THE ORDINAL CONVERGENCE AND GLIVENKO–CANTELLI TYPE THEOREMS IN $L_p(-\infty, \infty)$

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ABSTRACT. Let $F(t)$ be a distribution function and $F_n(t)$ the corresponding empirical distribution function. We find necessary and sufficient conditions for the ordinal convergence $\text{o-lim } F_n = F$ in the spaces $L_p(-\infty, \infty)$.

1. INTRODUCTION AND MAIN RESULTS

Let $\xi, \xi_1, \xi_2, \dots$ be independent identically distributed random variables in $\mathbf{R}^1$ with the distribution function $F(t)$. Consider the corresponding empirical distribution function

$$F_n(t) = \frac{1}{n} \sum_{i=1}^n I_{(-\infty, t)}(\xi_i), \quad -\infty < t < \infty.$$

Glivenko [1] proves that

$$\sup_{-\infty < t < \infty} |F_n(t) - F(t)| \rightarrow 0 \quad \text{a.s.}$$

(this is the well-known Glivenko–Cantelli theorem; it is sometimes called the main theorem of statistics). The shorthand “a.s.” here and in what follows means “almost surely”. Further studies of the asymptotic behavior of the random function $F_n(t)$ are related to the so-called empirical stochastic process

$$\beta_n(t) = \sqrt{n}(F_n(t) - F(t)), \quad -\infty < t < \infty.$$

Surveys of results in this direction can be found in [2]–[4].

We treat the stochastic processes

$$(1) \quad X_n(t) = I_{(-\infty, t)}(\xi_n) - F(t), \quad -\infty < t < \infty,$$

as random elements assuming values in the space $L_p(-\infty, \infty)$, $1 \leq p < \infty$. The well-known laws of large numbers in Banach spaces [5] imply that

$$(2) \quad \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |F_n(t) - F(t)|^p dt = 0 \quad \text{a.s.}$$

if

$$(3) \quad \mathbf{E} \|X_n\|_{L_p} < \infty.$$

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One can improve relation (2) as follows:

$$(4) \quad \begin{aligned} \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} \left| \sup_{n > m} F_n(t) - F(t) \right|^p dt &= 0, \\ \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} \left| \inf_{n > m} F_n(t) - F(t) \right|^p dt &= 0 \quad \text{a.s.} \end{aligned}$$

The ordinal law of large numbers is studied in the author papers [6, 7] for Banach lattices. The ordinal law of large numbers can be rewritten in the form of (4) for the random element X_n defined by equality (1) in the space $L_p(-\infty, \infty)$, $1 \leq p < \infty$. On the other hand, equalities (4) can be viewed as an ordinal variant of the Glivenko–Cantelli theorem in the space $L_p(-\infty, \infty)$.

Below are the main results of the current paper.

Theorem 1. *The empirical distribution function $F_n(t)$ satisfies the ordinal law of large numbers (4) for $1 < p < \infty$ if and only if*

$$(5) \quad \mathbb{E} |\xi|^{1/p} < \infty.$$

Theorem 2. *If $p = 1$ and $0 < F(t) < 1$ for all $t \in (-\infty, \infty)$, then the condition*

$$(6) \quad \int_0^\infty (1 - F(t)) \ln \left(\frac{1}{1 - F(t)} \right) dt + \int_{-\infty}^0 F(t) \ln \left(\frac{1}{F(t)} \right) dt < \infty$$

is sufficient for the ordinal law of large numbers (4).

Remark 1. The ordinal law of large numbers (4) holds if $|\xi| < C < \infty$ almost surely. This follows from Theorems 1 and 2. Another way to prove the same result is to use the Glivenko–Cantelli theorem.

Related to Theorem 2 is the question of whether or not condition (3) (or equivalent condition (5)) is sufficient for the ordinal law of large numbers (4) in the space $L_p(-\infty, \infty)$ if $p = 1$?

The answer to this question is negative. A corresponding example is given at the end of this paper.

2. ORDINAL LAW OF LARGE NUMBERS IN BANACH LATTICES

Let B be a separable Banach lattice equipped with a norm $\|\cdot\|$ and modulus $|\cdot|$ and let X_i , $i \geq 1$, be independent copies of a random element X assuming values in B , $S_n = \sum_{i=1}^n X_i$.

Along with the convergence in the norm, one can study the ordinal convergence in Banach lattices. We recall the corresponding definition (see [8, 9]).

A sequence (x_n) of elements of a Banach lattice B is called o -convergent to an element x , $x = o\text{-}\lim_{n \rightarrow \infty} x_n$, if there exists a sequence (v_n) such that $|x - x_n| < v_n$ and $v_n \downarrow 0$, that is, $v_1 \geq v_2 \geq \dots$ and $\inf_{n \geq 1} v_n = 0$.

We say that a random element X with $\mathbb{E} X = 0$ satisfies the ordinal law of large numbers if

$$(7) \quad o\text{-}\lim_{n \rightarrow \infty} \frac{S_n}{n} = 0 \quad \text{a.s.}$$

It is known that the ordinal convergence of a sequence (x_n) to an element x is equivalent to

$$\limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n = x,$$

where

$$\begin{aligned}\limsup_{n \rightarrow \infty} x_n &= \text{o-lim}_{m \rightarrow \infty} \left(\sup_{n \geq m} x_n \right), \\ \liminf_{n \rightarrow \infty} x_n &= \text{o-lim}_{m \rightarrow \infty} \left(\inf_{n \geq m} x_n \right).\end{aligned}$$

Thus equality (7) can be rewritten in the case of a separable, σ -complete, and ordinally continuous Banach lattice B as follows:

$$\lim_{m \rightarrow \infty} \left\| \sup_{n \geq m} \frac{S_n}{n} \right\| = \lim_{m \rightarrow \infty} \left\| \inf_{n \geq m} \frac{S_n}{n} \right\| = 0 \quad \text{a.s.}$$

This implies relation (4) for the space $L_p(-\infty, \infty)$.

The general results on the ordinal law of large numbers in Banach lattices obtained in the papers [6, 7] will essentially be used in the proof of equalities (4) below.

Theorem A ([7]). *Let B be a separable Banach lattice that does not contain uniformly ℓ_1^n and let X be a random element assuming values in B and such that $\mathbb{E} X = 0$. Then the following conditions are equivalent:*

- (i) *X satisfies the b-law of large numbers, that is,*

$$\lim_{n \rightarrow \infty} \frac{\|S_n\|}{n} = 0 \quad \text{a.s.};$$

- (ii) *X satisfies the ordinal law of large numbers (7);*
 (iii) *the first moment exists,*

$$(8) \quad \mathbb{E} \|X\| < \infty.$$

Let the element $\mathfrak{S}_\psi(X)$ of a Banach lattice be defined by

$$\begin{aligned}\mathfrak{S}_\psi(X) &= \sup(x \in K_\psi), \\ K_\psi &= (\mathbb{E} \eta X : \mathbb{E} \psi^*(\eta) \leq 1),\end{aligned}$$

where

$$\psi^*(t) = \sup_{s \in \mathbf{R}^1} (st - \psi(s))$$

is the convex conjugate function to the N -function $\psi(t)$ [9]. The element $\mathfrak{S}_\psi(X)$ is called the mean ψ -deviation of the random element X (see [6]).

Theorem B ([6]). *Let B be a separable 1-concave Banach lattice, X a random element in B such that $\mathbb{E} X = 0$, and $\psi(t) = |t| \ln(1 + |t|)$. Then X satisfies the ordinal law of large numbers (7) if the mean ψ -deviation $\mathfrak{S}_\psi(X)$ exists in B .*

We also need a general criterion for the ordinal law of large numbers. Put

$$\begin{aligned}\bar{X}_n &= X_n \mathbb{I}(\|X_n\| \leq n), & \tilde{X}_n &= X_n \mathbb{I}(\|X_n\| > n), \\ \bar{S}_n &= \sum_{i=1}^n \bar{X}_i, & \tilde{S}_n &= \sum_{i=1}^n \tilde{X}_i,\end{aligned}$$

where the symbol $\mathbb{I}(A)$ stands for the indicator of a random event A .

Theorem 3. *Let B be a separable σ -complete Banach lattice and let X be a random element in B that satisfies (8) and such that $\mathbb{E} X = 0$. Then the following conditions are equivalent:*

- (i) *X satisfies the ordinal law of large numbers (7);*

(ii) *the following moment exists:*

$$(9) \quad \mathbf{E} \left\| \sup_{n \geq 1} \left| \frac{\bar{S}_n}{n} \right| \right\| < \infty.$$

Proof of Theorem 3. The reasoning given in the paper [7] proves that if inequality (8) holds, then the condition

$$(10) \quad \left\| \sup_{n \geq 1} \left| \frac{S_n}{n} \right| \right\| < \infty$$

is necessary and sufficient for the ordinal law of large numbers (7).

Moreover if inequality (8) holds, then

$$\sum_{n \geq 1} \mathbf{P}(\|X_n\| \geq n) < \infty.$$

The Borel–Cantelli lemma implies that only a finite number of random elements $\tilde{X}_n$ are nonzero almost surely, that is,

$$\left\| \sup_{n \geq 1} \frac{|\tilde{S}_n|}{n} \right\| < \infty \quad \text{a.s.}$$

Since $S_n = \bar{S}_n + \tilde{S}_n$, condition (10) is equivalent to

$$(11) \quad Z = \left\| \sup_{n \geq 1} \frac{|\bar{S}_n|}{n} \right\| < \infty \quad \text{a.s.}$$

It remains to check the implication (11) $\Rightarrow$ (9), since the converse is obvious.

We follow the Yurinskiĭ method [10] and apply it to sums of independent random elements in Banach spaces. Put

$$\begin{aligned} Z_n &= \sup_{1 \leq k \leq n} \frac{|\bar{S}_k|}{k}, \quad Z_{n,i} = \sup_{1 \leq k \leq n} \frac{1}{k} \left| \sum_{j \leq k, j \neq i} \bar{X}_j \right|, \\ \zeta_i &= \mathbf{E}_i \|Z_n\| - \mathbf{E}_{i-1} \|Z_n\|, \\ \mathbf{E}_i \eta &= \mathbf{E}(\eta/F_i), \end{aligned}$$

where $F_i = \sigma(X_1, \dots, X_i)$ is the σ -algebra generated by the random elements $X_1, \dots, X_i$, and F_0 is the trivial σ -algebra.

Then

$$(12) \quad \|Z_n\| - \mathbf{E} \|Z_n\| = \sum_{i=1}^n \zeta_i.$$

Since the random variable ζ_i can be written as

$$\zeta_i = (\mathbf{E}_i - \mathbf{E}_{i-1})(\|Z_n\| - \|Z_{n,i}\|),$$

we apply the inequality

$$\|\|Z_n\| - \|Z_{n,i}\|\| \leq \left\| \frac{\bar{X}_i}{i} \right\|$$

and get

$$(13) \quad |\zeta_i| \leq \left\| \frac{\bar{X}_i}{i} \right\| + \mathbf{E} \left\| \frac{\bar{X}_i}{i} \right\|.$$

Note that (ζ_i) is a martingale difference; thus we derive from (13) that

$$\mathbb{E} \left| \sum_{i=1}^n \zeta_i \right|^2 \leq \sum_{i=1}^n \mathbb{E} |\zeta_i|^2 \leq C \sum_{i=1}^n \mathbb{E} \left\| \frac{\bar{X}_i}{i} \right\|^2.$$

The series on the right hand side can be estimated similarly to the proof of the Kolmogorov strong law of large numbers (see [11, Chapter VII, §8, proof of Theorem 1]). This leads to the following bound:

$$\sum_{i=1}^n \mathbb{E} \left\| \frac{\bar{X}_i}{i} \right\|^2 \leq C \mathbb{E} \|X\|.$$

We conclude from the latter inequality and representation (12) that the sequence of random variables

$$(\|Z_n\| - \mathbb{E} \|Z_n\|)$$

is stochastically bounded, that is,

$$(14) \quad \lim_{x \rightarrow \infty} \sup_{n \geq 1} \mathbb{P} (\|Z_n\| - \mathbb{E} \|Z_n\| > x) = 0.$$

The conditions of the theorem imply that the Banach lattice B is separable and σ -complete, whence it follows that the lattice is ordinally continuous ([8, p. 7]). Applying bound (11) we obtain as $n \rightarrow \infty$ that

$$\|Z_n\| \uparrow \|Z\| < \infty \quad \text{a.s.}$$

By Fatou's lemma,

$$\mathbb{E} \|Z_n\| \uparrow \mathbb{E} \|Z\|.$$

Were inequality (9) wrong, we would obtain

$$\mathbb{E} \|Z\| = \infty.$$

Then taking into account the preceding relations we get

$$\|\|Z_n\| - \mathbb{E} \|Z_n\|\| \xrightarrow{\mathbb{P}} \infty,$$

which contradicts (14). □

3. PROOFS OF THEOREMS 1 AND 2

We start with the proof of Theorem 1. According to Section 2, equalities (4) follow from the ordinal law of large numbers (7) in the space $L_p(-\infty, \infty)$ for the random element $X_n = (X_n(t), -\infty < t < \infty)$ defined by (1). The space $L_p(-\infty, \infty)$ is a separable ([9, Chapter 4, Section 3.3]), p -concave, and p -convex Banach lattice that does not uniformly contain ℓ_1^n in the case of $1 < p < \infty$ ([5, pp. 242, 248]). Thus Theorem A for $1 < p < \infty$ implies that the law of large numbers (4) is equivalent to condition (3).

Now we show that conditions (3) and (5) are equivalent. The definition of the random element X_n implies that

$$\begin{aligned}
 \mathbb{E} \|X_n\|_{L_p} &= \mathbb{E} \left(\int_{-\infty}^{\infty} (|1 - F(t)|^p \mathbb{I}(\xi < t) + |F(t)|^p \mathbb{I}(\xi \geq t)) dt \right)^{1/p} \\
 &= \mathbb{E} \left(\int_{\xi}^{\infty} |1 - F(t)|^p dt + \int_{-\infty}^{\xi} |F(t)|^p dt \right)^{1/p} \\
 (15) \quad &\leq \left(\int_0^{\infty} |1 - F(t)|^p dt + \int_{-\infty}^0 |F(t)|^p dt \right)^{1/p} \\
 &\quad + \mathbb{E} \left(\int_{-|\xi|}^0 |1 - F(t)|^p dt + \int_0^{|\xi|} |F(t)|^p dt \right)^{1/p} \\
 &\leq 2 \mathbb{E} |\xi|^{1/p} + \left(\int_0^{\infty} |1 - F(t)|^p dt + \int_{-\infty}^0 |F(t)|^p dt \right)^{1/p}.
 \end{aligned}$$

Since

$$\mathbb{E} |X_n(t)| = 2(1 - F(t))F(t),$$

we have for $d_\varepsilon > 0$ that

$$\begin{aligned}
 \mathbb{E} \|X_n\|_{L_p} &\geq \|\mathbb{E} |X_n|\|_{L_p} = 2 \left(\int_{-\infty}^{\infty} |(1 - F(t))F(t)|^p dt \right)^{1/p} \\
 (16) \quad &\geq 2 \left(|F(d_\varepsilon)|^p \int_{d_\varepsilon}^{\infty} |1 - F(t)|^p dt + |1 - F(-d_\varepsilon)|^p \int_{-\infty}^{-d_\varepsilon} |F(t)|^p dt \right)^{1/p}.
 \end{aligned}$$

For a given number $\varepsilon > 0$ we choose d_ε such that

$$1 - F(-d_\varepsilon) > 1 - \varepsilon, \quad F(d_\varepsilon) > 1 - \varepsilon.$$

Then bound (16) can be rewritten as follows:

$$\left(\int_{d_\varepsilon}^{\infty} |1 - F(t)|^p dt + \int_{-\infty}^{-d_\varepsilon} |F(t)|^p dt \right)^{1/p} \leq \frac{\mathbb{E} \|X_n\|_{L_p}}{2(1 - \varepsilon)}.$$

This together with (15) implies that

$$\mathbb{E} \|X_n\|_{L_p} \left(1 - \frac{1}{2(1 - \varepsilon)} \right) \leq 2 \mathbb{E} |\xi|^{1/p} + \left(\int_0^{d_\varepsilon} |1 - F(t)|^p dt + \int_{-d_\varepsilon}^0 |F(t)|^p dt \right)^{1/p}.$$

The latter inequality for $\varepsilon < 1/2$ proves the implication (5) $\Rightarrow$ (3).

Now we establish the converse implication. First we consider the case of $0 < F(0) < 1$. We again use bounds (15):

$$\begin{aligned}
 \mathbb{E} \|X_n\|_{L_p} &\geq \mathbb{E} \left(\mathbb{I}(\xi > 0) \int_0^{\xi} |F(t)|^p dt + \mathbb{I}(\xi \leq 0) \int_{\xi}^0 |1 - F(t)|^p dt \right)^{1/p} \\
 &\geq \mathbb{E} (\mathbb{I}(\xi > 0) |\xi| |F(0)|^p + \mathbb{I}(\xi \leq 0) |\xi| |1 - F(0)|^p)^{1/p} \\
 &\geq \min(F(0), 1 - F(0)) \mathbb{E} |\xi|^{1/p},
 \end{aligned}$$

whence the implication (3) $\Rightarrow$ (5) follows.

If the condition $0 < F(0) < 1$ fails, we choose a point x_0 instead of 0 such that $0 < F(x_0) < 1$ and repeat the above reasoning for the point x_0 .

Now we pass to the proof of Theorem 2. In view of Theorem B, the condition

$$(17) \quad \int_{-\infty}^{\infty} \mathfrak{S}_{\psi}(X_n(t)) dt < \infty$$

for $\psi(t) = |t| \ln(1 + |t|)$ is sufficient for the law of large numbers (4) if a Banach lattice is 1-concave.

We prove that condition (17) follows from (6). For this, we need an upper estimate for the mean deviation $\mathfrak{S}_{\psi}(X_n(t))$.

Since $\mathfrak{S}_{\psi}(\cdot)$ is a norm on the Orlicz space of random variables,

$$(18) \quad \begin{aligned} \mathfrak{S}_{\psi}(X_n(t)) &= \mathfrak{S}_{\psi}((1 - F(t))\mathbb{I}(\xi < t) + F(t)\mathbb{I}(\xi \geq t)) \\ &\leq (1 - F(t))\mathfrak{S}_{\psi}\mathbb{I}(\xi < t) + F(t)\mathfrak{S}_{\psi}\mathbb{I}(\xi \geq t). \end{aligned}$$

It is known ([9, Chapter 4, Section 3.6]) that the norm $\mathfrak{S}_{\psi}(\eta)$ is equivalent to the norm

$$\|\eta\|_{\psi,1} = \inf \left(\lambda > 0 : \mathbb{E} \psi \left(\frac{\eta}{\lambda} \right) \leq 1 \right),$$

which is easier to evaluate in the case under consideration.

Indeed

$$\|I(\xi < t)\|_{\psi,1} = \inf \left(\lambda > 0 : \psi \left(\frac{1}{\lambda} \right) F(t) \leq 1 \right) = \frac{1}{\psi^{-1}\left(\frac{1}{F(t)}\right)}.$$

We obtain similarly that

$$\|I(\xi \geq t)\|_{\psi,1} = \frac{1}{\psi^{-1}\left(\frac{1}{1-F(t)}\right)}.$$

Taking into account (18) we get

$$(19) \quad \mathfrak{S}_{\psi}(X_n(t)) \leq C \left(\frac{1 - F(t)}{\psi^{-1}\left(\frac{1}{F(t)}\right)} + \frac{F(t)}{\psi^{-1}\left(\frac{1}{1-F(t)}\right)} \right).$$

It is easy to check that

$$\psi^{-1}(t) \sim \frac{t}{\ln t} \quad \text{as } t \rightarrow \infty.$$

Therefore inequality (19) implies that

$$\mathfrak{S}_{\psi}(X_n(t)) \leq C(1 - F(t))F(t) \left(\ln \left(\frac{1}{F(t)} \right) + \ln \left(\frac{1}{1 - F(t)} \right) \right)$$

for large $|t|$. It is easy to see that the latter estimate and condition (6) imply condition (17).

4. EXAMPLE

Let ξ be a random variable with the distribution function $F(t)$ being continuous and such that

$$1 - F(t) = \frac{1}{t \ln t (\ln \ln t)^2}$$

for $t \geq e^e$. If we set $F(0) = 1/2$, linearly extrapolate $F(t)$ on the interval $0 < t < e^e$, and let $F(-t) = 1 - F(t)$ for negative t , then F is well defined for all real arguments. Let (ξ_i) be a sequence of independent copies of ξ and let $F_n(t)$ be the empirical distribution function constructed from the random variables $\xi_1, \dots, \xi_n$.

It is clear that the random variable ξ satisfies conditions (3) and (5) for $p = 1$. Thus it follows from the results of Section 3 that the law of large numbers (2) holds for the empirical distribution function $F_n(t)$. Let us prove that the ordinal law of large numbers (4) does not hold in the space $L_1(-\infty, \infty)$ for the random variable ξ .

By Theorem 3, it is sufficient to show that

$$\int_{-\infty}^{\infty} \mathbb{E} \sup_{n \geq 1} \frac{1}{n} \left| \sum_{i=1}^n X_i(t) \mathbb{I}(\|X_i\|_{L_1} \leq n) \right| dt = \infty,$$

where $X_i(t)$ are stochastic processes defined by equality (1). Note that condition (3) holds for the process $X_i(t)$ if $p = 1$.

It is easy to see that the latter integral is unbounded if

$$(20) \quad \int_{-\infty}^{\infty} \mathbb{E} \sup_{n \geq 1} \frac{1}{n} |X_n(t) \mathbb{I}(\|X_n\|_{L_1} \leq n)| dt = \infty.$$

Using the same method as in the proof of inequality (15) we write

$$\begin{aligned} \|X_n\|_{L_1} &\leq \int_0^{\infty} (1 - F(t)) dt + \int_{-\infty}^0 F(t) dt + \int_{\xi_n}^0 (1 - F(t)) dt \\ &\leq \mathbb{E} |\xi| + (1 - F(0)) |\xi_n| = \mathbb{E} |\xi| + \frac{1}{2} |\xi_n| \end{aligned}$$

for $\xi_n < 0$, and

$$\|X_n\|_{L_1} \leq \int_0^{\infty} (1 - F(t)) dt + \int_{-\infty}^0 F(t) dt + \int_0^{\xi_n} F(t) dt \leq \mathbb{E} |\xi| + F(0) |\xi_n| = \mathbb{E} |\xi| + \frac{1}{2} |\xi_n|$$

for $\xi_n \geq 0$. Therefore

$$(21) \quad \|X_n\|_{L_1} \leq \mathbb{E} |\xi| + \frac{1}{2} |\xi_n|$$

in either case. Since

$$\mathbb{I}(|\xi_n| \leq n) \leq \mathbb{I}\left(\mathbb{E} |\xi| + \frac{|\xi_n|}{2} \leq n\right)$$

for $n > 2 \mathbb{E} |\xi|$, the latter inequality and bound (21) imply that relation (20) follows from

$$(22) \quad \int_{-\infty}^{\infty} \mathbb{E} \sup_{n \geq 1} \frac{1}{n} |X_n(t)| \mathbb{I}(|\xi_n| \leq n) dt = \infty.$$

It is obvious that

$$|X_n(t)| \geq (1 - F(t)) \mathbb{I}(\xi_n < t),$$

whence

$$\begin{aligned} (23) \quad &\int_{-\infty}^{\infty} \mathbb{E} \sup_{n \geq 1} \frac{1}{n} |X_n(t)| \mathbb{I}(|\xi_n| \leq n) dt \\ &\geq (1 - F(0)) \int_{-\infty}^0 \mathbb{E} \sup_{n \geq 1} \frac{1}{n} \mathbb{I}(\xi_n \leq t) \mathbb{I}(|\xi_n| \leq n) dt. \end{aligned}$$

Now we estimate the expression under the integral sign on the right hand side of (23). Put

$$A_n = \{\xi_n \leq t, \xi_n > -n\}.$$

Then

$$\begin{aligned} (24) \quad \mathbb{E} \sup_{n \geq 1} \frac{I(A_n)}{n} &= \sum_{n > |t|} \frac{1}{n} \mathbb{P} \left(A_n \cap \left(\bigcap_{k=1}^{n-1} \bar{A}_k \right) \right) \\ &\geq \sum_{n > |t|} \frac{1}{n} (F(t) - F(-n)) \prod_{k=1}^{n-1} (1 - F(t) + F(-n)) \\ &\geq \sum_{n > |t|} \frac{1}{n} (F(t) - F(-n)) (1 - F(t))^{n-1} \end{aligned}$$

for $t < 0$. It is not complicated to obtain an upper bound for the following series if $t < -e^e$, namely

$$(25) \quad \sum_{n>|t|} \frac{1}{n} F(-n) = \sum_{n>|t|} \frac{1}{n^2 \ln n (\ln \ln n)^2} \leq \frac{1}{\ln |t| (\ln \ln |t|)^2} \sum_{n>|t|} \frac{1}{n^2} \sim F(t).$$

Since $\int_{-\infty}^0 F(t) dt < \infty$, we collect together estimates (23)–(25) and see that relation (22) follows from

$$(26) \quad \int_{-\infty}^0 \sum_{n>|t|} \frac{1}{n} F(t) (1 - F(t))^{n-1} dt = \infty.$$

Now we find the asymptotic behavior of the expression under the integral sign in (26) for large $|t|$. Using the Taylor expansion

$$\ln \frac{1}{1-x} = \sum_{n \geq 1} \frac{x^n}{n}, \quad -1 < x < 1,$$

we get for $t < -e^e$ that

$$(27) \quad \sum_{n \geq 1} \frac{1}{n} (1 - F(t))^n = \ln \frac{1}{F(t)} = \ln |t| + \ln \ln |t| + 2 \ln \ln \ln |t|.$$

Then we estimate the sum of the first terms of the series in (27):

$$(28) \quad \begin{aligned} \sum_{1 \leq n \leq |t|} \frac{1}{n} (1 - F(t))^n &= \int_1^{|t|} \frac{(1 - F(t))^x}{x} dx + O(1) \\ &= \ln |t| (1 - F(t))^{|t|} - \ln(1 - F(t)) \int_1^{|t|} \ln x (1 - F(t))^x dx + O(1) \\ &= d_1 + d_2 + O(1). \end{aligned}$$

Applying an elementary inequality $(1 - x^{-1})^x < e^{-1}$, $x > 1$, we write a bound for the first term in (28):

$$\begin{aligned} d_1 &= \ln |t| \left(1 - \frac{1}{|t| \ln |t| (\ln \ln |t|)^2} \right)^{|t|} \\ &\leq \ln |t| \exp \left\{ -1 / \ln |t| (\ln \ln |t|)^2 \right\} = \ln |t| \left(1 - \frac{1}{\ln |t| (\ln \ln |t|)^2} + o \left(\frac{1}{\ln |t| \ln \ln |t|} \right)^2 \right) \\ &= \ln |t| - \frac{1}{(\ln \ln |t|)^2} + o \left(\frac{1}{\ln \ln |t|} \right)^2. \end{aligned}$$

The second term in (28) is estimated in the following way:

$$d_2 \sim F(t) \int_1^{|t|} \ln x (1 - F(t))^x dx \leq \frac{1}{|t| \ln |t| (\ln \ln |t|)^2} \int_1^{|t|} \ln x dx \leq \frac{1}{(\ln \ln |t|)^2}.$$

These estimates together with (28) yield

$$\sum_{1 \leq n \leq |t|} \frac{1}{n} (1 - F(t))^n \leq \ln |t| + O(1),$$

whence

$$\sum_{n>|t|} \frac{1}{n} (1 - F(t))^n \geq \ln \ln |t| + O(\ln \ln \ln |t|)$$

by (27). This is what had to be proved, in fact. Indeed

$$\begin{aligned} \sum_{n>|t|} \frac{1}{n} F(t)(1-F(t))^{n-1} &\geq \frac{F(t)}{1-F(t)} (\ln \ln |t| + O(\ln \ln \ln |t|)) \\ &\geq \frac{1}{|t| \ln |t| \ln \ln |t|} + O\left(\frac{\ln \ln \ln |t|}{|t| \ln |t| (\ln \ln |t|)^2}\right). \end{aligned}$$

As a corollary of the latter inequality we prove that integral (26) is unbounded. We proved above that this means that equality (4) does not hold for the empirical distribution function $F_n(t)$ in the space $L_1(-\infty, \infty)$.

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