MODELLING A SOLUTION OF A HYPERBOLIC EQUATION WITH RANDOM INITIAL CONDITIONS

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ABSTRACT. A new method is proposed in this paper to construct models for solutions of boundary problems for hyperbolic equations with random initial conditions. We assume that initial conditions are strictly sub-Gaussian random fields (in particular, Gaussian random fields with zero mean). The models approximate solutions with a given accuracy and reliability in the uniform metric.

1. INTRODUCTION

We construct a model that approximates a solution of the boundary problem (3.1)–(3.3) for the hyperbolic equation with random initial conditions. The model is convenient to use when developing a software for computers. It approximates a solution with a given reliability and accuracy in the uniform metric.

We consider strictly sub-Gaussian random fields [1] to model initial conditions in problem (3.1)–(3.3). Note that Gaussian fields are a particular case of strictly sub-Gaussian random fields.

It is known that a solution of the boundary problem can be represented, under certain conditions, in the form of an infinite series, namely

$$U(x, t) = \sum_{k=1}^{\infty} \left(A_k \cos(\sqrt{\lambda_k} t) + B_k \sin(\sqrt{\lambda_k} t) \right) V_k(x),$$

where $V_k(x)$ are known functions and A_k and B_k are random variables whose joint distributions are known.

One can consider the following model for a solution of the boundary problem:

$$(1.1) \quad U(x, t, N) = \sum_{k=1}^N \left(A_k \cos(\sqrt{\lambda_k} t) + B_k \sin(\sqrt{\lambda_k} t) \right) V_k(x).$$

One can find the values of N for which $U(x, t, N)$ approximates the field $U(x, t)$ with a given reliability and accuracy (see [2]).

The main disadvantage of this method is that the random variables A_k and B_k are independent only for very special initial conditions. Therefore it is practically impossible to apply this method for large N .

A new method is proposed in this paper to construct a model for a solution of boundary problem (3.1)–(3.3). The idea of the method is, first, to model the initial conditions with a given accuracy and, second, to compute approximate values $\hat{A}_k$ and $\hat{B}_k$ of the

coefficients A_k and B_k , respectively, by using the model for the initial conditions. The finite sum

$$(1.2) \quad \hat{U}(x, t, N) = \sum_{k=1}^N \left(\hat{A}_k \cos(\sqrt{\lambda_k} t) + \hat{B}_k \sin(\sqrt{\lambda_k} t) \right) V_k(x)$$

is considered as a model for the solution. We find values of N and an accuracy of the approximation of A_k and B_k by $\hat{A}_k$ and $\hat{B}_k$ for which model (1.2) approximates the solution of the boundary problem with a given reliability and accuracy in the uniform metric.

The paper consists of five sections. We recall necessary definitions and results of the theory of strictly sub-Gaussian stochastic processes in Section 2. The main result, Theorem 3.2, is stated in Section 3. The proof of the theorem is given in Section 4, and some examples are considered in Section 5.

Note that all the results of the paper hold for the case where the initial conditions are zero mean Gaussian random fields. Some methods to model Gaussian and sub-Gaussian random fields can be found in the book [2].

2. STRICTLY SUB-GAUSSIAN RANDOM VARIABLES AND STRICTLY SUB-GAUSSIAN STOCHASTIC PROCESSES

Definition 2.1 ([1]). A random variable ξ is called sub-Gaussian if there exists a number $a \geq 0$ such that

$$\mathbb{E} \exp\{\lambda \xi\} \leq \exp\left\{\frac{a^2 \lambda^2}{2}\right\}$$

for all $\lambda \in \mathbf{R}$.

The set of all sub-Gaussian random variables is denoted by $\text{Sub}(\Omega)$.

Definition 2.2 ([1]). A sub-Gaussian random variable ξ is called strictly sub-Gaussian if $\tau^2(\xi) = \mathbb{E} \xi^2$, that is,

$$\mathbb{E} \exp\{\lambda \xi\} \leq \exp\left\{\frac{\lambda^2 \sigma^2}{2}\right\}$$

for all $\lambda \in \mathbf{R}$ where $\sigma^2 = \mathbb{E} \xi^2$; in this case, $\tau(\xi)$ is called the sub-Gaussian standard.

Definition 2.3 ([1]). A family of random variables $\Delta \in \text{Sub}(\Omega)$ is called strictly sub-Gaussian if

$$\tau^2\left(\sum_{i \in I} \lambda_i \xi_i\right) = \mathbb{E}\left(\sum_{i \in I} \lambda_i \xi_i\right)^2$$

for all finite sets of random variables $\{\xi_i, i \in I\}$ of Δ and all $\lambda_i \in \mathbf{R}$.

Lemma 2.1 ([2]). *Let Δ be a strictly sub-Gaussian family of random variables. Then the linear span of Δ considered in the space $L_2(\Omega)$ is a strictly sub-Gaussian family.*

Remark 2.1. In what follows, we consider strictly sub-Gaussian families that are closed in the space $L_2(\Omega)$.

Definition 2.4 ([1]). A stochastic process $X = \{X(t), t \in T\}$ is called strictly sub-Gaussian if the family of random variables $\{X(t), t \in T\}$ is strictly sub-Gaussian. Random processes $X = \{X_i(t), t \in T, i = 1, 2, \dots, m\}$ are called jointly strictly sub-Gaussian if the family of random variables $\{X_i(t), t \in T, i = 1, 2, \dots\}$ is strictly sub-Gaussian.

Remark 2.2. Properties of random variables and stochastic processes forming strictly sub-Gaussian families can be found in [1, 2]. In particular, independent strictly sub-Gaussian stochastic processes are jointly strictly sub-Gaussian. Another example of strictly sub-Gaussian stochastic processes is presented by Gaussian zero mean stochastic processes.

3. MAIN RESULT

Consider the following partial differential equation

$$(3.1) \quad \frac{\partial^2 u}{\partial t^2} = L(u),$$

where

$$L(u) = \sum_{i=1}^m \sum_{j=1}^m \frac{\partial}{\partial x_i} \left(a_{ij}(X) \frac{\partial u}{\partial x_j} \right) - a(X)u.$$

Assume that the coefficients a and a_{ij} of equation (3.1) are defined in a finite connected domain

$$G = \{0 \leq x_i \leq S_i, i = 1, 2, \dots, m\}$$

and are such that

$$a(X) \geq 0, \quad a_{ij} = a_{ji}, \quad \sum_{i,j=1}^m a_{ij} \gamma_i \gamma_j \geq \alpha \sum_{i=1}^m \gamma_i^2, \quad \alpha > 0.$$

We study the following problem [4]: find a solution of equation (3.1) in the cylinder $Q_T = G \times [0; T]$ subject to the initial conditions

$$(3.2) \quad u|_{t=0} = \xi(X), \quad \frac{\partial u}{\partial t} \Big|_{t=0} = \eta(X)$$

and the boundary condition

$$(3.3) \quad u|_S = 0, \quad t \in [0; T],$$

where $\{\xi(X), X \in G\}$ and $\{\eta(X), X \in G\}$ are independent strictly sub-Gaussian stochastic processes and S is the boundary of the domain G .

The solution of problem (3.1)–(3.3) is given by

$$(3.4) \quad u(X, t) = \sum_{k=1}^{\infty} \left(A_k \cos \sqrt{\lambda_k} t + B_k \sin \sqrt{\lambda_k} t \right) v_k(X),$$

where

$$A_k = \int_G \xi(X) v_k(X) dX, \quad B_k = \frac{1}{\sqrt{\lambda_k}} \int_G \eta(X) v_k(X) dX,$$

λ_k and $v_k(X)$ are eigenvalues and eigenfunctions of the Sturm–Liouville problem:

$$(3.5) \quad L(v) + \lambda v = 0, \quad v|_S = 0.$$

The following theorem is a particular case of Theorem 4.3 of [3].

Theorem 3.1. *Let $\xi(X)$, $\eta(X)$, $X \in G$, be jointly strictly sub-Gaussian stochastic processes (in particular, they can be independent) such that*

$$B(X, Y) = B_{0000}(X, Y) = \mathbb{E} \xi(X) \xi(Y), \\ R(X, Y) = R_{0000}(X, Y) = \mathbb{E} \eta(X) \eta(Y).$$

A twice continuously differentiable solution of problem (3.1)–(3.3) in the domain

$$0 \leq t \leq T, \quad G = \{0 \leq x_i \leq S_i, i = 1, \dots, m\}$$

that can be represented in the form of series (3.4) converging uniformly in probability exists almost surely if the following conditions are satisfied.

1) There exist continuous partial derivatives ($X, Y \in G$)

$$\begin{aligned} B_{i0i0}(X, Y) &= \frac{\partial^2 B(X, Y)}{\partial x_i \partial y_i}, \quad i = 1, \dots, m, \\ B_{ikik}(X, Y) &= \frac{\partial^4 B(X, Y)}{\partial x_i \partial y_i \partial x_k \partial y_k}, \quad i, k = 1, \dots, m, \\ R_{i0i0}(X, Y) &= \frac{\partial^2 R(X, Y)}{\partial x_i \partial y_i}, \quad i = 1, \dots, m, \end{aligned}$$

and

$$\begin{aligned} \sup_{\substack{|x_i - y_i| \leq h \\ i=1, \dots, m}} (B_z(X, X) + B_z(Y, Y) - 2B_z(X, Y))^{1/2} &\leq \frac{C_z}{|\ln h|^\delta}, \\ \sup_{\substack{|x_i - y_i| \leq h \\ i=1, \dots, m}} (R_{z_1}(X, X) + R_{z_1}(Y, Y) - 2R_{z_1}(X, Y))^{1/2} &\leq \frac{C_{z_1}}{|\ln h|^\delta} \end{aligned}$$

for sufficiently small h , where $\delta > \frac{1}{2}$, z equals $(0, 0, 0, 0)$ or $(i, 0, i, 0)$, $i = 1, \dots, m$, or (i, k, i, k) , $i, k = 1, \dots, m$, and z_1 equals $(0, 0, 0, 0)$ or $(i, 0, i, 0)$, $i = 1, \dots, m$.

2) The following series converge:

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} r_k r_l [|\mathbb{E} A_k A_l| + |\mathbb{E} B_k B_l| + 2|\mathbb{E} A_k B_k|] < \infty,$$

where $r_k = \max_{i,j=1, \dots, m} (\lambda_k v_k, v_{ki}, v_{kij})$,

$$v_k = \sup_{X \in G} |v_k(X)|, \quad v_{ki} = \sup_{X \in G} \left| \frac{\partial v_k(X)}{\partial x_i} \right|, \quad v_{kij} = \sup_{X \in G} \left| \frac{\partial^2 v_k(X)}{\partial x_i \partial x_j} \right|.$$

3) If $\sup_{X \in G} |v_l(X)| \leq \delta_{l00}$, then

$$\sup_{X \in G} \left| \frac{\partial v_l(X)}{\partial x_i} \right| \leq \delta_{li0}, \quad \sup_{X \in G} \left| \frac{\partial^2 v_l(X)}{\partial x_i \partial x_j} \right| \leq \delta_{lij}$$

and for all $\delta > \frac{1}{2}$ and $|h| < 1$,

$$\begin{aligned} \sup_{\substack{|x_k - y_k| < h \\ k=1, \dots, m}} |v_l(X) - v_l(Y)| &\leq \gamma_{l00} \frac{1}{|\ln h|^\delta}, \\ \sup_{\substack{|x_k - y_k| < h \\ k=1, \dots, m}} \left| \frac{\partial v_l(X)}{\partial x_i} - \frac{\partial v_l(Y)}{\partial x_i} \right| &\leq \gamma_{li0} \frac{1}{|\ln h|^\delta}, \quad i = 1, \dots, m, \\ \sup_{\substack{|x_k - y_k| < h \\ k=1, \dots, m}} \left| \frac{\partial^2 v_l(X)}{\partial x_i \partial x_j} - \frac{\partial^2 v_l(Y)}{\partial x_i \partial x_j} \right| &\leq \gamma_{lij} \frac{1}{|\ln h|^\delta}, \quad i, j = 1, \dots, m, \end{aligned}$$

and the following series converge:

$$\begin{aligned} \sum_{l=1}^{\infty} \left((\mathbb{E} A_l^2)^{1/2} + (\mathbb{E} B_l^2)^{1/2} \right) \lambda_l (\delta_{l00} + (\ln \lambda_l)^\delta \gamma_{l00}) &< \infty, \\ \sum_{l=1}^{\infty} \left((\mathbb{E} A_l^2)^{1/2} + (\mathbb{E} B_l^2)^{1/2} \right) (\delta_{li0} + (\ln \lambda_l)^\delta \gamma_{li0}) &< \infty, \quad i = 1, \dots, m, \\ \sum_{l=1}^{\infty} \left((\mathbb{E} A_l^2)^{1/2} + (\mathbb{E} B_l^2)^{1/2} \right) (\delta_{lij} + (\ln \lambda_l)^\delta \gamma_{lij}) &< \infty, \quad i, j = 1, \dots, m. \end{aligned}$$

If the assumptions of this theorem hold, then one can construct a model for a solution of problem (3.1)–(3.3) approximating the solution with a given reliability and accuracy in the uniform metric.

Let $\{\hat{\xi}(X), X \in G\}$ and $\{\hat{\eta}(X), X \in G\}$ be models of processes $\{\xi(X), X \in G\}$ and $\{\eta(X), X \in G\}$, respectively. Note that the models $\hat{\xi}(X)$ and $\hat{\eta}(X)$ are independent stochastic processes.

Put

$$\hat{A}_k = \int_G \hat{\xi}(X) v_k(X) dX, \quad \hat{B}_k = \frac{1}{\sqrt{\lambda_k}} \int_G \hat{\eta}(X) v_k(X) dX.$$

The sum

$$(3.6) \quad u^N(X, t) = \sum_{k=1}^N \left(\hat{A}_k \cos \sqrt{\lambda_k} t + \hat{B}_k \sin \sqrt{\lambda_k} t \right) v_k(X)$$

is called a model of the process $u(X, t)$.

Definition 3.1. Let a solution $u(X, t)$ of problem (3.1)–(3.3) be represented in the form of series (3.4). We say that a model $u^N(X, t)$ approximates $u(X, t)$ with a given reliability $1 - \gamma$ and accuracy δ in the uniform metric in the domain Q_T if

$$\mathbf{P} \left\{ \sup_{(X, t) \in Q_T} |u^N(X, t) - u(X, t)| > \delta \right\} \leq \gamma.$$

Put

$$\Delta_N(X, t, N) = u(X, t) - u^N(X, t) = u_N(X, t) + V_N(X, t),$$

where

$$u_N(X, t) = \sum_{k=N+1}^{\infty} \left(A_k \cos \sqrt{\lambda_k} t + B_k \sin \sqrt{\lambda_k} t \right) v_k(X),$$

$$V_N(X, t) = \sum_{k=1}^N \left((\hat{A}_k - A_k) \cos \sqrt{\lambda_k} t + (\hat{B}_k - B_k) \sin \sqrt{\lambda_k} t \right) v_k(X).$$

Below is the main result of the paper.

Theorem 3.2. *Let*

$$\{\xi(X), X \in G\} \quad \text{and} \quad \{\eta(X), X \in G\}$$

be independent SSub(Ω) processes. Let the models $\hat{\xi}(X)$ and $\hat{\eta}(X)$ be such that

$$\int_G \sqrt{\mathbf{E}(\hat{\xi}(X) - \xi(X))^2} dX < \chi, \quad \int_G \sqrt{\mathbf{E}(\hat{\eta}(X) - \eta(X))^2} dX < \chi.$$

Then the stochastic process $u^N(X, t)$ defined by (3.6) is a model of the stochastic process $u(X, t)$ that approximates it with reliability $1 - \gamma$ and accuracy δ in the uniform metric in the domain Q_T if γ and N are such that

$$\left(T^{1/2} + \sum_{i=1}^m S_i^{1/2} \right) A_N^2 \varepsilon_0^2(N) < \delta,$$

$$\frac{1}{2} \left(\frac{\delta^{1/3} \left(\delta^{2/3} - 3 \left(T^{1/2} + \sum_{i=1}^m S_i^{1/2} \right)^{2/3} A_N^{1/3} \varepsilon_0^{1/3}(N) \right)}{\varepsilon_0(N)} \right)^2 \geq \ln \frac{1}{\gamma},$$

where

$$\begin{aligned}
A_N &= \left(\left(\sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} A_k^2} (\nu_k + c_k \sqrt{\lambda_k}) \right)^2 + \left(\sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} B_k^2} (\nu_k + c_k \sqrt{\lambda_k}) \right)^2 \right)^{1/2} \\
&\quad + \chi \left(\left(\sum_{k=1}^N c_k (\nu_k + c_k \sqrt{\lambda_k}) \right)^2 + \left(\sum_{k=1}^N \frac{c_k}{\sqrt{\lambda_k}} (\nu_k + c_k \sqrt{\lambda_k}) \right)^2 \right)^{1/2}, \\
\varepsilon_0(N) &= \left(\left(\sum_{k=N+1}^{+\infty} c_k \sqrt{\mathbb{E} A_k^2} \right)^2 + \left(\sum_{k=N+1}^{+\infty} c_k \sqrt{\mathbb{E} B_k^2} \right)^2 \right)^{1/2} \\
&\quad + \chi \left(\left(\sum_{k=1}^N c_k^2 \right)^2 + \left(\sum_{k=1}^N \frac{c_k^2}{\sqrt{\lambda_k}} \right)^2 \right)^{1/2}.
\end{aligned}$$

Here the numbers λ_k denote the eigenvalues of the Sturm–Liouville problem

$$\nu_k = \sup_{X \in G} \sum_{i=1}^m \left| \frac{\partial v_k(X)}{\partial x_i} \right|, \quad c_k = \sup_{X \in G} |v_k(X)|.$$

Remark 3.1. The series involved in the definitions of A_N and $\varepsilon_0(N)$ converge in view of the assumptions of Theorem 3.1.

4. PROOF OF THEOREM 3.2

Proof. Since $\Delta_N(X, t, N)$ is a strictly sub-Gaussian stochastic process, we apply the results of the paper [3] and conclude that

$$(4.1) \quad \mathbb{P} \left\{ \sup_{(X,t) \in Q_T} |\Delta_N(X, t, N)| > \delta \right\} \leq 2\tilde{A}(\delta, \theta)$$

for all $0 < \theta < 1$, where

$$(4.2) \quad \tilde{A}(\delta, \theta) = \exp \left\{ - \frac{(\delta(1-\theta) - \frac{2}{\theta} I(\theta \varepsilon_0))^2}{2\varepsilon_0^2} \right\},$$

ε_0 is an arbitrary number such that

$$\begin{aligned}
\varepsilon_0 &\geq \sup_{(X,t) \in Q_T} \left(\mathbb{E} |\Delta_N(X, t, N)|^2 \right)^{1/2}, \\
(4.3) \quad I(\theta \varepsilon_0) &= \frac{1}{\sqrt{2}} \int_0^{\theta \varepsilon_0} \left(\ln \left(\frac{T}{2\sigma^{-1}(x)} + 1 \right) + \sum_{i=1}^m \ln \left(\frac{S_i}{2\sigma^{-1}(x)} + 1 \right) \right)^{1/2} dx,
\end{aligned}$$

and where $\sigma(h)$ is a continuous increasing function such that $\sigma(0) = 0$ and

$$\|X - X_1\| = \max_{i=1, \dots, m} |x_i - x_{i1}|,$$

$$\begin{aligned}
& \sup_{\substack{\|X-X_1\|\leq h \\ |t-t_1|\leq h}} \left(\mathbb{E} |\Delta_N(X, t, N) - \Delta_N(X_1, t_1, N)|^2 \right)^{1/2} \leq \sigma(h), \\
& \sup_{\substack{\|X-X_1\|\leq h \\ |t-t_1|\leq h}} \left(\mathbb{E} |u_N(X, t) + V_N(X, t) - u_N(X_1, t_1) - V_N(X_1, t_1)|^2 \right)^{1/2} \\
(4.4) \quad & \leq \sup_{\substack{\|X-X_1\|\leq h \\ |t-t_1|\leq h}} \left[\left(\mathbb{E} |u_N(X, t) - u_N(X_1, t_1)|^2 \right)^{1/2} \right. \\
& \quad \left. + \left(\mathbb{E} |V_N(X, t) - V_N(X_1, t_1)|^2 \right)^{1/2} \right],
\end{aligned}$$

and

$$\begin{aligned}
& \sup_{(X,t) \in Q_T} \left(\mathbb{E} |\Delta_N(X, t, N)|^2 \right)^{1/2} \\
(4.5) \quad & = \sup_{(X,t) \in Q_T} \left(\mathbb{E} |u_N(X, t) + V_N(X, t)|^2 \right)^{1/2} \\
& \leq \sup_{(X,t) \in Q_T} \left[\left(\mathbb{E} |u_N(X, t)|^2 \right)^{1/2} + \left(\mathbb{E} |V_N(X, t)|^2 \right)^{1/2} \right].
\end{aligned}$$

Since the stochastic processes $\xi(X)$ and $\eta(X)$ are independent, that is, A_k and B_l are independent, we obtain

$$\begin{aligned}
& \mathbb{E} |u_N(X, t) - u_N(X_1, t_1)|^2 \\
& = \mathbb{E} \left| \sum_{k=N+1}^{\infty} \left(A_k \cos(\sqrt{\lambda_k} t) + B_k \sin(\sqrt{\lambda_k} t) \right) v_k(X) \right. \\
& \quad \left. - \sum_{k=N+1}^{\infty} \left(A_k \cos(\sqrt{\lambda_k} t_1) + B_k \sin(\sqrt{\lambda_k} t_1) \right) v_k(X_1) \right|^2 \\
& = \mathbb{E} \left| \sum_{k=N+1}^{\infty} \left(A_k \left(\cos(\sqrt{\lambda_k} t) v_k(X) - \cos(\sqrt{\lambda_k} t_1) v_k(X_1) \right) \right. \right. \\
(4.6) \quad & \quad \left. \left. + B_k \left(\sin(\sqrt{\lambda_k} t) v_k(X) - \sin(\sqrt{\lambda_k} t_1) v_k(X_1) \right) \right) \right|^2 \\
& \leq \sum_{k=N+1}^{\infty} \sum_{l=N+1}^{\infty} |\mathbb{E} A_k A_l| \left| \cos(\sqrt{\lambda_k} t) v_k(X) - \cos(\sqrt{\lambda_k} t_1) v_k(X_1) \right| \\
& \quad \times \left| \cos(\sqrt{\lambda_l} t) v_l(X) - \cos(\sqrt{\lambda_l} t_1) v_l(X_1) \right| \\
& \quad + \sum_{k=N+1}^{\infty} \sum_{l=N+1}^{\infty} |\mathbb{E} B_k B_l| \left| \sin(\sqrt{\lambda_k} t) v_k(X) - \sin(\sqrt{\lambda_k} t_1) v_k(X_1) \right| \\
& \quad \times \left| \sin(\sqrt{\lambda_l} t) v_l(X) - \sin(\sqrt{\lambda_l} t_1) v_l(X_1) \right|
\end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{k=N+1}^{\infty} \sqrt{\mathbb{E} A_k^2} \left| v_k(X) \cos(\sqrt{\lambda_k} t) - v_k(X_1) \cos(\sqrt{\lambda_k} t_1) \right| \right)^2 \\
&\quad + \left(\sum_{k=N+1}^{\infty} \sqrt{\mathbb{E} B_k^2} \left| v_k(X) \sin(\sqrt{\lambda_k} t) - v_k(X_1) \sin(\sqrt{\lambda_k} t_1) \right| \right)^2 \\
&\leq \left(\sum_{k=N+1}^{\infty} \sqrt{\mathbb{E} A_k^2} \left| \cos(\sqrt{\lambda_k} t) (v_k(X) - v_k(X_1)) \right. \right. \\
&\quad \left. \left. + v_k(X) (\cos(\sqrt{\lambda_k} t) - \cos(\sqrt{\lambda_k} t_1)) \right| \right)^2 \\
&\quad + \left(\sum_{k=N+1}^{\infty} \sqrt{\mathbb{E} B_k^2} \left| \sin(\sqrt{\lambda_k} t) (v_k(X) - v_k(X_1)) \right. \right. \\
&\quad \left. \left. + v_k(X) (\sin(\sqrt{\lambda_k} t) - \sin(\sqrt{\lambda_k} t_1)) \right| \right)^2 \\
&\leq \left(\sum_{k=N+1}^{\infty} \sqrt{\mathbb{E} A_k^2} \left(|v_k(X) - v_k(X_1)| + |v_k(X)| \left| \cos(\sqrt{\lambda_k} t) - \cos(\sqrt{\lambda_k} t_1) \right| \right) \right)^2 \\
&\quad + \left(\sum_{k=N+1}^{\infty} \sqrt{\mathbb{E} B_k^2} \left(|v_k(X) - v_k(X_1)| + |v_k(X)| \left| \sin(\sqrt{\lambda_k} t) - \sin(\sqrt{\lambda_k} t_1) \right| \right) \right)^2.
\end{aligned}$$

Since $v_k(X)$ are the eigenfunctions of the Sturm–Liouville problem, the partial derivatives $\partial v_k(X)/\partial x_i$ exist, are continuous, and

$$\begin{aligned}
|v_k(X) - v_k(X_1)| &= \left| \sum_{i=1}^m \frac{\partial v_k(\tilde{X})}{\partial x_i} (x_i - x_{i_1}) \right| \\
&\leq \left| \sum_{i=1}^m \frac{\partial v_k(\tilde{X})}{\partial x_i} \right| |x_i - x_{i_1}| \leq \nu_k h,
\end{aligned}$$

where

$$\nu_k = \sup_{\tilde{X} \in G} \sum_{i=1}^m \left| \frac{\partial v_k(\tilde{X})}{\partial x_i} \right|.$$

It is easy to check that

$$\begin{aligned}
(4.7) \quad \left| \cos \sqrt{\lambda_k} t - \cos \sqrt{\lambda_k} t_1 \right| &\leq \left| 2 \sin \frac{\sqrt{\lambda_k}(t - t_1)}{2} \right| \\
&\leq 2 \left| \frac{\sqrt{\lambda_k}(t - t_1)}{2} \right| \\
&= \sqrt{\lambda_k} |t - t_1| \leq \sqrt{\lambda_k} h
\end{aligned}$$

and

$$(4.8) \quad \left| \sin(\sqrt{\lambda_k}t) - \sin(\sqrt{\lambda_k}t_1) \right| \leq \left| 2 \sin\left(\frac{\lambda_k(t-t_1)}{2}\right) \right| \leq \sqrt{\lambda_k}h$$

for $|t - t_1| < h$.

Then inequalities (4.6), (4.7), and (4.8) imply that

$$(4.9) \quad \begin{aligned} & \left(\mathbb{E} (u_N(X, t) - u_N(X_1, t_1))^2 \right)^{1/2} \\ & \leq \left[\left(\sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} A_k^2} (\nu_k h + c_k \sqrt{\lambda_k} h) \right)^2 \right. \\ & \quad \left. + \left(\sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} B_k^2} (\nu_k h + c_k \sqrt{\lambda_k} h) \right)^2 \right]^{1/2} \\ & = h \left[\left(\sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} A_k^2} (\nu_k + c_k \sqrt{\lambda_k}) \right)^2 \right. \\ & \quad \left. + \left(\sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} B_k^2} (\nu_k + c_k \sqrt{\lambda_k}) \right)^2 \right]^{1/2}. \end{aligned}$$

We also have

$$(4.10) \quad \begin{aligned} & \mathbb{E} |V_N(X, t) - V_N(X_1, t_1)|^2 \\ & = \mathbb{E} \left| \sum_{k=1}^N \left((\hat{A}_k - A_k) \cos(\sqrt{\lambda_k}t) + (\hat{B}_k - B_k) \sin(\sqrt{\lambda_k}t) \right) v_k(X) \right. \\ & \quad \left. - \sum_{k=1}^N \left((\hat{A}_k - A_k) \cos(\sqrt{\lambda_k}t_1) + (\hat{B}_k - B_k) \sin(\sqrt{\lambda_k}t_1) \right) v_k(X_1) \right|^2 \\ & = \mathbb{E} \left| \sum_{k=1}^N \left((\hat{A}_k - A_k) \left(\cos(\sqrt{\lambda_k}t) v_k(X) - \cos(\sqrt{\lambda_k}t_1) v_k(X_1) \right) \right. \right. \\ & \quad \left. \left. + (\hat{B}_k - B_k) \left(\sin(\sqrt{\lambda_k}t) v_k(X) - \sin(\sqrt{\lambda_k}t_1) v_k(X_1) \right) \right) \right|^2 \\ & \leq \sum_{k=1}^N \sum_{l=1}^N \left| \mathbb{E} (\hat{A}_k - A_k) (\hat{A}_l - A_l) \right| \left| \cos(\sqrt{\lambda_k}t) v_k(X) - \cos(\sqrt{\lambda_k}t_1) v_k(X_1) \right| \\ & \quad \times \left| \cos(\sqrt{\lambda_l}t) v_l(X) - \cos(\sqrt{\lambda_l}t_1) v_l(X_1) \right| \\ & \quad + \sum_{k=1}^N \sum_{l=1}^N \left| \mathbb{E} (\hat{B}_k - B_k) (\hat{B}_l - B_l) \right| \left| \sin(\sqrt{\lambda_k}t) v_k(X) - \sin(\sqrt{\lambda_k}t_1) v_k(X_1) \right| \\ & \quad \times \left| \sin(\sqrt{\lambda_l}t) v_l(X) - \sin(\sqrt{\lambda_l}t_1) v_l(X_1) \right| \end{aligned}$$

$$\begin{aligned}
& \leq \left(\sum_{k=1}^N \sqrt{\mathbb{E}(\hat{A}_k - A_k)^2} \left| \cos(\sqrt{\lambda_k} t) (v_k(X) - v_k(X_1)) \right. \right. \\
& \quad \left. \left. + v_k(X) \left(\cos(\sqrt{\lambda_k} t) - \cos(\sqrt{\lambda_k} t_1) \right) \right| \right)^2 \\
& + \left(\sum_{k=1}^N \sqrt{\mathbb{E}(\hat{B}_k - B_k)^2} \left| \sin(\sqrt{\lambda_k} t) (v_k(X) - v_k(X_1)) \right. \right. \\
& \quad \left. \left. + v_k(X) \left(\sin(\sqrt{\lambda_k} t) - \sin(\sqrt{\lambda_k} t_1) \right) \right| \right)^2 \\
& \leq \left(\sum_{k=1}^N \sqrt{\mathbb{E}(\hat{A}_k - A_k)^2} \right. \\
& \quad \left. \times \left(|v_k(X) - v_k(X_1)| + |v_k(X_1)| \left| \cos(\sqrt{\lambda_k} t) - \cos(\sqrt{\lambda_k} t_1) \right| \right) \right)^2 \\
& + \left(\sum_{k=1}^N \sqrt{\mathbb{E}(\hat{B}_k - B_k)^2} |v_k(X) - v_k(X_1)| \right. \\
& \quad \left. + |v_k(X_1)| \left| \sin(\sqrt{\lambda_k} t) - \sin(\sqrt{\lambda_k} t_1) \right| \right)^2 \\
& \leq h^2 \left[\left(\sum_{k=1}^N \sqrt{\mathbb{E}(\hat{A}_k - A_k)^2} (\nu_k + c_k \lambda_k) \right)^2 \right. \\
& \quad \left. + \left(\sum_{k=1}^N \sqrt{\mathbb{E}(\hat{B}_k - B_k)^2} (\nu_k + c_k \lambda_k) \right)^2 \right].
\end{aligned}$$

One can easily obtain that

$$\begin{aligned}
\mathbb{E}(\hat{A}_k - A_k)^2 &= \mathbb{E} \left(\int_G (\hat{\xi}(X) - \xi(X)) v_k(X) dX \right)^2 \\
&= \int_G \int_G \mathbb{E}(\hat{\xi}(X) - \xi(X)) (\hat{\xi}(Y) - \xi(Y)) v_k(X) v_k(Y) dX dY \\
(4.11) \quad &\leq \left(\int_G \sqrt{\mathbb{E}(\hat{\xi}(X) - \xi(X))^2} |v_k(X)| dX \right)^2 \\
&\leq c_k^2 \left(\int_G \sqrt{\mathbb{E}(\hat{\xi}(X) - \xi(X))^2} dX \right)^2 \leq (c_k \chi)^2.
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathbb{E}(\hat{B}_k - B_k)^2 &\leq \left(\frac{1}{\sqrt{\lambda_k}} \int_G \sqrt{\mathbb{E}(\hat{\eta}(X) - \eta(X))^2} |v_k(X)| dX \right)^2 \\
(4.12) \quad &\leq \left(\frac{c_k}{\sqrt{\lambda_k}} \chi \right)^2.
\end{aligned}$$

Thus we derive from (4.9), (4.10), (4.11), and (4.12) that $\sigma(h) = h \cdot A_N$, where

$$(4.13) \quad \begin{aligned} A_N = & \left(\left(\sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} A_k^2} (\nu_k + c_k \sqrt{\lambda_k}) \right)^2 \right. \\ & \left. + \left(\sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} B_k^2} (\nu_k + c_k \sqrt{\lambda_k}) \right)^2 \right)^{1/2} \\ & + \chi \left(\left(\sum_{k=1}^N c_k (\nu_k + c_k \sqrt{\lambda_k}) \right)^2 + \left(\sum_{k=1}^N \frac{c_k}{\sqrt{\lambda_k}} (\nu_k + c_k \sqrt{\lambda_k}) \right)^2 \right)^{1/2}. \end{aligned}$$

The series

$$\begin{aligned} & \sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} A_k^2} (\nu_k + c_k \sqrt{\lambda_k}), & \sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} B_k^2} (\nu_k + c_k \sqrt{\lambda_k}), \\ & \sum_{k=1}^N c_k (\nu_k + c_k \sqrt{\lambda_k}), & \sum_{k=1}^N \frac{c_k}{\sqrt{\lambda_k}} (\nu_k + c_k \sqrt{\lambda_k}) \end{aligned}$$

converge in view of the assumptions of Theorem 3.1.

It is easy to see that

$$(4.14) \quad \begin{aligned} \mathbb{E} |u_N(X, t)|^2 &= \mathbb{E} \left| \sum_{k=N+1}^{\infty} \left(A_k \cos(\sqrt{\lambda_k} t) + B_k \sin(\sqrt{\lambda_k} t) \right) v_k(X) \right|^2 \\ &= \left| \sum_{k=N+1}^{\infty} \sum_{l=N+1}^{\infty} \left(\mathbb{E} A_k A_l \cos(\sqrt{\lambda_k} t) \cos(\sqrt{\lambda_l} t) \right. \right. \\ & \quad \left. \left. + \mathbb{E} B_k B_l \sin(\sqrt{\lambda_k} t) \sin(\sqrt{\lambda_l} t) \right) v_k(X) v_l(X) \right|^2 \\ &\leq \sum_{k=N+1}^{\infty} \sum_{l=N+1}^{\infty} c_k c_l \sqrt{\mathbb{E} A_k^2} \sqrt{\mathbb{E} A_l^2} + \sum_{k=N+1}^{\infty} \sum_{l=N+1}^{\infty} c_k c_l \sqrt{\mathbb{E} B_k^2} \sqrt{\mathbb{E} B_l^2} \\ &\leq \left(\sum_{k=N+1}^{+\infty} c_k \sqrt{\mathbb{E} A_k^2} \right)^2 + \left(\sum_{k=N+1}^{+\infty} c_k \sqrt{\mathbb{E} B_k^2} \right)^2 \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} |V_N(X, t)|^2 &= \mathbb{E} \left| \sum_{k=1}^N \left((\hat{A}_k - A_k) \cos(\sqrt{\lambda_k} t) + (\hat{B}_k - B_k) \sin(\sqrt{\lambda_k} t) \right) v_k(X) \right|^2 \\ &= \left| \sum_{k=1}^N \sum_{l=1}^N v_k(X) v_l(X) \right. \\ & \quad \times \left(\mathbb{E} (\hat{A}_k - A_k) (\hat{A}_l - A_l) \cos(\sqrt{\lambda_k} t) \cos(\sqrt{\lambda_l} t) \right. \\ & \quad \left. \left. + \mathbb{E} (\hat{B}_k - B_k) (\hat{B}_l - B_l) \sin(\sqrt{\lambda_k} t) \sin(\sqrt{\lambda_l} t) \right) \right|^2 \end{aligned}$$

$$\begin{aligned}
(4.15) \quad & \leq \sum_{k=1}^N \sum_{l=1}^N |v_k(X)| \cdot |v_l(X)| \sqrt{\mathbb{E}(\hat{A}_k - A_k)^2} \sqrt{\mathbb{E}(\hat{A}_l - A_l)^2} \\
& + \sum_{k=1}^N \sum_{l=1}^N |v_k(X)| \cdot |v_l(X)| \sqrt{\mathbb{E}(\hat{B}_k - B_k)^2} \sqrt{\mathbb{E}(\hat{B}_l - B_l)^2} \\
& \leq \sum_{k=1}^N \sum_{l=1}^N c_k c_l \sqrt{\mathbb{E}(\hat{A}_k - A_k)^2} \sqrt{\mathbb{E}(\hat{A}_l - A_l)^2} \\
& + \sum_{k=1}^N \sum_{l=1}^N c_k c_l \sqrt{\mathbb{E}(\hat{B}_k - B_k)^2} \sqrt{\mathbb{E}(\hat{B}_l - B_l)^2} \\
& \leq \left(\sum_{k=1}^N c_k \sqrt{\mathbb{E}(\hat{A}_k - A_k)^2} \right)^2 + \left(\sum_{k=1}^N c_k \sqrt{\mathbb{E}(\hat{B}_k - B_k)^2} \right)^2 \\
& \leq \left(\sum_{k=1}^N c_k^2 \chi \right)^2 + \left(\sum_{k=1}^N \frac{c_k^2}{\sqrt{\lambda_k}} \chi \right)^2.
\end{aligned}$$

Thus we obtain from (4.14) and (4.15) that

$$\begin{aligned}
\varepsilon_0(N) = & \left(\left(\sum_{k=N+1}^{+\infty} c_k \sqrt{\mathbb{E} A_k^2} \right)^2 + \left(\sum_{k=N+1}^{+\infty} c_k \sqrt{\mathbb{E} B_k^2} \right)^2 \right)^{1/2} \\
& + \chi \left(\left(\sum_{k=1}^N c_k^2 \right)^2 + \left(\sum_{k=1}^N \frac{c_k^2}{\sqrt{\lambda_k}} \right)^2 \right)^{1/2}.
\end{aligned}$$

Substituting these values of $\sigma(h)$ and ε_0 in equality (4.3), we get for $z = \theta \varepsilon_0(N)$ that

$$\begin{aligned}
I(z) &= \frac{1}{\sqrt{2}} \int_0^z \left(\ln \left(\frac{T \cdot A_N}{2x} + 1 \right) + \sum_{i=1}^m \ln \left(\frac{S_i \cdot A_N}{2x} + 1 \right) \right)^{1/2} dx \\
&\leq \frac{1}{\sqrt{2}} \left[\int_0^z \left(\ln \left(\frac{T \cdot A_N}{2x} + 1 \right) \right)^{1/2} dx + \sum_{i=1}^m \int_0^z \ln \left(\left(\frac{S_i \cdot A_N}{2x} + 1 \right) \right)^{1/2} dx \right] \\
&\leq \left[\int_0^z \left(\frac{T \cdot A_N}{x} \right)^{1/2} dx + \sum_{i=1}^m \int_0^z \left(\frac{S_i \cdot A_N}{x} \right)^{1/2} dx \right] \\
&= \left(T^{1/2} + \sum_{i=1}^m S_i^{1/2} \right) A_N^{1/2} z^{1/2}.
\end{aligned}$$

Then equality (4.2) can be rewritten as

$$\tilde{A}(\delta, \theta) \leq \exp \left\{ -\frac{1}{2} \left(\frac{\delta(1-\theta) - \frac{2}{\theta^{1/2}} \left(T^{1/2} + \sum_{i=1}^m S_i^{1/2} \right) A_N^{1/2} \varepsilon_0(N)^{1/2}}{\varepsilon_0(N)} \right)^2 \right\}.$$

If

$$\left(T^{1/2} + \sum_{i=1}^m S_i^{1/2} \right) A_N^2 \varepsilon_0^2(N) < \delta,$$

then $\tilde{A}(\delta, \theta)$ attains its minimum at

$$\theta = \frac{\left(T^{1/2} + \sum_{i=1}^m S_i^{1/2}\right)^{2/3} A_N^{1/3} \varepsilon_0(N)^{1/3}}{\delta^{2/3}}.$$

Namely

$$\min_{\theta} \tilde{A}(\delta, \theta) = \left\{ \exp - \frac{1}{2} \left(\frac{\delta^{1/3} \left(\delta^{2/3} - 3 \left(T^{1/2} + \sum_{i=1}^m S_i^{1/2} \right)^{2/3} A_N^{1/3} \varepsilon_0^{1/3}(N) \right)}{\varepsilon_0(N)} \right)^2 \right\} \leq \gamma.$$

Therefore, given an accuracy δ , one can construct a model with reliability $1 - \gamma$ if

$$\frac{1}{2} \left(\frac{\delta^{1/3} \left(\delta^{2/3} - 3 \left(T^{1/2} + \sum_{i=1}^m S_i^{1/2} \right)^{2/3} A_N^{1/3} \varepsilon_0^{1/3}(N) \right)}{\varepsilon_0(N)} \right)^2 \geq \ln \frac{1}{\gamma},$$

$$\left(T^{1/2} + \sum_{i=1}^m S_i^{1/2} \right) A_N^2 \varepsilon_0^2 < \delta. \quad \square$$

5. EXAMPLES

Example 5.1. Consider the problem of the vibration of a homogeneous string if its ends are fixed immovable:

$$(5.1) \quad u_{xx} - u_{tt} = 0, \quad x \in [0, l], \quad t \in [0, T],$$

$$(5.2) \quad u_{t=0} = \xi(x), \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = \eta(x),$$

$$(5.3) \quad u|_{x=0} = 0, \quad u|_{x=l} = 0.$$

We assume that the initial deviations of points of the string and their initial velocities are independent strictly sub-Gaussian stochastic processes. The solution of problem (5.1)–(5.3) can be written as

$$(5.4) \quad u(x, t) = \sum_{k=1}^{\infty} \left(A_k \cos\left(\frac{k\pi}{l}t\right) + B_k \sin\left(\frac{k\pi}{l}t\right) \right) \sin\left(\frac{k\pi}{l}x\right),$$

where

$$A_k = \frac{2}{l} \int_0^l \xi(x) \sin\left(\frac{k\pi}{l}x\right) dx, \quad B_k = \frac{2}{k\pi} \int_0^l \eta(x) \sin\left(\frac{k\pi}{l}x\right) dx.$$

Let D denote the rectangle $[0, l] \times [0, T]$. Let $\hat{\xi}(x)$ and $\hat{\eta}(x)$ be models of the processes $\xi(x)$ and $\eta(x)$, respectively. The processes $\hat{\xi}(x)$ and $\hat{\eta}(x)$ are independent. Let a model of the process (5.4) be of the form

$$(5.5) \quad u^N(x, t) = \sum_{k=1}^N \left(\hat{A}_k \cos\left(\frac{k\pi}{l}t\right) + \hat{B}_k \sin\left(\frac{k\pi}{l}t\right) \right) \sin\left(\frac{k\pi}{l}x\right),$$

where

$$\hat{A}_k = \frac{2}{l} \int_0^l \hat{\xi}(x) \sin\left(\frac{k\pi}{l}x\right) dx, \quad \hat{B}_k = \frac{2}{k\pi} \int_0^l \hat{\eta}(x) \sin\left(\frac{k\pi}{l}x\right) dx.$$

Let

$$\Delta_N(x, t, N) = u(x, t) - u^N(x, t) = u_N(x, t) + v_N(x, t)$$

for

$$u_N(x, t) = \sum_{k=N+1}^{\infty} \left(A_k \cos\left(\frac{k\pi}{l}t\right) + B_k \sin\left(\frac{k\pi}{l}t\right) \right) \sin\left(\frac{k\pi}{l}x\right),$$

$$v_N(x, t) = \sum_{k=1}^N \left((\hat{A}_k - A_k) \cos\left(\frac{k\pi}{l}t\right) + (\hat{B}_k - B_k) \sin\left(\frac{k\pi}{l}t\right) \right) \sin\left(\frac{k\pi}{l}x\right).$$

Theorem 5.1. *Let $\{\xi(x), x \in [0, l]\}$ and $\{\eta(x), x \in [0, l]\}$ be independent $\text{SSub}(\Omega)$ stochastic processes. Let the models $\hat{\xi}(x)$ and $\hat{\eta}(x)$ be such that*

$$\frac{1}{l} \int_0^l \sqrt{\mathbb{E}(\hat{\xi}(x) - \xi(x))^2} dx < \Lambda, \quad \frac{1}{l} \int_0^l \sqrt{\mathbb{E}(\hat{\eta}(x) - \eta(x))^2} dx < \Lambda.$$

Then the stochastic process $u^N(X, t)$ defined by (5.5) is a model that approximates the stochastic process $u(X, t)$ with reliability $1 - \gamma$ and accuracy δ in the uniform metric in the domain D if

$$\left(T^{1/2} + l^{1/2} \right) A_N^2 \varepsilon_0^2(N) < \delta,$$

$$\frac{1}{2} \left(\frac{\delta^{1/3} \left(\delta^{2/3} - 3 \left(T^{1/2} + l^{1/2} \right)^{2/3} A_N^{1/3} \varepsilon_0^{1/3}(N) \right)}{\varepsilon_0(N)} \right)^2 \geq \ln \frac{1}{\gamma},$$

where

$$A_N = \frac{2\pi}{l} \left[\left(\left(\sum_{k=N+1}^{\infty} k \sqrt{\mathbb{E} A_k^2} \right)^2 + \left(\sum_{k=N+1}^{\infty} k \sqrt{\mathbb{E} B_k^2} \right)^2 \right)^{1/2} + 2\Lambda N \left[\frac{(N+1)^2}{4} + \frac{l^2}{\pi^2} \right]^{1/2} \right],$$

$$\varepsilon_0(N) = \left(\left(\sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} A_k^2} \right)^2 + \left(\sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} B_k^2} \right)^2 \right)^{1/2} + 2\Lambda \left(N + \frac{l^2}{\pi^2} \left(\sum_{k=1}^N \frac{1}{k} \right)^2 \right)^{1/2}.$$

Proof. This result follows from Theorem 3.2. One only needs to estimate A_N and $\varepsilon_0(N)$ in this case. Consider

$$\begin{aligned} & \mathbb{E} |u_N(x, t) - u_N(x_1, t_1)|^2 \\ &= \mathbb{E} \left| \sum_{k=N+1}^{\infty} \left(A_k \cos\left(\frac{k\pi}{l}t\right) + B_k \sin\left(\frac{k\pi}{l}t\right) \right) \sin\left(\frac{k\pi}{l}x\right) \right. \\ & \quad \left. - \sum_{k=N+1}^{\infty} \left(A_k \cos\left(\frac{k\pi}{l}t_1\right) + B_k \sin\left(\frac{k\pi}{l}t_1\right) \right) \sin\left(\frac{k\pi}{l}x_1\right) \right|^2 \\ &\leq \left(\sum_{k=N+1}^{\infty} \sqrt{\mathbb{E} A_k^2} \left| \cos\left(\frac{k\pi}{l}t\right) \sin\left(\frac{k\pi}{l}x\right) - \cos\left(\frac{k\pi}{l}t_1\right) \sin\left(\frac{k\pi}{l}x_1\right) \right| \right)^2 \\ & \quad + \left(\sum_{k=N+1}^{\infty} \sqrt{\mathbb{E} B_k^2} \left| \sin\left(\frac{k\pi}{l}t\right) \sin\left(\frac{k\pi}{l}x\right) - \sin\left(\frac{k\pi}{l}t_1\right) \sin\left(\frac{k\pi}{l}x_1\right) \right| \right)^2. \end{aligned}$$

It is easy to see that

$$\begin{aligned} & \left| \cos\left(\frac{k\pi}{l}t\right) \sin\left(\frac{k\pi}{l}x\right) - \cos\left(\frac{k\pi}{l}t_1\right) \sin\left(\frac{k\pi}{l}x_1\right) \right| \\ & \leq \left| \cos\left(\frac{k\pi}{l}t\right) - \cos\left(\frac{k\pi}{l}t_1\right) \right| + \left| \sin\left(\frac{k\pi}{l}x\right) - \sin\left(\frac{k\pi}{l}x_1\right) \right| \\ & \leq 2 \left| \sin\left(\frac{\frac{k\pi}{l}(t-t_1)}{2}\right) \right| + 2 \left| \sin\left(\frac{\frac{k\pi}{l}(x-x_1)}{2}\right) \right| \leq 2\frac{k\pi}{l}h. \end{aligned}$$

Similarly

$$\left| \sin\frac{k\pi}{l}t \sin\frac{k\pi}{l}x - \sin\frac{k\pi}{l}t_1 \sin\frac{k\pi}{l}x_1 \right| \leq 2\frac{k\pi}{l}h.$$

Thus

$$\begin{aligned} & \left[\mathbb{E} |u_N(x, t) - u_N(x_1, t_1)|^2 \right]^{1/2} \\ & \leq \frac{2\pi}{l}h \left[\left(\sum_{k=N+1}^{\infty} k\sqrt{\mathbb{E} A_k^2} \right)^2 + \left(\sum_{k=N+1}^{\infty} k\sqrt{\mathbb{E} B_k^2} \right)^2 \right]^{1/2}. \end{aligned}$$

Analogously

$$\begin{aligned} & \left[\mathbb{E} |v_N(x, t) - v_N(x_1, t_1)|^2 \right]^{1/2} \\ & \leq \frac{2\pi}{l}h \left[\left(\sum_{k=1}^N k\sqrt{\mathbb{E}(\hat{A}_k - A_k)^2} \right)^2 + \left(\sum_{k=1}^N k\sqrt{\mathbb{E}(\hat{B}_k - B_k)^2} \right)^2 \right]^{1/2}. \end{aligned}$$

We estimate the second moments:

$$\begin{aligned} \mathbb{E}(\hat{A}_k - A_k)^2 &= \mathbb{E} \left(\frac{2}{l} \int_0^l (\hat{\xi}(x) - \xi(x)) \sin \frac{k\pi}{l}x \, dx \right)^2 \leq \left(\frac{2}{l} \int_0^l \sqrt{\mathbb{E}(\hat{\xi}(x) - \xi(x))^2} \, dx \right)^2 \\ &\leq 4\Lambda^2. \end{aligned}$$

Similarly

$$\begin{aligned} \mathbb{E}(\hat{B}_k - B_k)^2 &\leq \frac{4l^2}{k^2\pi^2}\Lambda^2, \\ \left[\mathbb{E} |v_N(x, t) - v_N(x_1, t_1)|^2 \right]^{1/2} &\leq \frac{2\pi}{l}h \left[\left(\sum_{k=1}^N 2\Lambda k \right)^2 + \left(\sum_{k=1}^N \frac{2l}{\pi} \Lambda \right)^2 \right]^{1/2} \\ &= \frac{4\pi}{l}h\Lambda N \left[\frac{(N+1)^2}{4} + \frac{l^2}{\pi^2} \right]^{1/2}. \end{aligned}$$

Therefore $\sigma(h) = hA_N$ where

$$\begin{aligned} A_N &= \frac{2\pi}{l} \left[\left(\sum_{k=N+1}^{\infty} k\sqrt{\mathbb{E} A_k^2} \right)^2 + \left(\sum_{k=N+1}^{\infty} k\sqrt{\mathbb{E} B_k^2} \right)^2 \right]^{1/2} \\ &\quad + 2\Lambda N \left[\frac{(N+1)^2}{4} + \frac{l^2}{\pi^2} \right]^{1/2}. \end{aligned}$$

Reasoning in the same way, we get

$$\begin{aligned} \mathbb{E} |u_N(x, t)|^2 &\leq \left(\sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} A_k^2} \right)^2 + \left(\sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} B_k^2} \right)^2, \\ \mathbb{E} |v_N(x, t)|^2 &\leq \left(\sum_{k=1}^N \sqrt{\mathbb{E} (\hat{A}_k - A_k)^2} \right)^2 + \left(\sum_{k=1}^N \sqrt{\mathbb{E} (\hat{B}_k - B_k)^2} \right)^2 \\ &\leq 4\Lambda^2 \left(N + \frac{l^2}{\pi^2} \left(\sum_{k=1}^N \frac{1}{k} \right)^2 \right). \end{aligned}$$

Thus

$$\varepsilon_0(N) = \left(\left(\sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} A_k^2} \right)^2 + \left(\sum_{k=N+1}^{+\infty} \sqrt{\mathbb{E} B_k^2} \right)^2 \right)^{1/2} + 2\Lambda \left(N + \frac{l^2}{\pi^2} \left(\sum_{k=1}^N \frac{1}{k} \right)^2 \right)^{1/2}.$$

Applying Theorem 3.2, we complete the proof of Theorem 5.1. $\square$

Example 5.2. Let the assumptions of Theorem 5.1 hold and let $\eta(x) = 0$, $l = \pi$, $T = \pi$, and let $\xi(x)$ be a Gaussian stochastic process such that $\xi(x) = \sum_{j=0}^{\infty} \xi_j \sin(jx)$, where ξ_j are independent Gaussian random variables with $\mathbb{E} \xi_j = 0$ and $\mathbb{E} \xi_j^2 = b^j$. Here b is a number such that $0 < b < 1$.

Let $\hat{\xi}(x) = \hat{\xi}_M(x) = \sum_{j=0}^M \xi_j \sin(jx)$. Then

$$\mathbb{E} (\xi(x) - \hat{\xi}_M(x))^2 = \sum_{j=M+1}^{\infty} b^j \sin^2(jx) \leq \sum_{j=M+1}^{\infty} b^j = \frac{b^{M+1}}{1-b}.$$

Now given Λ , we choose M such that

$$\left(\frac{b^{M+1}}{1-b} \right)^2 < \Lambda,$$

that is,

$$M \leq \frac{\ln(\sqrt{\Lambda}(1-b))}{\ln(b)}.$$

In this case, $A_k = \frac{2}{\pi} \int_0^\pi \xi(x) \sin(kx) dx = \xi_k$; that is, $\mathbb{E} A_k^2 = b^k$. Hence

$$\begin{aligned} A_N &= 2 \left(\sum_{k=N+1}^{\infty} k (\sqrt{b})^k \right) + 4\Lambda N \frac{N+1}{2} \\ &= 2 \frac{(\sqrt{b})^{N+2} \left((N+1)(1-\sqrt{b}) \right) - \sqrt{b}}{(1-\sqrt{b})^2} + 4\Lambda N \frac{N+1}{2}, \\ \varepsilon_0(N) &= \sum_{k=N+1}^{\infty} (\sqrt{b})^k + 2\Lambda N^{1/2} \leq \frac{(\sqrt{b})^{N+1}}{1-b} + 2\Lambda N^{1/2} = \bar{\varepsilon}_0(N). \end{aligned}$$

Therefore if N and Λ are such that

$$A_N \bar{\varepsilon}_0(N) < \left(\frac{\delta}{2\sqrt{\pi}} \right)^{1/2} \quad \text{and} \quad \frac{\delta - 3 \cdot 2^{2/3} \delta^{1/3} \pi^{1/3} A_N^{1/3} (\bar{\varepsilon}_0(N))^{1/3}}{\bar{\varepsilon}_0(N)} \geq 2 \ln \left(\frac{1}{\gamma} \right),$$

then the model approximates a solution with a given accuracy and reliability.

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