ON THE PROPERTIES OF THE SECOND MOMENT OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL-FUNCTIONAL EQUATIONS WITH VARYING COEFFICIENTS

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ABSTRACT. Sufficient conditions for the mean square stability of solutions of linear stochastic differential-functional Itô–Skorokhod equations with unbounded aftereffect are obtained in the paper. The critical case is also studied.

1. ASYMPTOTIC MEAN SQUARE STABILITY

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and

$$\{\mathcal{F}_t, t \geq 0\}, \quad \mathcal{F}_t \subset \mathcal{F},$$

a current of minimal σ -algebras. Further let $\mathbb{R}$ be the real one-dimensional Euclidean space equipped with the norm $|\cdot|$, and D_0 the Skorokhod space of functions $\{\varphi(\theta)\} \subset \mathbb{R}$ defined on $(-\infty, 0]$. Every function $\varphi(\theta)$ of D_0 has no discontinuities of the second kind, has the left limit at every point of discontinuity, is right continuous, and has the limit as $\theta \rightarrow -\infty$ [4].

Let $\{x(t) \equiv x(t, \omega)\} \subset \mathbb{R}$ be a stochastic process defined for $t \geq 0$ by the stochastic differential-functional Itô–Skorokhod equation

$$(1) \quad dx(t) = a(t, x_t) dt + b(t, x_t) dw(t) + \int_{\mathbb{R}} g(t, x_t, u) \bar{v}(dt, du),$$

$$(2) \quad x(t) = \varphi(t) \quad \text{for all } t \in (-\infty, 0]$$

(see [1], [2], [5]) where $\varphi \in D_0$; in what follows the trajectory $\{x(t)\} \subset \mathbb{R}$ up to the moment $t \geq 0$ is denoted by $x_t \equiv \{x(t + \theta), -\infty < \theta \leq 0\}$; $\{w(t) \equiv w(t, \omega)\} \subset \mathbb{R}$ is a one-dimensional Wiener process; $\bar{v}(t, A) \equiv v(t, A) - t\Pi(A)$, $A \subset \mathbb{R}$, is a centered Poisson measure in $\mathbb{R}$ with parameter $t\Pi(A) \equiv \mathbb{E} \{\nu(t, A)\}$ where $\{w(t)\}$ and $\{\bar{v}(t, A)\}$ are independent and $\mathcal{F}_t$ -measurable for $t \geq 0$.

The coefficients a , b , and g are linear functionals for any $t \geq 0$ defined on $\mathbb{R}_+ \times D_0$, $\mathbb{R}_+ \times D_0$, and $\mathbb{R}_+ \times D_0 \times \mathbb{R}$, respectively. We treat D_0 as a metric space with the Skorokhod metric ρ_D (see [4, Chapter VI, §5]).

To facilitate the discussion of the behavior of stochastic processes $\{x(t)\} \subset \mathbb{R}$ without discontinuities of the second kind, a simpler metric is often considered (see [3]). This metric is generated by the seminorm

$$(3) \quad \|\varphi\|_* \equiv \left\{ \int_{-\infty}^0 |\varphi(\theta)|^2 \mathbb{K}(d\theta) \right\}^{1/2}$$

where $\mathbb{K}(\cdot)$ is some finite measure defined on the Borel sets of $(-\infty, 0]$, that is,

$$\mathbb{K}(-\infty, 0) = \mathbb{K} < \infty.$$

Definition 1. Let L_t be the σ -algebra of Borel sets on $(-\infty, t]$. A separable stochastic process $\{x(t)\} \subset \mathbb{R}$ defined by relation (2) for $t \in (-\infty, 0]$, measurable with respect to the σ -algebra $\mathcal{F}_t \times L_t$, and satisfying for all $t \geq 0$ the integral Itô-Skorokhod equation

$$(4) \quad x(t) = \varphi(0) + \int_0^t a(s, x_s) ds + \int_0^t b(s, x_s) dw(s) + \int_0^t \int_{\mathbb{R}} g(s, x_s, u) \bar{v}(ds, du)$$

with probability one, is called a solution of the stochastic differential-functional equation (1) with initial condition (2).

We denote by H_0 the space of $\mathcal{F}_0$ -measurable functions $\varphi: (-\infty, 0] \times \Omega \rightarrow \mathbb{R}$ equipped with the norm

$$(5) \quad \|\varphi\|_0 \equiv \left\{ \sup_{-\infty < \theta \leq 0} \mathbb{E} \left\{ |\varphi(\theta)|^2 \right\} \right\}^{1/2}.$$

Assume that $a, b: \mathbb{R}_+ \times D_0 \rightarrow \mathbb{R}$ and $g: \mathbb{R}_+ \times D_0 \times \mathbb{R} \rightarrow \mathbb{R}$ are functionals, measurable with respect to all their arguments and such that

$$(6) \quad |a(t, \varphi)|^2 + |b(t, \varphi)|^2 + \int_{\mathbb{R}} |g(t, \varphi, u)|^2 \Pi(du) \leq L \int_{-\infty}^0 (1 + |\varphi(\theta)|^2) dK(\theta),$$

$$(7) \quad \begin{aligned} & |a(t\varphi) - a(t, \psi)|^2 + |b(t, \varphi) - b(t, \psi)|^2 + \int_{\mathbb{R}} |g(t, \varphi, u) - g(t, \psi, u)|^2 \Pi(du) \\ & \leq L \int_{-\infty}^0 (|\varphi(\theta) - \psi(\theta)|^2) dK(\theta) \end{aligned}$$

for all $t \geq 0$ where $L > 0$ is a constant.

Then according to [4, Chapter II, §1, Theorems 3 and 4] and [3, Theorem 6.2.1], in the space D_0 there exists a unique (up to stochastic equivalence) solution of the problem (1), (2) such that

$$(8) \quad \begin{aligned} & \mathbb{E} \left\{ \sup_{0 \leq s \leq T} |x(s)|^2 / \mathcal{F}_t \right\} \leq A (1 + \|\varphi\|_0^2), \\ & \mathbb{E} \left\{ \sup_{t \leq s \leq t+h} |x(s) - x(t)|^2 / \mathcal{F}_t \right\} \leq B (1 + \|\varphi\|_0^2) h \end{aligned}$$

where $A > 0$ and $B > 0$ are constants depending on $T > 0$, $L > 0$, and $K > 0$.

Denote by $\{h(t, s)\} \subset \mathbb{R}$ the fundamental solution of the deterministic equation

$$(9) \quad dy(t) = a(t, y_t) dt$$

with the initial function η such that $\eta(t) = 0$ for $t < s$ and $\eta(t) = 1$ for $t = s$. Using $\{h(t, s)\}$, the solution of the problem (1), (2) can be rewritten in the integral form [8]:

$$(10) \quad x(t) = y(t) + \int_0^t h(t, s) b(s, x_s) dw(s) + \int_0^t \int_{\mathbb{R}} h(t, s) g(s, x_s, u) \bar{v}(ds, du)$$

where $\{y(t)\}$ is a solution of equation (9) with nonrandom initial function $\{\varphi(t)\}$ (see (2)).

Now we obtain sufficient conditions for the mean square asymptotic stability of the trivial solution of the problem (1), (2).

Theorem 1. Assume that

- 1) the trivial solution of equation (9) is exponentially stable;

2) the fundamental solution of equation (9) is such that

$$(11) \quad d = \overline{\lim}_{t \rightarrow \infty} d(t) = \overline{\lim}_{t \rightarrow \infty} \int_0^t \left[b^2(t, h_t(\theta, s)) + \int_{\mathbb{R}} g^2(t, h_t(\theta, s), u) \Pi(du) \right] ds < 1;$$

3) the functionals $a(t, \cdot)$, $b(t, \cdot)$, and $g(t, \cdot, u)$ are uniformly bounded with respect to t and $u \in \mathbb{R}$ in the norm (5) of the space D_0 where $\Pi(du) = du/|u|^2$.

Then the trivial solution of the problem (1), (2) is mean square asymptotically stable.

Proof. Using equation (10) at the moment $t + \theta$ and applying the linear operators $b(t, \cdot)$ and $g(t, \cdot, u)$ [6] we obtain

$$\begin{aligned} b(t, x_t) &= b(t, y_t) + \int_0^t b(t, h_t(\theta, s)) b(s, x_s) dw(s) \\ &\quad + \int_0^t \int_{\mathbb{R}} b(t, h_t(\theta, s)) g(s, x_s, u) \bar{v}(ds, du), \\ g(t, x_t, u) &= g(t, y_t, u) + \int_0^t g(t, h_t(\theta, s), u) b(s, x_s) dw(s) \\ &\quad + \int_0^t \int_{\mathbb{R}} g(t, h_t(\theta, s), u) g(s, x_s, u) \bar{v}(ds, du). \end{aligned}$$

Now we square both sides of the latter two equations, then take the mathematical expectation ($\mathbb{E}\{\cdot\}$), and use some properties of the stochastic integral [3]. As a result we get

$$\begin{aligned} \mu_b(t) &= b^2(t, y_t) + \int_0^t b^2(t, h_t(\theta, s)) \mu_b(s) ds + \int_0^t \int_{\mathbb{R}} b^2(t, h_t(\theta, s)) \mu_g(s, u) \Pi(du) ds, \\ \int_{\mathbb{R}} \mu_g(t, u) \Pi(du) &= \int_{\mathbb{R}} g^2(t, y_t, u) \Pi(du) + \int_0^t \int_{\mathbb{R}} g^2(t, h_t(\theta, s), u) \mu_b(s) \Pi(du) ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} g^2(t, h_t(\theta, s), u) \mu_g(s, u_1) \Pi(du) \Pi(du_1) ds \end{aligned}$$

where

$$\mu_b(t) \equiv \mathbb{E} \{ b^2(t, x_t) \}, \quad \mu_g(t, u) \equiv \mathbb{E} \{ g^2(t, x_t, u) \}.$$

After the summation of these equations we get

$$(12) \quad m(t) = f(t) + \int_0^t \left[b^2(t, h_t(\theta, s)) + \int_{\mathbb{R}} g^2(t, h_t(\theta, s), u) \Pi(du) \right] m(s) ds$$

where

$$m(t) \equiv \mu_b(t) + \int_{\mathbb{R}} \mu_g(t, u) \Pi(du), \quad f(t) \equiv b^2(t, y_t) + \int_{\mathbb{R}} g^2(t, y_t, u) \Pi(du).$$

According to the definition of sup, for a given $\varepsilon > 0$ we find $T > 0$ such that $\sup_{s \geq T} d(s) \leq d + \varepsilon$ for $t \geq T$. Put

$$M(t, T) \equiv \max_{T \leq s \leq t} m(s).$$

Taking into account (11) we obtain

$$\begin{aligned} M(t, T) &\leq \max_{T \leq \tau \leq t} f(\tau) + \max_{T \leq \tau \leq t} \int_0^T \left[b^2(\tau, h_\tau(\theta, s)) + \int_{\mathbb{R}} g^2(\tau, h_\tau(\theta, s), u) \Pi(du) \right] m(s) ds \\ &\quad + \max_{T \leq \tau \leq t} d(\tau) M(t, T). \end{aligned}$$

The contraction principle [6] implies that

$$\overline{\lim}_{t \rightarrow \infty} M(t, T) < \infty.$$

Relation (11) yields

$$(13) \quad m(t) \leq f(t) + \int_0^{T_1} \left[b^2(t, h_t(\theta, s)) + \int_{\mathbb{R}} g^2(t, h_t(\theta, s), u) \Pi(du) \right] ds \\ + \sup_{\tau \geq T_1} d(\tau) M(t, T_1)$$

for all t and T_1 such that $t \geq T_1$.

The assumptions of Theorem 1 allow one to apply the Lebesgue dominated convergence theorem [6], since $m(t) < \infty$. Thus

$$\overline{\lim}_{t \rightarrow \infty} \left\{ f(t) + \int_0^{T_1} \left[b^2(t, h_t(\theta, s)) + \int_{\mathbb{R}} g^2(t, h_t(\theta, s), u) \Pi(du) \right] ds \right\} = 0.$$

Relation (12) implies that

$$\overline{\lim}_{t \rightarrow \infty} m(t) \leq (d + \varepsilon) \overline{\lim}_{t \rightarrow \infty} M(t, T_1) \leq (d + \varepsilon) \overline{\lim}_{t \rightarrow \infty} m(t).$$

If $\varepsilon > 0$ is sufficiently small, then it follows from (11) that $\overline{\lim}_{t \rightarrow \infty} m(t) = 0$.

Now equation (10) yields

$$\mathbb{E} \{x^2(t)\} = y^2(t) + \int_0^t h^2(t, s) m(s) ds.$$

It is clear that for any $\varepsilon > 0$ there exists $T > 0$ such that $m(t) < \varepsilon$ for all $t \geq T$. Thus assumption 2) of Theorem 1 implies that

$$\mathbb{E} \{x^2(t)\} \leq y^2(t) + \int_0^T h^2(t, s) m(s) ds + p\varepsilon.$$

Now, using assumption 1) of Theorem 1, we pass to the limit in the last inequality and obtain

$$\overline{\lim}_{t \rightarrow \infty} \mathbb{E} \{x^2(t)\} \leq p\varepsilon,$$

which means that $\lim_{t \rightarrow \infty} \mathbb{E} \{x^2(t)\} = 0$.

Note that $\mathbb{E} \{x^2(t)\}$ continuously depends on the initial function $\{\varphi(\theta)\}$, since the operators defined by the integral equation for $\mathbb{E} \{x^2(t)\}$ are bounded (see [6]). This completes the proof of Theorem 1. $\square$

2. THE CRITICAL CASE

The critical case is considered in [1, 2, 8] for the solution of a stochastic differential-functional equation with aftereffect. Consider the problem of the mean square stability of a trivial solution of the stationary stochastic differential-functional equation with unbounded aftereffect:

$$(14) \quad dz(t) = a(z_t) dt + b(z_t) dw(t) + \int g(z_t, u) \bar{v}(dt, du)$$

for the critical case

$$(15) \quad \int_0^\infty \left[b^2(h_t) + \int_{\mathbb{R}} g^2(h_t, u) \Pi(du) \right] dt = 1.$$

It is known [10] that there exists an initial function $\{\varphi(\theta)\}$ for the stochastic differential-functional equation (14) such that

$$\lim_{t \rightarrow \infty} \mathbb{E} \{x^2(t)\} \neq 0 \quad (\neq \infty).$$

Let a stochastic process $x \in \mathbb{R}$ be defined by the equation

$$(16) \quad dx(t) = a(x_t) dt + (1 + \beta(t))b(x_t) dw(t) + \int_{\mathbb{R}} (1 + \gamma(t))g(x_t, u) \bar{v}(dt, du),$$

$$(17) \quad x(\theta) = \varphi(\theta) \quad \text{for all } \theta \in (-\infty, 0],$$

where $\varphi \in D_0$; a , b , and g are functionals defined on D_0 and $D_0 \times \mathbb{R}$, respectively; $\{w(t)\}$ is a homogeneous Wiener process; and $\bar{v}(t, A)$ is a centered Poisson measure with parameter $t\Pi(A)$. We assume that $w(t)$ and $\bar{v}$ are independent.

We denote by $\mathcal{N}_1$ the set of scalar functions $\{\alpha(t)\}$ continuous on $[0, \infty)$ and such that $\int_0^\infty |2\alpha(t) + \alpha^2(t)| dt < \infty$, and by $\mathcal{N}_2$, the set of scalar functions $\Delta(t) \geq 0$ continuous on $[0, \infty)$ and such that $\int_0^\infty (2\Delta(t) + \Delta^2(t)) dt = \infty$.

Theorem 2. *Assume that*

- 1) *the trivial solution of equation (9) is exponentially stable;*
- 2) *condition (15) holds.*

Then the trivial solution of the stochastic differential-functional equation (15) is

- I) *mean square stable if $\alpha(t) \equiv \max\{\beta(t), \gamma(t)\} \in \mathcal{N}_1$ is decreasing;*
- II) *mean square unstable if $\alpha(t) \equiv \min\{\beta(t), \gamma(t)\} \in \mathcal{N}_2$ is increasing.*

The symbol $\max\{\beta(t), \gamma(t)\}$ stands for the supremum of functions $\{\alpha(t)\}$ such that

$$\alpha(t) \geq \beta(t) \quad \text{and} \quad \alpha(t) \geq \gamma(t) \quad \text{for all } t \geq 0,$$

while $\min\{\beta(t), \gamma(t)\}$ stands for the infimum of functions $\{\alpha(t)\}$ such that

$$\bar{\alpha}(t) \leq \beta(t) \quad \text{and} \quad \bar{\alpha}(t) \leq \gamma(t) \quad \text{for all } t \geq 0.$$

Proof. Let $\beta \in \mathcal{N}_1 \vee \mathcal{N}_2$. If $\{h(t)\}$ is the fundamental solution of equation (9), then a solution of problem (16), (17) can be represented in the form

$$(18) \quad \begin{aligned} x(t) = y(t) &+ \int_0^t h(t-s)(1 + \beta(s))b(x_s) dw(s) \\ &+ \int_0^t \int_{\mathbb{R}} h(t-s)(1 + \gamma(s))g(x_s, u) \bar{v}(ds, du) \end{aligned}$$

(see [1]) where $\{y(t)\}$ is a solution of stationary equation (9) constructed for initial condition (17).

We apply the operators $(1 + \beta(t))b(\cdot)$ and $(1 + \gamma(t))g(\cdot, u)$ to both sides of equation (16). Then we proceed in the way that led to equation (11) and obtain for all $t \geq 0$ that

$$(19) \quad m(t) = f(t) + \int_0^t \left[(1 + \beta(t))^2 b^2(h_{t-s}) + (1 + \gamma(t))^2 \int_{\mathbb{R}} g^2(h_{t-s}, u) \Pi(du) \right] m(s) ds$$

where

$$m(t) \equiv (1 + \beta(t))^2 \mu_b(t) + (1 + \gamma(t))^2 \int_{\mathbb{R}} \mu_g(t, u) \Pi(du);$$

$$\mu_b(t) \equiv \mathbb{E} \{b^2(x_t)\}; \quad \mu_g(t, u) \equiv \mathbb{E} \{g^2(x_t, u)\};$$

$$f(t) \equiv (1 + \beta(t))^2 b^2(y_t) + (1 + \gamma(t))^2 \int_{\mathbb{R}} g^2(y_t, u) \Pi(du).$$

I) Let $\beta \in \mathcal{N}_1$ and $\gamma \in \mathcal{N}_1$ be decreasing. We check that $\{m(t)\}$ is bounded. First we show that $\{\overline{m}(t)\}$ is bounded and is a solution of the equation

$$(20) \quad \overline{m}(t) = f(t) + \int_0^t (1 + \alpha(s))^2 k(t-s) \overline{m}(s) ds$$

where $\alpha(t) \equiv \max\{\beta(t), \gamma(t)\}$ for all $t \geq 0$, and $k(t) \equiv b^2(h_t) + \int_{\mathbb{R}} g^2(h_t, u) \Pi(du)$. It is clear that solutions of equations (19) and (20) are such that $m(t) \leq \overline{m}(t)$. Further, in view of the first assumption of Theorem 2 the functions $\{f(t)\}$ and $\{k(t)\}$ are such that

$$(21) \quad f(t) + k(t) \leq N e^{-\varepsilon t} \quad \text{for all } t \geq 0$$

where $N > 0$ and $\varepsilon > 0$ are some constants.

Applying the Laplace transform we rewrite equation (17) as follows:

$$(22) \quad \overline{m}(t) = f(t) + \int_0^t H(t-s) H(t-s) f(s) ds + \int_0^t H(t-s) \tau(s) \overline{m}(s) ds$$

(see [7]) where $\tau(s) = 2\alpha(s) + \alpha^2(s)$,

$$H(t) \equiv k(t) + \int_0^t k(t-s) k(s) ds + \int_0^t \int_0^s k(t-s) k(s-s_1) k(s_1) ds ds_1 + \dots$$

Indeed, applying the Laplace transform to equation (22) we obtain

$$M(\lambda) = F(\lambda) + \overline{K}(\lambda) M(\lambda) + \overline{K}(\lambda) L \{\tau(s) \overline{m}(s)\}$$

(see [7]) where $\overline{K}$ and $\overline{m}$ are the Laplace transforms of K and m , respectively. Thus

$$M(\lambda) = \frac{F(\lambda)}{1 - \overline{K}(\lambda)} + \frac{\overline{K}(\lambda)}{1 - \overline{K}(\lambda)} L \{\tau(s) \overline{m}(s)\}.$$

Expanding the fraction $1/(1 - \overline{K}(\lambda))$ into the series for $\lambda \neq 0$ we obtain from assumption 2) of Theorem 2 that

$$M(\lambda) = F(\lambda) + F(\lambda) [\overline{K}(\lambda) + \overline{K}^2(\lambda) + \dots] + L \{\tau(s) \overline{m}(s)\} [\overline{K}(\lambda) + \overline{K}^2(\lambda) + \dots].$$

Applying the inverse Laplace transform we reduce $\overline{m}(t)$ to the right-hand side of (22).

Since $\int_0^\infty k(t) dt = 1$, we apply the Laplace transform and show that $H(t) = C + \delta(t)$ where $C = \text{const} > 0$ and $\{\delta(t)\}$ is a continuous function on $[0, \infty)$ such that

$$(23) \quad \lim_{t \rightarrow \infty} \delta(t) = 0.$$

Then

$$\overline{m}(t) \leq B + A \int_0^t |\tau(s)| \overline{m}(s) ds \quad \text{for all } t \geq 0$$

where $B \equiv \sup_{t \geq 0} \int_0^t H(t-s) f(s) ds \in (0, \infty)$ and $A \equiv \sup_{t \geq 0} H(t) \in (0, \infty)$. It follows from the Gronwall–Bellman lemma [4] that

$$\overline{m}(t) \leq B \exp \left\{ A \int_0^t |\tau(s)| ds \right\} \quad \text{for all } t \geq 0.$$

Let $\alpha \in \mathcal{N}_1$. Then

$$\int_0^\infty |\tau(s)| ds < \infty,$$

whence $\sup_{t \geq 0} \overline{m}(t) < +\infty$ and $\sup_{t \geq 0} m(t) < +\infty$. Squaring both sides of equation (16) and applying the operator of the mathematical expectation we get

$$(24) \quad \mu(t) = y^2(t) + \int_0^t h^2(t-s) m(s) ds \quad \text{for all } t \geq 0$$

where $\mu(t) \equiv \mathbb{E} \{x^2(t)\}$ and $\{m(t)\}$ is the function defined above. Note that

$$(25) \quad y^2(t) + h^2(t) \leq \mathbb{E} \exp(-\varepsilon_1 t) \quad \text{for all } t \geq 0$$

for some positive constants $M > 0$ and $\varepsilon_1 > 0$.

According to (20), $\sup_{t \geq 0} m(t) < \infty$ and $\int_0^\infty |\tau(s)| ds < \infty$. Relation (24) implies that $\sup_{t \geq 0} \mu(t) < \infty$. This completes the proof of the first part of the theorem, since the function $\varphi \in D_0$ is arbitrary and the mean square stability of solutions of linear systems is equivalent to the mean square boundedness of every solution (see [2]).

II) Consider the increasing function $\bar{\alpha}(t) \equiv \min \{\beta(t), \gamma(t)\} \in \mathcal{N}_2$. We show that the function $\{m(t)\}$ defined by equation (19) is unbounded. First we check that the function $\bar{m}(t) \leq m(t)$ is unbounded and is a solution of the equation

$$\bar{m}(t) = f(t) + \int_0^t (1 + \bar{\alpha}(s))^2 k(t-s) \bar{m}(s) ds$$

or, equivalently, of the equation

$$(26) \quad \bar{m}(t) = f(t) + \int_0^t H(t-s) f(s) ds + \int_0^t H(t-s) \tau(s) \bar{m}(s) ds$$

where $\{f(t)\}$, $\{\tau(t)\}$, $\{k(t)\}$, and $\{H(t)\}$ are defined above.

Using assumption 2) we first consider the initial deterministic function $\varphi \in D_0$ for the solution $\{y(t)\}$ of equation (9) such that

$$(27) \quad \int_0^\infty f(t) dt > 0.$$

Consider equation (26). Using (18), (19), and (27) we get

$$\lim_{t \rightarrow \infty} \int_0^t H(t-s) f(s) ds > 0,$$

whence

$$(28) \quad \lim_{t \rightarrow \infty} \int_0^t H(t-s) \tau(s) \left[\int_0^s H(s-s_1) f(s_1) ds_1 \right] ds = \infty,$$

since $\int_0^\infty \tau(s) ds = \infty$ (the latter integral is infinite in view of $\bar{\alpha} \in \mathcal{N}_2$).

Taking (27) and integral equality (26) into account we obtain

$$\bar{m}(t) \geq \int_0^t H(t-s) f(s) ds + \int_0^t H(t-s) \tau(s) \left[\int_0^s H(s-s_1) f(s_1) ds_1 \right] ds$$

for all $t \geq 0$.

Then it follows from (28) that $\lim_{t \rightarrow \infty} \bar{m}(t) = \infty$. Hence $\lim_{t \rightarrow \infty} m(t) = \infty$, since $\bar{m} \leq m(t)$. Relation (24) leads to $\lim_{t \rightarrow \infty} \mu(t) = \infty$, which completes the proof of part 2) of Theorem 2. $\square$

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