EXISTENCE AND UNIQUENESS OF THE SOLUTION OF A STOCHASTIC DIFFERENTIAL EQUATION, DRIVEN BY FRACTIONAL BROWNIAN MOTION WITH A STABILIZING TERM

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ABSTRACT. A stochastic differential equation driven by a Wiener process and fractional Brownian motion is considered in the paper. We prove the existence and uniqueness of the solution if the equation contains a certain stabilizing term.

1. INTRODUCTION

Consider the following stochastic equation written in integral form:

$$(1.1) \quad \begin{aligned} X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dW_s + \int_0^t c(s, X_s) dB_s^H \\ + \varepsilon \int_0^t c(s, X_s) dV_s, \quad t \in [0, T], \end{aligned}$$

where $a, b, c: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable functions, V and W are independent standard Brownian motions, $\varepsilon > 0$, and B^H is a fractional Brownian motion independent of W and V with the Hurst parameter $H \in (3/4, 1]$. We assume that the coefficients of this equation are such that all stochastic integrals on the right hand side of (1.1) are well defined.

The integral $\varepsilon \int_0^t c(s, X_s) dV_s$ in (1.1) is considered as a stabilizing term. This term allows us to prove the existence and uniqueness of the solution adapted to the flow $\mathcal{F}'_t$, $t \geq 0$, where

$$(1.2) \quad \mathcal{F}'_t = \sigma \{X_0, W_s, (\varepsilon V_s + B_s^H) \mid s \in [0, t]\}.$$

The results of this paper remain true for the case of $b = 0$. The case of $b = 0$ and $\varepsilon = 0$ corresponds to the equation containing a fractional Brownian motion only (an equation without a standard Brownian motion). Conditions for the existence and uniqueness of a solution for the latter equation are considered in the paper [1]. One of these conditions is that the derivative $\partial_x c(s, x)$ exists. We avoid this and some other conditions by introducing a stabilizing term. Conditions for the existence and uniqueness of a solution of a semilinear equation with $\varepsilon = 0$, $b(s, x) = bx$, and $c(s, x) = cx$ are obtained in [2].

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2. MAIN PART

2.1. Existence and uniqueness of solutions of some auxiliary equations. According to the paper [3], the process $M_t^{H,\varepsilon} := \varepsilon V_t + B_t^H$ considered with respect to the flow $\mathcal{F}_t$, $t \in [0, T]$, where

$$\mathcal{F}_t = \sigma \{ \varepsilon V_s + B_s^H, s \in [0, t] \},$$

is equivalent to the standard Brownian motion. Since X_0 , W , B^H , and V are independent, this implies that the process $M_t^{H,\varepsilon}$ is equivalent to the standard Brownian motion with respect to the flow $\mathcal{F}_t'$. Using Hitsuda's results [4], one can represent this process in the following form:

$$\begin{aligned} M_t^{H,\varepsilon} &= \varepsilon \left(V_t + \frac{1}{\varepsilon} B_t^H \right) \\ (2.1) \quad &= \varepsilon \left(V_t' + \int_0^t \int_0^s r_\varepsilon(s, u) dV_u' ds \right) \\ &= \varepsilon V_t' + \int_0^t \int_0^s \varepsilon r_\varepsilon(s, u) dV_u' ds, \end{aligned}$$

where V' is another standard Brownian motion adapted to the flow $\mathcal{F}_t$ ($\mathcal{F}_t'$) and where $r_\varepsilon \in \mathbf{L}^2([0, T]^2)$ is a corresponding kernel dependent on ε and H .

Using (2.1) we rewrite equation (1.1) as follows:

$$\begin{aligned} (2.2) \quad X_t &= X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dW_s + \varepsilon \int_0^t c(s, X_s) dV_s' \\ &\quad + \int_0^t c(s, X_s) \int_0^s \varepsilon r_\varepsilon(s, u) dV_u' ds. \end{aligned}$$

Let

$$(2.3) \quad R_t^\varepsilon := \int_0^t \left(\int_0^s \varepsilon r_\varepsilon(s, u) dV_u' \right)^2 ds, \quad t \in [0, T].$$

Define a sequence of stopping times τ_N , $N \in \mathbb{N}$, by

$$(2.4) \quad \tau_N := \inf \{ t \in [0, T] : R_t^\varepsilon > N \} \wedge T.$$

We set $\tau_N := T$ if $\{ t \in [0, T] : R_t^\varepsilon > N \} = \emptyset$.

Now we prove that $\tau_N \rightarrow T$ with probability one.

Lemma 2.1. *Let a process R_t^ε , $\varepsilon > 0$, $t \in [0, T]$, and stopping times τ_N , $N \in \mathbb{N}$, be defined by (2.3) and (2.4), respectively. Then for all $\varepsilon > 0$,*

$$\tau_N \rightarrow T$$

with probability one.

Proof. We evaluate the probability of the event that $\tau_N < T$:

$$\begin{aligned} \mathbb{P}\{\tau_N < T\} &= \mathbb{P}\left\{ \int_0^T \left(\int_0^s \varepsilon r_\varepsilon(s, u) dV_u' \right)^2 ds > N \right\} \\ &\leq \frac{\varepsilon^2}{N} \mathbb{E} \int_0^T \left(\int_0^s r_\varepsilon(s, u) dV_u' \right)^2 ds \\ &= \frac{\varepsilon^2}{N} \int_0^T \int_0^s r_\varepsilon^2(s, u) du ds \\ &\leq \frac{\varepsilon^2}{N} \int_0^T \int_0^T r_\varepsilon^2(s, u) du ds \leq \frac{\varepsilon^2 K_\varepsilon}{N}, \end{aligned}$$

where

$$K_\varepsilon = \|r_\varepsilon\|_{\mathbf{L}^2([0,T]^2)}^2 < \infty.$$

Thus $\mathbb{P}\{\tau_N < T\} \rightarrow 0$ as $N \rightarrow \infty$, whence Lemma 2.1 follows, since τ_N are nondecreasing in N . $\square$

Remark 2.2. We proved, in fact, that for all $\omega \in \Omega_0$ where the event Ω_0 is such that $\mathbb{P}(\Omega_0) = 1$, there exists a number $N = N(\omega)$ for which $\tau_{N(\omega)} = T$.

Without loss of generality we assume that $\varepsilon = 1$ and $r(t, s) := r_1(t, s)$ in what follows. Consider the following family of equations:

$$(2.5) \quad \begin{aligned} X_t^N = X_0 &+ \int_0^t a(s, X_s^N) ds + \int_0^t b(s, X_s^N) dW_s + \int_0^t c(s, X_s^N) dV'_s \\ &+ \int_0^{t \wedge \tau_N} c(s, X_s^N) \int_0^s r(s, u) dV'_u ds. \end{aligned}$$

We prove that the solution X_t^N of equation (2.5) exists for all $N \in \mathbb{N}$.

Theorem 2.3. *Assume that*

1. *The functions $a(\cdot, 0)$, $b(\cdot, 0)$, and $c(\cdot, 0)$ are bounded by some constant L in the interval $[0, T]$.*
2. *There exists an increasing function $l(s): [0, T] \rightarrow \mathbb{R}$ such that*

$$(2.6) \quad \begin{aligned} &(a(s, x) - a(s, y))^2 + (b(s, x) - b(s, y))^2 + (c(s, x) - c(s, y))^2 \\ &\leq l(s)(x - y)^2 \end{aligned}$$

for all $x, y \in \mathbb{R}$.

If X_0 is $\mathcal{F}'_0$ -measurable and $\mathbb{E} X_0^2 < \infty$, then equation (2.5) possesses an $\mathcal{F}'_t$ -measurable solution X_t^N such that $\mathbb{E}(X_t^N)^2 < \infty$.

Proof. Following the idea of the proof of Theorem 3 in Chapter 4, Section 1 of [5], we construct the solution by successive approximations. Put $X_t^{N,0} := X_0$ and

$$(2.7) \quad \begin{aligned} X_t^{N,n+1} = X_0 &+ \int_0^t a(s, X_s^{N,n}) ds + \int_0^t b(s, X_s^{N,n}) dW_s + \int_0^t c(s, X_s^{N,n}) dV'_s \\ &+ \int_0^{t \wedge \tau_N} c(s, X_s^{N,n}) \int_0^s r(s, u) dV'_u ds. \end{aligned}$$

If $X_t^{N,n}$ is defined for some n , is $\mathcal{F}'_t$ -measurable, and $\mathbb{E}(X_t^{N,n})^2$ is bounded on $[0, T]$, then the integrals on the right hand side of (2.7) are well defined and are $\mathcal{F}'_t$ -measurable. Now we prove that $\mathbb{E}(X_t^{N,n+1})^2$ is bounded in $[0, T]$. Indeed,

$$\begin{aligned} \left(X_t^{N,n+1} - X_t^{N,0}\right)^2 &\leq 4 \left(\int_0^t a(s, X_s^{N,n}) ds \right)^2 + 4 \left(\int_0^t b(s, X_s^{N,n}) dW_s \right)^2 \\ &\quad + 4 \left(\int_0^t c(s, X_s^{N,n}) dV'_s \right)^2 \\ &\quad + 4 \left(\int_0^{t \wedge \tau_N} c(s, X_s^{N,n}) \int_0^s r(s, u) dV'_u ds \right)^2, \end{aligned}$$

$$\begin{aligned}
\mathbb{E} \left(\int_0^t a(s, X_s^{N,n}) ds \right)^2 &\leq T \int_0^t \mathbb{E} (a(s, X_s^{N,n}))^2 ds \\
&\leq 2T \int_0^t a^2(s, 0) ds + 2T \int_0^t \mathbb{E} (a(s, X_s^{N,n}) - a(s, 0))^2 ds \\
&\leq 2(LT)^2 + 2Tl(T) \int_0^T \mathbb{E} (X_s^{N,n})^2 ds, \\
\mathbb{E} \left(\int_0^t b(s, X_s^{N,n}) dW_s \right)^2 &\leq 2 \int_0^t b^2(s, 0) ds + 2 \int_0^t \mathbb{E} (b(s, X_s^{N,n}) - b(s, 0))^2 ds \\
&\leq 2TL^2 + 2l(T) \int_0^T \mathbb{E} (X_s^{N,n})^2 ds.
\end{aligned}$$

Similarly,

$$\mathbb{E} \left(\int_0^t c(s, X_s^{N,n}) dV'_s \right)^2 \leq \left(2TL^2 + 2l(T) \int_0^T \mathbb{E} (X_s^{N,n})^2 ds \right).$$

By the definition of the stopping times τ_N we have

$$\begin{aligned}
&\mathbb{E} \left(\int_0^{t \wedge \tau_N} c(s, X_s^{N,n}) \int_0^s r(s, u) dV'_u ds \right)^2 \\
&\leq \mathbb{E} \left[\int_0^{t \wedge \tau_N} (c(s, X_s^{N,n}))^2 ds \int_0^{t \wedge \tau_N} \left(\int_0^s r(s, u) dV'_u \right)^2 ds \right] \\
&\leq N \mathbb{E} \left(\int_0^t c(s, X_s^{N,n}) ds \right)^2 \leq N \left(2(LT)^2 + 2Tl(T) \int_0^T \mathbb{E} (X_s^{N,n})^2 ds \right)
\end{aligned}$$

in view of Hölder's inequality. The latter bounds imply that there exists a constant C_N such that

$$\mathbb{E} (X_t^{N,n+1})^2 \leq C_N \left(\mathbb{E} X_0^2 + \int_0^t \mathbb{E} (X_s^{N,n})^2 ds + L^2 \right).$$

Since the process X_0 is $\mathcal{F}'_t$ -measurable and $\mathbb{E}(X_t^{N,0})^2 = \mathbb{E} X_0^2$, the processes $X_t^{N,n}$ are well defined for all n , are $\mathcal{F}'_t$ -measurable, and the second moments $\mathbb{E}(X_t^{N,n})^2$ are bounded.

Further,

$$\begin{aligned}
\mathbb{E} (X_t^{N,n+1} - X_t^{N,n})^2 &\leq 4 \mathbb{E} \left(\int_0^t (a(s, X_s^{N,n}) - a(s, X_s^{N,n-1})) ds \right)^2 \\
&\quad + 4 \mathbb{E} \left(\int_0^t (b(s, X_s^{N,n}) - b(s, X_s^{N,n-1})) dW_s \right)^2 \\
&\quad + 4 \mathbb{E} \left(\int_0^t (c(s, X_s^{N,n}) - c(s, X_s^{N,n-1})) dV'_s \right)^2 \\
&\quad + 4 \mathbb{E} \left(\int_0^{t \wedge \tau_N} (c(s, X_s^{N,n}) - c(s, X_s^{N,n-1})) \int_0^s r(s, u) dV'_u ds \right)^2.
\end{aligned}$$

Using the definition of the stopping times τ_N and condition 2 we prove that there exists a constant $K_{N,T}$ such that

$$\begin{aligned}
\mathbb{E} (X_t^{N,n+1} - X_t^{N,n})^2 &\leq K_{N,T} \int_0^t \mathbb{E} (X_s^{N,n} - X_s^{N,n-1})^2 ds, \quad n \geq 1, \\
\mathbb{E} (X_t^{N,1} - X_t^{N,0})^2 &\leq K_{N,T} \int_0^t \mathbb{E} (X_s^{N,0})^2 ds = K_{N,T} t \mathbb{E} X_0^2
\end{aligned}$$

by Hölder's inequality. This implies that

$$\mathbb{E} \left(X_t^{N,n+1} - X_t^{N,n} \right)^2 \leq (K_{N,T})^{n+1} \frac{T^{n+1}}{(n+1)!} \mathbb{E} X_0^2.$$

Since

$$\sum_{n \in \mathbb{N}} \sup_{t \in [0, T]} \left(\mathbb{E} \left(X_t^{N,n+1} - X_t^{N,n} \right)^2 \right)^{1/2} < \infty,$$

the stochastic process $X_t^{N,n}$ converges in the mean square sense to X_t^N as $n \rightarrow \infty$. Moreover,

$$(2.8) \quad \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \mathbb{E} \left(X_t^{N,n} - X_t^N \right)^2 = 0.$$

Note that the process X_t^N is $\mathcal{F}_t'$ -measurable.

Now we prove that (2.5) holds almost surely. Equality (2.5) holds almost surely if the integrals on the right hand side of (2.7) converge in the mean square sense to the corresponding integrals on the right hand side of (2.5) as $n \rightarrow \infty$. We prove this convergence for the integrals involving the stopping time τ_N . The proof for other integrals is the same. We have

$$\begin{aligned} & \mathbb{E} \left(\int_0^{t \wedge \tau_N} (c(s, X_s^{N,n}) - c(s, X_s^N)) \int_0^s r(s, u) dV'_u ds \right)^2 \\ & \leq TN\varepsilon^2 \int_0^T \mathbb{E} (c(s, X_s^{N,n}) - c(s, X_s^N))^2 ds \\ & \leq Tl(T)N \int_0^T \mathbb{E} (X_s^{N,n} - X_s^N)^2 ds. \end{aligned}$$

The expression on the right hand side approaches zero in view of (2.8). $\square$

The following result provides conditions for the uniqueness of a solution of equation (2.5).

Theorem 2.4. *Let the coefficients of equation (2.5) be such that, for any $r \geq 0$, there exists a constant l_r for which*

$$(a(s, x) - a(s, y))^2 + (b(s, x) - b(s, y))^2 + (c(s, x) - c(s, y))^2 \leq l_r(x - y)^2$$

for $|x| \leq r$, $|y| \leq r$, and $s \leq r$. If an $\mathcal{F}_t'$ -adapted solution of equation (2.5) exists in the interval $[0, T]$, then it is unique.

Proof. Let X_t^N and $\hat{X}_t^N$ be two solutions of equation (2.5). For every $r > 0$, we consider the stopping times $\xi_r := \inf\{t \in [0, T] : |X_t^N| + |\hat{X}_t^N| \geq r\} \wedge r \wedge T$. Then

$$\begin{aligned} X_{t \wedge \xi_r}^N - \hat{X}_{t \wedge \xi_r}^N &= \int_0^{t \wedge \xi_r} \left(a(s, X_s^N) - a(s, \hat{X}_s^N) \right) ds \\ &+ \int_0^{t \wedge \xi_r} \left(b(s, X_s^N) - b(s, \hat{X}_s^N) \right) dW_s \\ &+ \int_0^{t \wedge \xi_r} \left(c(s, X_s^N) - c(s, \hat{X}_s^N) \right) dV'_s \\ &+ \int_0^{t \wedge \xi_r \wedge \tau_N} \left(c(s, X_s^N) - c(s, \hat{X}_s^N) \right) \int_0^s r(s, u) dV'_u ds. \end{aligned}$$

Using the properties of stochastic integrals, Hölder's inequality, and boundedness of the second moments of ξ_r , we obtain that

$$\begin{aligned} \mathbb{E} \left(\int_0^{t \wedge \xi_r} \left(a(s, X_s^N) - a(s, \hat{X}_s^N) \right) ds \right)^2 &\leq T l_r \mathbb{E} \int_0^{t \wedge \xi_r} \left(X_s^N - \hat{X}_s^N \right)^2 ds, \\ \mathbb{E} \left(\int_0^{t \wedge \xi_r} \left(b(s, X_s^N) - b(s, \hat{X}_s^N) \right) dW_s \right)^2 &\leq l_r \mathbb{E} \int_0^{t \wedge \xi_r} \left(X_s^N - \hat{X}_s^N \right)^2 ds, \\ \mathbb{E} \left(\int_0^{t \wedge \xi_r} \left(c(s, X_s^N) - c(s, \hat{X}_s^N) \right) dW_s \right)^2 &\leq l_r \mathbb{E} \int_0^{t \wedge \xi_r} \left(X_s^N - \hat{X}_s^N \right)^2 ds \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E} \left(\int_0^{t \wedge \xi_r \wedge \tau_N} \left(c(s, X_s^N) - c(s, \hat{X}_s^N) \right) \int_0^s r(s, u) dV'_u ds \right)^2 \\ &\leq l_r N \mathbb{E} \int_0^{t \wedge \xi_r} \left(X_s^N - \hat{X}_s^N \right)^2 ds \end{aligned}$$

in view of the assumptions of the theorem. Thus

$$\mathbb{E} \left(X_{t \wedge \xi_r}^N - \hat{X}_{t \wedge \xi_r}^N \right)^2 \leq L_{N,T} \mathbb{E} \int_0^{t \wedge \xi_r} \left(X_s^N - \hat{X}_s^N \right)^2 ds$$

for some $L_{N,T}$. This implies that

$$(2.9) \quad \mathbb{E} \left(X_t^N - \hat{X}_t^N \right)^2 \mathbb{I}_{\{\xi_r > t\}} \leq L_{N,T} \int_0^t \mathbb{E} \left(X_s^N - \hat{X}_s^N \right)^2 \mathbb{I}_{\{\xi_r > s\}} ds.$$

Now we obtain from the Gronwall lemma that

$$\mathbb{E} \left(X_t^N - \hat{X}_t^N \right)^2 \mathbb{I}_{\{\xi_r > t\}} = 0.$$

Since

$$\mathbb{I}_{\{\xi_r > t\}} \rightarrow 1 \quad \text{as } r \rightarrow \infty,$$

we have

$$X_t^N = \hat{X}_t^N \quad \text{for } t \in [0, T].$$

The solutions are continuous at the point T , whence we get that $X_T^N = \hat{X}_T^N$. The theorem is proved. $\square$

Lemma 2.5. *Let the assumptions of Theorem 2.4 hold. We also assume that, for some constant L ,*

$$(2.10) \quad (a(s, x))^2 + (b(s, x))^2 + (c(s, x))^2 \leq L(1 + x^2).$$

If a solution of equation (2.5) exists, then there exists a constant K such that

$$(2.11) \quad \mathbb{E} \left(\sup_{s \leq t} \left(X_s^N \right)^2 \mid X_0 \right) \leq K (X_0^2 + 1) e^{Kt}.$$

Proof. We estimate $\mathbb{E}((X_t^N)^2 \mid X_0)$:

$$\begin{aligned}
 \mathbb{E}((X_t^N)^2 \mid X_0) &\leq 5 \mathbb{E} \left[X_0^2 + \left(\int_0^t a(s, X_s^N) ds \right)^2 + \left(\int_0^t b(s, X_s^N) dW_s \right)^2 \right. \\
 &\quad + \left(\int_0^t c(s, X_s^N) dV'_s \right)^2 \\
 &\quad \left. + \left(\int_0^{t \wedge \tau_N} c(s, X_s^N) \int_0^s r(s, u) dV'_u ds \right)^2 \mid X_0 \right] \\
 (2.12) \quad &\leq 5 \mathbb{E} \left[X_0^2 + t \int_0^t a(s, X_s^N)^2 ds + \int_0^t b(s, X_s^N)^2 ds \right. \\
 &\quad \left. + \int_0^t c(s, X_s^N)^2 ds + N \int_0^t c(s, X_s^N)^2 ds \mid X_0 \right] \\
 &\leq 5 \left(X_0^2 + L(T+1+N) \int_0^t (1 + \mathbb{E}((X_s^N)^2 \mid X_0)) ds \right).
 \end{aligned}$$

Thus the Gronwall lemma implies that

$$1 + \mathbb{E}((X_t^N)^2 \mid X_0) \leq (5X_0^2 + 1) e^{5L(1+N+T)t}.$$

Using the latter bound, the following inequalities

$$\begin{aligned}
 \mathbb{E} \left(\sup_{s \leq t} \left(\int_0^s a(u, X_u^N) du \right)^2 \mid X_0 \right) &\leq t \int_0^t \mathbb{E} \left(a(s, X_s^N)^2 \mid X_0 \right) ds, \\
 \mathbb{E} \left(\sup_{s \leq t} \left(\int_0^s b(u, X_u^N) dW_u \right)^2 \mid X_0 \right) &\leq 4 \int_0^t \mathbb{E} \left(b(s, X_s^N)^2 \mid X_0 \right) ds, \\
 \mathbb{E} \left(\sup_{s \leq t} \left(\int_0^s c(u, X_u^N) dV'_u \right)^2 \mid X_0 \right) &\leq 4 \int_0^t \mathbb{E} \left(c(s, X_s^N)^2 \mid X_0 \right) ds, \\
 \mathbb{E} \left(\sup_{s \leq t} \left(\int_0^{s \wedge \tau_N} c(u, X_u^N) \int_0^u r(u, v) dV'_v du \right)^2 \mid X_0 \right) &\leq Nt \int_0^t \mathbb{E} \left(c(s, X_s^N)^2 \mid X_0 \right) ds
 \end{aligned}$$

and (2.10) and (2.12) we prove (2.11). $\square$

Theorem 2.6. *Let the assumptions of Theorem 2.4 hold for the coefficients of equation (2.5). We also assume condition (2.10). If X_0 is $\mathcal{F}'_0$ -measurable, then equation (2.5) possesses an $\mathcal{F}'_t$ -measurable solution X_t^N in $[0, T]$.*

Proof. Let $\phi_n(x) = x$ for $|x| < n$ and $\phi_n(x) = nx/|x|$ for $x \geq n$. Also let

$$a_n(t, x) = a(t, \phi_n(x)), \quad b_n(t, x) = b(t, \phi_n(x)), \quad b_n(t, x) = b(t, \phi_n(x)).$$

Denote by X_t^{Nn} a solution of the equation

$$\begin{aligned}
 X_t^{Nn} &= \phi_n(X_0) + \int_0^t a_n(s, X_s^{Nn}) ds + \int_0^t b_n(s, X_s^{Nn}) dW_s \\
 (2.13) \quad &+ \int_0^t c_n(s, X_s^{Nn}) dV'_s + \int_0^{t \wedge \tau_N} c_n(s, X_s^{Nn}) \int_0^s r(s, u) dV'_u ds.
 \end{aligned}$$

The existence of an $\mathcal{F}'_t$ -measurable solution follows from Theorem 2.3. Indeed, condition 1 follows from (2.10), while condition 2 holds for $l(s) = l_n$ if $s \leq n$ and for $l(s) = l_s$ if

$s \geq n$. Put $\zeta_n = \inf\{t \in [0, T]: X_t^{Nn} > n\} \wedge T$. It is easy to see that

$$X_t^{Nn_1} = X_t^{Nn}$$

for $n_1 > n$ and all $t < \zeta_n$. Lemma 2.5 implies that there exists a function $g(t)$ that does not depend on n and such that

$$\mathbb{E} \left(\sup_{s \leq t} (X_s^{Nn})^2 \mid X_0 \right) \leq g(t) (1 + \phi_n^2(X_0)).$$

Thus

$$(2.14) \quad \mathbb{P} \{ \zeta_n \leq t \mid X_0 \} \leq \frac{1}{n^2} \mathbb{E} \left(\sup_{s \leq t} (X_s^{Nn})^2 \mid X_0 \right) \leq \frac{g(t)(1 + X_0^2)}{n^2}.$$

Since $X_t^{Nn_1} = X_t^{Nn}$ for $n_1 > n$ and $t < \zeta_n$, we obtain $\zeta_{n_1} \geq \zeta_n$. Therefore $\zeta_n \rightarrow T$ as $n \rightarrow \infty$ almost surely by (2.14). This means that, for all $t \in [0, T)$, there exists a number n such that $\zeta_n > t$, whence $X_t^{Nn_1} = X_t^{Nn}$ for all $n_1 > n$. Put $X_t^N := X_t^{Nn}$ for $t < \zeta_n$. If $t < \zeta_n$, then

$$\begin{aligned} a_n(t, X_t^{Nn}) &= a(t, X_t^N), \\ b_n(t, X_t^{Nn}) &= b(t, X_t^N), \\ c_n(t, X_t^{Nn}) &= c(t, X_t^N). \end{aligned}$$

Hence X_t^N , $t \in [0, T)$, satisfies equation (2.5). The solution is extended to the point T by continuity. $\square$

Remark 2.7. The assumptions of Theorem 2.6 imply those of Theorem 2.4. Therefore the solution of equation (2.5) is unique.

2.2. Existence and uniqueness of the initial equation. We prove that the assumptions of Theorem 2.6 imply that a solution of equation (1.1) exists, is adapted to the flow $\mathcal{F}_t'$, and is unique.

Theorem 2.8. *Let the assumptions of Theorem 2.6 hold. Then there exists a unique solution of equation (1.1).*

Proof. First we note that definition (2.4) of the stopping times τ_N , $N \in \mathbb{N}$, Theorem 2.6, and Remark 2.7 imply for all $N_1, N_2 \in \mathbb{N}$ that solutions $X_t^{N_1}$ and $X_t^{N_2}$ coincide in the interval $[0, \tau_{N_1} \wedge \tau_{N_2}]$. Then we get from Lemma 2.1 that, for all $t \in [0, T)$, there exists a number N such that

$$X_t^{N_1} = X_t^N$$

for all $N_1 \geq N$. We put $X_t = X_t^N$ for $t < \tau_N$ and $X_T := X_{T-}$.

Now we show that X_t , $t \in [0, T]$, satisfies equation (1.1). Indeed, equations (1.1) and (2.2) are equivalent with respect to the flow $\mathcal{F}_t'$ and equation (2.5) coincides with equation (2.2) up to the moment τ_N . If N is such that $\tau_N > t$, then X_t satisfies equation (2.5). Thus X_t satisfies equation (1.1). $\square$

Remark 2.9. Instead of the term $\int_0^t b(s, X_s) dW_s$ one can use an arbitrary finite sum $\sum_{i=1}^K \int_0^t b_i(s, X_s) d(W_i)_s$, $K \in \mathbb{N} \cup \{0\}$, in equation (1.1) if standard Brownian motions W_i , $i = 1, \dots, K$, are jointly independent and do not depend on both B^H and V .

Remark 2.10. Equation (1.1) has a unique solution in every finite interval $[0, T]$.

3. CONCLUDING REMARKS

We considered equation (1.1) driven by fractional Brownian motion containing a certain stabilizing term. We proved the existence and uniqueness of the solution of this equation.

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