ANALYTICAL PROBLEMS OF THE ASYMPTOTIC BEHAVIOR OF MARKOV FUNCTIONALS. I

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ABSTRACT. Some results are given for the asymptotic behavior of Markov functionals of a homogeneous ergodic Markov process.

1. INTRODUCTION

Let $X(t)$ be a homogeneous Markov ergodic process [1] with either discrete or continuous time, and let $X(t)$ take values in a phase space $(E, \mathcal{B})$. For the sake of simplicity we assume that E is a complete separable metric space, that $\mathcal{B}$ is the σ -algebra of Borel sets, and that $X(t)$ is a stochastically continuous process in the case of continuous time. The transient probability and invariant probability distribution of the process $X(t)$ are denoted by $P(t, x, A)$, $t \geq 0$, $x \in E$, $A \in \mathcal{B}$, and $\pi(A)$, $A \in \mathcal{B}$, respectively.

Consider a family $\xi_\varepsilon(t)$ of Markov functionals that depends on a small parameter $\varepsilon > 0$ and is asymptotically degenerate. The latter means that

$$\begin{aligned} (1) \quad & \xi_\varepsilon(0) = \xi(0), \quad \varepsilon > 0, \\ (2) \quad & \lim_{\varepsilon \rightarrow 0} \mathbb{P}_{x,i} \{ \xi_\varepsilon(t) \neq i \} = 0 \end{aligned}$$

for all $x \in E$, $i \in I$, and $t \geq 0$. Here $\mathbb{P}_{x,i}$ is the regular conditional probability given $X(0) = x$, $\xi(0) = i$.

Conditions (1)–(2) are equivalent to the condition that the joint distribution of the vector $\{X(t), \xi_\varepsilon(t)\}$ converges as $\varepsilon \rightarrow 0$ to the distribution of the vector $\{X(t), \xi(0)\}$.

Below we consider Markov functionals with a countable space of states $I = \{1, 2, \dots\}$. In the case of continuous time, we assume that the trajectories of $\{X(t), \xi(t)\}$ are right continuous and have limits on the left with probability one.

The process $X(t)$ is assumed to be aperiodic in the case of continuous time; that is,

$$(3) \quad \lim_{t \rightarrow \infty} \int_E \varphi(y) P_t(x, dy) = \int_E \pi(dy) \varphi(y)$$

for all $x \in E$ and all continuous bounded functions $\varphi(y)$. If the time is discrete, we additionally require that

$$(4) \quad \lim_{k \rightarrow \infty} P_k(x, A) = \pi(A)$$

for all $x \in E$ uniformly in $A \in \mathcal{B}$.

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One can prove that there exists a Markov moment τ and a set $D \in \mathcal{B}$ such that $\pi(D) > 0$ and

$$\begin{aligned} (5) \quad & \mathbb{P}_x\{X(\tau) \in A\} = \pi_D(A), \\ (6) \quad & \sup_{x \in D} [1 - \mathbb{P}_{x,i}\{\xi_\varepsilon(\tau) = i\}] \xrightarrow{\varepsilon \rightarrow 0} 0, \\ (7) \quad & \sup_{x \in D} \mathbb{P}(\tau, \tau > T) \xrightarrow{T \rightarrow \infty} 0, \\ (8) \quad & \tau \geq 1 \quad \mathbb{P}_x\text{-almost surely} \end{aligned}$$

for all $i \in I$, where $\pi_D(A) = \pi(AD)/\pi(D)$.

Put $m = \mathbb{P}_{\pi_D, i}\tau$. Here and in what follows,

$$\mathbb{P}_{\pi_D, i}(\cdot) = \int_D \pi_D(dx) \mathbb{P}_{x, i}(\cdot).$$

2. MAIN RESULTS

The first result of this section deals with the case of continuous time.

Theorem 1. *Let conditions (1), (2), and (3) hold. Assume also that*

$$(9) \quad \sup_{\varepsilon > 0} \mathbb{P}_{x, i}\{\xi_\varepsilon(t) = i, \xi_\varepsilon(s) \neq i\} \xrightarrow{s \rightarrow t} 0$$

for all $t \geq 0$, $x \in E$, and $i \in I$. If there exists a matrix $C = \|c_{ij}\|_{i,j=1}^\infty$ such that

$$(10) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon m} [\mathbb{P}_{\pi_D, i}\{\xi_\varepsilon(\tau) = i\} - 1] = c_{ii},$$

$$(11) \quad \lim_{\varepsilon \rightarrow 0} \sum_{j \neq i} \left[\frac{1}{\varepsilon m} \mathbb{P}_{\pi_D, i}\{\xi_\varepsilon(\tau) = j\} - c_{ij} \right] = 0,$$

$$(12) \quad \sum_j c_{ij} = 0, \quad \sup_i |c_{ii}| < \infty,$$

then

$$(13) \quad \mathbb{P}_{x, i}[\varphi(X(t)), \xi_\varepsilon(t) = j] - p_{ij}(u) \int_E \pi(dy) \varphi(y) \xrightarrow[\substack{\varepsilon \rightarrow 0 \\ t \rightarrow \infty \\ \varepsilon t \rightarrow u}]{} 0$$

for all $x \in E$, $i \in I$, and all continuous bounded functions $\varphi(y)$ where $p_{ij}(u)$ is the entry (i, j) of the matrix e^{uC} , $C = \|c_{ij}\|_{i,j=1}^\infty$.

Remark 1. Assume that

$$(14) \quad -\infty < c < c_{ii} \quad \text{for all } i \in I$$

instead of (12). Put $-c_{ii} = \lambda_i$ and $c_{ij} = \lambda_i \pi_{ij}$ for $i \neq j$. If $\lambda_i > 0$, then we let $\pi_{ii} = 0$; otherwise, that is, if $\lambda_i = 0$, we let $\pi_{ij} = 0$ for $i \neq j$ and $\pi_{ii} = 1$ for all i . Then

$$p_{ij}(u) = \sum_{n=0}^{\infty} p_{ij}^{(n)}(u)$$

under assumption (13) where

$$\begin{aligned} p_{ij}^0(u) &= \delta^{ij} e^{-\lambda_i u}, \\ p_{ij}^{(n+1)}(u) &= \sum_{k \neq i} \int_0^u \lambda_i e^{-\lambda_i(u-s)} \pi_{ik} p_{kj}^{(n)}(s) ds, \quad n = 0, 1, 2, \dots \end{aligned}$$

Proof. Fix a bounded continuous function $\varphi(x) \geq 0$ and a state $l \in \{1, 2, \dots\}$. Let

$$f_\varepsilon^i(x, t) = \mathbb{P}_{x,i}[\varphi(X(t)), \xi_\varepsilon(t) = l].$$

By the law of total probability,

$$(15) \quad \begin{aligned} f_\varepsilon^i(x, t) &= \mathbb{P}_{x,i}[\varphi(X(t)), \xi_\varepsilon(t) = l, t < \tau] \\ &+ \int_E \int_0^t \sum_j \mathbb{P}_{x,i}\{X(\tau) \in dy, \tau \in du, \xi_\varepsilon(\tau) = j\} f_\varepsilon^j(y, t - u). \end{aligned}$$

For $i \neq j$, put

$$\begin{aligned} Q_\varepsilon^i(x, dy \times du) &= \mathbb{P}_{x,i}\{X(\tau) \in dy, \tau \in du, \xi_\varepsilon(\tau) = i\}, \\ Q_\varepsilon^{ij}(x, dy \times du) &= \mathbb{P}_{x,i}\{X(\tau) \in dy, \tau \in du, \xi_\varepsilon(\tau) = j\}, \\ g_\varepsilon^i(x, t) &= \mathbb{P}_{x,i}[\varphi(X(t)), \xi_\varepsilon(t) = l, t < \tau]. \end{aligned}$$

Now equation (15) can be rewritten as follows

$$(16) \quad f_\varepsilon^i(x, t) = g_\varepsilon^i(x, t) + Q_\varepsilon^i * f_\varepsilon^i(x, t) + \sum_{j \neq i} Q_\varepsilon^{ij} * f_\varepsilon^j(x, t).$$

This implies that

$$(17) \quad f_\varepsilon^i(x, t) = H_\varepsilon^i * g_\varepsilon^i(x, t) + \sum_{j \neq i} R_\varepsilon^{ij} * f_\varepsilon^j(x, t),$$

where $H_\varepsilon^i(x, dy \times dt)$ is the potential of the kernel $Q_\varepsilon^i(x, dy \times dt)$, that is,

$$\begin{aligned} H_\varepsilon^i(x, dy \times dt) &= \sum_{r=0}^{\infty} (Q_\varepsilon^i)^{*r}(x, dy \times dt), \\ R_\varepsilon^{ij}(x, dy \times dt) &= H_\varepsilon^i * Q_\varepsilon^{ij}(x, dy \times dt), \quad i \neq j. \end{aligned}$$

First we consider the case of

$$(18) \quad \sup_{x \in E} \sum_{j \neq i} R_\varepsilon^{ij}(x, E \times [0, \infty)) \leq r < 1$$

for all sufficiently small $\varepsilon > 0$ and all $i \in I$.

Put

$$h_\varepsilon^i(x, t) = H_\varepsilon^i * f_\varepsilon^i(x, t)$$

and define the sequence of functions $h_{\varepsilon,k}^i(x, t)$ by

$$(19) \quad \begin{aligned} h_{\varepsilon,0}^i(x, t) &= h_\varepsilon^i(x, t), \\ h_{\varepsilon,k+1}^i(x, t) &= \sum_{j \neq i} R_\varepsilon^{ij} * h_{\varepsilon,k}^j(x, t). \end{aligned}$$

Starting with

$$f_{\varepsilon,0}^i(x, t) = f_\varepsilon^i(x, t)$$

we similarly define the sequence of functions $f_{\varepsilon,k}^i(x, t)$. It is clear that

$$(20) \quad f_\varepsilon^i(x, t) = \sum_{k=0}^{N-1} h_{\varepsilon,k}^i(x, t) + f_{\varepsilon,N}^i(x, t)$$

for all natural numbers N . Fix an arbitrary number $\delta > 0$ and choose a natural number $N = N(\delta)$ so large that

$$(21) \quad \sup_i \sup_{x \in E} \sup_{t \geq 0} f_{\varepsilon,N}^i(x, t) < \delta$$

for all sufficiently small $\varepsilon > 0$. Such a number N exists in view of condition (18). In what follows we need the following auxiliary result.

Fix $\alpha \geq 0$, $i \neq j$, and put

$$\begin{aligned}\widehat{Q}_{\varepsilon,\alpha}^i(x, A) &= \int_0^\infty e^{-\alpha s/t} Q_\varepsilon^i(x, A \times ds), \\ \widehat{Q}_{\varepsilon,\alpha}^{ij}(x, A) &= \int_0^\infty e^{-\alpha s/t} Q_\varepsilon^{ij}(x, A \times ds), \\ \widehat{H}_{\varepsilon,\alpha}^i(x, A) &= \int_0^\infty e^{-\alpha s/t} H_\varepsilon^i(x, A \times ds), \\ \widehat{R}_{\varepsilon,\alpha}^{ij}(x, A) &= \int_0^\infty e^{-\alpha s/t} R_\varepsilon^{ij}(x, A \times ds).\end{aligned}$$

It is clear that

$$(22) \quad \widehat{Q}_\varepsilon^i(x, A) \leq \pi_D(A),$$

$$(23) \quad \lim_{\varepsilon \rightarrow 0} \widehat{Q}_{\varepsilon,\alpha}^i(x, A) = \pi_D(A)$$

uniformly in $x \in D$, $A \in \mathcal{B}$.

Denote by $\lambda_{\varepsilon,\alpha}^i$ the Perron root of the kernel $\widehat{Q}_{\varepsilon,\alpha}^i$, and by $e_{\varepsilon,\alpha}^i(x)$ and $\rho_{\varepsilon,\alpha}^i(A)$ the eigenfunction and eigenmeasure, respectively, that correspond to the Perron root $\lambda_{\varepsilon,\alpha}^i$, that is,

$$\begin{aligned}\int_D \widehat{Q}_{\varepsilon,\alpha}^i(x, dy) e_{\varepsilon,\alpha}^i(y) &= \lambda_{\varepsilon,\alpha}^i e_{\varepsilon,\alpha}^i(x), \quad x \in D, \\ \int_D \rho_{\varepsilon,\alpha}^i(dx) \widehat{Q}_{\varepsilon,\alpha}^i(x, A) &= \lambda_{\varepsilon,\alpha}^i \rho_{\varepsilon,\alpha}^i(A), \quad A \in \mathcal{B}.\end{aligned}$$

For the sake of brevity we put

$$e_\varepsilon^i(x) = e_{\varepsilon,0}^i(x), \quad \rho_\varepsilon^i(A) = \rho_{\varepsilon,0}^i(A)$$

and assume that

$$(24) \quad \rho_\varepsilon^i(D) = 1, \quad \int_D \rho_\varepsilon^i(dx) e_\varepsilon^i(x) = 1,$$

$$(25) \quad \begin{aligned}\int_D \rho_{\varepsilon,\alpha}^i(dx) e_\varepsilon^i(x) &= 1, \quad \alpha > 0, \\ \int_D \rho_{\varepsilon,\alpha}^i(dx) e_{\varepsilon,\alpha}^i(x) &= 1, \quad \alpha > 0.\end{aligned}$$

Equality (23) and the theorem on the perturbation of separated parts of the spectrum [3, Chapter 4, §4, Theorem 3.16] imply that

$$\begin{aligned}\lambda_{\varepsilon,\alpha}^i &\rightarrow 1, \\ e_{\varepsilon,\alpha}^i(x) &\rightarrow 1\end{aligned}$$

uniformly in $x \in D$, and

$$\rho_{\varepsilon,\alpha}^i(A) \xrightarrow{\varepsilon \rightarrow 0} \pi_D(A)$$

uniformly in $A \in \mathcal{B}$ for all $\alpha \geq 0$.

Finally we put

$$\rho_\varepsilon^{ij}(A) = \frac{1}{1 - \lambda_{\varepsilon,0}^i} \rho_\varepsilon^i Q_\varepsilon^{ij}(A) = \frac{1}{1 - \lambda_{\varepsilon,0}^i} \int_D \rho_\varepsilon^i(dx) Q_\varepsilon^{ij}(x, A)$$

for $i \neq j$, where

$$Q_\varepsilon^{ij}(x, A) = \widehat{Q}_{\varepsilon,0}^{ij}(x, A) = \mathbb{P}_{x,i}\{X(\tau) \in A, \xi_\varepsilon(\tau) = j\}.$$

□

Lemma 1. *If all the assumptions of Theorem 1 hold, then*

$$\sum_{j \neq i} \left| \widehat{R}_{\varepsilon, \alpha}^{ii}(x, A) - \frac{c_{ii}}{c_{ii}u - \alpha} \rho_{\varepsilon}^{ij}(A) \right| \xrightarrow[\varepsilon t \rightarrow u]{t \rightarrow \infty, \varepsilon \rightarrow 0} 0$$

for all $u \geq 0$ and $\alpha \geq 0$ uniformly in $x \in D$, $A \in \mathcal{B}$.

Proof. The kernels $\widehat{Q}_{\varepsilon, \alpha}^i$, $\widehat{Q}_{\varepsilon, \alpha}^{ij}$, $\widehat{H}_{\varepsilon, \alpha}^i$, and $\widehat{R}_{\varepsilon, \alpha}^i$ are such that

$$\begin{aligned} \widehat{H}_{\varepsilon, \alpha}^i &= (1 - Q_{\varepsilon, \alpha}^i)^{-1}, \\ \widehat{R}_{\varepsilon, \alpha}^{ij} &= \widehat{H}_{\varepsilon, \alpha}^i \widehat{Q}_{\varepsilon, \alpha}^{ij}. \end{aligned}$$

It is also clear that

$$\begin{aligned} \widehat{Q}_{\varepsilon, \alpha}^i(x, A) &= \mathbb{P}_{x, i} \left[e^{-\alpha\tau/t}, X(\tau) \in A, \xi_{\varepsilon}(\tau) = i \right], \\ \widehat{Q}_{\varepsilon, \alpha}^{ij}(x, A) &= \mathbb{P}_{x, i} \left[e^{-\alpha\tau/t}, X(\tau) \in A, \xi_{\varepsilon}(\tau) = j \right]. \end{aligned}$$

According to the theorem on the isolated eigenvalue,

$$\widehat{H}_{\varepsilon, \alpha}^i(x, A) - \frac{1}{1 - \lambda_{\varepsilon, \alpha}^i} e_{\varepsilon, \alpha}^i(x) \rho_{\varepsilon, \alpha}^i(A) \xrightarrow{\varepsilon \rightarrow 0} I(x, A) - \pi_D(A)$$

uniformly in $x \in D$ and $A \in \mathcal{B}$ (see [1, Chapter 3, §3, Theorem 1]). This together with (23) implies that

$$(26) \quad \sum_{j \neq i} \left[\widehat{R}_{\varepsilon, \alpha}^{ij}(x, A) - \frac{m\varepsilon}{1 - \lambda_{\varepsilon, \alpha}^i} e_{\varepsilon, \alpha}^i(x) \rho_{\varepsilon, \alpha}^{ij}(A) \right] \xrightarrow{\varepsilon \rightarrow 0} 0$$

uniformly in $x \in D$ and $A \in \mathcal{B}$, where

$$\rho_{\varepsilon, \alpha}^{ij}(A) = \frac{1}{m\varepsilon} \rho_{\varepsilon, \alpha}^i Q_{\varepsilon, \alpha}^{ij}(A).$$

Now relations (22) and (23) imply that

$$(27) \quad 1 - \lambda_{\varepsilon, \alpha}^i \sim 1 - \mathbb{P}_{\pi_D, i} \left[e^{-\alpha\tau/t}, \xi_{\varepsilon}(\tau) = i \right]$$

as $\varepsilon \rightarrow 0$. Applying the latter relation for $\alpha = 0$, we obtain from (10) that

$$(28) \quad \frac{1 - \lambda_{\varepsilon, 0}^i}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} -mc_{ii}.$$

The right hand side of (27) for $\alpha > 0$ is equal to

$$(29) \quad 1 - \mathbb{P}_{\pi_D, i} \{ \xi_{\varepsilon}(\tau) = i \} + \mathbb{P}_{\pi_D, i} \left[\left(1 - e^{-\alpha\tau/t} \right), \xi_{\varepsilon}(\tau) = i \right].$$

The second term in (29) divided by α/t approaches m as $t \rightarrow \infty$. Thus the whole expression (29) equals

$$(30) \quad 1 - \mathbb{P}_{\pi_D, i} \{ \xi_{\varepsilon}(\tau) = i \} + m \frac{\alpha}{t} + o\left(\frac{1}{t}\right).$$

Using equality (10) we evaluate the limit:

$$\lim_{\substack{t \rightarrow \infty \\ \varepsilon \rightarrow 0 \\ \varepsilon t \rightarrow u}} t \left[1 - \mathbb{P}_{\pi_D, i} \{ \xi_{\varepsilon}(\tau) = i \} + m \frac{\alpha}{t} + o\left(\frac{1}{t}\right) \right] = m(-c_{ii}u + \alpha),$$

whence

$$(31) \quad t(1 - \lambda_{\varepsilon, \alpha}^i) \xrightarrow[\varepsilon t \rightarrow u]{t \rightarrow \infty, \varepsilon \rightarrow 0} m(-c_{ii}u + \alpha).$$

The next step is to prove that

$$(32) \quad t [\rho_{\varepsilon,\alpha}^i(A) - \rho_\varepsilon^i(A)] \xrightarrow[\substack{\varepsilon \rightarrow 0 \\ t \rightarrow \infty}]{} \alpha m \pi_D(A) - \alpha \mathbb{P}_{\pi_D}[\tau, X(\tau) \in A]$$

uniformly in $A \in \mathcal{B}$.

By the definition of $\rho_{\varepsilon,\alpha}^i$ and ρ_ε^i we have

$$(33) \quad (\rho_{\varepsilon,\alpha}^i - \rho_\varepsilon^i) (Q_\varepsilon^i - \lambda_\varepsilon^i) = \rho_{\varepsilon,\alpha}^i (Q_\varepsilon^i - \widehat{Q}_{\varepsilon,\alpha}^i) + (\lambda_{\varepsilon,\alpha}^i - \lambda_\varepsilon^i) \rho_{\varepsilon,\alpha}^i$$

where $Q_\varepsilon^i = \widehat{Q}_{\varepsilon,0}^i$ and $\lambda_\varepsilon^i = \lambda_{\varepsilon,0}^i$.

According to (31) and (28), we have

$$(34) \quad t [\lambda_{\varepsilon,\alpha}^i - \lambda_\varepsilon^i] \xrightarrow[\substack{\varepsilon \rightarrow 0 \\ t \rightarrow \infty \\ \varepsilon t \rightarrow u}]{} -m\alpha$$

for $u > 0$. Furthermore, put

$$\overline{Q}_\varepsilon^i(x, A) = \mathbb{P}_{x,i}[\tau, X(\tau) \in A, \xi_\varepsilon(\tau) = i].$$

Since

$$Q_\varepsilon^i(x, A) - \widehat{Q}_{\varepsilon,\alpha}^i(x, A) = \mathbb{P}_{x,i} \left[\left(1 - e^{-\alpha\tau/t} \right), X(\tau) \in A, \xi_\varepsilon(\tau) = i \right],$$

we get

$$\begin{aligned} 0 &\leq \alpha \overline{Q}_\varepsilon^i(x, A) - t [Q_\varepsilon^i(x, A) - \widehat{Q}_{\varepsilon,\alpha}^i(x, A)] \\ &\leq \mathbb{P}_{x,i} \left[\left(\alpha\tau - \frac{1 - e^{-\alpha\tau/t}}{\alpha/t} \right), X(\tau) \in A, \xi_\varepsilon(\tau) = i \right] \\ &\leq \mathbb{P}_{x,i} \left(\alpha\tau - \frac{1 - e^{-\alpha\tau/t}}{\alpha/t} \right). \end{aligned}$$

In view of (7), the last expression approaches zero uniformly in $x \in D$ as $t \rightarrow \infty$. Moreover, (6) and (7) imply

$$\overline{Q}_\varepsilon^i(x, A) \xrightarrow[\varepsilon \rightarrow 0]{} \overline{Q}(x, A) = \mathbb{P}_x[\tau, X(\tau) \in A]$$

uniformly in $x \in D$ and $A \in \mathcal{B}$. Thus,

$$(35) \quad t [Q_\varepsilon^i(x, A) - \widehat{Q}_{\varepsilon,\alpha}^i(x, A)] \xrightarrow[\substack{t \rightarrow \infty \\ \varepsilon \rightarrow 0}]{} \alpha \overline{Q}(x, A)$$

uniformly in $x \in D$ and $A \in \mathcal{B}$.

Furthermore, let V_ε be the generalized inverse operator of $Q_\varepsilon^i - \lambda_\varepsilon^i$; that is,

$$(36) \quad V_\varepsilon = \frac{1}{2\pi i} \oint (z - Q_\varepsilon^i)^{-1} \frac{dz}{z - \lambda_\varepsilon^i},$$

where $C = \{z \in \mathbb{C}: |z| = \frac{1}{3}\}$ is the circle of radius $\frac{1}{3}$ in the complex plane centered at the origin. Since $\lambda_\varepsilon^i > \frac{1}{3}$ for sufficiently small ε , the operator V_ε is well defined. Therefore

$$(37) \quad (Q_\varepsilon^i - \lambda_\varepsilon^i) V_\varepsilon(x, A) = I(x, A) - e_\varepsilon(x) \rho_\varepsilon(A).$$

It is also clear that

$$V_\varepsilon(x, A) \xrightarrow[\varepsilon \rightarrow 0]{} \pi_D(A) - I(x, A)$$

uniformly in $x \in D$ and $A \in \mathcal{B}$.

Using (24) and (25) we obtain from (33) and (37) that

$$(38) \quad t [\rho_{\varepsilon,\alpha}^i(A) - \rho_\varepsilon^i(A)] = t \rho_{\varepsilon,\alpha}^i (Q_\varepsilon^i - Q_{\varepsilon,\alpha}^i) V_\varepsilon(A) + t (\lambda_{\varepsilon,\alpha}^i - \lambda_\varepsilon^i) \rho_{\varepsilon,\alpha}^i V_\varepsilon(A).$$

Combining (35), (34), and (38) and collecting the like terms we prove (32).

We see from (26), (31), and (32) that the lemma follows from

$$(39) \quad \sup_{x \in D} \sup_{A \in \mathcal{B}} \sum_{j \neq i} t \left| Q_\varepsilon^{ij}(x, A) - \widehat{Q}_{\varepsilon, \alpha}^{ij}(x, A) \right| \xrightarrow[\varepsilon \rightarrow 0]{t \rightarrow \infty} 0$$

and

$$(40) \quad \sup_{\varepsilon > 0} \sum_{j \neq i} \rho_\varepsilon^{ij}(D) < \infty.$$

We start with the proof of (39). The sum in (39) is nonnegative and does not exceed

$$\sum_{j \neq i} t \mathbb{P}_{x, i} \left[\left(1 - e^{-\alpha \tau / t} \right), \xi_\varepsilon(\tau) = j \right].$$

In turn, the last expression does not exceed

$$\alpha \mathbb{P}_x[\tau, \tau > T] + \alpha T [1 - \mathbb{P}_{x, i}\{\xi_\varepsilon(\tau) = i\}]$$

for any $T > 0$. The first term in the last expression approaches zero uniformly in $x \in D$ as $T \rightarrow \infty$ in view of (14), while the second term, for every fixed $T > 0$, approaches zero uniformly in $x \in D$ as $\varepsilon \rightarrow 0$. Relation (39) is proved.

To prove (40) note that

$$\begin{aligned} \sum_{j \neq i} \rho_\varepsilon^{ij}(D) &\leq \frac{1}{1 - \lambda_\varepsilon^i} \int_D \rho_\varepsilon^i(dx) [1 - \mathbb{P}_{x, i}\{\xi_\varepsilon(\tau) = i\}] \\ &= \frac{1}{1 - \lambda_\varepsilon^i} [\rho_\varepsilon^i(D) - \rho_\varepsilon^i Q_\varepsilon^i(D)] = 1. \end{aligned}$$

In fact, we obtained a stronger result, namely

$$(41) \quad \limsup_{\varepsilon \rightarrow 0} \sum_{j \neq i} \rho_\varepsilon^{ij}(D) \leq 1.$$

The lemma is proved. $\square$

Lemma 2. *Suppose the assumptions of Lemma 1 hold and a bounded function $\Psi(x, t)$ is measurable with respect to the variables $x \in D$ and $t \geq 0$. We also assume that*

$$(42) \quad \sup_{x \in D} |\Psi(x, t + s) - \Psi(x, t)| \xrightarrow[s \rightarrow 0]{\substack{\varepsilon \rightarrow 0 \\ t \rightarrow \infty \\ \varepsilon t \rightarrow u}} 0$$

for all $t \geq 0$. Then

$$(43) \quad \sum_{j \neq i} \left| \int_D \int_0^t R_\varepsilon^{ij}(x, dy \times t ds) \Psi(y, s) - c_{ij} u \int_0^t e^{c_{ii} u s} \Psi(y, s) ds \right| \xrightarrow[\varepsilon \rightarrow 0]{\substack{\varepsilon \rightarrow 0 \\ t \rightarrow \infty \\ \varepsilon t \rightarrow u}} 0$$

uniformly in $x \in D$ and $0 \leq t \leq T$, where $T > 0$, $u > 0$, and $i \in I$ are arbitrary.

Proof. Note that relation (42) holds uniformly in t belonging to an arbitrary finite interval in view of compactness, that is,

$$(44) \quad \sup_{t \in [0, T]} \sup_{x \in D} |\Psi(x, t + s) - \Psi(x, t)| \xrightarrow[s \rightarrow 0]{} 0$$

for all $T > 0$. Lemma 1 and the continuity theorem for the Laplace transform imply

$$(45) \quad \sum_{j \neq i} \left| \int_D \int_0^t R_\varepsilon^{ij}(x, dy \times [ts_1, ts_2]) \Psi(y) - \int_D \rho_\varepsilon^{ij}(dy) \Psi(y) \int_{s_1}^{s_2} (-c_{ii}) u e^{c_{ii} u s} ds \right| \xrightarrow[\varepsilon \rightarrow 0]{\substack{\varepsilon \rightarrow 0 \\ t \rightarrow \infty \\ \varepsilon t \rightarrow u}} 0$$

for all $0 \leq s_1 \leq s_2$ and all bounded $\mathcal{B}$ -measurable functions $\Psi(y)$.

For a fixed $\delta > 0$, put

$$\begin{aligned}\Psi_{\delta}^{+}(x, t) &= \sup_{k\delta \leq s \leq k\delta + \delta} \Psi(x, s), & k\delta \leq t \leq k\delta + \delta, \\ \Psi_{\delta}^{-}(x, t) &= \inf_{k\delta \leq s \leq k\delta + \delta} \Psi(x, s), & k\delta \leq t \leq k\delta + \delta.\end{aligned}$$

Since the function $\Psi(x, t)$ is continuous in t , the functions $\Psi_{\delta}^{\pm}(x, t)$ are measurable with respect to all their arguments. Relation (44) implies that for all $T > 0$,

$$(46) \quad \sup_{t \in [0, T]} \sup_{x \in D} |\Psi_{\delta}^{\pm}(x, t) - \Psi(x, t)| \xrightarrow{\delta \rightarrow 0} 0.$$

Now relation (45) implies that

$$\sum_{j \neq i} \left| \int_D \int_0^t R_{\varepsilon}^{ij}(x, dy \times t ds) \Psi_{\delta}^{\pm}(y, s) - \int_D \rho_{\varepsilon}^{ij}(dy) \int_0^t (-c_{ii}) u e^{c_{ii}us} \Psi_{\delta}^{\pm}(y, s) ds \right| \xrightarrow[\substack{\varepsilon \rightarrow 0 \\ t \rightarrow \infty \\ \varepsilon t \rightarrow u}]{} 0$$

for all $\delta > 0$. The limit

$$(47) \quad \sum_{j \neq i} \left| \rho_{\varepsilon}^{ij}(D) - \frac{c_{ij}}{(-c_{ii})} \right| \xrightarrow{\varepsilon \rightarrow 0} 0$$

exists by (10) and (11). Hence

$$\sum_{j \neq i} \left| \int_D \int_0^t R_{\varepsilon}^{ij}(x, dy + t ds) \Psi_{\delta}^{\pm}(y, s) - c_{ij} u \int_0^t e^{c_{ii}us} \Psi_{\delta}^{\pm}(y, s) ds \right| \xrightarrow[\substack{\varepsilon \rightarrow 0 \\ t \rightarrow \infty \\ \varepsilon t \rightarrow u}]{} 0.$$

This together with (46) implies (43). The lemma is proved. $\square$

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