$$

where $R_c^{\min}(t,\,.\,)$ *is the convex hull of the function*

$$(5.25)\qquad \begin{aligned} R^{\min}(t,\,.\,) &= \min\{R^+(t,\,.\,), R^-(\,.\,)\}, \\ R^+(t,z) &= am(m-1)z^2 + m(z-c_+)^2(a+b_+m) + t, \\ R^-(z) &= am(m-1)z^2 + m(z-c_-)^2(a+b_-m). \end{aligned}$$

When the right-hand side of (5.24) is integrated over the domain Ω, the central term vanishes. Therefore, the formula for the relaxed functional looks like this:

$$(5.26)\qquad \begin{aligned} \mathfrak{J}[u,t] &= \int_\Omega \mathcal{F}(\nabla u, t)\,dx, \quad u \in \mathbb{H}, \quad t \in \mathbb{R}, \\ \mathcal{F}(\nabla u, t) &= a\frac{|\operatorname{curl} u|^2}{4} + R_c^{\min}(t,z), \quad z = \frac{\operatorname{div} u}{m}, \end{aligned}$$

which indicates its convexity in the argument $u \in \mathbb{H}$.

The explicit formula for a quasiconvex hull allows us to characterize the set of all solutions of problem (5.9) and specify what proportion among them solves problem (3.10). For this, we need the following set:

$$(5.27)\quad \mathbb{Y}_t'' = \Big\{u \in \mathbb{H},\, \chi \in \mathbb{Z}'' : \operatorname{curl} u = 0,\, \operatorname{div} u = \alpha(\widehat{Q}(t))(\chi - \widehat{Q}(t)),\, \frac{1}{|\Omega|}\int_\Omega \chi\,dx = \widehat{Q}(t)\Big\}.$$

We recall that, by Lemma 5.2, the function $\widehat{u}_t$ is a solution of problem (5.4) if and only if it is the first component of a pair $\widehat{u}_t, \widehat{\chi}_t \in \mathbb{Y}_t'$ (see (3.27)).

If $[c(a+bm)] \neq 0$, then the function $\widehat{Q}(t)$ is single-valued, and $\alpha(\widehat{Q}(t))) \neq 0$. Hence, in this case, in the pair $u, \chi \in \mathbb{Y}_t''$ each of its components is uniquely determined by the other one. If $[c(a+bm)] = 0$ and $t \neq t^*(= t_\pm)$, then $\widehat{Q}(t)$ is still single-valued, but $\alpha(\widehat{Q}(t)) = 0$. In this case the set $\mathbb{Y}_t''$ is exhausted by the pairs $u = 0$, $\chi = 1$ for $t < t^*$ and $u = 0$, $\chi = 0$ for $t > t^*$. In the case where $[c(a+bm)] = 0$ and $t = t^*$, the set $\mathbb{Y}_t''$

consists of the pairs for which $u = 0$ and χ is an arbitrary element of $\mathbb{Z}''$, because $\widehat{Q}(t^*)$ is a multivalued function and the number $\alpha(\widehat{Q}(t^*))$ equals zero.

Theorem 5.4. *For the energy density* (3.12) *satisfying condition* (3.14), *the function* $\breve{u}_t$ *is a solution of problem* (5.9) *if and only if it is the first component of a pair* $\breve{u}_t$, $\breve{\chi}_t$ *belonging to the set* $\mathbb{Y}''_t$.

Examples of pairs $u, \chi \in \mathbb{Y}''_t$ for $t \in (t_-, t_+) \neq \varnothing$ can be constructed in various ways. We consider two of them.
(1) Given any function p,

$$p \in \overset{\circ}{W}{}^2_2(\Omega) \quad \text{such that} \quad \frac{\Delta p(x)}{\alpha(\widehat{Q}(t))} \in [-\widehat{Q}(t), 1 - \widehat{Q}(t)] \text{ a.e. on } \Omega \tag{5.28}$$

(note that for $t \in (t_-, t_+) \neq \varnothing$ we have $\alpha(\widehat{Q}(t)) \neq 0$ and $\widehat{Q}(t) \in (0,1)$), we calculate functions u and χ by the following formulas:

$$u = \nabla p, \quad \chi = \frac{\Delta p}{\alpha(\widehat{Q}(t))} + \widehat{Q}(t). \tag{5.29}$$

Clearly, the pairs (5.28), (5.29) belong to $\mathbb{Y}''_t$, and among these pairs there are pairs with $\chi \in \mathbb{Z}'$ and pairs with $\chi \notin \mathbb{Z}'$.
(2) Fix an open set $\omega \subset \Omega$. Using its arbitrary Vitali covering and the number $\widehat{Q}(t)$, in ω we can construct a solution $\widehat{u}_t$, $\widehat{\chi}_t$ of problem (3.18). We extend the function $\widehat{u}_t$ by zero and the function $\widehat{\chi}_t$ by the constant $\widehat{Q}(t)$ to the set $\Omega \setminus \omega$ closed in Ω. Obviously, the resulting pair $\breve{u}_t$, $\breve{\chi}_t$ lies in $\mathbb{Y}''_t$ and if $|\omega| \neq |\Omega|$, then it is not a solution of problem (3.10).

When passing from the original problem (3.10) to the relaxed one (5.9), we get the existence theorem for equilibrium states (for the densities (3.12), (3.14) equilibrium states already exist), but we lose information about the distribution of phases, the function $\widehat{\chi}_t$. Theorem 5.4 compensates for this loss. In the case where $\breve{\chi}_t \notin \mathbb{Z}'$, this function describes the phase fraction with index + in a two-phase relaxed state of equilibrium.

Another approach to the passage from pure two-phase states to the phase mixture was suggested in Lemma 4.4. Note that (Lemma 4.5), for the densities (3.12), (3.14) these two methods coincide only if $t = t_\pm = t^*$. As has already been noted, in the other cases for these densities, the method of Lemma 4.4 does not yield a mixture of phases in the state of equilibrium.

5.3. Quasiconvex hull in the anisotropic case. Now we obtain certain information about the quasiconvex hull for the general form (3.6) of the energy densities of a two-phase elastic medium and make conclusions about the equilibrium states of a relaxed functional.

We start with deriving a two-sided estimate for a quasiconvex hull. To formulate our next theorem, we need the following notation. We set

$$\begin{aligned} t^*(M) &= -[< A(e(M) - \zeta), e(M) - \zeta >], \quad M \in R^{m \times m}, \\ \mu_1^-(M) &= t^*(M) - \frac{|[A(e(M) - \zeta)]|^2}{\nu}, \\ \mu_2^-(M) &= t^*(M) + \frac{|[A(e(M) - \zeta)]|^2}{\nu}, \\ \mu_1^+(M) &= t^*(M) - \frac{\nu \operatorname{tr}^2[A(e(M) - \zeta)]}{m^2}, \\ \mu_2^+(M) &= t^*(M) + \frac{\nu \operatorname{tr}^2[A(e(M) - \zeta)]}{m^2}. \end{aligned} \tag{5.30}$$

Obviously, $\mu_2^{\pm}(M) \geq \mu_1^{\pm}(M)$, and equality is possible only if $M \in \mathbb{R}^{m\times m}$ satisfies which

$$\operatorname{tr}[A(e(M)-\zeta)] = 0, \quad [A(e(M)-\zeta)] = 0 \tag{5.31}$$

for the signs $+$ and $-$, respectively.

Let functions $P_{\pm}(t, M)$, $t \in \mathbb{R}$, $M \in \mathbb{R}^{m\times m}$, be defined by the formulas

$$\begin{aligned}
&\text{if } M \text{ satisfies } \mu_1^+(M) < \mu_2^+(M), \text{ then}\\
&\qquad P_+(t,M) = 1 \text{ for } t \leq \mu_1^+(M), \qquad P_+(t,M) = 0 \text{ for } t \geq \mu_2^+(M),\\
&\qquad P_+(t,M) = \frac{\mu_2^+(M) - t}{\mu_2^+(M) - \mu_1^+(M)} \text{ for } t \in (\mu_1^+(M), \mu_2^+(M));\\
&\text{if } M \text{ satisfies } \mu_1^+(M) = \mu_2^+(M) = t^*(M), \text{ then}\\
&\qquad P_+(t,M) = 1 \text{ for } t < t^*(M), \qquad P_+(t,M) = 0 \text{ for } t > t^*(M),\\
&\qquad P_+(t^*(M), M) \text{ is an arbitrarily number in the interval } [0,1];
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
&\text{for } M \text{ satisfying } \mu_1^-(M) < \mu_2^-(M),\\
&\qquad P_-(t,M) = 1 \text{ for } t \leq \mu_1^-(M), \qquad P_-(t,M) = 0 \text{ for } t \geq \mu_2^-(M),\\
&\qquad P_-(t,M) = \frac{\mu_2^-(M) - t}{\mu_2^-(M) - \mu_1^-(M)} \text{ for } t \in (\mu_1^-(M), \mu_2^-(M));\\
&\text{if } M \text{ satisfies } \mu_1^-(M) = \mu_2^-(M) = t^*(M), \text{ then}\\
&\qquad P_-(t,M) = 1 \text{ for } t < t^*(M), \qquad P_-(t,M) = 0 \text{ for } t > t^*(M),\\
&\qquad P_-(t^*(M), M) \text{ is an arbitrary number in the interval } [0,1].
\end{aligned} \tag{5.33}$$

The functions $P_{\pm}(t, M)$ are involved in the definitions of the functions $\mathcal{F}_{\pm}(M, t)$:

$$\begin{aligned}
\mathcal{F}_+(M,t) &= P_+(t,M)(F^+(M)+t) + (1-P_+(t,M))F^-(M)\\
&\qquad - \frac{\mu_2^+(M) - \mu_1^+(M)}{2} P_+(t,M)(1-P_+(t,M)),\\
\mathcal{F}_-(M,t) &= P_-(t,M)(F^+(M)+t) + (1-P_-(t,M))F^-(M)\\
&\qquad - \frac{\mu_2^-(M) - \mu_1^-(M)}{2} P_-(t,M)(1-P_-(t,M)),
\end{aligned} \tag{5.34}$$

where the energy density of the deformation $F^{\pm}(M)$ are as in (3.6).

Since

$$F^+(M) - F^-(M) = -t^*(M), \tag{5.35}$$

the fact that the functions $P_{\pm}(t, M)$ are multivalued does not prevent the functions $\mathcal{F}_{\pm}(M, t)$ from being single-valued and continuous.

Theorem 5.5. *For the function* (5.2) *constructed by the energy densities* (3.6)*, the quasiconvex hull* (5.6) *admits the estimate*

$$\mathcal{F}_-(M,t) \leq \mathcal{F}(M,t) \leq \mathcal{F}_+(M,t), \qquad M \in \mathbb{R}^{m\times m}, \quad t \in \mathbb{R}. \tag{5.36}$$

Now we establish some properties of solutions of the relaxed problem (5.9). In the formulation of the next theorem we use the numbers $\mu_{1,2}^-$ defined by (3.35) and the temperatures of phase transitions $t_{\pm}$ for the energy functional $I_0[u, \chi, t]$, for which estimate (3.38) is proved.

Theorem 5.6. (1) *The function $\breve{u}_t = 0$ is a solution of problem* (5.9) *for each value of the temperature t.*
(2) *For given $t \in (t_-, t_+) \neq \varnothing$, if the function $\breve{u}_t = 0$ is the only solution of problem* (5.9)*, then for such t problem* (3.10) *is solvable.*

(3) *The only solution of problem* (5.9) *for* $t \notin [\mu_1^-, \mu_2^-]$ *is the function* $\breve{u}_t = 0$.

The anisotropy of the energy densities $F^{\pm}(M)$ can arise as a result of the anisotropy of the tensors of elastic moduli $A^{\pm}$, or the tensors of residual deformation $\zeta^{\pm}$. We calculate the quasiconvex hull for the densities

$$\begin{gathered} F^{\pm}(M) = |e(M) - c_{\pm}\lambda \otimes \lambda|^2, \\ M \in \mathbb{R}^{m\times m}, \quad c_{\pm} \in \mathbb{R}, \quad c_+ \neq c_-, \quad \lambda \in \mathbb{R}^m, \quad |\lambda| = 1, \end{gathered} \tag{5.37}$$

with the anisotropic residual strain tensors $\zeta^{\pm} = c_{\pm}\lambda \otimes \lambda$.

To formulate the theorem, we introduce the following notation:

$$\begin{aligned} &H^+(z,t) = (z - c_+)^2 + t, \quad H^-(z) = (z - c_-)^2, \\ &H^{\min}(z,t) = \min\{H^+(z,t), H^-(z)\}, \\ &H_c^{\min}(\,.\,,t) \text{ is the convex hull of the function } H^{\min}(\,.\,,t). \end{aligned} \tag{5.38}$$

Theorem 5.7. *For the quasiconvex hull of the energy densities* (5.37)*, we have the formula*

$$\begin{aligned} \mathcal{F}(M,t) &= |e(M)|^2 - z^2(M) + H_c^{\min}(z(M),t), \\ z(M) &= (e(M)\lambda, \lambda). \end{aligned} \tag{5.39}$$

As in the proof of formula (5.26), we rewrite expression (5.39) in the form

$$\begin{aligned} \mathcal{F}(\nabla u,t) = \frac{|\operatorname{curl} u|^2}{4} &+ (\operatorname{div} u)^2 - z^2(\nabla u) \\ &+ H_c^{\min}(z(\nabla u),t) + (u^i_{x_j}u^j_{x_i} - (\operatorname{div} u)^2), \\ z(\nabla u) = (e(\nabla u)\lambda, \lambda).& \end{aligned} \tag{5.40}$$

Since the last term on the right-hand side of (5.40) vanishes when we integrate over Ω, we obtain the following identity for the relaxed functional:

$$\begin{aligned} &\mathfrak{J}[u,t] = \int_\Omega \mathcal{F}(\nabla u,t)\,dx, \quad u \in \mathbb{H}, \quad t \in \mathbb{R}, \\ &\mathcal{F}(\nabla u,t) = \frac{|\operatorname{curl} u|^2}{4} + (\operatorname{div} u)^2 - z^2(\nabla u) + H_c^{\min}(z(\nabla u),t), \\ &z(\nabla u) = (e(\nabla u)\lambda, \lambda), \end{aligned} \tag{5.41}$$

which has a form similar to the expression (5.26) for the relaxed functional of an isotropic two-phase elastic medium.

The knowledge of an explicit expression for the quasiconvex hull is not necessary for description of the set of all solutions of problem (5.9). The following theorem gives an example of such a situation.

Theorem 5.8. *For any* $t \in \mathbb{R}$*, the function* $\breve{u}_t = 0$ *is a unique solution of problem* (5.9) *for the energy densities* (3.45) *in any domain* $\Omega \subset R^m$ *with arbitrary* $a_{\pm}$ *when* $k = 1$ *and* $a_+ = a_-$ *when* $1 < k < m$.

Theorems 5.6 and 5.8 show that for the densities as in Theorem 5.8 and $t \in (t_-, t_+) \neq \varnothing$ (the numbers $t_{\pm}$ are defined by (3.46)), problem (3.9) has no solutions, because the two-phase equilibrium states can be realized only with a nonzero displacement field.

5.4. Examples of limit points as $\sigma \to 0$ for equilibrium displacement fields. Our goal here is to describe, on the basis of the above results, the possible behavior as $\sigma \to 0$ of the equilibrium displacement fields $\widehat{u}_{t,\sigma}$ for two model problems: isotropic (with the energy densities (3.12) under condition (3.14)) and anisotropic (with the energy densities (3.45)). In the statement of the theorems below, we use formulas (3.24) for the phase transition temperatures $t_{\pm}$, the notation (3.27) for the set $\mathbb{Y}'_t$ and (5.13) for the equilibrium states $\widehat{u}_t^{\min}$, $\widehat{\chi}_t^{\min}$, where $\sigma = 0$ and the phase interface boundary area is minimal.

Theorem 5.9. *Let the energy densities be given by formulas* (3.12), (3.14)*, and let $\Omega = B_R \subset \mathbb{R}^m$, $m \geq 2$, where B_R is the ball of radius R centered at the origin. Then the following is true for $t \in (t_-, t_+)$:*
(a) *no first component $\widehat{u}_t$ of a pair $\widehat{u}_t, \widehat{\chi}_t \in \mathbb{Y}'_t$ constructed by a Vitali covering of the ball B_R, can be a limit point of any sequence $\widehat{u}_{t,\sigma_n}$ as $\sigma_n \to 0$;*
(b) *the limit point of any sequence $\widehat{u}_{t,\sigma_n}$ as $\sigma_n \to 0$ can only be the first component $\widehat{u}_t$ of some pair $\widehat{u}_t^{\min}, \widehat{\chi}_t^{\min} \in \mathbb{Y}'_t$.*

Theorem 5.10. *Let the strain energy density be given by formulas* (3.45)*, and let the restrictions of Theorem* 5.8 *be satisfied. Then in an arbitrarily domain Ω for any equilibrium states $\widehat{u}_{t,\sigma}$, $\widehat{\chi}_{t,\sigma}$, $\sigma > 0$, as $\sigma \to 0$ we have $\widehat{u}_{t,\sigma} \rightharpoondown 0$ in the space $\mathbb{H}$.*

5.5. Bibliographical notes. The basic results of this section were presented in the papers [35, 44–46]. The theory of relaxed problems and quasiconvex hulls can be found in the monographs [57, 61] and the paper [58] useful in the sense of ideas. In this section we did not touch upon the question on the Γ-convergence of the energy functional of a two-phase elastic medium as $\sigma \to 0$. The main points of the theory of Γ-convergence were presented in [8]. Their partial transfer to the theory of phase transitions was realized, in particular, in [5]. The available numerous results on the calculation of quasi-convex hulls for various applied problems can be found in [53, 54, 57–59, 61].

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St. Petersburg State University, University Embankment 7/9, 199034 St. Petersburg, Russia

Email address: `victor.osmolovskii@gmail.com`

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