THE SUPREMUM OF A MARTINGALE RELATED TO A BRANCHING RANDOM WALK

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ABSTRACT. Let W_n , $n \geq 1$, be a standard martingale constructed from a supercritical branching random walk where the number of individuals in a generation is allowed to be infinite with a positive probability. We find the behavior of $P\{\sup_n W_n > x\}$ as $x \rightarrow \infty$ under certain conditions.

1. INTRODUCTION

Let $S(\cdot)$ be a point process on a real line with points $\infty > A_1 \geq A_2 \geq \dots$, that is,

$$S(B) := \sum_i 1_{\{A_i \in B\}}$$

for an arbitrary Borel set $B \subset \mathbb{R}$. Assume that a population starts with a progenitor and its position coincides with the origin. The progenitor splits into a random number of offspring constituting the first generation. The positions of the members of the first generation on the real line are determined by a copy of the point process S . The members of the first generation (independently of each other) split into their offspring constituting the second generation. The positions of the members of the population are shifted with respect to the position of the progenitor according to independent copies of S . This procedure continues until the extinction of the population or forever. For $n = 1, 2, \dots$, we denote by $S^{(n)}$ a point process describing the positions on the real line of the members of the generation n . The sequence of the point processes $S^{(n)}$, $n = 1, 2, \dots$, is called a branching random walk.

We assign a finite sequence of positive real numbers to each member of the population (the empty sequence corresponds to the progenitor). The rule for assigning a sequence (a label) u to a member x of the population is as follows: $u = (j_1, \dots, j_m)$ corresponds to a member which is the child j_m of the child j_{m-1} of $\dots$ of the child j_1 of the progenitor. In particular, the labels $u = (i)$ correspond to children of the progenitor. The length of a sequence u is denoted by $|u|$. The notation $|u| = m$ means that an individual x with the label u lives in the generation m . In what follows, each individual is identified with its label u . In particular, the position on the line of an individual u is denoted by A_u . Note that the distributions of sequences $\{A_u : |u| = 1\}$ and $\{A_i : i = 1, 2, \dots\}$ are the same.

The majority of papers related to branching random walks make use of the condition $M := S(\mathbb{R}) < \infty$ almost surely (see [5] for a survey of the topic). In this paper, we allow M to be nonrandom or random, finite or infinite with a positive probability. It

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is important for our consideration that the population does not become extinct with a positive probability, that is, the corresponding branching random walk is supercritical. For this reason, we assume that $\mathbf{E} M > 1$ if $\mathbf{P}\{M < \infty\} = 1$.

Let there exist a number $\gamma > 0$ such that

$$m(\gamma) := \mathbf{E} \sum_{|u|=1} e^{\gamma A_u} < \infty.$$

We introduce the following sequence of random variables:

$$W_n := \frac{1}{m^n(\gamma)} \sum_{|u|=n} e^{\gamma A_u}, \quad n = 1, 2, \dots$$

Note that $\mathbf{E} W_n = 1$, $n = 1, 2, \dots$. Let $\mathcal{F}_n = \sigma(S^{(1)}, \dots, S^{(n)})$ be the σ -algebra that contains all the information about the first n generations. The sequence $(W_n, \mathcal{F}_n)$, $n = 1, 2, \dots$, is a (nonnegative) martingale and therefore converges almost surely. If a martingale converges almost surely to a random variable W that is positive with a positive probability, then the martingale converges to W in the mean sense, too. Moreover

$$(1) \quad W = \frac{1}{m^n(\gamma)} \sum_{|u|=n} e^{\gamma A_u} W^{(u)}$$

where $\{W^{(u)} : |u| = n\}$ are (conditionally) independent copies of W given $\mathcal{F}_n$.

Put $Y_i := e^{\gamma A_i} / m(\gamma)$. Let Z be a random variable whose distribution is determined from the equality

$$(2) \quad \mathbf{E} f(Z) = \mathbf{E} \sum_{i=1}^M Y_i f(Y_i)$$

that holds for an arbitrary bounded Borel function f . Note that the distribution of the random variable Z does not have an atom at zero.

Theorem 1. *Let there exist a number $b > 1$ such that $\mathbf{E} \sum_{i=1}^M Y_i^b = 1$,*

$$\mathbf{E} \sum_{i=1}^M Y_i^b \ln^+ Y_i < \infty,$$

and $\mathbf{E}(\sum_{i=1}^M Y_i)^b < \infty$. If $\ln Z$ has a nonarithmetical distribution, then there exists a constant $C \in (0, \infty)$ such that

$$\lim_{x \rightarrow \infty} x^b \mathbf{P} \left\{ \sup_n W_n > x \right\} = C.$$

If the distribution of $\ln Z$ is arithmetical with step δ , then there is a periodic function $C(x)$ with period δ such that

$$\lim_{k \rightarrow \infty} e^{(\delta k + x)b} \mathbf{P} \left\{ \sup_n W_n > e^{\delta k + x} \right\} = C(x)$$

for an arbitrary $x \in \mathbb{R}$.

Remark. 1. The constant C and function $C(x)$ can be written explicitly; however the corresponding expressions are cumbersome (see the proof of Theorem 1).

2. Let the assumptions of Theorem 1 hold. Then W_n converges in the mean sense to a random variable W (we obtain this result in the course of the proof of Theorem 1). Following the ideas of the proof of Theorem 1 and of Proposition 7 of [5], one can check that the limit relations of Theorem 1 hold for the tail distribution of W for similar

(different, however) constant C and function $C(x)$. The claim of Proposition 7 of [5] that

$$\lim_{x \rightarrow \infty} x^b \mathbf{P}\{W > x\} = \text{const}$$

irrespective of the distribution of $\ln Z$ is wrong. In fact, the constants are different for arithmetical and nonarithmetical distributions of $\ln Z$. Similar wrong conclusions appeared in the previous papers, too, namely in the proposition of [4] and Proposition 1.2 of [6]. All these results have to be corrected in the way described above.

2. PROOF OF THEOREM 1

Lemma 1. *If W_n converges in the mean sense to a random variable W , then there are constants $0 < r < 1$ and $D = D(r) > 1$ such that*

$$\mathbf{P}\{W > t\} \leq \mathbf{P}\left\{\sup_n W_n > t\right\} \leq D \mathbf{P}\{W > rt\}$$

for $t > 1$.

Proof. This result follows from Lemma 2 of [1] if $M < \infty$. Our proof is simpler and does not require any assumption imposed on M .

We only need to check the second inequality. For $t > 1$ define the random events

$$E_n := \left\{ \max_{0 \leq k \leq n-1} W_k \leq t, W_n > t \right\}, \quad n = 1, 2, \dots$$

Then

$$\mathbf{P}\left\{\sup_n W_n > t\right\} = \sum_{n=1}^{\infty} \mathbf{P}\{E_n\}$$

and

$$(3) \quad \mathbf{P}\{W > rt\} \geq \mathbf{P}\left\{W > rt, \sup_n W_n > t\right\} = \sum_{n=1}^{\infty} \mathbf{P}\{W > rt \mid E_n\} \mathbf{P}\{E_n\}$$

for an arbitrary $r > 0$. By Q_n , $n = 1, 2, \dots$, we denote the random event

$$\left\{ \frac{1}{m^n(\gamma)} \sum_{|u|=n} \frac{e^{\gamma A_u} (W^{(u)} \wedge d)}{W_n} > r \right\}.$$

The set $\{u: |u| = n\}$ can be ordered, since $S^{(n)}$ is a locally finite integer-valued measure. Denote by $\{\alpha_{i,n}, i = 1, 2, \dots\}$ the realizations of

$$\left\{ \frac{1}{m^n(\gamma)} \frac{e^{\gamma A_u}}{W_n} : |u| = n \right\}.$$

Note that $\sum_{i=1}^{\infty} \alpha_{i,n} = 1$. Using inequality (1), we obtain for an arbitrary fixed $d > 0$ that

$$(4) \quad \begin{aligned} \mathbf{P}\{W > rt \mid E_n\} &\geq \mathbf{P}\left\{ \frac{1}{m^n(\gamma)} \sum_{|u|=n} \frac{e^{\gamma A_u} (W^{(u)} \wedge d)}{W_n} > \frac{rt}{W_n} \mid E_n \right\} \\ &\geq \mathbf{P}\{Q_n \mid E_n\} = (\mathbf{P}\{E_n\})^{-1} \mathbf{E} \mathbf{P}\{Q_n \mid \mathcal{F}_n\} 1_{E_n} \\ &= (\mathbf{P}\{E_n\})^{-1} \mathbf{E} \mathbf{P}\left\{ \sum_{i=1}^{\infty} \alpha_{i,n} (W^{(i)} \wedge d - r) > 0 \right\} 1_{E_n}. \end{aligned}$$

The random variables $W^{(i)}$, $i = 1, 2, \dots$, in the latter equality of (4) are independent copies of W .

Let η be an arbitrary random variable with $\mathbf{E} \eta > 0$. The Cauchy–Schwarz inequality implies that

$$(5) \quad \mathbf{P}\{\eta > 0\} \geq \frac{(\mathbf{E} \eta)^2}{\mathbf{E} \eta^2}.$$

Since $\mathbf{E} W = 1$, one can fix $r \in (0, 1)$ and choose $d = d(r)$ such that $\mathbf{E}(W \wedge d - r) > 0$. Then we apply inequality (5) to the random variables $\eta_n := \sum_{i=1}^{\infty} \alpha_{i,n}(W^{(i)} \wedge d - r)$ with r and d specified above. We obtain

$$\mathbf{P}\{\eta_n > 0\} \geq \frac{(\mathbf{E}(W \wedge d - r))^2}{\mathbf{E}(W \wedge d - r)^2} =: \frac{1}{D} \quad \text{almost surely.}$$

Substituting this result in (4), we get

$$\mathbf{P}\{W > rt \mid E_n\} \geq 1/D.$$

Taking into account (3), we complete the proof of the lemma. $\square$

Lemma 2. *Let W_n converge in the mean sense to a random variable W , and let the distribution of the random variable $\overline{W}$ be such that*

$$\mathbf{P}\{\overline{W} \in dx\} = x \mathbf{P}\{W \in dx\}.$$

Then

$$(6) \quad \overline{W} \stackrel{d}{=} B\overline{W} + Q,$$

where (B, Q) is a random vector that does not depend on $\overline{W}$ and whose distribution is such that

$$(7) \quad \mathbf{E} k(B, Q) = \mathbf{E} \sum_{i=1}^M Y_i k\left(Y_i, \sum_{j \neq i} Y_j W^{(j)}\right)$$

for all bounded Borel functions k of two arguments. In particular, the distribution of B coincides with that of Z defined by (2). Given $\mathcal{F}_1$, the random variables $W^{(j)}$, $j = 1, 2, \dots$, in (7) are independent copies of W .

Proof. This result follows from Lemma 4.1 of [7] if $M < \infty$ almost surely. Our proof is new and simpler. The index i in the sums and products appearing in the proof below runs from 1 to M .

Let $\varphi(s)$ be the Laplace transform of a random variable W . Equality (1) implies for $n = 1$ that

$$(8) \quad \begin{aligned} \varphi(s) &= \mathbf{E} \prod_i \varphi(sY_i), \\ \varphi'(s) &= \mathbf{E} \sum_i \varphi'(sY_i) Y_i \prod_{j \neq i} \varphi(sY_j). \end{aligned}$$

We denote the Laplace transform of the random variable $\overline{W}$ by $\varphi_{\overline{W}}(s)$. Then

$$\varphi_{\overline{W}}(s) = -\varphi'(s).$$

Now we evaluate the Laplace transform of the right hand side of (6). Using equalities (7) for $k(u, v) = e^{-sXu - sv}$ where s and x are fixed, we obtain

$$\mathbf{E} e^{-sxB - sQ} = \mathbf{E} \left(\sum_i Y_i e^{-sXY_i - s \sum_{j \neq i} Y_j W^{(j)}} \right) = \mathbf{E}(\mathbf{E}(\cdot \mid \mathcal{F}_1)) = \mathbf{E} \sum_i Y_i e^{-sXY_i} \prod_{j \neq i} \varphi(sY_j).$$

Further

$$\begin{aligned}
\mathbb{E} e^{-s(B\overline{W}+Q)} &= \int_0^\infty \mathbb{E} e^{-sx B - sQ} d\mathbb{P}\{\overline{W} \leq x\} \\
&= \mathbb{E} \sum_i Y_i \int_0^\infty e^{-sx Y_i} d\mathbb{P}\{\overline{W} \leq x\} \prod_{j \neq i} \varphi(s Y_j) \\
&= \mathbb{E} \sum_i (-\varphi'(s Y_i) Y_i) \prod_{j \neq i} \varphi(s Y_j).
\end{aligned}$$

Taking into account (8), we conclude that

$$-\varphi'(s) = \mathbb{E} e^{-s\overline{W}} = \mathbb{E} e^{-s(B\overline{W}+Q)}.$$

This proves (6).

Put $k(u, v) := f(u)$. Then we obtain from (7) that

$$\mathbb{E} f(B) = \mathbb{E} \sum_i Y_i f(Y_i).$$

Thus $B \stackrel{d}{=} Z$ and the lemma is proved. $\square$

Proof of Theorem 1. Let

$$N(x) := \int_0^\infty \frac{1}{z} \mathbb{P} \left\{ \sup_n W_n > \frac{x}{z} \right\} d\mathbb{P}\{Z \leq z\}.$$

The measure U defined by $U(dz) := z^{b-1} \mathbb{P}\{Z \in dz\}$ is a probability measure, since $\int_0^\infty U(dz) = \mathbb{E} Z^{b-1} = \mathbb{E} \sum_{i=1}^M Y_i^b = 1$. Now

$$x^b \mathbb{P} \left\{ \sup_n W_n > x \right\} = \int_0^\infty \frac{x^b}{z^b} \mathbb{P} \left\{ \sup_n W_n > \frac{x}{z} \right\} dU(z) + x^b \left[\mathbb{P} \left\{ \sup_n W_n > x \right\} - N(x) \right].$$

Integrate this equality over the interval $[0, e^y]$ and then multiply the result by e^{-y} . Then we get

$$\begin{aligned}
(9) \quad e^{-y} \int_0^{e^y} x^b \mathbb{P} \left\{ \sup_n W_n > x \right\} dx &= e^{-y} \int_0^{e^y} \int_0^\infty \frac{x^b}{z^b} \mathbb{P} \left\{ \sup_n W_n > \frac{x}{z} \right\} dU(z) dx \\
&\quad + e^{-y} \int_0^{e^y} x^b \left[\mathbb{P} \left\{ \sup_n W_n > x \right\} - N(x) \right] dx.
\end{aligned}$$

Put

$$\begin{aligned}
P(y) &= e^{-y} \int_0^{e^y} x^b \mathbb{P} \left\{ \sup_n W_n > x \right\} dx, \\
Q(y) &= e^{-y} \int_0^{e^y} x^b \left[\mathbb{P} \left\{ \sup_n W_n > x \right\} - N(x) \right] dx.
\end{aligned}$$

Since

$$\begin{aligned}
&e^{-y} \int_0^{e^y} \int_0^\infty \frac{x^b}{z^b} \mathbb{P} \left\{ \sup_n W_n > \frac{x}{z} \right\} dU(z) dx \\
&= e^{-y} \int_0^\infty dU(z) z \int_0^{e^y/z} v^b \mathbb{P} \left\{ \sup_n W_n > v \right\} dv \\
&= \int_{-\infty}^\infty e^{-(y-t)} dU(e^t) \int_0^{e^{y-t}} v^b \mathbb{P} \left\{ \sup_n W_n > v \right\} dv = \int_{-\infty}^\infty P(y-t) dU(e^t),
\end{aligned}$$

we see that (9) is the renewal equation

$$P(y) = \int_{-\infty}^{\infty} P(y-t) dU(e^t) + Q(y).$$

The function $q(x) := \mathbb{E}(\sum_{i=1}^M Y_i^x)$ is convex in the interval where it is finite. Thus condition $q(1) = q(b) = 1$ implies for $x \in (1, b)$ that

$$(10) \quad \mathbb{E} \sum_{i=1}^M Y_i^x < 1.$$

Applying the Jensen inequality, we get $\mathbb{E} \ln Z = \mathbb{E}(\sum_{i=1}^M Y_i \ln Y_i) < 0$. The condition $\mathbb{E}(\sum_{i=1}^M Y_i)^b < \infty$ implies that $\mathbb{E} \sum_{i=1}^M Y_i \ln^+ Y_i < \infty$. Considering the case of $\beta_1 = 1$ in Theorem 2 of [5], we conclude from Lemma A.5 of [5] that the martingale W_n converges in the mean sense to the random variable W . Moreover, Proposition 4 of [5] yields

$$(11) \quad \mathbb{E} W^a < \infty \quad \text{for all } a \in (1, b).$$

Now we describe the asymptotic behavior of $P(x)$ as $x \rightarrow \infty$ with the help of the renewal theorem on the whole axis. First we prove that the function $Q(y)$ is directly Riemann integrable. Put

$$g(x) := x^b \left[\mathbb{P} \left\{ \sup_n W_n > x \right\} - N(x) \right].$$

Then

$$Q(y) = e^{-y} \int_0^{e^y} g(x) dx = e^{-y} \int_{-\infty}^y e^t g(e^t) dt.$$

According to Lemma 9.2 of [2] the function $Q(y)$ is directly Riemann integrable if

$$0 < \int_{-\infty}^{\infty} g(e^t) dt < \infty$$

or equivalently if

$$(12) \quad 0 < I(b) := \int_0^{\infty} y^{b-1} \left[\mathbb{P} \left\{ \sup_n W_n > y \right\} - N(y) \right] dy < \infty.$$

Choose an arbitrary $a \in (1, b)$. Inequality (10) implies that $\mathbb{E} Z^{a-1} = \mathbb{E} \sum_{i=1}^M Y_i^a < 1$. We also have $\mathbb{E}(\sup_n W_n)^a < \infty$ from (11) and Lemma 1. Therefore

$$\begin{aligned} \int_0^{\infty} y^{a-1} \mathbb{P} \left\{ \sup_n W_n > y \right\} dy &= \frac{1}{a} \int_0^{\infty} y^a d\mathbb{P} \left\{ \sup_n W_n < y \right\} = \frac{1}{a} \mathbb{E} \left(\sup_n W_n \right)^a < \infty, \\ \int_0^{\infty} y^{a-1} N(y) dy &= \frac{1}{a} \mathbb{E} \left(\sup_n W_n \right)^a \mathbb{E} Z^{a-1} < \infty, \\ 0 < I(a) &= \frac{1}{a} \mathbb{E} \left(\sup_n W_n \right)^a (1 - \mathbb{E} Z^{a-1}) < \infty. \end{aligned}$$

Since $W \leq \sup_n W_n$ almost surely, $\mathbb{E} W^a \leq \mathbb{E}(\sup_n W_n)^a$. Moreover, Lemma 1 implies that there are constants $r \in (0, 1)$ and $D > 1$ such that

$$\mathbb{E} \left(\sup_n W_n \right)^a \leq \frac{D}{r^a} \mathbb{E}(W \vee r)^a.$$

Putting

$$K_a := \mathbb{E} W^a (1 - \mathbb{E} Z^{a-1}),$$

we have

$$(13) \quad K_a \leq aI(a) \leq \frac{D}{r^a} K_a.$$

Recall the known inequality

$$(14) \quad (2^{s-1} \wedge 1)(x^s + y^s) \leq (x + y)^s \leq (2^{s-1} \vee 1)(x^s + y^s), \quad x, y \geq 0, s > 0.$$

This inequality holds, since the function $x \rightarrow x^s$ is concave for $s \in (0, 1]$ and convex for $s > 1$. To estimate K_a , we use equality (6).

For $a - 1 \in (0, 1)$ we have

$$\begin{aligned} \mathbb{E} W^a &= \mathbb{E} \overline{W}^{a-1} = \mathbb{E} (B\overline{W} + Q)^{a-1} \stackrel{(14)}{\leq} \mathbb{E} B^{a-1} \mathbb{E} W^a + \mathbb{E} Q^{a-1}, \\ K_a &\leq \mathbb{E} Q^{a-1}. \end{aligned}$$

Given $b_1 \in (1, b)$, we choose $\varepsilon > 0$ such that

$$\mathbb{E} B^{b_1-1} (1 + \varepsilon^{-1})^{b_1-1} < 1.$$

Assume that $d := \mathbb{P}\{\varepsilon Q \geq B\overline{W}\} = 0$. Then $\overline{W} \leq (1 + \varepsilon^{-1})B\overline{W}$ in distribution and this contradicts the property of ε . Thus $d > 0$. If $\varepsilon y \geq x > 0$ and $h \in (0, 1)$, then

$$(x + y)^h - x^h \geq h(x + y)^{h-1}y \geq h(1 + \varepsilon)^{h-1}y^h$$

by the Rolle theorem. Hence

$$K_a \geq \mathbb{E} \left((B\overline{W} + Q)^{a-1} - B^{a-1}\overline{W}^{a-1} \right) 1_{\{\varepsilon Q \geq B\overline{W}\}} \geq (a-1)(1 + \varepsilon)^{a-2} \mathbb{E} Q^{a-1} 1_{\{\varepsilon Q \geq B\overline{W}\}}.$$

If $a - 1 \geq 1$, then we use the inequality

$$\begin{aligned} (u + v)^{a-1} &= u^{a-1} + (a-1) \int_0^v (u + t)^{a-2} dt \leq u^{a-1} + (a-1)(u + v)^{a-2}v \\ &\stackrel{(14)}{\leq} u^{a-1} + (a-1) (2^{a-3} \vee 1) (u^{a-2}v + v^{a-1}) \end{aligned}$$

for $u, v \geq 0$ and Hölder's inequality for $p := (a-1)/(a-2)$, $q := a-1$. Finally we obtain

$$\begin{aligned} K_a &\leq \text{const} (\mathbb{E} W^{a-1} \mathbb{E} B^{a-2} Q + \mathbb{E} Q^{a-1}) \\ &\leq \text{const} \left(\mathbb{E} W^{a-1} (\mathbb{E} B^{a-1})^{1/p} (\mathbb{E} Q^{a-1})^{1/q} + \mathbb{E} Q^{a-1} \right) \\ &\leq \text{const} \left(\mathbb{E} W^{a-1} (\mathbb{E} Q^{a-1})^{1/q} + \mathbb{E} Q^{a-1} \right). \end{aligned}$$

Moreover

$$\mathbb{E} W^a = \mathbb{E} \overline{W}^{a-1} = \mathbb{E} (B\overline{W} + Q)^{a-1} \stackrel{(14)}{\geq} \mathbb{E} B^{a-1} \mathbb{E} W^a + \mathbb{E} Q^{a-1}, \quad K_a \geq \mathbb{E} Q^{a-1}.$$

Note that $\mathbb{E} Q^{b-1} > 0$ and Lemma 4.2 of [7] yields

$$\mathbb{E} Q^x \leq \begin{cases} \mathbb{E} \left(\sum_{i=1}^M Y_i \right)^{x+1}, & x \in (0, 1], \\ \mathbb{E} W^x \mathbb{E} \left(\sum_{i=1}^M Y_i \right)^{x+1}, & x > 1, \end{cases}$$

provided $\mathbb{E} Q^x < \infty$. Taking into account the bounds proved for K_a , one can pass to the limit in inequality (13) as a approaches b along a certain subsequence, since the expectations $\mathbb{E} W^{b-1}$ and $\mathbb{E} \left(\sum_{i=1}^M Y_i \right)^b$ are finite by the assumptions of the theorem. As a result we prove (12), whence we obtain that the function Q is directly Riemann integrable.

Since

$$\int_{-\infty}^{\infty} t dU(e^t) = \int_0^{\infty} t^{b-1} \ln t \mathbb{P}\{Z \in dt\} = \mathbb{E} Z^{b-1} \ln Z = \mathbb{E} \sum_{i=1}^M Y_i^b \ln Y_i < \infty,$$

the renewal theorem, for the real line implies that

$$\lim_{x \rightarrow \infty} P(x) = \left(\mathbb{E} \sum_{i=1}^M Y_i^b \ln Y_i \right)^{-1} \int_{-\infty}^{\infty} Q(y) dy = \left(\mathbb{E} \sum_{i=1}^M Y_i^b \ln Y_i \right)^{-1} I(b) =: C \stackrel{(12)}{=} (0, \infty)$$

if the distribution of the random variable $\ln Z$ is nonarithmetical. This implies in particular that

$$\lim_{t \rightarrow \infty} t^b \mathbb{P} \left\{ \sup_n W_n > t \right\} = C.$$

Using the renewal theorem, one can prove only that

$$\lim_{n \rightarrow \infty} P(x + \delta n) = \left(\mathbb{E} \sum_i Y_i^b \ln Y_i \right)^{-1} \sum_{k=-\infty}^{\infty} Q(x + \delta k) =: C(x)$$

if the distribution of $\ln Z$ is arithmetical with step $\delta > 0$. Put

$$G_b(x) := x^b \mathbb{P} \left\{ \sup_n W_n > x \right\}.$$

Similarly to the proof of Theorem 2 in [3] we obtain for real $x_1 \geq x_2 > 0$ that

$$\lim_{k \rightarrow \infty} \int_{e^{x_2}}^{e^{x_1}} e^{\delta k b} s^b \mathbb{P} \left\{ \sup_n W_n > e^{\delta k} s \right\} ds = e^{x_1} C(x_1) - e^{x_2} C(x_2).$$

Since the function $\mathbb{P} \{ \sup_n W_n > x \}$ is monotone, the integrand in the latter equality is bounded if s is fixed. This implies that there are two numbers b_1 and b_2 such that $0 < b_1 \leq G_b(x) \leq b_2 < \infty$. According to the selection theorem there exists a sequence $k_1, k_2, \dots \rightarrow \infty$ such that $e^{\delta k b} s^b \mathbb{P} \{ \sup_n W_n > e^{\delta k} s \}$ converges to some function $C^*(s)$ along this sequence. The function $C^*(x)/x^b$ is nonincreasing as a limit of nonincreasing functions. Therefore it has an at most countable set of points of discontinuity. The equality

$$\int_{e^{x_2}}^{e^{x_1}} C^*(s) ds = e^{x_1} C(x_1) - e^{x_2} C(x_2)$$

uniquely determines the function $C^*(x)$ at all points of continuity and this proves that the definition of C^* does not depend on the sequence $k_1, k_2, \dots \rightarrow \infty$. Therefore

$$G_b(e^{x+\delta k}) = e^{(x+\delta k)b} \mathbb{P} \left\{ \sup_n W_n > e^{x+\delta k} \right\} \rightarrow C^*(e^x) = C(x)$$

as $k \rightarrow \infty$ for all x except (possibly) for a countable set. The Cantor diagonalization method proves that the latter equality holds for all $x > 0$. The theorem is proved. $\square$

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