A GENERALIZATION OF KARAMATA'S THEOREM ON THE ASYMPTOTIC BEHAVIOR OF INTEGRALS

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ABSTRACT. A generalization of Karamata's theorem on the asymptotic behavior of integrals of regularly varying functions with oscillating components is obtained in the paper.

1. INTRODUCTION

Jován Karamata [6, 7] introduced the notion of regularly varying (RV) functions and proved a number of fundamental theorems for them. The theorem on the asymptotic behavior of integrals of RV functions is one of those results. This theorem found many applications in probability theory (see, for example, [5, 3]). Below we generalize this theorem to a certain class of functions with a nondegenerate group of regular points; see [4].

Let $\mathbf{R}$ be the set of real numbers, $\mathbf{R}_+$ the set of positive numbers, $\mathbf{Z}$ the set of integers, and $\mathbf{N}$ the set of natural numbers.

We assume that $A > 0$ and let $\mathbb{F}_+(A)$ be the set of positive and (Lebesgue) measurable functions $f = (f(x), x \geq A)$.

A function $f \in \mathbb{F}_+(A)$ is called *regularly varying (at infinity)* in the Karamata sense if the limit

$$\kappa_f(\lambda) = \lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)}$$

exists and is positive and finite for all $\lambda > 0$.

We say that an RV function f is *slowly varying* (SV) if

$$\kappa_f(\lambda) = 1 \quad \text{for all } \lambda > 0.$$

If f is an RV function, then there is a real number ρ such that

$$\kappa_f(\lambda) = \lambda^\rho, \quad \lambda > 0.$$

The number ρ is called the *index* of the function f . The index $\rho = 0$ characterizes SV functions.

Any RV function f of index ρ is represented in the following form:

$$f(x) = x^\rho \ell(x), \quad x \geq A,$$

where ℓ is a corresponding SV function.

A measurable real function $\varphi(x)$, $x \geq A$, is called *locally integrable* if it is (Lebesgue) integrable on every interval $[a, b] \subset [A, \infty)$.

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Recall that any of the following three relations: $\varphi(x) \sim_{x \rightarrow \infty} \psi(x)$, $\varphi(x) \sim \psi(x)$, and $\varphi \sim \psi$ as $x \rightarrow \infty$ means that

$$\lim_{x \rightarrow \infty} \frac{\varphi(x)}{\psi(x)} = 1.$$

We say that the functions φ and ψ are *asymptotically equivalent* (at infinity) in this case.

There are two parts of the Karamata theorem, namely the forward and backward Karamata theorems (see [6, 7] for continuous RV functions and [3, 5, 8] for locally integrable RV functions). In this paper, we consider the case of $\rho > -1$.

Forward theorem. *Let f be a locally integrable regularly varying function of index $\rho > -1$. Then*

$$(1) \quad \int_A^x f(t) dt \underset{x \rightarrow \infty}{\sim} \frac{xf(x)}{\rho + 1}.$$

Backward theorem. *Let f be a locally integrable function belonging to $\mathbb{F}_+(A)$. If there is a number $\gamma \in (0, \infty)$ such that*

$$(2) \quad \int_A^x f(t) dt \underset{x \rightarrow \infty}{\sim} \frac{xf(x)}{\gamma},$$

then f is an RV function of index $\rho = \gamma - 1$.

The aim of the paper is to generalize the above two theorems to the case of functions $f \in \mathbb{F}_+(A)$ such that

$$(3) \quad f(x) = x^\rho \ell(x) H(\ln x), \quad x \geq A,$$

where $\rho \in \mathbf{R}$, $\ell = (\ell(x), x \geq A)$ is a slowly varying function and

$$H = (H(u), u \in \mathbf{R})$$

is a positive continuous periodic function.

The class of functions introduced above is denoted by Φ .

2. DEFINITIONS AND AUXILIARY RESULTS

Consider some properties of functions of the class Φ (that is, the functions represented in the form (3)). First of all, the functions of this class are positive and measurable. Thus all the factors in representation (3) are also positive and measurable functions.

The function H on the right hand side of (3) is positive, continuous, and periodic. This function is constant; that is, there exists a constant $c > 0$ such that $H(x) = c$ for all $x \in \mathbf{R}$ if and only if f is a regularly varying function. In this case, H can be removed from representation (3) by adjoining this constant to the function ℓ . Otherwise, H is not a slowly varying function and one cannot neglect it in representation (3).

In what follows we assume (if the opposite is not stated) that

$$(4) \quad H(0) = 1.$$

Condition (4) does not restrict the class Φ , since the function $f \in \Phi$ can be represented as follows:

$$f(x) = x^\rho \ell_0(x) H_0(\ln x), \quad x \geq A,$$

where $H_0(u) = H(u)/H(0)$ and $\ell_0(x) = H(0)\ell(x)$. It is clear that $H_0(0) = 1$ and ℓ_0 is a slowly varying function.

The function

$$r(x) = x^\rho \ell(x), \quad x \geq A,$$

is called the *regular (RV) component* of f ; the index ρ of the function r is called the *index* of the function f ($\rho_f = \rho$). It is clear that

$$\rho = \lim_{x \rightarrow \infty} \frac{\ln f(x)}{\ln x}.$$

The function ℓ is called the *slowly varying (SV) component* of f . Note that the function ℓ is allowed to be discontinuous.

The function H is called the *oscillating function* or *oscillating component* of f .

If $\varphi = (\varphi(x), x \in \mathbf{R})$ is a continuous periodic function, then we denote by $S_{\text{per}}(\varphi)$ the set of periods of the function φ ; respectively, by $S_{\text{pper}}(\varphi)$ we denote the set of positive periods of the function φ .

The number

$$T(\varphi) = \inf S_{\text{pper}}(\varphi)$$

is called the *oscillating characteristic* of the function φ . Since the function φ is continuous, either $T(\varphi) = 0$ (in which case the function φ is *constant*) or $T(\varphi) > 0$. In the latter case, $T(\varphi)$ is the least positive period of the function φ , that is,

$$S_{\text{per}}(\varphi) = \{nT(\varphi), n \in \mathbf{Z}\}.$$

If H is the oscillating component of a function f , then its characteristic $T(H)$ is also called the *oscillating characteristic* of the function f and is denoted by $T(H_f)$.

The class of functions $f \in \Phi$ such that $T(H_f) > 0$ is denoted by Φ_+ . Note that the members of the class Φ_+ are not regularly varying; they are so-called functions with a nondegenerate group of regular points; see [4].

The class

$$\Phi_0 = \Phi \setminus \Phi_+$$

contains functions f of Φ that are represented in the form (3) with functions H constant, whence, under condition (4),

$$H(x) \equiv 1.$$

Thus Φ_0 coincides with the class of RV functions.

If a function f belongs to the class Φ and is of index $\rho > -1$, then

$$(5) \quad \int_A^\infty f(x) dx = \infty.$$

Indeed,

$$\int_A^\infty r(x) dx = \infty$$

for an arbitrary RV function $r(x)$, $x \geq A$, of index $\rho > -1$ (see, for example, [3, 5, 8]). Thus (5) follows from the inequality

$$\int_A^\infty f(x) dx = \int_A^\infty x^\rho \ell(x) H_l(x) dx \geq \kappa \int_A^\infty x^\rho \ell(x) dx = \infty,$$

where $\kappa = \inf_{x \in \mathbf{R}} H(x) > 0$.

3. MAIN RESULTS

We now state two theorems (forward and backward ones) generalizing Karamata's theorem to functions belonging to the class Φ . The proofs will be given in further sections where we use and develop some of the methods of the papers [6, 7] and [3, 5, 8].

We start with a generalization of the forward theorem; see (1).

Theorem 3.1 (Forward theorem). *Let $f \in \Phi$ be a locally integrable function of index $\rho > -1$ whose oscillating component and characteristic are denoted by H and $T(H) = T$, respectively. Then there is a positive continuous periodic function $D = (D(x), x \in \mathbf{R})$ that depends on ρ and H and satisfies*

$$(6) \quad \int_A^x f(t) dt \underset{x \rightarrow \infty}{\sim} xD(\ln x)f(x).$$

Moreover,

- 1) $T(DH) = T(D) = T(H) = T$, where $T(D)$ is the oscillating characteristic of the function D and $T(DH)$ is the oscillating characteristic of the function

$$DH = (D(x)H(x), x \in \mathbf{R}).$$

- 2) If f is an RV function, that is, $H(x) \equiv 1$ and $T = 0$, then

$$(7) \quad D(x) = D_{H,\rho}(x) \equiv \frac{1}{\rho + 1}.$$

- 3) If $f \in \Phi_+$, that is, $T > 0$, then

$$(8) \quad D(x) = D_{H,\rho}(x) = \frac{\int_0^{T\{x/T\}} e^{(\rho+1)y} H(y) dy + C}{H(x) \exp(T(\rho+1)\{x/T\})}, \quad x \geq 0,$$

where

$$C = C_{H,\rho} = \frac{\int_0^T e^{(\rho+1)y} H(y) dy}{e^{T(\rho+1)} - 1}$$

and $\{x/T\}$ stands for the fractional part of the number x/T .

We now turn to a generalization of the backward theorem (see (2)).

Theorem 3.2 (Backward theorem). *Let f be a locally integrable function of the class $\mathbb{F}_+(A)$. If there is a positive continuous periodic function $B = (B(x), x \in \mathbf{R})$ with oscillating characteristic $T(B)$ such that*

$$(9) \quad \int_A^x f(t) dt \underset{x \rightarrow \infty}{\sim} xB(\ln x)f(x),$$

then the function f belongs to the class Φ , its index ρ is > -1 , and its oscillating characteristic is $H = (H(x), x \in \mathbf{R})$.

Moreover,

- 1) $T(BH) = T(H) = T(B)$, where $T(H)$ is the oscillating characteristic of the function H and $T(BH)$ is the oscillating characteristic of the function

$$BH = (B(x)H(x), x \in \mathbf{R}).$$

- 2) If $T(B) = 0$, that is, $B(x) \equiv \beta > 0$, then

$$(10) \quad \rho = \frac{1}{\beta} - 1$$

and $H(x) \equiv 1$, that is, f is an RV function of index (10).

- 3) If $T(B) > 0$, then f belongs to the class Φ_+ ,

$$(11) \quad \rho = \frac{1}{T(B)} \int_0^{T(B)} \frac{du}{B(u)} - 1,$$

and

$$H(x) = \frac{B(0)}{B(x)} \exp \left(\int_0^x \left(\frac{1}{B(t)} - (B^{-1})_{av} \right) dt \right), \quad x \geq 0,$$

where

$$(B^{-1})_{av} = \frac{B(0)}{T(B)} \int_0^{T(B)} \frac{du}{B(u)}.$$

Remark 3.1. Relation (9) implies that

$$\liminf_{x \rightarrow \infty} xf(x) > 0$$

and

$$\int_A^\infty f(t) dt = \infty.$$

Remark 3.2. Relations (6) and (9) uniquely determine the functions D and B , respectively. For example, if (6) holds for two positive continuous periodic functions D_1 and D_2 , then $D_1 \sim D_2$, whence $D_1 = D_2$. Thus $D = B$ in Theorems 3.1 and 3.2, and the relations that hold for the function B also hold for the function D , and *vice versa*.

Remark 3.3. Putting $H(x) \rightarrow 1$, $x \in \mathbf{R}$, and $T \rightarrow 0$ in (8), we get

$$D_{H,\rho}(x) \rightarrow \frac{1}{\rho+1}, \quad x \in \mathbf{R},$$

that is, equalities (7) and (8) are compatible. Similar compatibility holds for relations (10) and (11).

Remark 3.4. Let $I(x) = \int_A^x f(t) dt$, $x > A$. Theorem 3.1 implies that the function I belongs to the class Φ if so does the function f and $\rho_f > -1$. Moreover $\rho_I = \rho_f + 1$, $H_I = H_f B$, and $T(H_I) = T(H_f)$ in this case.

4. LEMMA ON THE ASYMPTOTIC BEHAVIOR OF INTEGRALS

We start with the following result for functions of the class Φ . With minor changes, its proof follows the lines of that of the Karamata theorem for regularly varying functions (see, for example, [3, 5]. For the sake of convenience we set $H_l(x) = H(\ln x)$.

Lemma 4.1. *Let $f \in \Phi$ be a locally integrable function of index $\rho > -1$ whose SV component is denoted by ℓ and whose oscillating component is denoted by H (see (3)). Then*

$$(12) \quad \int_A^x f(t) dt \underset{x \rightarrow \infty}{\sim} \ell(x) \int_A^x t^\rho H_l(t) dt.$$

Proof. Let

$$J(x) = \int_{A/x}^1 y^\rho \left(\frac{\ell(xy)}{\ell(x)} - 1 \right) H_l(xy) dy, \quad x \geq A,$$

and

$$J_\varepsilon(x) = \int_\varepsilon^1 y^\rho \left(\frac{\ell(xy)}{\ell(x)} - 1 \right) H_l(xy) dy, \quad x \geq (A/\varepsilon),$$

for $\varepsilon \in (0, 1)$.

Since H is a positive continuous periodic function and $\rho > -1$, we conclude that, for all $\varepsilon \in (0, 1)$ and $x \geq (A/\varepsilon)$,

$$(13) \quad |J_\varepsilon(x)| \leq K \int_\varepsilon^1 y^\rho \left| \frac{\ell(xy)}{\ell(x)} - 1 \right| dy \leq K \sup_{y \in [\varepsilon, 1]} \left| \frac{l(xy)}{l(x)} - 1 \right|,$$

where

$$K = \frac{\sup_{x \in \mathbf{R}} H(x)}{\rho + 1} < \infty.$$

According to the uniform convergence theorem for slowly varying functions (see, for example, [8]),

$$\lim_{x \rightarrow \infty} \sup_{y \in [\varepsilon, 1]} \left| \frac{\ell(xy)}{\ell(x)} - 1 \right| = 0$$

for all $\varepsilon \in (0, 1)$.

This together with (13) implies that

$$(14) \quad \lim_{x \rightarrow \infty} J_\varepsilon(x) = 0$$

for all $\varepsilon \in (0, 1)$. Further,

$$|J(x) - J_\varepsilon(x)| \leq \int_{A/x}^\varepsilon y^\rho \left| \frac{\ell(xy)}{\ell(x)} - 1 \right| H_l(xy) dy \leq K \left(\varepsilon^{\rho+1} + (\rho+1) \int_{A/x}^\varepsilon y^\rho \frac{\ell(xy)}{\ell(x)} dy \right)$$

for all $\varepsilon \in (0, 1)$ and $x \geq (A/\varepsilon)$.

Choosing $\delta \in (0, \rho+1)$, the Potter theorem (see, for example, [3]) implies that there is a number $x(\delta) > 0$ such that

$$\int_{A/x}^\varepsilon y^\rho \frac{\ell(xy)}{\ell(x)} dy \leq 2 \int_{A/x}^\varepsilon y^{\rho-\delta} dy \leq \frac{2\varepsilon^{\rho+1-\delta}}{\rho+1-\delta}$$

for $x \geq \max\{x(\delta), (A/\varepsilon)\}$. Thus

$$\limsup_{x \rightarrow \infty} |J(x) - J_\varepsilon(x)| \leq K \left(\varepsilon^{\rho+1} + \frac{2(\rho+1)\varepsilon^{\rho+1-\delta}}{\rho+1-\delta} \right)$$

for all $\varepsilon \in (0, 1)$, whence

$$(15) \quad \limsup_{\varepsilon \rightarrow 0} \limsup_{x \rightarrow \infty} |J(x) - J_\varepsilon(x)| = 0,$$

since $\rho+1-\delta > 0$.

Relations (14) and (15) imply that

$$\lim_{x \rightarrow \infty} J(x) = 0,$$

whence

$$(16) \quad \int_{A/x}^1 y^\rho \ell(xy) H_l(xy) dy \underset{x \rightarrow \infty}{\sim} \ell(x) \int_{A/x}^1 y^\rho H_l(xy) dy$$

in view of

$$\liminf_{x \rightarrow \infty} \int_{A/x}^1 y^\rho H_l(xy) dy > 0.$$

To complete the proof of the lemma we note that

$$\int_A^x f(t) dt = \int_A^x t^\rho \ell(t) H_l(t) dt = x^{\rho+1} \int_{A/x}^1 y^\rho \ell(xy) H_l(xy) dy, \quad x \geq A,$$

according to representation (3). In view of asymptotics (16),

$$\begin{aligned} \int_A^x f(t) dt &= x^{\rho+1} \int_{A/x}^1 y^\rho \ell(xy) H_l(xy) dy \\ &\sim x^{\rho+1} \ell(x) \int_{A/x}^1 y^\rho H_l(xy) dy = \ell(x) \int_A^x t^\rho H_l(t) dt \end{aligned}$$

as $x \rightarrow \infty$ and relation (12) follows. $\square$

Remark 4.1. Since $\rho > -1$ in Lemma 4.1, equality (5) implies that relation (12) is equivalent to the condition that, for all $x_1 \geq A$ and $x_2 > 0$, the integrals

$$\int_{x_1}^x f(t) dt \underset{x \rightarrow \infty}{\sim} \ell(x) \int_{x_2}^x t^\rho H_1(t) dt$$

are well defined.

Remark 4.2. Lemma 4.1 with $H(x) \equiv 1$ implies the forward Karamata theorem for RV functions (see (1)).

5. THE FIRST STEP OF THE PROOF OF THE FORWARD THEOREM FOR FUNCTIONS BELONGING TO THE CLASS Φ_+

Since Lemma 4.1 implies the forward Karamata theorem for RV functions, its generalization for the class Φ follows from that for the class Φ_+ . The following result is the first step of the corresponding proof; the rest of the proof of the forward theorem is given in Section 7.

Lemma 5.1. *Let $f \in \Phi_+$ be a locally integrable function of index $\rho > -1$ and let its oscillating component and oscillating characteristic be H and $T(H) = T$, respectively. Then*

$$(17) \quad \int_A^x f(t) dt \underset{x \rightarrow \infty}{\sim} x D(\ln x) f(x),$$

where $(D(x), x \in \mathbf{R})$ is a positive continuous periodic function such that

$$D(x) = D_{H,\rho}(x) = \frac{\int_0^{T\{x/T\}} e^{(\rho+1)y} H(y) dy + C}{H(x) \exp(T(\rho+1)\{x/T\})}, \quad x \geq 0,$$

$$C = C(H, \rho) = \frac{\int_0^T e^{(\rho+1)y} H(y) dy}{e^{T(\rho+1)} - 1};$$

here $\{x/T\}$ stands for the fractional part of the number $x/T(H)$. Moreover,

$$(18) \quad T(D) \leq T(H),$$

where $T(D)$ is the oscillating characteristic of the function D .

Proof. Lemma 4.1 together with relation (5) and Remark 4.1 implies that

$$(19) \quad \int_A^x f(t) dt \underset{x \rightarrow \infty}{\sim} \ell(x) U_1(x),$$

where ℓ is the SV component of the function f (see representation (3)) and

$$U_1(x) = \int_1^x t^\rho H(\ln t) dt, \quad x \geq 1.$$

It is clear that

$$U_1(x) = \int_0^{\ln x} e^{(\rho+1)u} H(u) du = V(\ln x),$$

where

$$V(x) = \int_0^x e^{(\rho+1)u} H(u) du = \sum_{k=0}^{[x/T]-1} \int_{kT}^{(k+1)T} e^{(\rho+1)u} H(u) du + R(x),$$

$$R(x) = \int_{T[x/T]}^x e^{(\rho+1)u} H(u) du,$$

and $[x/T]$ is the integer part of the number x/T .

Since H is a periodic function with period T ,

$$\int_{kT}^{(k+1)T} e^{(\rho+1)u} H(u) du = \int_0^T e^{(\rho+1)(y+kT)} H(y) dy = e^{kT(\rho+1)} \int_0^T e^{(\rho+1)y} H(y) dy$$

for all $k \in \mathbf{N}$. Thus

$$V(x) = C \left(e^{T(\rho+1)[x/T]} - 1 \right) + R(x),$$

where

$$C = \frac{\int_0^T e^{(\rho+1)y} H(y) dy}{e^{T(\rho+1)} - 1} > 0.$$

Moreover,

$$R(x) = \int_{T[x/T]}^x e^{(\rho+1)u} H(u) du = e^{T(\rho+1)[x/T]} \int_0^{T\{x/T\}} e^{(\rho+1)u} H(u) du.$$

After simple algebra we obtain

$$(20) \quad U_1(x) = x^{\rho+1} \Theta(\ln x) - C, \quad x \geq 1,$$

with

$$\Theta(x) = \frac{\int_0^{T\{x/T\}} e^{(\rho+1)y} H(y) dy + C}{\exp(T(\rho+1)\{x/T\})}, \quad x \geq 0.$$

The function Θ is positive for $x \geq 0$ and periodic with period T . It is straightforward to check that Θ is continuous. Denote by $\tilde{\Theta} = (\tilde{\Theta}(x), x \in \mathbf{R})$ the continuous periodic extension of this function to the whole axis $\mathbf{R}$.

By the assumption of Lemma 5.1, $\rho > -1$, and thus equality (20) implies that the limit $\lim_{x \rightarrow \infty} U_1(x)$ is infinite and

$$U_1(x) \underset{x \rightarrow \infty}{\sim} x^{\rho+1} \tilde{\Theta}(\ln x).$$

Taking into account asymptotics (19), we prove relation (17) with

$$D(x) = \frac{\tilde{\Theta}(x)}{H(x)}.$$

It remains to note that $(D(x), x \in \mathbf{R})$ is a positive continuous periodic function with period T , whence inequality (18) follows. $\square$

6. THE PROOF OF THE BACKWARD THEOREM

Theorem 6.1 below allows one to complete the proof of Theorem 3.2. It also contains some complements to that theorem. In the proof of Theorem 6.1 we use Lemma 5.1.

The proof of Theorem 3.2 will be given after the proof of Theorem 6.1.

Note that the rest of the proof of Theorem 3.1 uses Theorem 3.2 (see Section 7).

Theorem 6.1. *Let f be a locally integrable function belonging to $\mathbb{F}_+(A)$. If there are a number $\gamma \in (0, \infty)$ and a positive continuous periodic function $(\Gamma(x), x \in \mathbf{R})$ whose oscillating characteristic is $T(\Gamma) = T$ with $\Gamma(0) = 1$ and*

$$(21) \quad \int_A^x f(t) dt \underset{x \rightarrow \infty}{\sim} \frac{x \Gamma(\ln x) f(x)}{\gamma},$$

then f belongs to the class Φ , its index ρ is -1 , and its oscillating component and characteristic are $H = (H(x), x \in \mathbf{R})$ and $T(H) = T$, respectively.

Moreover,

1)

$$(22) \quad T(\Gamma H) = T(H) = T,$$

where $T(H)$ is the oscillating characteristic of the function H , and $T(\Gamma H)$ is the oscillating characteristic of the function $\Gamma H = (\Gamma(x)H(x), x \in \mathbf{R})$;

2) $T = 0$ if and only if $H(x) \equiv 1$ and

$$(23) \quad \rho = \gamma - 1,$$

that is, if and only if f is a regularly varying function of index (23);

3) $T > 0$ if and only if $f \in \Phi_+$, $\rho = \gamma (\Gamma^{-1})_{av} - 1$, and

$$H(x) = \frac{1}{\Gamma(x)} \exp \left(\gamma \int_0^x \left(\frac{1}{\Gamma(t)} - (\Gamma^{-1})_{av} \right) dt \right), \quad x \geq 0,$$

where

$$(\Gamma^{-1})_{av} = \frac{1}{T} \int_0^T \frac{du}{\Gamma(u)}.$$

Proof. Put

$$(24) \quad b(x) = \frac{xf(x)\Gamma(\ln x)}{\int_A^x f(t) dt}, \quad x > A.$$

According to the asymptotic formula (21),

$$(25) \quad \lim_{x \rightarrow \infty} b(x) = \gamma.$$

Equality (24) implies that

$$\frac{b(x)}{x\Gamma(\ln x)} = \frac{f(x)}{\int_A^x f(t) dt}.$$

Integrating both sides, we see that

$$\int_{A+1}^x \frac{b(t) dt}{t\Gamma(\ln t)} = \ln \left(\int_A^x f(t) dt \right) - \ln \left(\int_A^{A+1} f(t) dt \right)$$

for all $x > A + 1$. Thus

$$\int_A^x f(t) dt = a \cdot \exp \left(\int_{A+1}^x \frac{b(t) dt}{t\Gamma(\ln t)} \right), \quad x > A + 1,$$

where

$$a = \int_A^{A+1} f(t) dt.$$

This equality together with (24) implies the following representation:

$$f(x) = \frac{ab(x)}{x\Gamma(\ln x)} \exp \left(\int_{A+1}^x \frac{b(t) dt}{t\Gamma(\ln t)} \right), \quad x \geq A + 1.$$

For convenience, we rewrite this representation as follows:

$$f(x) = \frac{ab(x)}{x\Gamma(\ln x)} \exp \left(\int_A^x \frac{b(t) - \gamma}{t\Gamma(\ln t)} dt \right) \exp \left(\gamma \int_A^x \frac{dt}{t\Gamma(\ln t)} \right).$$

For $x > A + 1$, put

$$\ell_1(x) = ab(x)c(x),$$

where

$$c(x) = \exp \left(\int_A^x \frac{b(t) - \gamma}{t\Gamma(\ln t)} dt \right).$$

According to (25),

$$\lim_{t \rightarrow \infty} \frac{(b(t) - 1)}{\Gamma(\ln t)} = 0,$$

whence we conclude that $(c(x), x > A+1)$ is an SV function by the integral representation theorem for slowly varying functions (see [3, 8]). The function $(\ell_1(x), x > A+1)$ also is an SV function as a product of SV functions.

Moreover,

$$\int_{A+1}^x \frac{dt}{t\Gamma(\ln t)} = \int_{x_0}^{\ln x} \frac{du}{\Gamma(u)} = \int_0^{\ln x} \frac{du}{\Gamma(u)} - \int_0^{x_0} \frac{du}{\Gamma(u)}$$

for $x \geq A+1$, where $x_0 = \ln(A+1)$. Hence

$$(26) \quad f(x) = \frac{\ell(x)}{x\Gamma(\ln x)} M(\ln x), \quad x \geq A+1,$$

where

$$M(x) = \exp\left(\gamma \int_0^x \frac{dt}{\Gamma(t)}\right), \quad x \geq 0,$$

and ℓ is a slowly varying function such that

$$\ell(x) = \ell_1(x) \exp\left(-\gamma \int_0^{x_0} \frac{dt}{\Gamma(t)}\right), \quad x > A+1.$$

Now we define a number $(\Gamma^{-1})_{av}$ and a positive continuous periodic function $\tilde{h} = (\tilde{h}(x), x \in \mathbf{R})$. Let

$$(\Gamma^{-1})_{av} = \frac{1}{\Gamma(0)} = 1$$

if $T = 0$, that is, if the function Γ is constant, and let

$$(\Gamma^{-1})_{av} = \frac{1}{T} \int_0^T \frac{dt}{\Gamma(t)}$$

if $T > 0$. Moreover, let

$$\tilde{h}(x) \equiv 0$$

if $T = 0$, and let

$$\tilde{h}(x) = \int_0^x \left(\frac{1}{\Gamma(t)} - (\Gamma^{-1})_{av} \right) dt, \quad x \geq 0,$$

if $T > 0$. Note that T is a period of $\tilde{h}$. Then

$$\int_0^x \frac{dt}{\Gamma(t)} = x (\Gamma^{-1})_{av} + \int_0^x \left(\frac{1}{\Gamma(t)} - (\Gamma^{-1})_{av} \right) dt = x (\Gamma^{-1})_{av} + \tilde{h}(x), \quad x \geq 0,$$

whence it follows that

$$M(x) = \exp\left(x\gamma (\Gamma^{-1})_{av} + \gamma \tilde{h}(x)\right), \quad x \geq 0.$$

Taking into account equality (26), we represent the function f as follows:

$$(27) \quad f(x) = x^\rho \ell(x) H(\ln x), \quad x > A+1,$$

where

$$\begin{aligned} \rho &= \gamma (\Gamma^{-1})_{av} - 1, \\ H(x) &= \exp(h(x)), \quad x \in \mathbf{R}, \end{aligned}$$

and

$$(28) \quad h(x) = \gamma \tilde{h}(x) - \ln \Gamma(x), \quad x \in \mathbf{R}.$$

Since H is a positive continuous periodic function with period T and $H(0) = 1$, equality (3) holds, which implies that the function f belongs to the class Φ and is of index

$$(29) \quad \rho = \gamma (\Gamma^{-1})_{av} - 1 > -1.$$

If $T = 0$, then $\Gamma(x) \equiv 1$. This implies that $H(x) \equiv 1$, that is, f is a regularly varying function of index $\rho = \gamma - 1$.

Let $H(x) \equiv 1$. Then representation (28) implies that the positive continuous periodic function Γ is a continuously differentiable solution of the following first order linear differential equation with constant coefficients:

$$\frac{d\Gamma(x)}{dx} = \gamma (\Gamma^{-1})_{av} \Gamma(x) - \gamma, \quad x \in \mathbf{R}.$$

This equation holds if and only if $\Gamma(x) \equiv 1$, that is, $T = 0$.

Therefore, the equality $T = 0$ holds if and only if f is a regularly varying function of index $\rho = \gamma - 1$. This proves statement 2) of Theorem 6.1.

In its turn, the inequality $T > 0$ holds if and only if the function H is not constant. In the latter case, H is a positive continuous periodic function with period T . Thus

$$(30) \quad 0 < T(H) \leq T.$$

Hence $T > 0$ if and only if $f \in \Phi_+$. This together with (27)–(29) implies statement 3) of Theorem 6.1.

To complete the proof of Theorem 6.1, one needs to prove equality (22). If f is a regularly varying function, then $H(x) \equiv \Gamma(x) \equiv \Gamma(x)H(x) \equiv 1$. This means that $T(H) = T(\Gamma) = T(\Gamma H) = 0$ and thus (22) follows.

Now let $f \in \Phi_+$. Since H is the oscillating component of the function f , Lemma 5.1 implies that if condition (29) holds, then there is a positive continuous periodic function D for which relation (17) holds and

$$(31) \quad T(D) \leq T(H).$$

Condition (21) implies that relation (17) also holds for the positive continuous periodic function

$$D_1 = \frac{1}{\gamma} \Gamma.$$

Therefore, two positive continuous periodic functions D and D_1 are asymptotically equivalent, which happens only if these functions coincide, whence we conclude that

$$T(D) = T(D_1) = T.$$

Taking into account (30) and (31), we get

$$(32) \quad T(H) = T.$$

Now consider the function $\Gamma H = (\Gamma(x)H(x), x \in \mathbf{R})$. We derive from (28) that

$$\Gamma(x)H(x) = \exp \left(\int_0^x \left(\frac{1}{\Gamma(t)} - (\Gamma^{-1})_{av} \right) dt \right), \quad x \geq 0.$$

Since $f \in \Phi_+$, that is, $T > 0$, the positive periodic function ΓH has period T . We show that the function ΓH does not have a smaller period. This is the case if the function

$$\Psi(x) = \ln(\Gamma(x)H(x)) = \int_0^x \left(\frac{1}{\Gamma(t)} - (\Gamma^{-1})_{av} \right) dt, \quad x \geq 0,$$

does not have a period smaller than T . To the contrary, assume that there is a number $\tau \in (0, T)$ such that

$$\Psi(x + \tau) = \Psi(x)$$

for all $x > 0$. The function Γ is positive and continuous, thus the latter equality implies that

$$\Gamma(x + \tau) = \left(\frac{d\Psi(x + \tau)}{dx} \right)^{-1} = \left(\frac{d\Psi(x)}{dx} \right)^{-1} = \Gamma(x)$$

for all $x > 0$. This contradicts the assumption that T is the minimal positive period of the function Γ . Thus $T(\Gamma H) = T$ and this together with (32) proves (22).

Theorem 6.1 is completely proved. $\square$

Proof of Theorem 3.2. Theorem 3.2 follows from Theorem 6.1 with

$$\Gamma(x) = \frac{B(x)}{B(0)} \quad \text{and} \quad \gamma = \frac{1}{B(0)}. \quad \square$$

7. THE PROOF OF THE FORWARD THEOREM

Now we are in a position to complete the proof of Theorem 3.1.

Proof. Statement 2) follows from Lemma 4.1, statement 1) follows from statement 1) of Theorem 3.2, and statement 3) follows from Lemma 5.1. $\square$

8. CONCLUDING REMARKS

A generalization of Karamata's theorem on the asymptotic behavior of integrals for RV functions of index $\rho > -1$ is obtained in the paper for the case of varying upper limit of integration. A generalization is also given for functions with oscillating components. The results of the paper allow one to study Abelian and Tauberian theorems for functions with oscillating components, which are important for some problems in probability theory.

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