AN ESTIMATE OF RUIN PROBABILITIES FOR LONG RANGE DEPENDENCE MODELS

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ABSTRACT. We obtain an estimate of the ruin probability for the case where a company buys a risky asset (security) that is a semimartingale with absolutely continuous characteristics with respect to the Lebesgue measure. This result is applied to some long range dependence models, in particular to the fractional Brownian motion and mixed models.

1. INTRODUCTION

Consider the model for the risk process of an insurance company whose portfolio consists of two components. The first component R_1 is classical, namely

$$R_1(t) = x + ct - \sum_{i=1}^{N_t} X_i,$$

where $x \geq 0$ is the initial capital (reserve) of the insurance company, the constant $c \geq 0$ is the premium rate, X_i , $i \geq 1$, are independent identically distributed random variables (they model the payoff process), N_t is a Poisson process, independent of X_i ; the intensity of N_t is λ . The second component reflects the assumption that the company may buy a security whose price is S_t (a random variable that is positive with probability one) and for which there exists (in a certain sense) a stochastic differential (the precise description of R_2 is given below). The company may, of course, buy a nonrisky asset with a fixed interest rate. However, this option is nonessential for our approach; thus we assume that there is no nonrisky asset in the market. Let $\hat{\varphi}_t$, $t \geq 0$, be the amount of money invested by the insurance company to the risky asset at the moment t . Consider the filtration

$$\mathcal{F}_t = \sigma \{R_1(u), S_u, 0 \leq u \leq t\}$$

and assume that $\hat{\varphi}_t$ is an $\mathcal{F}_t$ -predictable process for which there exists the stochastic integral

$$R_2(t) = \int_0^t \hat{\varphi}_s dS_s, \quad t \geq 0.$$

The definition and construction of this integral will be given below. Thus the capital of the company is equal to

$$(1) \quad Y(t, x, \hat{\varphi}) = R_1(t) + R_2(t) = x + ct - \sum_{i=1}^{N_t} X_i + \int_0^t \hat{\varphi}_s dS_s.$$

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For the sake of convenience we consider $\varphi_s = \hat{\varphi}_s S_s$ instead of $\hat{\varphi}_s$. If $\hat{\varphi}_s$ is an $\mathcal{F}_s$ -predictable process and S_s is a continuous process, then φ_s is an $\mathcal{F}_s$ -predictable process. The corresponding capital is denoted by $Y(t, x, \varphi)$. Our aim is to consider models for the risky asset with the long range dependence structure and obtain estimates of the corresponding ruin probabilities.

Section 2 is devoted to the general semimartingale model of the risky asset whose components have absolutely continuous characteristics with respect to the Lebesgue measure. Estimates of the ruin probability and optimal strategy φ_s , $s \geq 0$, are also found. A particular case of the model is considered in [1] for the risky asset described by the geometric Brownian motion, that is,

$$dS_t = S_t(a dt + b dW_t).$$

Section 3 is devoted to semimartingale long range dependence models of the risky asset. We apply the results of Section 2 to this case.

Section 4 deals with the case where the model of the risky asset is not a semimartingale. In particular, we consider the fractional Brownian motion or mixed fractional Brownian motion for some values of the Hurst parameter. Since the zero strategy is optimal in both cases, we consider the approximations of these models for which better estimates of the ruin probability are available as compared to the zero strategy φ_s .

2. AN ESTIMATE OF THE RUIN PROBABILITY AND THE OPTIMAL STRATEGY IN THE GENERAL SEMIMARTINGALE MODEL

Consider a complete probability space with a filtration

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$$

and assume that all the processes are adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$. Let the risky asset

$$S = S_t, \quad t \geq 0,$$

in representation (1) be a semimartingale such that

$$(2) \quad dS_t = S_t(dM_t + dA_t), \quad t \geq 0,$$

where M_t is a square integrable continuous martingale whose square characteristic $\langle M \rangle_t$ is absolutely continuous with respect to the Lebesgue measure, that is,

$$\langle M \rangle_t = \int_0^t \beta_s ds, \quad \int_0^t |\beta_s| ds < \infty, \quad t > 0, \quad \text{a.s.}$$

We also assume that the variation of the process A_t is integrable, namely,

$$A_t = \int_0^t \alpha_s ds, \quad \int_0^t |\alpha_s| ds < \infty, \quad t > 0, \quad \text{a.s.}$$

Note that all the results of this section remain true if we assume that

$$\mathcal{F}_t = \sigma \{M_s, A_s, N_s, R_1(s), 0 \leq s \leq t\}.$$

The portfolio corresponding to an $\mathcal{F}_s$ -predictable strategy φ_s is of the form

$$\int_0^t \varphi_s dM_s + \int_0^t \varphi_s dA_s.$$

Assume that

$$(A) \quad \mathbb{E} \int_0^t \varphi_s^2 \beta_s ds < \infty, \quad \int_0^t |\varphi_s \alpha_s| ds < \infty.$$

Then

$$\int_0^t \varphi_s dM_s$$

is the Itô integral with respect to a square integrable martingale, and thus this integral itself is a square integrable martingale;

$$\int_0^t \varphi_s dA_s$$

is the Lebesgue–Stieltjes integral. Therefore the portfolio of the insurance company is given by

$$(3) \quad Y(t, x, \varphi) = x + ct - \sum_{i=1}^{N_t} X_i + \int_0^t \varphi_s dM_s + \int_0^t \varphi_s dA_s.$$

Throughout the paper we assume that the processes M , A , and φ do not depend on X_i , $i \geq 1$, and N .

Consider the positive stochastic process

$$\mathcal{Z}(t, x, \varphi, r) = \exp \{-rY(t, x, \varphi)\}, \quad r > 0.$$

The problem is to find conditions under which $\mathcal{Z}(\cdot, x, \varphi, r)$ has either super- or sub-martingale properties. Put

$$h(r) = \mathbb{E} \exp \{rX_1\} - 1, \\ f(t, \varphi, r) = \lambda h(r) - cr + \frac{r^2}{2} \varphi_t^2 \beta_t - r\varphi_t \alpha_t.$$

Theorem 1. *Let*

$$(B) \quad \begin{aligned} &1) \quad \mathbb{E} \exp \{rX_1\} < \infty; \\ &2) \quad f(t, \varphi, r) \leq 0, \quad t \geq 0, \text{ a.s.}; \\ &3) \quad \mathbb{E} \exp \left\{ \frac{r^2}{2} \int_0^t \varphi_s^2 \beta_s ds \right\} < \infty, \quad t \geq 0, \end{aligned}$$

for some $r > 0$ and some strategy $\{\varphi_t, t \geq 0\}$. Then the process $\{\mathcal{Z}(t, x, \varphi, r), t \geq 0\}$ is an $\mathcal{F}_t$ -supermartingale. If inequality (B), 2) becomes an equality, then the process $\{\mathcal{Z}(t, x, \varphi, r), t \geq 0\}$ is an $\mathcal{F}_t$ -martingale.

Proof. Assumption (B), 3) is, in fact, the Novikov condition. It implies that

$$\mathbb{E} \exp \left\{ -r \int_0^t \varphi_s dM_s - \frac{r^2}{2} \int_0^t \varphi_s^2 \beta_s ds \right\} = 1, \quad t \geq 0.$$

Moreover, the process

$$P_t := \exp \left\{ -r \int_0^t \varphi_s dM_s - \frac{r^2}{2} \int_0^t \varphi_s^2 \beta_s ds \right\}$$

is a martingale under this condition. It follows from assumption (B), 1) that

$$\mathbb{E} \exp \left\{ r \sum_{i=1}^{N_t} X_i \right\} = \exp \{ \lambda h(r)t \},$$

whence

$$\begin{aligned}
\mathbb{E} \mathcal{Z}(t, 0, \varphi, r) &= e^{-rct} \mathbb{E} \exp \left\{ r \sum_{i=1}^{N_t} X_i \right\} \mathbb{E} \exp \left\{ -r \int_0^t \varphi_s dM_s - r \int_0^t \varphi_s \alpha_s ds \right\} \\
&= \exp \{ -rct + h(r)\lambda t \} \mathbb{E} \exp \left\{ -r \int_0^t \varphi_s dM_s - \frac{r^2}{2} \int_0^t \varphi_s^2 \beta_s ds \right\} \\
&\quad \times \exp \left\{ \frac{r^2}{2} \int_0^t \varphi_s^2 \beta_s ds - r \int_0^t \varphi_s \alpha_s ds \right\} \\
&\leq 1
\end{aligned}$$

for all $t > 0$ in view of representation (2), assumption (B), 2), and since $\{X_i, i \geq 1, N\}$ and $\{M, A, \varphi\}$ are independent. Thus $\mathcal{Z}(t, x, \varphi, r)$ is an integrable process. Since

$$\sum_{i=N_{t_1}+1}^{N_{t_2}} X_i$$

is independent of $\mathcal{F}_{t_1}$ and (M, A, φ) , we have for $0 \leq t_1 < t_2$ that

$$\begin{aligned}
&\mathbb{E} [\mathcal{Z}(t_2, x, \varphi, r) / \mathcal{F}_{t_1}] \\
&= \mathcal{Z}(t_1, x, \varphi, r) \exp \{ -rc(t_2 - t_1) \} \\
&\quad \times \mathbb{E} \left[\exp \left\{ -r \int_{t_1}^{t_2} \varphi_s dM_s - \frac{r^2}{2} \int_{t_1}^{t_2} \varphi_s^2 \beta_s ds \right\} \right. \\
&\quad \left. \times \exp \left\{ \frac{r^2}{2} \int_{t_1}^{t_2} \varphi_s^2 \beta_s ds - r \int_{t_1}^{t_2} \varphi_s \alpha_s ds \right\} / \mathcal{F}_{t_1} \right] \\
&= \mathcal{Z}(t_1, x, \varphi, r) \\
&\quad \times \mathbb{E} \left[\exp \left\{ -r \int_{t_1}^{t_2} \varphi_s dM_s - \frac{r^2}{2} \int_{t_1}^{t_2} \varphi_s^2 \beta_s ds \right\} \exp \left\{ \int_{t_1}^{t_2} f(s, \varphi, r) ds \right\} / \mathcal{F}_{t_1} \right] \\
&\leq \mathcal{Z}(t_1, x, \varphi, r) \\
&\quad \times \mathbb{E} \left[\exp \left\{ -r \int_{t_1}^{t_2} \varphi_s dM_s - \frac{r^2}{2} \int_{t_1}^{t_2} \varphi_s^2 \beta_s ds \right\} / \mathcal{F}_{t_1} \right] = \mathcal{Z}(t_1, x, \varphi, r)
\end{aligned}$$

according to assumptions (B), 2) and (B), 3). $\square$

Remark 1. It will be seen from the further estimates for the ruin probability that one should choose the parameter r as large as possible. One of the restrictions is that r should be greater than the (unique) root of the equation $\lambda h(r) = rc$, since the estimate involving this root can be obtained for $\varphi_s \equiv 0$ (that is, if no money is invested in the risky asset).

Now we determine the strategies $\{\varphi_s, s \geq 0\}$ for which conditions (B), 2) and (B), 3) hold and find the maximal possible value of the parameter $r > 0$.

Theorem 2. *Let the functions α and β be such that*

- 1) $\beta(s) > 0, s \geq 0$;
- (C) 2) *there exists $c_1 > 0$ such that $\operatorname{ess\,inf}_{s \geq 0} \frac{\alpha_s^2}{2\beta_s} \geq c_1$ almost surely,*

where $\operatorname{ess\,inf}_{s \geq 0} = \sup_{A: \lambda(A)=0} \inf_{s \in \mathbb{R} \setminus A}$ and λ is the Lebesgue measure on $\mathbb{R}_+$. Then there exists a nonzero predictable strategy $\{\varphi_s, s \geq 0\}$ such that assumptions (B), 2) and (B), 3) hold. Moreover one can put $r = \hat{r}$ where $\hat{r}$ is a unique root of the equation $\lambda h(r) = cr + c_1$.

Proof. First we choose a strategy to satisfy condition (B), 2); that is, we choose φ_s such that $f(t, \varphi, \hat{r}) \leq 0$. The latter inequality is equivalent to

$$(4) \quad \frac{\hat{r}^2}{2} \varphi_s^2 \beta_s - \hat{r} \varphi_s \alpha_s + c_1 \leq 0, \quad \text{where } c_1 = \lambda h(\hat{r}) - c\hat{r}.$$

The discriminant of the left-hand side of (3) is equal to $D(s, \omega) = \hat{r}^2 \alpha_s^2 - 2\hat{r}^2 c_1 \beta_s \geq 0$ for $\alpha_s^2 / \beta_s \geq c_1$. Thus $D(s, \omega) \geq 0 \pmod{\lambda \times P}$ if $\text{ess inf}_{s \geq 0} \alpha_s^2 / (2\beta_s) \geq c_1 > 0$. Then we choose

$$\varphi_s \in \left[\frac{\alpha_s - \sqrt{\alpha_s^2 - 2c_1 \beta_s}}{\hat{r} \beta_s}, \frac{\alpha_s + \sqrt{\alpha_s^2 - 2c_1 \beta_s}}{\hat{r} \beta_s} \right]$$

for s such that $D(s, \omega) \geq 0$ and $\varphi_s = 0$ for other points. Note that assumption (B), 2) is satisfied for such a choice. To satisfy condition (B), 3) put

$$(5) \quad \varphi_s = \frac{\alpha_s - \sqrt{\alpha_s^2 - 2c_1 \beta_s}}{\hat{r} \beta_s} = \frac{2c_1}{\hat{r}(\alpha_s^2 - 2c_1 \beta_s)}.$$

Then

$$\hat{r} \varphi_s \alpha_s - c_1 = \frac{2c_1 \alpha_s}{\alpha_s + \sqrt{\alpha_s^2 - 2c_1 \beta_s}} - c_1 \leq c_1,$$

and inequality (4) implies that

$$\frac{\hat{r}^2}{2} \varphi_s^2 \beta_s \leq \hat{r} \varphi_s \alpha_s - c_1 \leq c_1,$$

whence (B), 3) follows. $\square$

Remark 2. Since $h(0) = 0$ and the function $h(r)$ is convex and increasing, the equation $\lambda h(r) - cr = c_1$ has a unique root $\hat{r} = \hat{r}(c_1)$ for $c_1 > 0$; $\hat{r}(c_1)$ is increasing with respect to c_1 . We noticed in Remark 1 that $\hat{r}$ should be chosen as large as possible; that is, we should choose the largest possible number c_1 . This procedure is described in Theorem 2. If $\text{ess inf}_{s \geq 0} \alpha_s^2 / (2\beta_s) = 0$ with a positive probability, then c_1 , as well as the root $\hat{r} = \hat{r}(0)$ of the equation $\lambda h(\hat{r}) - c\hat{r} = 0$ considered in Theorem 3, does not give any additional “gain” as compared to the zero strategy φ_s .

Now let $\tau_x^\varphi = \inf\{t > 0: Y(t, x, \varphi) \leq 0\}$ be the ruin moment for an insurance company.

Theorem 3. *Let assumptions (C) of Theorem 2 hold. Assume that the strategy φ_s is chosen according to (5). Then*

$$P\{\tau_x^\varphi < \infty\} \leq e^{-\hat{r}x},$$

where $\hat{r} = \hat{r}(c_1)$ is a unique root of the equation $\lambda h(\hat{r}) = c\hat{r} + c_1$.

Remark 3. Putting $M_t = bW_t$ and $A_t = at$ we obtain a result proved in the paper [1]:

$$\frac{d_s^2}{2\beta_s} = \frac{a^2}{2b^2}$$

and $P\{\tau_x^\varphi < \infty\} = e^{-\hat{r}x}$, where $\hat{r}$ is a unique root of the equation

$$\lambda h(r) = cr + \frac{a^2}{2b^2}, \quad \varphi_s = \frac{\alpha_s}{\hat{r} \beta_s} = \frac{a}{\hat{r} b^2}.$$

Proof. Put $\tilde{Z}(t, x, \varphi, \hat{r}) = Z(t \wedge \tau_x, x, \varphi, \hat{r})$. According to Theorem 1, $\tilde{Z}$ is a supermartingale with respect to the filtration $(\mathcal{F}_t, t \geq 0)$. Then

$$\begin{aligned} e^{-\hat{r}x} &= \tilde{Z}(0, x, \varphi, \hat{r}) \\ &\geq E \tilde{Z}(t, x, \varphi, \hat{r}) = E Z(\tau_x, x, \varphi, \hat{r}) I\{\tau_x < t\} + E Z(t, x, \varphi, \hat{r}) I\{\tau_x \geq t\} \\ &\geq E Z(\tau_x, x, \varphi, \hat{r}) I\{\tau_x < t\} \end{aligned}$$

for $t \geq 0$. On the other hand,

$$\lim_{t \rightarrow \infty} \mathbb{E} \mathcal{Z}(\tau_x, x, \varphi, \hat{r}) I\{\tau_x < t\} = \mathbb{E} \mathcal{Z}(\tau_x, x, \varphi, \hat{r}) I\{\tau_x < \infty\}.$$

Thus

$$e^{-\hat{r}x} \geq \mathbb{E} [\mathcal{Z}(\tau_x, x, \varphi, \hat{r}) / \tau_x < \infty] \mathbb{P}\{\tau_x < \infty\},$$

whence

$$\mathbb{P}\{\tau_x < \infty\} \leq \frac{e^{-\hat{r}x}}{\mathbb{E}[\mathcal{Z}(\tau_x, x, \varphi, \hat{r}) / \tau_x < \infty]} \leq e^{-\hat{r}x}. \quad \square$$

Remark 4. Similarly to Theorems 4 and 6 of [1] one can prove the following result. Let a random variable X_1 be such that

$$(6) \quad p := \sup_{y \geq 0} \mathbb{E} \left[e^{-r(y-X_1)} / X_1 > y \right] < \infty$$

and let a strategy φ_s be such that $\frac{1}{2}r^2\varphi_s^2\beta_s - r\varphi_s\alpha_s + \lambda h(r) - cr \geq 0$ for all $s \geq 0$ and some $r > 0$. Then the process $\tilde{\mathcal{Z}}(t, x, \varphi, r)$ is a uniformly integrable submartingale and $\mathbb{P}\{\tau_x^\varphi < \infty\} \geq Ce^{-rx}$, where $C = p^{-1} > 0$. This explains why assumptions (B), 2) and B), 3) determine the optimal strategy. Moreover if inequality (6) holds, then the stopped process $\tilde{Y}(t, x, \varphi) := Y(t \wedge \tau_x^\varphi, x, \varphi)$ converges to ∞ on the set $\{\tau_x^\varphi = \infty\}$ as $t \rightarrow \infty$ whatever strategy φ is.

3. AN ESTIMATE OF THE RUIN PROBABILITY FOR SEMIMARTINGALE MODELS WITH LONG RANGE DEPENDENCE

3.1. A wide class of processes of the semimartingale type with long range dependence structure can be constructed as the convolution of some smooth kernel with the Wiener process. We consider a slightly modified example of such processes introduced in the paper [2]; namely, we consider the processes of the following form:

$$R_t^\zeta = \int_0^t \zeta(t-s) dW_s, \quad t \geq 0,$$

where W_t is a Wiener process; $\zeta: \mathbb{R} \rightarrow \mathbb{R}$, $\zeta(x) = 0$ for $x < 0$, and $\zeta(x) = \zeta(0) + \int_0^x \psi(y) dy$ for $x \geq 0$; $\psi \in L^2(\mathbb{R}_+)$; $\zeta(0) \neq 0$. Then

$$(7) \quad R_t^\zeta = \zeta(0)W_t + \int_0^t \int_0^s \psi(s-u) dW_u ds.$$

This result can be proved by using the stochastic variant of the Fubini theorem. Equality (7) is the canonical representation of the long range dependence semimartingale R_t^ζ (see [2]).

Now let the dynamics of the price of the risky asset be described by the equation

$$(8) \quad dS_t = S_t(dR_t^\zeta + c_2 dt), \quad \text{where } c_2 > 0.$$

Using the notation of Section 1, $\alpha_s = \int_0^s \psi(s-u) dW_u + c$ and $\beta_s = \zeta^2(0)$. Let us check the main assumption (C), 2). Since the stochastic process $\alpha_s = \int_0^s \psi(s-u) dW_u + c$ is Gaussian, $\text{ess inf}_{s \geq 0} \alpha^2(s) / (2\beta_s) = 0$ with positive probability. Thus the zero strategy φ_s is optimal in this case.

Now we consider some modifications of model (8).

(a) We substitute the stochastic process

$$\alpha_s^1 = \left(\int_0^s \psi(s-u) dW_u \right) \wedge (c_1 - c_2) + c_2$$

for α_s where $c_1 > 0$ is some constant. Then $\mathbb{P}\{\tau_x^\varphi < \infty\} \leq e^{-\hat{r}x}$ according to Theorems 2 and 3, where $\hat{r}$ is a unique root of the equation

$$\lambda h(r) = cr + \frac{c_1^2}{2\zeta^2(0)},$$

and

$$\varphi_s = \frac{\alpha_s^1 - \sqrt{(\alpha_s^1)^2 - 2c_1\zeta^2(0)}}{\hat{r}\zeta^2(0)}.$$

(b) We consider the stochastic process $\alpha_s^2 = \int_0^s \psi(s-u) dB_u$ instead of α_s , where B_s is a nondecreasing process, $B_0 \geq 0$, $\psi \geq 0$, and the above integral exists for all $s \geq 0$. Let, for example,

$$\alpha_s^2 = \int_0^s \psi(s-u) |W_u| du.$$

It is obvious that the corresponding process

$$R_t := \alpha W_t + \int_0^t \int_0^s \psi(s-u) |W_u| du ds, \quad \alpha \neq 0,$$

is a long range dependence process and if, similar to (8), the price for the risky asset changes according to $dS_t = S_t(dR_t + c_3 dt)$, then $\mathbb{P}\{\tau_x^\varphi < \infty\} \leq e^{-\hat{r}x}$, where $\hat{r}$ is a unique root of the equation $\lambda h(r) = cr + c_3^2/(2\alpha^2)$.

3.2. Another class of models for the price of the risky asset with long range dependence is represented by mixed models of the following form:

$$dS_t = S_t \left(d\mathbf{M}_t^{\beta,H} + \alpha dt \right),$$

where $\mathbf{M}_t^{\beta,H} = W_t + \beta W_t^H$, $\beta \neq 0$, and W_t^H is an independent of W_t fractional Brownian motion with the Hurst index $H \in (\frac{1}{2}, 1)$, that is, a Gaussian process with stationary increments and such that

$$W_0^H = 0, \quad \mathbb{E} W_t^H = 0, \quad \mathbb{E} W_t^H W_s^H = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}).$$

The process W_t^H with $H > \frac{1}{2}$ is called a process with long memory, since

$$\mathbb{E} W_t^H (W_{t+k}^H - W_k^H) \sim k^{2H-2}, \quad t \rightarrow \infty,$$

and $\sum_{k=1}^\infty k^{2H-2} = \infty$. In this case, $S_t = S_0 \exp\{W_t + \alpha W_t^H - \frac{1}{2}t\}$, $\mathbb{E} S_t^2 < \infty$, and

$$\int_0^t S_s d\mathbf{M}_s^{\beta,H} = \int_0^t S_s dW_s + \beta \int_0^t S_s dW_s^H,$$

where $\int_0^t S_s dW_s$ is the Itô stochastic integral that is a square integrable martingale, and $\int_0^t S_s dW_s^H$ is the pathwise integral defined as the limit of integral Riemann–Stieltjes sums (see [3]).

According to Theorems 4, 2, and 4.15 of [2], for all $H \in (\frac{3}{4}, 1)$, $\alpha \in \mathbb{R} \setminus \{0\}$, and $T > 0$ there exists a Volterra kernel $r = r(t, s) \in L^2[0, T]^2$ and a Wiener process B_t , $t \in [0, T]$, in the space $(\Omega, \mathcal{F}, \mathbb{P})$ such that B is linearly represented via $\mathbf{M}^{H,\beta}$, does not depend on $(N, X_i, i \geq 1)$, and

$$\mathbf{M}^{H,\beta} = B_t + \int_0^t \int_0^s r(s, u) dB_u ds.$$

Therefore this model is similar to the model considered in Section 3.1. Further reasoning is the same (with obvious changes) as that used for the cases (a) and (b) of Section 3.1.

4. FRACTIONAL BROWNIAN MODEL AND MIXED MODEL
FOR THE PRICE OF A SECURITY OF A NONSEMIMARTINGALE TYPE

First we consider the model

$$dS_t = S_t (b dW_t^H + a dt), \quad H \in \left(\frac{1}{2}, 1\right).$$

Then $S_t = S_0 \exp\{at + bW_t^H\}$. It is known that there exists an arbitrage strategy for this model (see [2, 4]). The strategies obtained in the papers [2, 4] allow one to claim that

$$\int_0^t \varphi_s (a ds + b dW_s^H) \geq 0 \quad \text{a.s.}$$

The latter inequality does not help to improve an estimate for the ruin probability as compared to the zero strategy $\varphi_s \equiv 0$. The same reasoning concerns the mixed model

$$dS_t = S_t (b_1 dW_t + b_2 dW_t^H + a dt).$$

This model is not a semimartingale for $b_2 \neq 0$ and $H \in (\frac{1}{2}, \frac{3}{4}]$. Consider the following approximation of the process W_t^H . Following [4], we represent the process W_t^H as follows:

$$(9) \quad W_t^H = \int_0^t s^{H-1/2} dY_s,$$

where

$$Y_t = c_H \int_0^t (t-s)^{H-1/2} s^{1/2-H} dW_s = c_H \int_0^t \left[\int_s^t (u-s)^{H-3/2} du \right] s^{1/2-H} dW_s$$

and $c_H > 0$ is a constant. A formal application of the Fubini theorem to the right-hand side of (9) yields

$$Y_t = c_H \int_0^t \left[\int_0^u (u-s)^{H-3/2} s^{1/2-H} dW_s \right] du.$$

The inner integral in the latter equality does not exist, since

$$\int_0^u (u-s)^{2H-3} s^{1-2H} ds = \infty.$$

Thus we consider the following approximation for $0 < \alpha < 1$:

$$Y_t^\alpha = c_H \int_0^t \left[\int_0^{\alpha u} (u-s)^{H-3/2} s^{1/2-H} dW_s \right] du.$$

Then

$$W_t^{H,\alpha} := \int_0^t s^{H-1/2} dY_s^\alpha = c_H \int_0^t s^{H-1/2} \int_0^{\alpha s} (s-v)^{H-3/2} v^{1/2-H} dW_v ds$$

is an approximation of the fractional Brownian motion: $\mathbb{E}(W_t^H - W_t^{H,\alpha})^2 \rightarrow 0$, $\alpha \rightarrow 1$.

Consider the following model for the price of the risky asset:

$$dS_t = S_t (b dW_t^{H,\alpha} + c_4 dt).$$

Since $W_t^{H,\alpha}$ is a continuous process of bounded variation, we construct the strategy

$$\begin{aligned} \varphi_t^K &= K t^{1/2-H} \left(\int_0^{\alpha t} (t-z)^{H-3/2} z^{1/2-H} dW_z + c_4 \right)^{-1} \\ &\times I \left\{ \int_0^{\alpha t} (t-z)^{H-3/2} z^{1/2-H} dW_z + c_4 \neq 0 \right\}, \quad K > 0. \end{aligned}$$

Then the risk process becomes of a simpler form:

$$Y(t, x, \varphi) = x + ct - \sum_{i=1}^{N_t} X_i + Kt,$$

whence

$$\mathbf{P}\{\tau_x^\varphi < \infty\} \leq e^{-\hat{r}(K)x},$$

where $\hat{r}(K)$ is a unique root of the equation $\lambda h(r) = (c + K)r$, since $K > 0$ is arbitrary and

$$\hat{r}(K) \rightarrow \infty \quad \text{and} \quad \mathbf{P}\{\tau_x^\varphi < \infty\} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

One can improve the estimate for the ruin probability in the mixed model by considering the approximation

$$dS_t = S_t \left(b_1 dW_t + b_2 dW_t^{H,\alpha} + c dt \right).$$

Then one can apply the method of Section 3.1 to the sum $b_2 dW_t^{H,\alpha} + c dt$.

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