TWO-BOUNDARY PROBLEMS FOR A RANDOM WALK WITH NEGATIVE GEOMETRIC JUMPS

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ABSTRACT. Two-boundary problems for a random walk with negative geometric jumps are considered, and the corresponding results for a usual semicontinuous random walk are generalized for them. The following results are obtained: the probability distribution of ruin is found and expressed in terms of the lower and upper boundaries; formulas are given for the joint distribution of the infimum, supremum, and the walk itself at an arbitrary time instance; the transient probabilities and ergodic distribution are evaluated for the process describing the evolution of the random walk with two boundaries.

Two-boundary problems for random walks and stochastic processes have several applications in the queue theory, storage and inventory theories, reliability theory, and in many other fields.

Two-boundary problems have been studied for semicontinuous random walks and for semicontinuous stochastic processes. Several methods are known for solving those problems, namely combinatorial [1], resolvent [2]–[6], factorization [7], and renewal [8] methods.

In this paper we solve two-boundary problems for random walks with negative geometric jumps. This model is a generalization of a usual model of semicontinuous random walks.

1. MAIN NOTATION

Let $\alpha \in \{0, 1, \dots\}$ be a nonnegative integer-valued random variable,

$$\mathbb{E}[\theta^\alpha] = \sum_{i=0}^{\infty} a_i \theta^i, \quad a_i = \mathbb{P}[\alpha = i], \quad \mathbb{P}[\alpha > 1] > 0, \quad |\theta| \leq 1.$$

Consider the random variable

$$\xi = \alpha - \beta, \quad \xi \in \{0, \pm 1, \dots\} = \mathbf{Z},$$

where $\beta \in \{1, 2, \dots\}$ is a positive integer-valued random variable distributed geometrically with parameter $b \in [0, 1)$, that is,

$$\mathbb{P}[\beta = n] = (1 - b)b^{n-1}, \quad n > 0.$$

It is clear that

$$\mathbb{E}[\theta^\xi] = \mathbb{E}[\theta^\alpha] \frac{(1 - b)/\theta}{1 - b/\theta} \stackrel{\text{def}}{=} \sum_{i=-\infty}^{\infty} p_i \theta^i, \quad |\theta| = 1.$$

An easy evaluation yields the distribution of the random variable ξ :

$$(1) \quad p_i = \mathbb{P}[\xi = i] = (1 - b)b^{-i-1} \mathbb{E}[b^\alpha; \alpha > i], \quad i \in \mathbf{Z}.$$

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Some of the properties of the distribution p_i , $i \in \mathbf{Z}$, are listed in the following result.

Lemma 1. *For all $i \in \mathbf{Z}$,*

$$(2) \quad \begin{aligned} bp_i - p_{i-1} &= \begin{cases} 0, & i < 0, \\ -(1-b)a_i, & i \geq 0, \end{cases} \\ b\mathbf{P}[\xi \geq i] - \mathbf{P}[\xi \geq i-1] &= -(1-b)\mathbf{P}[\alpha \geq i], \\ b\mathbf{P}[\xi \leq i] - \mathbf{P}[\xi \leq i-1] &= -(1-b)\mathbf{P}[\alpha \leq i], \end{aligned}$$

where $\mathbf{P}[\alpha = m] = 0$ for $m < 0$.

These equalities follow from the explicit expression (1) for the distribution of the random variable ξ .

Let $\{\xi, \xi'_i\}$, $i > 0$, be a sequence of independent identically distributed random variables and let ξ_n , $n \geq 0$, be a random walk such that

$$\xi_0 = 0, \quad \xi_n = \xi'_1 + \cdots + \xi'_n, \quad n > 0.$$

The moment generating function of increments of the random walk ξ_n , $n \geq 0$, satisfies

$$\sum_{n=0}^{\infty} t^n \mathbf{E}[\theta^{\xi_n}] = \frac{1}{1 - t \mathbf{E}[\theta^{\xi}]} = \left\{ 1 - t \mathbf{E}[\theta^{\alpha}] \frac{(1-b)/\theta}{1-b/\theta} \right\}^{-1}, \quad |\theta| = 1, \quad t \in [0, 1].$$

The equation

$$1 - \frac{b}{\theta} - (1-b)\frac{t}{\theta} \mathbf{E}[\theta^{\alpha}] = 0$$

has a unique root $\theta = c(t)$ in the domain $|\theta| < 1$. Moreover the root is positive and $b < c(t) < 1$. This result is well known for semicontinuous random walks [9]; the proof for the case under consideration is similar.

Definition 1. A sequence $R_k(t)$, $k > 0$, is called the resolvent of the random walk ξ_n , $n \geq 0$, if

$$(3) \quad R(\theta, t) = \sum_{k=1}^{\infty} \theta^k R_k(t) = \left[\frac{b}{\theta} + (1-b)\frac{t}{\theta} \mathbf{E}[\theta^{\alpha}] - 1 \right]^{-1}, \quad |\theta| < c(t).$$

Since the right-hand side of (3) is analytic for $|\theta| < c(t)$, it can be expanded in a power series. In what follows we need the first term of the sequence $R_k(t)$, $k > 0$:

$$R_1(t) = (b + (1-b)t \mathbf{P}[\alpha = 0])^{-1}.$$

Definition 2. A sequence R_k , $k > 0$, is called the potential of the random walk ξ_n , $n \geq 0$, if

$$(4) \quad R(\theta) = \sum_{k=1}^{\infty} \theta^k R_k = \left[\frac{b}{\theta} + (1-b)\frac{1}{\theta} \mathbf{E}[\theta^{\alpha}] - 1 \right]^{-1}, \quad |\theta| < c(1) \leq 1.$$

The following two equalities are used later when solving two-boundary problems.

Lemma 2. *For all $k > 0$,*

$$(5) \quad R_k(t) + t(1-b)b^{-k-1} \sum_{i=1}^k R_i(t)b^i \mathbf{E}[b^{\alpha}; \alpha \leq k-i] = b^{-k},$$

$$(6) \quad S_k(t) + t(1-b)b^{-k-1} \sum_{i=1}^k S_i(t)b^i \mathbf{E}[b^{\alpha}; \alpha \leq k-i] = \frac{1}{1-b}[b^{-k} - 1],$$

where

$$S_k(t) = \sum_{i=1}^k R_i(t).$$

Proof. Denote the left-hand side of (5) by A_k , $k > 0$. Then

$$\sum_{k=1}^{\infty} \theta^k A_k = R(\theta, t) + t(1-b) \frac{1}{b} R(\theta, t) \frac{\mathbb{E}[\theta^\alpha]}{1-\theta/b}, \quad |\theta| < b.$$

It follows from the definition of the resolvent (3) that

$$R(\theta, t) \left[\frac{b}{\theta} + (1-b) \frac{t}{b} \mathbb{E}[\theta^\alpha] - 1 \right] = 1, \quad |\theta| < c(t),$$

whence

$$\sum_{k=1}^{\infty} \theta^k A_k = \frac{\theta}{b} \left\{ 1 - \frac{\theta}{b} \right\}^{-1}, \quad |\theta| < b.$$

Comparing the coefficients for θ^k , $k > 0$, on both sides of the latter equality, we get $A_k = b^{-k}$, $k > 0$. Equality (6) is proved similarly. $\square$

Now we turn to solving the two-boundary problems.

2. RUIN PROBLEM

Fix an integer $N > 1$ and put $\mathbb{P}[\xi_0 = k] = 1$, $k \in \{1, \dots, N-1\}$. Let

$$\tau_k = \inf\{n > 0: \xi_n \notin \{1, \dots, N-1\}\}$$

be the first moment when the random walk ξ_n , $n \geq 0$, exits the set $\{1, \dots, N-1\}$. We also introduce the first moments when the random walk ξ_n , $n \geq 0$, exits the set $\{1, \dots, N-1\}$ through the upper and lower boundary, respectively:

$$\begin{aligned} \tau_k^+ &= \inf\{n > 0: \xi_n \notin \{1, \dots, N-1\}, \xi_{\tau_k} \geq N\}, \\ \tau_k^- &= \inf\{n > 0: \xi_n \notin \{1, \dots, N-1\}, \xi_{\tau_k} \leq 0\}. \end{aligned}$$

Theorem 1. *Let*

$$\mathbb{E}[t^{\tau_k^-}] = \mathbb{E}[t^{\tau_k}; \xi_{\tau_k} \leq 0], \quad \mathbb{E}[t^{\tau_k^+}] = \mathbb{E}[t^{\tau_k}; \xi_{\tau_k} \geq N]$$

be the moment generating functions of the "ruin" times τ_k^- and τ_k^+ , $k \in \{1, \dots, N-1\}$. Then

$$\begin{aligned} \mathbb{E}[t^{\tau_k^-}] &= \frac{b^N}{1-b} \frac{R_{N-k}(t)}{\hat{R}_N(b, t)}, \\ (7) \quad \mathbb{E}[t^{\tau_k^+}] &= 1 - \frac{R_{N-k}(t)}{\hat{R}_N(b, t)} \left[\frac{b^N}{1-b} + (1-b)(1-t)\hat{S}_N(b, t) \right] + (1-b)(1-t)S_{N-k}(t), \end{aligned}$$

where $R_k(t)$, $k > 0$, is the resolvent (3) of the random walk ξ_n , $n \geq 0$,

$$\hat{R}_N(b, t) = \sum_{i=N}^{\infty} b^i R_i(t), \quad S_k(t) = \sum_{i=1}^k R_i(t), \quad \hat{S}_N(b, t) = \sum_{i=N}^{\infty} b^i S_i(t).$$

Corollary 1. *Let*

$$\mathbb{P}_k^- = \mathbb{P}[\xi_{\tau_k} \leq 0] \quad \text{and} \quad \mathbb{P}_k^+ = \mathbb{P}[\xi_{\tau_k} \geq N]$$

be the “ruin” probabilities corresponding to the lower and upper boundaries, respectively. Denote by $\mathbb{E}[\tau_k]$ the mean “ruin” time. Then

$$\begin{aligned} P_k^- &= \frac{b^N}{1-b} \frac{R_{N-k}}{\hat{R}_N(b)}, & P_k^+ &= 1 - \frac{b^N}{1-b} \frac{R_{N-k}}{\hat{R}_N(b)} = 1 - P_k^-, \\ \mathbb{E}[\tau_k] &= b^N \frac{R_{N-k}}{\hat{R}_N(b)} S_{N-1} + R_{N-k} - (1-b)S_{N-k}, \end{aligned}$$

where R_k , $k > 0$, is the potential of the random walk ξ_n , $n \geq 0$,

$$\hat{R}_N(b) = \sum_{i=N}^{\infty} b^i R_i, \quad S_k = \sum_{i=1}^k R_i.$$

Proof of Theorem 1. Let $\mathbb{P}[\xi_0 = N - k] = 1$, $k \in \{1, \dots, N - 1\}$. By the full probability formula, the functions

$$\varphi_k(t) = \mathbb{E} \left[t^{\tau_{N-k}^-} \right], \quad k \in \{1, \dots, N - 1\}, \quad t \in [0, 1),$$

satisfy the following system of equations:

$$(8) \quad \varphi_k(t) = t \mathbb{P}[\xi \leq k - N] + t \sum_{i=1}^{N-1} \varphi_i(t) p_{k-i}, \quad k \in \{1, \dots, N - 1\}.$$

System (8) is regular, that is,

$$t \sum_{i=1}^{N-1} p_{k-i} < 1 - tb^{N-2} \mathbb{E}[b^\alpha].$$

Therefore it has a unique solution (see [10]). We are going to find this solution in an explicit form. When solving this problem we follow a method that was useful for solving other two-boundary problems.

The functions $b\varphi_{k+1}(t)$ are such that

$$b\varphi_{k+1}(t) = tb \mathbb{P}[\xi \leq k + 1 - N] + tb \sum_{i=1}^{N-1} \varphi_i(t) p_{k+1-i}, \quad k \in \{0, \dots, N - 2\}.$$

Subtracting this equation from (8) and applying (2) we get

$$(9) \quad b\varphi_{k+1}(t) - \varphi_k(t) = -t(1-b) \sum_{i=1}^{k+1} \varphi_i(t) a_{k+1-i}, \quad k \in \{1, \dots, N - 2\}.$$

Consider equation (9) for all $k \geq 1$:

$$(10) \quad b\varphi_{k+1}(t) - \varphi_k(t) = -t(1-b) \sum_{i=1}^{k+1} \varphi_i(t) a_{k+1-i}, \quad k \geq 1,$$

and solve it by using the moment generating function method. Put

$$\Phi(\theta, t) = \sum_{k=1}^{\infty} \theta^k \varphi_k(t), \quad |\theta| < c(t).$$

Rewriting equation (10) for the corresponding moment generating functions we obtain

$$\Phi(\theta, t) \left(\frac{b}{\theta} + (1-b) \frac{t}{\theta} \mathbb{E}[\theta^\alpha] - 1 \right) = \varphi_1(t) \frac{1}{R_1(t)}, \quad |\theta| < c(t).$$

By the definition of the resolvent (3) we get from the latter equation that

$$(11) \quad \varphi_k(t) = \varphi_1(t) \frac{R_k(t)}{R_1(t)}, \quad k \geq 1.$$

The function $\varphi_k(t)$ in (11) satisfies equation (10) for all $k \geq 1$, thus it is a solution of equation (9) for $k \in \{1, \dots, N-2\}$. A solution of initial equation (8) belongs to the set of solutions of equation (9) for which the function $\varphi_1(t)$ is a free parameter. It remains to find a function $\varphi_1^*(t)$ such that $\varphi_k(t)$ defined by (11) satisfies equation (8). Substituting the function $\varphi_k(t)$ from (11) into the initial equation (8) and using Lemma 2 we get

$$\varphi_1^*(t) = \frac{b^N}{1-b} \frac{R_1(t)}{\hat{R}_N(b, t)}, \quad \hat{R}_N(b, t) = \sum_{i=N}^{\infty} R_i(t) b^i.$$

Thus

$$\varphi_k(t) = \mathbb{E} \left[t^{\tau_{N-k}} \right] = \frac{b^N}{1-b} \frac{R_k(t)}{\hat{R}_N(b, t)}, \quad k \in \{1, \dots, N-1\},$$

and the first equality of Theorem 1 is proved.

Now we prove the second equality. Let $\mathbb{P}[\xi_0 = N-k] = 1$, $k \in \{1, \dots, N-1\}$. By the full probability formula, the functions

$$\psi_k(t) = \mathbb{E} \left[t^{\tau_{N-k}}; \xi_{\tau_{N-k}} \geq N \right], \quad t \in [0, 1),$$

satisfy the system of equations

$$(12) \quad \psi_k(t) = t \mathbb{P}[\xi \geq k] + t \sum_{i=1}^{N-1} \psi_i(t) p_{k-i}, \quad k \in \{1, \dots, N-1\}.$$

The equation for $b\psi_{k+1}(t)$ is given by

$$b\psi_{k+1}(t) = tb \mathbb{P}[\xi \geq k+1] + tb \sum_{i=1}^{N-1} \psi_i(t) p_{k+1-i}, \quad k \in \{0, \dots, N-2\}.$$

Subtracting this equation from (12) we obtain that for all $k \geq 1$,

$$(13) \quad b\psi_{k+1}(t) - \psi_k(t) = -t(1-b) \sum_{i=1}^{k+1} \psi_i(t) a_{k+1-i} - t(1-b) \mathbb{P}[\alpha \geq k+1], \quad k \geq 1.$$

Put

$$\Psi(\theta, t) = \sum_{k=1}^{\infty} \theta^k \psi_k(t), \quad |\theta| < c(t).$$

Rewriting equation (13) for the corresponding moment generating functions and using the definition of the resolvent (3) we get

$$\Psi(\theta, t) = \frac{\theta}{1-\theta} + \frac{\psi_1(t)-1}{R_1(t)} R(\theta, t) + (1-b)(1-t) \frac{\theta}{1-\theta} R(\theta, t), \quad |\theta| < c(t).$$

Comparing the coefficients for θ^k , $k > 0$, on both sides of the latter equality we obtain

$$(14) \quad \psi_k(t) = 1 + \frac{\psi_1(t)-1}{R_1(t)} R_k(t) + (1-b)(1-t) S_{k-1}(t), \quad k \geq 1.$$

To find a function $\psi_1^*(t)$ such that $\psi_k(t)$ defined by (14) satisfies the initial system of equations (12), we substitute the right-hand side of (14) into the system of equations (12) and apply Lemma 2. Then we obtain the following relation:

$$\frac{\psi_1^*(t)-1}{R_1(t)} = -\frac{1}{\hat{R}_N(b, t)} \left[\frac{b^N}{1-b} + (1-b)(1-t) \hat{S}_N(b, t) \right] + (1-b)(1-t),$$

where

$$\hat{R}_N(b, t) = \sum_{i=N}^{\infty} b^i R_i(t), \quad \hat{S}_N(b, t) = \sum_{i=N}^{\infty} b^i S_i(t).$$

Therefore

$$\psi_k(t) = 1 - \frac{R_k(t)}{\hat{R}_N(b, t)} \left[\frac{b^N}{1-b} + (1-b)(1-t)\hat{S}_N(b, t) \right] + (1-b)(1-t)S_k(t),$$

$$k \in \{1, \dots, N-1\},$$

and the second equality of Theorem 1 is proved.

Further

$$\sum_{n=0}^{\infty} t^n \mathbb{P}[\tau_{N-k} > n] = \frac{1 - \varphi_k(t) - \psi_k(t)}{1-t} = \frac{(1-b)\hat{S}_N(b, t)}{\hat{R}_N(b, t)} R_k(t) - (1-b)S_k(t).$$

Since

$$(1-b)\hat{S}_N(b, t) = \hat{R}_N(b, t) + b^N S_{N-1}(t),$$

we have

$$\sum_{n=0}^{\infty} t^n \mathbb{P}[\tau_{N-k} > n] = b^N \frac{R_k(t)}{\hat{R}_N(b, t)} S_{N-1}(t) + R_k(t) - (1-b)S_k(t),$$

whence equality (7) follows. Corollary 1 follows from Theorem 1 for $t = 1$. $\square$

3. THE JOINT DISTRIBUTION OF $\inf\{\xi_0, \dots, \xi_n\}$, $\sup\{\xi_0, \dots, \xi_n\}$, AND ξ_n , $n \geq 0$

Assume that $\mathbb{P}[\xi_0 = 0] = 1$ and consider the random variables

$$\mu_n^+ = \sup\{\xi_0, \dots, \xi_n\}, \quad \mu_n^- = \inf\{\xi_0, \dots, \xi_n\}.$$

Let k and r be positive integers, $R_m(t) = 0$, $m \leq 0$.

Theorem 2. *The joint distribution*

$$\mathbb{P}_{(-r, k)}^n[i] = \mathbb{P}[-r < \mu_n^-, \xi_n = i, \mu_n^+ < k], \quad -r < i < k,$$

of the random variables μ_n^- , ξ_n , and μ_n^+ satisfies

$$\sum_{n=0}^{\infty} t^n \mathbb{P}_{(-r, k)}^n[i] = b^N \frac{R_k(t)}{\hat{R}_N(b, t)} R_{i+r}(t) + b R_{i+1}(t) - R_i(t), \quad N = k + r.$$

Proof of Theorem 2. Fix $N > 1$ and assume that $\mathbb{P}[\xi_0 = N - k] = 1$, $k \in \{1, \dots, N - 1\}$. Let

$$\tau_{N-k} = \inf\{n > 0: \xi_n \notin \{1, \dots, N - 1\}\}$$

be the first moment when the random walk ξ_n , $n \geq 0$, exits the set $\{1, \dots, N - 1\}$. Denote by

$$Q_{k,d}(n) = \mathbb{P}[\xi_n = N - d, \tau_{N-k} > n], \quad d \in \{1, \dots, N - 1\},$$

the transient probability of the random walk ξ_n , $n > 0$, on the interval $0 \leq n < \tau_{N-k}$.

By the full probability formula, the moment generating functions

$$Q_{k,d}^t = \sum_{n=0}^{\infty} t^n Q_{k,d}(n), \quad k, d \in \{1, \dots, N - 1\}, \quad t \in [0, 1),$$

satisfy the system of equations

$$(15) \quad Q_{k,d}^t = \delta_{k,d} + t \sum_{i=1}^{N-1} p_{k-i} Q_{i,d}^t, \quad k \in \{1, \dots, N - 1\},$$

where $\delta_{k,r}$ is the Kronecker symbol. The corresponding equation for $bQ_{k+1,d}^t$ is

$$bQ_{k+1,d}^t = b\delta_{k+1,d} + tb \sum_{i=1}^{N-1} p_{k+1-i} Q_{i,d}^t, \quad k \in \{0, \dots, N-2\}.$$

Subtracting this equation from (15) we obtain that for all $k > 0$,

$$(16) \quad bQ_{k+1,d}^t - Q_{k,d}^t = b\delta_{k+1,d} - \delta_{k,d} - t(1-b) \sum_{i=1}^{k+1} a_{k+1-i} Q_{i,d}^t.$$

Consider the moment generating function

$$Q_d^t(\theta) = \sum_{k=1}^{\infty} \theta^k Q_{k,d}^t, \quad |\theta| < c(t).$$

Rewriting (16) for the corresponding moment generating functions we obtain

$$Q_d^t(\theta) = \left(\frac{Q_{1,d}^t}{R_1(t)} - b\delta_{1,d} \right) R(\theta, t) + b\theta^{d-1} R(\theta, t) - \theta^d R(\theta, t), \quad |\theta| < c(t),$$

where $R(\theta, t)$ is the moment generating function of resolvent (3) for the random walk ξ_n , $n > 0$. Comparing the coefficients for θ^k , $k > 0$, and putting $R_m(t) = 0$, $m \leq 0$, we get

$$(17) \quad Q_{k,d}^t = \left(\frac{Q_{1,d}^t}{R_1(t)} - b\delta_{1,d} \right) R_k(t) + bR_{k-d+1}(t) - R_{k-d}(t).$$

We look for a function $Q_{1,d}^t = \tilde{Q}_{1,d}^t$ for which solution (17) of the infinite system (16) is also a solution of the initial system. Substituting expression (17) into the initial system (15) we get by Lemma 2 that

$$\frac{\tilde{Q}_{1,d}^t}{R_1(t)} - b\delta_{1,d} = b^N \frac{R_{N-d}(t)}{\hat{R}_N(b, t)}.$$

Combining this expression and (17) we obtain

$$(18) \quad Q_{k,d}^t = b^N \frac{R_{N-d}(t)}{\hat{R}_N(b, t)} R_k(t) + bR_{k-d+1}(t) - R_{k-d}(t), \quad k, d \in \{1, \dots, N-1\}.$$

Now put $P[\xi_0 = 0] = 1$, and let k and r be positive integers, $N = k + r$. Let

$$\tau = \inf\{n > 0: \xi_n \notin \{-r+1, \dots, k-1\}\}$$

be the first moment when the random walk ξ_n , $n > 0$, exits the set

$$\{-r+1, \dots, k-1\}.$$

Denote by

$$P_{(-r,k)}^n[i] = P[\xi_n = i, \tau > n], \quad -r < i < k,$$

the transient probability for the random walk ξ_n on the interval $0 \leq n < \tau$. Since the random walk ξ_n , $n \geq 0$, is homogeneous, we find from (18) that

$$\sum_{n=0}^{\infty} t^n P_{(-r,k)}^n[i] = b^N \frac{R_k(t)}{\hat{R}_N(b, t)} R_{i+r}(t) + bR_{i+1}(t) - R_i(t), \quad -r < i < k.$$

Since

$$P[\tau > n] = P[-r < \mu_n^-, \mu_n^+ < k],$$

we have

$$P_{(-r,k)}^n[i] = P[-r < \mu_n^-, \xi_n = i, \mu_n^+ < k], \quad -r < i < k,$$

whence Theorem 2 follows. $\square$

4. A RANDOM WALK ξ_n , $n \geq 0$, AND TWO ABSORBING BOUNDARIES

Fix a positive integer N and consider a random sequence $X_n \in \{0, 1, \dots, N\}$, $n \geq 0$, whose one-step transient probabilities are given by

$$(19) \quad k \rightarrow \begin{cases} i, & \mathbf{P}[\xi = i - k], \quad i \in \{1, \dots, N-1\}, \\ 0, & \mathbf{P}[\xi \leq -k], \\ N, & \mathbf{P}[\xi \geq N - k] \end{cases}$$

for all $k \in \{0, \dots, N\}$.

The homogeneous (discrete time) Markov process X_n , $n \geq 0$, describes the evolution of the random walk ξ_n , $n \geq 0$, in the strip $0, \dots, N$ with two reflecting boundaries with absorption (the upper boundary is at the level N , while the lower one is at the level 0).

If the process X_n , $n \geq 0$, reaches the boundary 0 at a moment n_0 , it stays there for a random time $\nu(0)$ until the first positive jump of the random walk ξ_{n-n_0} , $n \geq n_0$, occurs. Then the process is “reflected” and appears at a state $i \in \{1, \dots, N-1\}$ with probability $\mathbf{P}[\xi = i]$ (the probability of its appearing on the upper boundary is $\mathbf{P}[\xi \geq N]$). It is clear that $\nu(0)$ is a geometric random variable and

$$\mathbf{P}[\nu(0) = n] = \mathbf{P}[\xi > 0](\mathbf{P}[\xi \leq 0])^{n-1}, \quad n > 0.$$

If the process X_n , $n \geq 0$, reaches the upper boundary N at a moment n_N , it stays there for a random time $\nu(N)$ until the first negative jump of the random walk ξ_{n-n_N} , $n \geq n_N$, occurs. Then the process is “reflected” and appears at a state $i \in \{1, \dots, N-1\}$ with probability $\mathbf{P}[\xi = i - N]$ (the probability of its appearing on the lower boundary is $\mathbf{P}[\xi \leq -N]$). It is clear that $\nu(N)$ is a geometric random variable and

$$\mathbf{P}[\nu(N) = n] = \mathbf{P}[\xi < 0](\mathbf{P}[\xi \geq 0])^{n-1}, \quad n > 0.$$

Random walks with two reflecting boundaries are useful for queueing systems having a bounded number of waiting places as well as in storage and inventory theories, and in other applied probability fields.

4.1. Hitting the boundaries. Let $\mathbf{P}[X_0 = k] = 1$, and let

$$\tau_k^0 = \inf\{n > 0: X_n = 0\}, \quad k \in \{1, \dots, N\},$$

be the first time when the process X_n , $n \geq 0$, reaches the lower boundary 0. Also let

$$\tau_k^N = \inf\{n > 0: X_n = N\}, \quad k \in \{0, \dots, N-1\},$$

be the first time when the process X_n , $n \geq 0$, reaches the upper boundary N .

Theorem 3. *The moment generating functions of the random variables τ_k^0 and τ_k^N and their means $\mathbf{E}[\tau_k^0]$ and $\mathbf{E}[\tau_k^N]$ satisfy*

$$\begin{aligned} \mathbf{E}[t^{\tau_k^0}] &= \frac{b^N}{1-b} \frac{1 + (1-b)(1-t)S_{N-k}(t)}{b^N/(1-b) + (1-b)(1-t)\hat{S}_N(b, t)}, \\ \mathbf{E}[\tau_k^0] &= (1-b) \left[b^{-N} \hat{R}_N(b) + S_{N-1} - S_{N-k} \right], \\ \sum_{n=0}^{\infty} t^n \mathbf{P}[\tau_k^N > n] &= (1-b) \left\{ \frac{\hat{R}_{N+1}(b, t)}{\hat{R}_{N+1}(b, t) - b\hat{R}_N(b, t)} R_{N-k}(t) - S_{N-k}(t) \right\}, \\ \mathbf{E}[\tau_k^N] &= (1-b) \left\{ \frac{\hat{R}_{N+1}(b)}{\hat{R}_{N+1}(b) - b\hat{R}_N(b)} R_{N-k} - S_{N-k} \right\}, \end{aligned}$$

where

$$\begin{aligned}\hat{R}_N(b, t) &= \sum_{i=N}^{\infty} b^i R_i(t), & \hat{R}_N(b) &= \sum_{i=N}^{\infty} b^i R_i = \hat{R}_N(b, 1), \\ \hat{S}_N(b, t) &= \sum_{i=N}^{\infty} b^i S_i(t), & S_k(t) &= \sum_{i=1}^k R_i(t).\end{aligned}$$

Proof of Theorem 3. Let $\mathbf{P}[X_0 = N - k] = 1$. Using the transient probabilities (19) of the process X_n , $n \geq 0$, we see that the moment generating function

$$\tilde{\varphi}_k(t) = \mathbf{E} \left[t^{\tau_{N-k}^0} \right], \quad t \in (0, 1), \quad k \in \{0, \dots, N-1\},$$

satisfies the system of equations

$$(20) \quad \begin{aligned}\tilde{\varphi}_k(t) &= t \mathbf{P}[\xi \leq k - N] + t \tilde{\varphi}_0(t) \mathbf{P}[\xi \geq k] + t \sum_{i=1}^{N-1} \tilde{\varphi}_i(t) p_{k-i}, \\ &k \in \{0, \dots, N-1\},\end{aligned}$$

by the full probability formula. System (20) is strongly regular, that is,

$$t \mathbf{P}[\xi \geq k] + t \sum_{i=1}^{N-1} \mathbf{P}[\xi = k - i] = t \mathbf{P}[\xi > k - N] < 1 - t b^{N-1} \mathbf{E}[b^\alpha].$$

Therefore it has a unique solution (see [10]).

Using the transient probabilities (19) of the process X_n , $n \geq 0$, we prove that the moment generating function

$$\tilde{\psi}_k(t) = \mathbf{E} \left[t^{\tau_{N-k}^N} \right], \quad k \in \{1, \dots, N\}, \quad t \in [0, 1),$$

satisfies the system of equations

$$(21) \quad \tilde{\psi}_k(t) = t \mathbf{P}[\xi \geq k] + t \mathbf{P}[\xi \leq k - N] \tilde{\psi}_N(t) + t \sum_{i=1}^N \tilde{\psi}_i(t) p_{k-i}, \quad k \in \{1, \dots, N\},$$

by the full probability formula. The latter system is strongly regular, since

$$t \mathbf{P}[\xi < k - N] + t \mathbf{P}[k - N \leq \xi \leq k - 1] < t \mathbf{P}[\xi \leq N - 1] < 1 - t \mathbf{E} [1 - b^{\alpha-N}; \alpha > N].$$

Thus it has a unique solution (see [10]).

Applying to systems (20) and (21) the method used in the preceding sections for solving corresponding systems of equations, we complete the proof of Theorem 3. $\square$

4.2. Transient probabilities and ergodic distribution of the process X_n , $n \geq 0$.

Let $N > 0$ and let ν_t be a geometric random variable with parameter t : $\mathbf{P}[\nu_t = n] = (1 - t)t^n$, $n \geq 0$, $t \in [0, 1)$.

Theorem 4. *Let $\mathbf{P}[X_0 = k] = 1$, $k \in \{0, \dots, N-1\}$. Then*

$$\begin{aligned}\mathbf{P}[X_{\nu_t} = i] &= \frac{b^N}{1-b} \frac{R_{i+1}(t) - R_i(t)}{\hat{R}_{N+1}(b, t)} [1 + (1-b)(1-t)S_{N-k}(t)] \\ &\quad + (1-t) [bR_{i-k+1}(t) - R_{i-k}(t)], \quad i \in \{0, \dots, N-1\},\end{aligned}$$

$$\mathbf{P}[X_{\nu_t} = N] = \frac{\hat{R}_{N+1}(b, t) - b\hat{R}_N(b, t)}{(1-b)\hat{R}_{N+1}(b, t)} [1 + (1-b)(1-t)S_{N-k}(t)] - (1-t)R_{N-k}(t).$$

Moreover there exists the ergodic distribution

$$\Pi_i = \lim_{n \rightarrow \infty} \mathbf{P}[X_n = i], \quad i \in \{0, \dots, N\},$$

of the process X_n , $n \geq 0$, such that

$$\Pi_i = \frac{b^{N+1} R_{i+1} - R_i}{1 - b \hat{R}_{N+1}(b)}, \quad i \in \{0, \dots, N-1\}, \quad \Pi_N = \frac{\hat{R}_{N+1}(b) - b \hat{R}_N(b)}{(1-b) \hat{R}_{N+1}(b)}.$$

Proof of Theorem 4. Let $P[X_0 = N - k] = 1$, $k \in \{0, \dots, N\}$. Denote by

$$P_{kr}(n) = P[X_n = N - r], \quad r \in \{0, \dots, N\},$$

the transient probability of the process X_n , $n \geq 0$. By the full probability formula and (19), the functions

$$P_{k,r}^t = \sum_{n=0}^{\infty} t^n P_{kr}(n), \quad t \in [0, 1],$$

satisfy the system

$$(22) \quad P_{k,r}^t = \delta_{k,r} + t P_{N,r}^t P[\xi < k - N] + t P_{0,r}^t P[\xi \geq k] + t \sum_{i=1}^N P_{i,r}^t p_{k-i},$$

$$k, r \in \{0, \dots, N\}.$$

This problem is the most complicated among similar problems considered in this paper. Nevertheless the method used for solving the preceding two-boundary problems works in this case, too.

The functions $b P_{k+1,r}^t$, $k \in \{-1, \dots, N-1\}$, satisfy the following equations:

$$b P_{k+1,r}^t = b \delta_{k+1,r} + t b P_{N,r}^t P[\xi < k + 1 - N] + t b P_{0,r}^t P[\xi \geq k + 1] + t b \sum_{i=1}^N P_{i,r}^t p_{k+1-i}.$$

Subtracting (22) from the latter equality we get that for $k \in \{0, \dots, N-1\}$,

$$(23) \quad b P_{k+1,r}^t - P_{k,r}^t = b \delta_{k+1,r} - \delta_{k,r} - t(1-b) P_{0,r}^t P[\alpha \geq k + 1] - t(1-b) \sum_{i=1}^{k+1} P_{i,r}^t a_{k+1-i}.$$

Put

$$P_r(\theta, t) = \sum_{k=0}^{\infty} \theta^k P_{k,r}^t, \quad |\theta| < c(t).$$

Solving equations (23) by the moment generating function method we obtain that for all $k \geq 0$,

$$P_r(\theta, t) = \frac{P_{0,r}^t}{1 - \theta} [1 + (1-b)(1-t)R(\theta, t)] + b(1 - \delta_{r,0})\theta^{r-1}R(\theta, t) - \theta^r R(\theta, t),$$

$$|\theta| < c(t).$$

Comparing the coefficients for θ^k , $k \geq 0$, on both sides of this equality, we see that

$$(24) \quad P_{k,r}^t = P_{0,r}^t [1 + (1-b)(1-t)S_k(t)] + b(1 - \delta_{r,0})R_{k+1-r}(t) - R_{k-r}(t),$$

$$R_m(t) = 0, \quad m \leq 0.$$

Now we find a function $P_{0,r}^*(t)$ such that $P_{k,r}^t$ defined by (24) for $k \in \{0, \dots, N\}$ is a solution of the initial system (22). Substituting the right-hand side of (24) into (22) and applying Lemma 2 we prove that

$$P_{00}^*(t) = \frac{\hat{R}_{N+1}(b, t) - b \hat{R}_N(b, t)}{(1-b)(1-t) \hat{R}_{N+1}(b, t)},$$

$$P_{0r}^*(t) = \frac{b^{N+1} R_{N+1-r}(t) - R_{N-r}(t)}{1-b (1-t) \hat{R}_{N+1}(b, t)}, \quad r \in \{1, \dots, N\},$$

where

$$\hat{R}_N(b, t) = \sum_{i=N}^{\infty} b^i R_i(t).$$

Substituting $P_{0r}^*(t)$ into (24) we find that for all $k, r \in \{0, \dots, N\}$,

$$\begin{aligned} P_{k,r}^t &= \frac{b^{N+1}}{1-b} \frac{R_{N+1-r}(t) - R_{N-r}(t)}{(1-t)\hat{R}_{N+1}(b, t)} [1 + (1-b)(1-t)S_k(t)] \\ &\quad + bR_{k+1-r}(t) - R_{k-r}(t), \quad r \in \{1, \dots, N\}, \\ (25) \quad P_{k,0}^t &= \frac{\hat{R}_{N+1}(b, t) - b\hat{R}_N(b, t)}{(1-t)(1-b)\hat{R}_{N+1}(b, t)} [1 + (1-b)(1-t)S_k(t)] - R_k(t). \end{aligned}$$

Since

$$P[X_{\nu_t} = i] = (1-t)P_{N-k, N-i}^t, \quad k, i \in \{0, \dots, N\},$$

the first two identities of Theorem 4 follow from (25).

The process X_n , $n \geq 0$, is a Markov chain with a finite set of states $\{0, \dots, N\}$. If $P[\alpha > 1] > 0$, then there exists n_0 such that $\min_{k,r} P_{k,r}(n_0) > 0$. Then (see [11]) there exists the ergodic distribution

$$\Pi_i = \lim_{n \rightarrow \infty} P[X_n = i], \quad i \in \{0, \dots, N\}.$$

To get the ergodic distribution in an explicit form, one needs to evaluate the limits

$$\Pi_i = \lim_{t \rightarrow 1} (1-t) P_{k, N-i}^t, \quad i \in \{0, \dots, N\}.$$

Multiplying equalities (25) by $(1-t)$ and passing to the limits as $t \rightarrow 1$, we get by changing $r \rightarrow N-i$ that

$$\Pi_i = \frac{b^{N+1}}{1-b} \frac{R_{i+1} - R_i}{\hat{R}_{N+1}(b)}, \quad i \in \{0, \dots, N-1\}, \quad \Pi_N = \frac{\hat{R}_{N+1}(b) - b\hat{R}_N(b)}{(1-b)\hat{R}_{N+1}(b)}$$

where R_k , $k > 0$, is potential (4) of the random walk ξ_n , $n \geq 0$. Theorem 4 is proved. $\square$

Remark 1. Putting $b = 0$ in our theorems and corollaries we obtain the corresponding results for a lower semicontinuous random walk ξ_n , $n \geq 0$, whose increments have the moment generating function

$$\sum_{n=0}^{\infty} t^n E[\theta^{\xi_n}] = \left\{ 1 - \frac{t}{\theta} E[\theta^{\alpha}] \right\}^{-1}, \quad |\theta| = 1, \quad t \in (0, 1).$$

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