ON THE ORDER LAW OF THE ITERATED LOGARITHM

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I. K. MATSAK

ABSTRACT. We study the classical laws of the iterated logarithm due to Kolmogorov and Hartman–Wintner for random variables assuming values in Banach lattices.

1. INTRODUCTION

Let (ξ_n) be a sequence of independent random variables such that $\mathbb{E}\xi_n = 0$ and $\mathbb{E}\xi_n^2 = \sigma^2$, let (b_n) be a sequence of real numbers, and let $V_n = \sum_{i=1}^n b_i^2$. We say that a sequence (ξ_n) obeys the law of the iterated logarithm if

$$(1) \quad \begin{aligned} \limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^n b_i \xi_i}{\chi(V_n)} &= \sigma \quad \text{a.s.}, \\ \liminf_{n \rightarrow \infty} \frac{\sum_{i=1}^n b_i \xi_i}{\chi(V_n)} &= -\sigma \quad \text{a.s.}, \end{aligned}$$

where “a.s.” is the abbreviation for “almost surely”, $\chi(t) = (2tLL(t))^{1/2}$, and

$$L(t) = \max(1, \ln(t))$$

for $t > 0$.

A survey of results on the law of the iterated logarithm (1), its generalizations for Banach spaces, and relevant references can be found in [1]–[3].

Denote by B a Banach lattice equipped with a norm $\|\cdot\|$ and module $|\cdot|$. For a sequence (x_n) of elements of a σ -complete Banach lattice we define the upper and lower limits as follows:

$$\begin{aligned} \limsup_{n \rightarrow \infty} x_n &= \inf_m \left(\sup_{n \geq m} x_n \right), \\ \liminf_{n \rightarrow \infty} x_n &= \sup_m \left(\inf_{n \geq m} x_n \right). \end{aligned}$$

In what follows we make use of notions and some results of the theory of Banach lattices that can be found in the books [4]–[6].

The mean quadratic deviation of a random element and some of its generalizations can be defined for a Banach lattice (see [7]).

Consider a random element X assuming values in B . The mean p -deviation, $1 < p < \infty$, of a random element X is defined by

$$\Delta_p X = \sup(x \in K_p),$$

where

$$K_p = \left(x = \mathbb{E} \zeta X : \mathbb{E} |\zeta|^{p'} \leq 1 \right), \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

The element $\Delta X = \Delta_2 X$ of a Banach lattice is called the mean quadratic deviation of a random element X . The mean quadratic deviation coincides with the usual pointwise mean quadratic deviation

$$\Delta X = \left((\mathbb{E} |X(t)|^2)^{1/2}, t \in T \right)$$

in the case of a Banach function lattice if $\mathbb{E} X = 0$.

Let (X_n) be a sequence of independent random elements in B such that $\mathbb{E} X_n = 0$ and

$$(2) \quad \Delta(X_n) = \Delta X, \quad n = 1, 2, \dots,$$

and put $S_n = \sum_{i=1}^n b_i X_i$. The following relations are natural generalizations of the law of the iterated logarithm for the case of Banach lattices:

$$(3) \quad \begin{aligned} \limsup_{n \rightarrow \infty} \frac{S_n}{\chi(V_n)} &= \Delta X \quad \text{a.s.}, \\ \liminf_{n \rightarrow \infty} \frac{S_n}{\chi(V_n)} &= -\Delta X \quad \text{a.s.} \end{aligned}$$

Relations (3) are called the order law of the iterated logarithm in a Banach lattice B .

The order law of the iterated logarithm for identically distributed random variables was studied in the paper [8]. The current paper continues investigations originated in [8].

Remark 1. Condition (2) holds if $\mathbb{E} X_n = 0$, all X_n have the same covariance operator R , H_R is a Hilbert subspace of the space B associated to R [3], and the closed unit ball of the space H_R is order bounded in B (see [7]).

2. KOLMOGOROV'S LAW OF THE ITERATED LOGARITHM AND ITS GENERALIZATIONS

The following assertion generalizes the classical Kolmogorov theorem ([1, Chapter X, §1, Theorem 1]) to the case of Banach lattices.

Theorem 1. *Let B be a separable Banach lattice that does not contain ℓ_∞^n uniformly. Let X_n , $n \geq 1$, be independent random elements in B such that $\mathbb{E} X_n = 0$, $V_n \rightarrow \infty$ as $n \rightarrow \infty$, and condition (2) hold. Assume that there is a positive element $u \in B$ and a sequence of positive numbers (M_n) such that*

$$(4) \quad M_n = o \left(\left(\frac{V_n}{LL(V_n)} \right)^{1/2} \right),$$

$$(5) \quad |b_n X_n| \leq M_n \cdot u \quad \text{a.s.}$$

Then the order law of the iterated logarithm (3) holds.

Remark 2. Assertion (iv) of Proposition 3 in §3 shows that Theorem 1 does not hold for Banach lattices that uniformly contain ℓ_∞^n .

To prove Theorem 1 we need the following auxiliary result.

Lemma 1. *Let (b_n) be a sequence of real numbers, $V_n \rightarrow \infty$ as $n \rightarrow \infty$, $1 \leq r < \infty$, and let (ξ_n) be a sequence of independent random variables such that $\mathbb{E} \xi_n = 0$ and $\mathbb{E} \xi_n^2 = \sigma^2$. Assume that there exists a sequence $(\widehat{M}_n)$ of positive real numbers such that*

$$(6) \quad \sup_{n \geq 1} \widehat{M}_n b_n \left(\frac{V_n}{LL(V_n)} \right)^{-1/2} = H < \infty,$$

$$(7) \quad |\xi_n| \leq \widehat{M}_n \quad \text{a.s.}$$

Then

$$(8) \quad \left(\mathbb{E} \sup_{n \geq 1} \left| \frac{\sum_{k=1}^n b_k \xi_k}{\chi(V_n)} \right|^r \right)^{1/r} \leq C_r \cdot H,$$

where the constant $C_r < \infty$ does not depend on (ξ_n) (however, it may depend on the sequence (b_n)).

Proof of Lemma 1. It is known that inequalities (6) and (7) yield

$$\sup_{n \geq 1} \left| \frac{\sum_{k=1}^n b_k \xi_k}{\chi(V_n)} \right| < \infty \quad \text{a.s.}$$

(see [16]). This implies that the following two conditions are equivalent:

$$(9) \quad \mathbb{E} \sup_{n \geq 1} \left| \frac{\sum_{k=1}^n b_k \xi_k}{\chi(V_n)} \right|^r < \infty,$$

$$(10) \quad \mathbb{E} \sup_{n \geq 1} \left| \frac{b_n \xi_n}{\chi(V_n)} \right|^r < \infty$$

(see [3, p. 159]).

It is easy to check that

$$(11) \quad \left| \frac{b_n \xi_n}{\chi(V_n)} \right| \leq \frac{H}{\sqrt{2}} \quad \text{a.s.}$$

under the assumptions of the lemma, whence (10) follows. Therefore inequality (9) holds, too.

Let $(\Omega, \mathcal{A}, \mathbb{P}) = (\prod_{n=1}^{\infty} \Omega_n, \prod_{n=1}^{\infty} \mathcal{A}_n, \prod_{n=1}^{\infty} \mathbb{P}_n)$ be the product of probability spaces $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$, and let Σ be the family of all sequences (ξ_n) of independent random variables defined on Ω and such that, for every n , the random variable ξ_n depends only on $\omega_n \in \Omega_n$. Denote by Σ_K the space of sequences $(\xi_n) \in \Sigma$ satisfying the conditions

$$\mathbb{E} \xi_n = 0, \quad \mathbb{E} \xi_1^2 = \mathbb{E} \xi_2^2 = \dots = \mathbb{E} \xi_n^2 = \dots,$$

$$\|(\xi_n)\|_K = \sup_{n \geq 1} \|\xi_n\|_{\infty} b_n \left(\frac{V_n}{LL(V_n)} \right)^{-1/2} < \infty,$$

where $\|\xi_n\|_{\infty} = \inf(\alpha: \mathbb{P}(\omega \in \Omega_n: |\xi_n(\omega)| > \alpha) = 0)$.

Note that Σ_K is a Banach space with respect to the norm $\|\cdot\|_K$ (Σ_K is the direct sum of Banach spaces that are equivalent to $L^{\infty}(\Omega_n)$).

On the space Σ_K we consider the new norm

$$\|(\xi_n)\|_{K,1} = \|(\xi_n)\|_K + \left(\mathbb{E} \sup_{n \geq 1} \left| \frac{\sum_{k=1}^n b_k \xi_k}{\chi(V_n)} \right|^r \right)^{1/r}.$$

Since conditions (9) and (10) are equivalent and estimate (11) holds, the following two conditions

$$\|(\xi_n)\|_K < \infty$$

and

$$\|(\xi_n)\|_{K,1} < \infty$$

are equivalent. Therefore $\|(\cdot)\|_K$ and $\|(\cdot)\|_{K,1}$ are Banach norms on the space Σ_K . It is clear that $\|(\cdot)\|_K \leq \|(\cdot)\|_{K,1}$, whence we immediately obtain the equivalence of these norms:

$$(12) \quad \|(\cdot)\|_K \leq \|(\cdot)\|_{K,1} \leq C \|(\cdot)\|_K.$$

If a sequence (ξ_n) satisfies conditions (6) and (7), then $\|((\xi_n))\|_K \leq H$. The latter estimate together with (12) implies (8). $\square$

Proof of Theorem 1. Since Banach spaces are order isomorphic to Banach ideal spaces, we restrict ourselves to the case where B is a separable Banach ideal space on a measurable space (T, Λ, μ) . Moreover, one can assume that the Banach ideal space B is q -concave for some $q < \infty$ (see [4], [5]) and that $\mu(T) = 1$.

We prove the first equality in the law of the iterated logarithm (3); the second can be checked analogously. Let

$$\begin{aligned} X &= (X(t), t \in T) \in B \quad \text{a.s.}, \\ \Delta X &= (\sigma(t), t \in T) \in B, \\ U_m &= \left(U_m(t) = \sup_{n \geq m} \frac{S_n(t)}{\chi(V_n)}, t \in T \right), \\ U &= U_1. \end{aligned}$$

Following the method of [8], it is sufficient to show that

$$\text{o-lim}_{m \rightarrow \infty} U_m = \Delta X \quad \text{a.s.}$$

The latter equality follows from

$$(13) \quad \mu \left(t \in T : \lim_{m \rightarrow \infty} U_m(t) = \sigma(t) \right) = 1 \quad \text{a.s.}$$

and

$$(14) \quad \mathbb{E} \|U\|^q < \infty.$$

The Kolmogorov law of the iterated logarithm implies that

$$\lim_{m \rightarrow \infty} U_m(t) = \sigma(t) \quad \text{a.s.}$$

almost everywhere on T provided conditions (4) and (5) hold. Applying Fubini's theorem, we derive (13) from the latter equality.

To check (14) we use the following estimate:

$$(15) \quad (\mathbb{E} \|Y\|^q)^{1/q} \leq D_q \|\Delta_q Y\|,$$

where Y is a random element assuming values in a q -concave Banach lattice (see [7]). We obtain from (8) and (15) that

$$(\mathbb{E} \|U\|^q)^{1/q} \leq D_q \left\| \left(\mathbb{E} \sup_{n \geq 1} \left| \frac{S_n(t)}{\chi(V_n)} \right|^q \right)^{1/q} \right\| \leq C_q D_q H \|u\|,$$

whence (14) follows. $\square$

A similar method allows one to prove the following result.

Proposition 1. *Let B be a separable and q -concave, $2 < q < \infty$, Banach lattice. Assume that (X_n) is a sequence of independent random elements in B that satisfy condition (2), $\mathbb{E} X_n = 0$, and $V_n \rightarrow \infty$ as $n \rightarrow \infty$. Let there exist a positive element $u \in B$ and a number $\delta > 0$ such that*

$$\begin{aligned} \Delta_q(X_n) &\leq u, \\ b_n^2 &= O \left(V_n (L(V_n))^{-(1+\delta)/(q/2-1)} \right). \end{aligned}$$

Then the order law of the iterated logarithm (3) holds.

Consider a sequence (ε_i) of independent symmetric Bernoulli random variables,

$$P(\varepsilon_i = \pm 1) = \frac{1}{2}.$$

It is shown in [9] that

$$(16) \quad \limsup_{n \rightarrow \infty} \left| \frac{\sum_{i=1}^n \varepsilon_i b_i}{\chi(V_n)} \right| \leq 1 \quad \text{a.s.}$$

if $V_n \rightarrow \infty$. We prove a similar result for Banach ideal spaces. Assume that a Banach ideal space B contains a unit element $I = (I(t) \equiv 1, t \in T)$, and a sequence $(x_n) = (x_n(t), t \in T)$ is such that

$$(17) \quad A_n(t) = \sum_{i=1}^n |x_i(t)|^2 \rightarrow \infty, \quad n \rightarrow \infty,$$

almost everywhere on T , and

$$(18) \quad |x_1| \geq \delta I$$

for some $\delta > 0$.

Theorem 2. (i) *Let B be a separable Banach ideal space containing a unit element I , and assume that B does not contain ℓ_∞^n uniformly. Let a sequence (x_n) satisfy conditions (17) and (18). Then there exists a nonrandom positive element $I_x \in B$ such that*

$$\limsup_{n \rightarrow \infty} \left| \frac{\sum_{i=1}^n \varepsilon_i x_i}{\chi(A_n)} \right| = I_x \quad \text{a.s.}$$

and $I_x \leq I$.

(ii) *If additionally*

$$(19) \quad A_n^{-1}(t) \sum_{i=1}^n |x_i(t)|^2 J(t, G_i) \rightarrow 0, \quad n \rightarrow \infty,$$

for all $d > 0$ and everywhere on T where

$$J(t, G) = \begin{cases} 1, & t \in G, \\ 0, & t \notin G, \end{cases} \quad G_i = \left(t \in T: |x_i(t)| > d(A_i(t)/LL(A_i(t)))^{1/2} \right),$$

then $I_x = I$.

It is known that inequality (16) becomes an equality if (19) holds (see [10]).

The following auxiliary result is the main tool in the proof of Theorem 2. Put

$$V(n, \lambda) = \lambda + \sum_{i=1}^n b_i^2, \quad s_n = \sum_{i=1}^n \varepsilon_i b_i,$$

$$\Theta = \sup_{n \geq 1} \frac{|s_n|}{\chi(V(n, \lambda))},$$

and let $\Gamma(t)$ be the gamma function.

Lemma 2. *If $\lambda > e$ and $1 \leq r < \infty$, then*

$$(20) \quad E \Theta^r \leq (2\lambda)^{r/2} + \frac{\pi^2 r}{3} \left(\frac{\lambda}{\ln \ln \lambda} \right)^{r/2} \Gamma\left(\frac{r}{2} - 1\right).$$

Proof of Lemma 2. The well-known exponential inequality (see, for example, Chapter 3, §4 in [1])

$$\mathbf{P} \left(s_n > t \left(\sum_{i=1}^n b_i^2 \right)^{1/2} \right) \leq \exp \left(-\frac{t^2}{2} \right)$$

implies

$$(21) \quad \mathbf{P}(s_n > t\chi(V(n, \lambda))) \leq (\ln V(n, \lambda))^{-t^2}.$$

Put $\Theta_n = \{i \in N : \lambda^n \leq V(i, \lambda) < \lambda^{n+1}\}$. Considering only nonempty terms in the sequence (Θ_n) we obtain the subsequence (Θ_{n_k}) . Further let

$$s_n^* = \max_{1 \leq i \leq n} (s_i),$$

$$\alpha_k = \min(i : i \in \Theta_{n_k}), \quad \beta_k = \max(i : i \in \Theta_{n_k}).$$

Estimate (21) and Levy's inequality $\mathbf{P}(s_n^* > t) \leq 2\mathbf{P}(s_n > t)$ together with

$$\lambda^{n_k} \leq V(\alpha_k, \lambda) \leq V(\beta_k, \lambda) \leq \lambda^{n_k+1}$$

imply

$$(22) \quad \begin{aligned} \mathbf{P}(\Theta > t) &\leq \sum_{k=1}^{\infty} \mathbf{P} \left(\sup_{i \in \Theta_{n_k}} \frac{|s_i|}{\chi(V(i, \lambda))} > t \right) \leq \sum_{k=1}^{\infty} \mathbf{P} \left(\sup_{i \in \Theta_{n_k}} |s_i| > t\chi(V(\alpha_k, \lambda)) \right) \\ &\leq 2 \sum_{k=1}^{\infty} \mathbf{P}(s_{\beta_k}^* > t\chi(V(\alpha_k, \lambda))) \leq 4 \sum_{k=1}^{\infty} \mathbf{P}(s_{\beta_k} > t\chi(V(\alpha_k, \lambda))) \\ &\leq 4 \sum_{k=1}^{\infty} (\ln V(\alpha_k, \lambda))^{-t^2/\lambda} \leq 4(\ln \lambda)^{-t^2/\lambda} \sum_{k=1}^{\infty} (n_k)^{-t^2/\lambda} \\ &\leq 4(\ln \lambda)^{-t^2/\lambda} \zeta \left(\frac{t^2}{\lambda} \right), \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function.

Further

$$(23) \quad \mathbf{E} \Theta^r = r \int_0^{\infty} t^{r-1} \mathbf{P}(\Theta > t) dt \leq (2\lambda)^{r/2} + r \int_{\sqrt{2\lambda}}^{\infty} t^{r-1} \mathbf{P}(\Theta > t) dt$$

(see [11], Chapter V, §6). Now we substitute estimate (22) into the latter integral

$$\begin{aligned} r \int_{\sqrt{2\lambda}}^{\infty} t^{r-1} \mathbf{P}(\Theta > t) dt &\leq 4r \int_{\sqrt{2\lambda}}^{\infty} t^{r-1} (\ln \lambda)^{-t^2/\lambda} \zeta \left(\frac{t^2}{\lambda} \right) dt \\ &\leq 4r\zeta(2) \int_{\sqrt{2\lambda}}^{\infty} t^{r-1} (\ln \lambda)^{-t^2/\lambda} dt \leq \frac{\pi^2}{3} r \left(\frac{\lambda}{\ln \ln \lambda} \right)^{r/2} \Gamma \left(\frac{r}{2} - 1 \right), \end{aligned}$$

whence (20) follows by (21). $\square$

3. THE LAW OF THE ITERATED LOGARITHM FOR IDENTICALLY DISTRIBUTED RANDOM ELEMENTS

This section contains some sharpening of the results of [8].

Let (X_n) be a sequence of independent copies of a random element X such that $\mathbf{E} X = 0$, and let $S_n = \sum_{i=1}^n X_i$. We say that a random element X satisfies the order law of the iterated logarithm if (3) holds with $b_n \equiv 1$ and $V_n \equiv n$.

It is known (see [8]) that a random element X satisfies the order law of the iterated logarithm in a q -concave Banach lattice, $1 \leq q < \infty$, if

$$(24) \quad \Delta_{\psi} X \quad \text{exists,}$$

where $\psi(t) = t^2$ for $1 \leq q < 2$, and $\psi(t) = |t|^q \ln(1 + |t|)$ for $2 \leq q < \infty$ (here $\Delta_\psi X$ is the mean ψ -deviation of the random element X (see [7])).

There exists a random element in the space ℓ_2 for which condition (24) cannot be weakened if $q = 2$ (see [8]). This is the case for $1 \leq q < 2$ also, in view of the one-dimensional law of the iterated logarithm.

In contrast, condition (24) can be improved for the case of $q > 2$.

Proposition 2. *Let $2 < q < \infty$, and let B be a separable q -concave Banach lattice. Assume that X is a random element taking values in B and such that $\mathbb{E} X = 0$. If $\Delta_q X$ exists, then*

$$\mathbb{E} \left\| \sup_{n \geq 1} \frac{|S_n|}{\chi(n)} \right\|^q \leq C_q \|\Delta_q X\|^q$$

and X satisfies the order law of the iterated logarithm.

To prove Proposition 2 we use the argument of [8] completed by the following auxiliary result.

Lemma 3. *Let $\xi, \xi_1, \xi_2, \dots$ be independent identically distributed random variables. If $r > 1$, then*

$$\mathbb{E} \sup_{n \geq 1} \left| \frac{1}{n} \sum_{i=1}^n \xi_i \right|^r \leq C_r \mathbb{E} |\xi|^r.$$

The last lemma is a particular case of Lemma 2 in [7].

Assume that there exists a random variable $\tau \in L_2(\Omega)$ and an element $u \in B$ such that

$$(25) \quad |X| \leq \tau \cdot u.$$

It is known that condition (25) is sufficient for the existence of the mean square deviation ΔX of the random element X (see [7]).

Proposition 3. *Assume that B is a separable Banach lattice, X is a random element with values in B , and $\mathbb{E} X = 0$.*

- (i) *If B does not contain ℓ_∞^n uniformly and there exists a random variable $\tau \in L_2(\Omega)$ and an element $u \in B$ such that inequality (25) holds, then the random element X satisfies the order law of the iterated logarithm, and moreover*

$$\left(\mathbb{E} \left\| \sup_{n \geq 1} \frac{|S_n|}{\chi(n)} \right\|^p \right)^{1/p} \leq C(p, B) \cdot \Delta(\tau) \cdot \|u\|$$

for $0 < p < 2$.

- (ii) *If the assumptions of assertion (i) hold with $\tau \in L_\psi(\Omega)$ and $\psi(t) = |t^2| \ln(1 + |t|)$, then*

$$\left(\mathbb{E} \left\| \sup_{n \geq 1} \frac{|S_n|}{\chi(n)} \right\|^2 \right)^{1/2} \leq C(B) \cdot \Delta_\psi(\tau) \cdot \|u\|.$$

- (iii) *If the assumptions of assertion (i) hold with $\tau \in L_p(\Omega)$ and $p > 2$, then*

$$\left(\mathbb{E} \left\| \sup_{n \geq 1} \frac{|S_n|}{\chi(n)} \right\|^p \right)^{1/p} \leq C(p, B) \cdot \Delta_p(\tau) \cdot \|u\|.$$

- (iv) *If a Banach lattice B uniformly contains ℓ_∞^n , then there exists a random element X in B and an element $u \in B$ such that*

$$(26) \quad |X| \leq u \quad \text{a.s.}$$

and the order law of the iterated logarithm does not hold for the random element X .

Proof of Proposition 3. Assertions (i) and (ii) are proved in [8]. Assertion (iii) follows from the results of [8] and Lemma 2.

Proof of (iv). A counterexample to the central limit theorem for Banach spaces uniformly containing ℓ_∞^n is constructed in the paper [12]. A modification of the method of [12] allows one to construct an example for assertion (iv) (see also [13]).

If the space B uniformly contains ℓ_∞^n , then for all n and $\delta > 0$ there exists a sequence $(x_i)_1^n$ of pairwise disjoint elements in B such that

$$\max_{1 \leq i \leq n} |a_i| \leq \left\| \sum_{i=1}^n a_i x_i \right\| \leq (1 + \delta) \max_{1 \leq i \leq n} |a_i|$$

for every family of numbers $(a_i)_1^n$ (see [4, p. 91]). This implies (see [14]) that there exists a sequence of positive elements (x_i) such that

$$\|x_i\| = (\ln \ln(i + 7))^{-1}$$

and the series $\sum_{i=1}^\infty x_i$ converges unconditionally.

Put

$$u = \sum_{i=1}^\infty x_i, \quad X = \sum_{i=1}^\infty \xi_i x_i,$$

where (ξ_i) is a sequence of independent random variables such that

$$P(\xi_i = 1) = P(\xi_i = -1) = \frac{1}{\ln(i + 7)}, \quad P(\xi_i = 0) = 1 - \frac{2}{\ln(i + 7)}.$$

It is obvious that

$$|X| \leq \sum_{i=1}^\infty |x_i| = \sum_{i=1}^\infty x_i = u \quad \text{a.s.},$$

that is, condition (26) holds.

Let (X_k) and $(\xi_{ki})_{k=1}^\infty$ be sequences of independent copies of the random elements X and ξ_i , respectively,

$$X_k = \sum_{i=1}^\infty \xi_{ki} x_i.$$

Assume that the random element X satisfies the order law of the iterated logarithm. Then the inequality

$$\sup_{n \geq 1} \left\| \frac{S_n}{\chi(n)} \right\| \leq \left\| \sup_{n \geq 1} \frac{|S_n|}{\chi(n)} \right\| \quad \text{a.s.}$$

implies

$$(27) \quad \sup_{n \geq 1} \left\| \frac{S_n}{\chi(n)} \right\| < \infty \quad \text{a.s.}$$

Put

$$N_n = [\exp(n \ln n + \ln \ln n)],$$

$$W_{ni} = \bigcap_{k=1}^n \{\xi_{ki} = 1\}, \quad W_n = \bigcup_{i=n}^{N_n} W_{ni}.$$

It is shown in [13] that

$$(28) \quad \liminf_{n \rightarrow \infty} P(W_n) \geq 1 - \frac{1}{e}.$$

Further we apply estimate (27) and Levy's inequality [15],

$$\begin{aligned}
 (29) \quad 0 &= \lim_{n \rightarrow \infty} \mathbf{P} \left(\sup_{m \geq 1} \frac{\|S_m\|}{\chi(m)} > \ln n \right) \geq \liminf_{n \rightarrow \infty} \mathbf{P} \left(\frac{\|S_n\|}{\chi(n)} > \ln n \right) \\
 &\geq \frac{1}{2} \liminf_{n \rightarrow \infty} \mathbf{P} \left(\max_{n \leq i \leq N_n} \left\| \sum_{k=1}^n \xi_{ki} x_i \right\| > \ln n \right).
 \end{aligned}$$

If $\omega \in W_n$, then

$$\max_{n \leq i \leq N_n} \left\| \frac{\sum_{k=1}^n \xi_{ki} x_i}{\chi(n)} \right\| \geq \frac{n}{\chi(n) \ln \ln(N_n + 7)} \sim \frac{n^{1/2}}{\sqrt{2} \ln \ln n \ln n} \geq \ln n$$

for sufficiently large n , whence we get that

$$\liminf_{n \rightarrow \infty} \mathbf{P}(W_n) = 0$$

by (29). The latter equality contradicts (28). $\square$

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KYIV STATE UNIVERSITY FOR TECHNOLOGY AND DESIGN, NEMIROVICH-DANCHENKO STREET 2, KYIV 02011, UKRAINE

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