A SUBGEOMETRIC ESTIMATE OF THE STABILITY FOR TIME-HOMOGENEOUS MARKOV CHAINS

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ABSTRACT. Estimates for the stability of time-homogeneous Markov chains are obtained with the help of the coupling method. The results are proved for both the uniform metric and $\|\cdot\|_v$.

1. INTRODUCTION

In this paper, we consider the stability of time-homogeneous Markov chains.

Let $(E, \mathcal{E})$ be a measurable space and X and X' two independent chains defined in this space. The transition probabilities of the chains are $P(x, A)$ and $P'(x, A)$, respectively. In order to obtain an estimate of the stability, we assume that the chains are close to each other in the sense that

$$(1) \quad P = (1 - \varepsilon)Q + \varepsilon R, \quad P' = (1 - \varepsilon)Q + \varepsilon R',$$

where Q is the “common” part of two transition probabilities. The smallness of the parameter ε measures the closeness of two chains. We show that as $\varepsilon \rightarrow 0$, the difference of the transition probabilities after a sufficient number of steps tends to zero not faster than ε does.

We also obtain an estimate for the nonuniform metric $\|\cdot\|_v$. The proof of this result uses the methods introduced in the proof of Theorem 2.1 in [2]. However, an essentially different result is studied in [2], and this explains why we repeat some reasoning from [2] for the case under consideration.

A thorough investigation of the stability of time-homogeneous Markov chains is carried out in [1]. However, the operator techniques are used in [1], while the key method of this paper is the coupling. This method allows us to generalize the results of [1] to the case of nonhomogeneous Markov chains.

For a measure μ on $(E, \mathcal{E})$ and a mapping $v: E \rightarrow \mathbb{R}$, we introduce the norm

$$\|\mu\|_v = \sup_{|g| \leq v} \left| \int_E \mu(dx) g(x) \right|.$$

In particular, if $v = 1$, then

$$\|\mu\| = |\mu|(E).$$

Recall that the mapping $Q: E \times \mathcal{E} \rightarrow \mathbb{R}$ is a transition kernel if $Q(x, \cdot)$ is a measure on $\mathcal{E}$ for all $x \in E$ and $Q(\cdot, A)$ is an $\mathcal{E}$ -measurable function for all $A \in \mathcal{E}$. The product

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of two kernels is given by

$$PQ(x, A) = \int_E Q(x, dy)P(y, A).$$

By Q^k we denote the k -fold product of Q by itself.

We define a measurable function $Qf: E \rightarrow \mathbb{R}$ by

$$Qf(x) = \int_E Q(x, dy)f(y),$$

where $f: E \rightarrow \mathbb{R}$ is an arbitrary measurable function.

Given a measure $\mu \in b\mathcal{E}_+$, we define another measure μQ by

$$\mu Q(A) = \int_E \mu(dx)Q(x, A).$$

2. MAIN RESULTS

The following conditions are used below to derive estimates:

- (A1) Minorization condition (see [3, Chapter 5]): There are a set $C \in \mathcal{E}$, a probability measure ν , and a constant $\alpha > 0$ such that

$$Q(x, \cdot) \geq \alpha\nu(\cdot)$$

for all $x \in C$.

- (A2) For a stochastic kernel Q , there exists a finite invariant measure π such that

$$K = \sum_{k \geq 0} k \|\nu Q^k - \pi\| < \infty,$$

$$K(x) = \sum_{k \geq 0} \|Q^k(x, \cdot) - \pi(\cdot)\| < \infty \quad \text{for all } x \in E.$$

We introduce the following kernels:

$$(2) \quad P_\alpha(x, dy) = \frac{P(x, dy) - \alpha\nu(dy)}{1 - \alpha}, \quad P'_\alpha(x, dy) = \frac{P'(x, dy) - \alpha\nu(dy)}{1 - \alpha},$$

$$(3) \quad \begin{aligned} T(x, x'; dy, dy') &= (1 - \alpha)\mathbb{I}_{C \times C}(x, x')P_\alpha(x, dy)P'_\alpha(x', dy') \\ &\quad + \mathbb{I}_{(C \times C)^c}(x, x')P(x, dy)P'(x, dy). \end{aligned}$$

Put

$$(4) \quad \lambda(dy, dy') = \int_E \int_{E \times E} \pi(dx)R(x, dy) \times R'(x, dy').$$

Consider the following condition:

- (A3) Set

$$(5) \quad m = \sum_{k \geq 0} k \lambda T^k(E, E).$$

Then m is finite.

This condition implies that

$$(6) \quad \bar{L} = \sum_{k \geq 0} \lambda T^k(E, E) < \infty.$$

Now we introduce the transition probability

$$(7) \quad \hat{P}(x, x'; dy, dy') = T(x, x'; dy, dy') + \alpha\mathbb{I}_{C \times C}(x, x')\nu(dy \cap dy').$$

Our proof is based on the coupling method. This method can be described in short as follows. Two independent chains start. If they belong to $C \times C$, then we toss a coin

that results in a head with probability α or a tail with probability $1 - \alpha$. If the coin lands a tail up, then the chains make the next step with probabilities P_α and P'_α and further develop independently. If the coin lands a head up, then the chains are pasting together and develop according to the transition probability Q and initial distribution ν . At every consecutive step, the pasting chains disconnect with probability ε and form two independent chains with the initial distribution $R(x, dy)R'(x, dy')$.

Then the above definitions can be stated in terms of the coupling; namely, T is the probability that the chains do not couple, while $T^k(x, x')$ is the probability that the chains do not couple during k steps if they started disconnected from the point (x, x') . Further, $\bar{L}$ is the mathematical expectation of the coupling moment for a disconnected chain with initial distribution λ . Finally, $\hat{P}$ describes the evolution of a disconnected chain.

Theorem 2.1. *Let conditions (A1), (A2), and (A3) hold.*

Put $\varepsilon_0 = (m + 5K)^{-1}$. Then

$$(8) \quad \sup_{n > 0, A \in \mathcal{E}} |P^n(x, A) - P'^n(x, A)| \leq \varepsilon M(x, \varepsilon)$$

for all $x \in E$ and $\varepsilon < \varepsilon_0$. Moreover,

$$M(x, \varepsilon) = K(x) + \bar{L} + \left(K(x) + 1 + \frac{2K(1 - \varepsilon m) + 3}{1 - \varepsilon/\varepsilon_0} \right) (\varepsilon K + \bar{L})$$

and $M(x, \varepsilon) \leq (\bar{L} + 1)(K(x) + 2K + 5)$ starting with a certain small positive number ε_1 . This means that $M(x, \varepsilon)$ is bounded from above and the upper bound does not depend on ε for sufficiently small ε .

The following reasoning is needed to prove a similar result in the nonuniform metric.

The transition probability $\hat{P}$ (see equality (7)) determines the probability $\hat{\mathbb{P}}_{x, x'}$ and expectation $\hat{\mathbb{E}}_{x, x'}$. Put

$$(9) \quad \sigma_{C \times C} = \inf(n \geq 0; (X_n, X'_n) \in C \times C).$$

Let the functions $\alpha(u)$ and $\beta(u)$ and a real number $\rho \in (0, 1)$ be such that

$$(10) \quad \alpha(u)\beta(v) \leq \rho u + (1 - \rho)v.$$

The pair of functions $(cu^\beta, c_1u^{1-\beta})$, for example, satisfies the latter condition.

Consider the sequences $r(n)$ of real numbers such that $r(n) \geq n$. Let

$$R(n) = \sum_{k=0}^n r(k).$$

Assume that

(B1)

$$\bar{R} = \sum_{n \geq 0} \alpha(R(n))^{-1} < \infty.$$

In particular, if $r(n) = n$ and $\beta > 1/2$, then $R(n) \sim n^2$ and $\alpha(R(n))^{-1} \sim n^{-2\beta}$, whence we derive that the series in condition (B1) converges.

Consider the function $U(x, y) = \hat{\mathbb{E}}_{x, y}[R(\sigma_{C \times C})]$ and introduce two further conditions:

(U1) For all $x, y \in E$ $U(x, y) < \infty$,

$$(11) \quad b_U = \sup_{x, y \in C} \hat{P}U(x, y) < \infty, \quad \lambda U < \infty.$$

(V1) There exists a function $v: E \times E \rightarrow [0, \infty)$ such that

$$(12) \quad b_V = \sup_{x, y \in C} \hat{P}V(x, y) < \infty, \quad \lambda V < \infty$$

for all $x, y \in E$ with $V(x, y) < \infty$, where

$$V(x, y) = \mathbb{E}_{x, y} \left[\sum_{k=0}^{\sigma_C \times C} v(X_k, X'_k) \right].$$

We also assume that there is a function $g: X \rightarrow \mathbb{R}$ such that

$$(13) \quad g(x) + g(y) \leq \beta(V(x, y)).$$

For α defined in condition (A1), let

$$(14) \quad M_U = \sup_{n \geq 1} \left[\left(b_U r(n) \frac{1 - \alpha}{\alpha} - R(n + 1) \right)_+ \right],$$

$$(15) \quad M_V = b_V \frac{1 - \alpha}{\alpha}.$$

Theorem 2.2. *Let conditions (A1), (A2), and (A3) hold. We also assume that (B1), (U1), and (V1) hold. Let a function g satisfy inequality (13) and let $\varepsilon_0 = (m + 5K)^{-1}$. Then*

$$(16) \quad \sup_{n > 0, A \in \mathcal{E}} \|P^n(x, A) - P'^n(x, A)\|_g \leq \varepsilon M_g(x, \varepsilon)$$

for all $x \in E$ and all $\varepsilon < \varepsilon_0$, where $\bar{L}_g = (\rho(\lambda U + M_u) + (1 - \rho)(\lambda V + M_v)) \bar{R}$, $\bar{R}$ is defined in condition (B2), and

$$M_g(x, \varepsilon) = K(x) + \bar{L}_g + \left(K(x) + 1 + \frac{2K(1 - \varepsilon m) + 3}{1 - \varepsilon/\varepsilon_0} \right) (\varepsilon K + \bar{L}_g).$$

Moreover, $M_g(x, \varepsilon) \leq (\bar{L}_g + 1)(K(x) + 2K + 5)$ for $\varepsilon < \varepsilon_1$, where ε_1 is a sufficiently small number.

3. MAIN LEMMAS

Now we formally define the coupling procedure.

Consider the space $Z = (E \times E \times \{0, 1, 2\})$ and a Markov chain $(Z_n, n \geq 0)$ defined on Z . We define $Z_n = (X1_n, X2_n, d_n)$ as follows: $d_0 = 0$, $X1_0 = X_0$, and $X2_0 = X'_0$; if Z_n is already defined, then Z_{n+1} is defined as follows.

If $d_n = 0$ and $(X_n, X'_n) \in C \times C$, then we toss a coin landing head up with probability α and tail up otherwise. If the coin lands head up, then we put $d_{n+1} = 1$ and $X1_{n+1} = X2_{n+1} = X$, where $X \sim \nu(\cdot)$. Otherwise, if the coin lands tail up, then we set $d_{n+1} = 0$ and $(X1_{n+1}, X2_{n+1}) \sim T(X1_n, X2_n, (\cdot, \cdot))$.

If $d_n = 0$ and $(X_n, X'_n) \notin C \times C$, then we set $d_{n+1} = 0$ and

$$(X1_{n+1}, X2_{n+1}) \sim P \times P'(\cdot, \cdot).$$

If $d_n \in \{1, 2\}$, then we toss another coin landing head up with probability ε . If the coin lands head up, then we set $Z_{n+1} = (X, X', 0)$, where $X \sim R(X1_n, \cdot)$ and $X' \sim R'(X2_n, \cdot)$. Otherwise, if the coin lands tail up, we set $Z_{n+1} = (X, X, 2)$, where $X \sim Q(X1_n)$.

Denote by $\bar{\mathbb{P}}$ and $\bar{\mathbb{E}}$ the probability measure and the corresponding expectation generated by the chain $(Z_n, n \geq 0)$.

Below we write the transition probability for the chain Z_n :

$$\begin{aligned}
\bar{P}(x, x', 0; A \times A' \times \{0\}) &= (1 - \alpha)T(x, x'; A \times A'), \\
\bar{P}(x, x', 0; A \times A' \times \{1\}) &= \alpha\nu(A \cap A'), \\
\bar{P}(x, x', 0; E \times E \times \{2\}) &= 0, \\
\bar{P}(x, x', 1; A \times A' \times \{0\}) &= \varepsilon I_{\{x=x'\}} R(x, A) R'(x', A'), \\
\bar{P}(x, x', 1; A \times A' \times \{1\}) &= 0, \\
\bar{P}(x, x', 1; A \times A' \times \{2\}) &= (1 - \varepsilon) I_{\{x=x'\}} Q(x, A \cap A'), \\
\bar{P}(x, x', 2; A \times A' \times D) &= \bar{P}(x, x', 1; A \times A' \times D).
\end{aligned}$$

We introduce the following notation:

$$(17) \quad h_n(x) = \sum_{k=1}^{n-1} (1 - \varepsilon) \varepsilon \int Q^{k-1}(x, dy) R(y, dz) R'(y, dz') T^{n-k}(z, z'),$$

$$(18) \quad h_n = \int \nu(dx) h_n(x),$$

$$(19) \quad g_n = \sum_{k=1}^{n-1} (1 - \varepsilon) \varepsilon \int \int \nu(dx) Q^{k-1}(x, y) R(y, dz) R'(y, dz') T^{n-k}(z, z'),$$

$$(20) \quad a_n(x) = \sum_{k=1}^{n-1} (1 - \varepsilon) \varepsilon \alpha \int Q^{k-1}(x, dy) R(y, dz) R'(y, dz') T^{n-k}(z, z', C \times C),$$

$$(21) \quad g_n = \sum_{k=1}^{n-1} (1 - \varepsilon) \varepsilon \alpha \int \nu Q^{k-1}(x, dy) R(y, dz) R'(y, dz') T^{n-k}(z, z', C \times C),$$

$$(22) \quad u_n = \sum_{t \geq 0} (g^{*t})_n,$$

$$(23) \quad s_n = \int \pi(dy) R(y, dz) R'(y, dz') T^n(z, z', E, E) = \lambda T^n(E, E),$$

$$S_n = \sum_{k \geq n} s_k.$$

Therefore, $\bar{L} = \sum_{k \geq 0} s_k$ and $m = \sum_{k \geq 0} S_k = \sum_{k \geq 0} k s_k$.

The above variables have a clear probabilistic meaning. Namely, h_n ($h_n(x)$) is the probability that the chains with initial distribution ν (δ_x) start in couple, then disconnect, and after that do not couple again up to the moment n .

Further, g_n is the probability that the chains with initial distribution ν start in couple, then disconnect, and after that couple again at the moment n . The sequence u_n represents the renewals generated by g_n .

Finally, s_n is the probability that the chains with distribution λ start disconnected and do not couple during the time n .

We denote by τ the first pasting moment and by τ_1 the moment of the next coupling.

Remark 3.1. The term $T^k(x, x')$ appeared in the above definitions should be understood as $T^k(x, x', E, E)$. Sometimes we write $h_n(x, dy, dy')$ and $\bar{L}(dy, dy')$. In this case, the operators $T^k(x, x')$ appeared in the definition of $h_n(x)$ and $\bar{L}$ should be understood as $T^k(x, x', dy, dy')$.

Lemma 3.1.

$$\begin{aligned} \sup_{n \geq 0} \|\delta_x P^n - \delta_x P'^n\| &\leq h_n(x) + \left(\sup_{n \geq 0} a_n(x) + \sup_{n \geq 1} u_n \right) \sum_{k \geq 0} h_k, \\ \sup_{n \geq 0} \|\delta_x P^n - \delta_x P'^n\|_v &\leq \int h_n(x, dy, dy')(v(x) + v(y)) \\ &\quad + \left(\sup_{n \geq 0} a_n(x) + \sup_{n \geq 1} u_n \right) \int \sum_{k \geq 0} h_k(dy, dy')(v(x) + v(y)). \end{aligned}$$

Proof. We have

$$\begin{aligned} P(x, A) &= \bar{P}(x, x', d; A \times E \times \{0, 1, 2\}), \\ P'(x', A') &= \bar{P}(x, x', d; E \times A' \times \{0, 1, 2\}) \end{aligned}$$

for all $d \in \{0, 1, 2\}$.

It is sufficient to prove the lemma for $d \in \{0, 1\}$, since the transition probabilities for $d = 2$ coincide with those for $d = 1$:

$$\begin{aligned} \bar{P}(x, x', 0; A \times E \times \{0, 1, 2\}) &= \bar{P}(x, x', 0; A \times E \times \{0\}) + \bar{P}(x, x', 0; A \times E \times \{1\}) \\ &= (1 - \varepsilon)T(x, x'; A \times E) + \varepsilon\nu(cr(A, E)) \\ &= (1 - \varepsilon) \frac{P(x, A) - \varepsilon\nu(A)}{1 - \varepsilon} \frac{P'(x', E) - \varepsilon\nu(E)}{1 - \varepsilon} + \varepsilon\nu(A) \\ &= (1 - \varepsilon) \frac{P(x, A) - \varepsilon\nu(A)}{1 - \varepsilon} + \varepsilon\nu(A) \\ &= P(x, A), \\ \bar{P}(x, x, 1; A \times E \times \{0, 1, 2\}) &= \bar{P}(x, x, 1; A \times E \times \{0\}) + \bar{P}(x, x, 1; A \times E \times \{2\}) \\ &= \alpha R(x, A)R'(x, E) + (1 - \alpha)Q(x, A) \\ &= \alpha R(x, A) + (1 - \alpha)Q(x, A) \\ &= P(x, A). \end{aligned}$$

The equalities for P' are obtained similarly.

Now let $\phi \in \mathcal{E}_+$, $d \in \{0, 1, 2\}$, and $x' \in E$. Then

$$P\phi(x) = \int_{E \times E \times \{0, 1, 2\}} \bar{P}(x, x', d; dx_1 \times dx'_1 \times i) \phi(x_1).$$

It is sufficient to check the above equality for $\phi(x) = I_A(x)$, where $A \in \mathcal{E}$:

$$\begin{aligned} (24) \quad &\int_{E \times E \times \{0, 1, 2\}} \bar{P}(x, x', d; dx_1 \times dx'_1 \times i) I_A(x_1) \\ &= \bar{P}(x, x', 1; A \times E \times \{0, 1, 2\}) = P(x, A) = P\phi(x). \end{aligned}$$

Then

$$P^n \phi(x) = \int_{E \times E \times \{0, 1, 2\}} \bar{P}^n(x, x', d; dx_n \times dx'_n \times i) \phi(x_n)$$

for an arbitrary $d \in \{0, 1, 2\}$ and all $x' \in E$ and $\phi \in \mathcal{E}_+$.

Lemma 3.1 for $n = 1$ follows from (24). Assume that the statement of Lemma 3.1 holds for some n and check it for $n + 1$:

$$\begin{aligned}
& \int_{E \times E \times \{0,1,2\}} \bar{P}^n(x, x', d; dx_n \times dx'_n \times i_n) \int \bar{P}(x_n, x'_n, i_n; dx_{n+1} \times dx'_{n+1} \times i_{n+1}) \phi(x_{n+1}) \\
&= \int_{E \times E \times \{0,1,2\}} \bar{P}^n(x, x', d; dx_n \times dx'_n \times i_n) P\phi(x_n) \\
&= P^{n+1}\phi(x), \\
|P^n(x, A) - P'^n(x, A)| &= |\bar{P}^n(x, x, 1; A \times E \times \{0, 1, 2\} - \bar{P}^n(x, x, 1; E \times A \times \{0, 1, 2\})|, \\
&|\bar{P}^n(x, x, 1; A \times E \times \{1\} - \bar{P}^n(x, x, 1; E \times A \times \{1\})| \\
&= \left| \int \bar{P}(x, x, 1; dx_1 \times dx_2 \times \{0, 1, 2\}) \cdots \int \varepsilon \nu(A \cap E) \right. \\
&\quad \left. - \int \bar{P}(x, x, 1; dx_1 \times dx_2 \times \{0, 1, 2\}) \cdots \int \varepsilon \nu(E \cap A) \right| \\
&= 0, \\
&|\bar{P}^n(x, x, 1; A \times E \times \{2\} - \bar{P}^n(x, x, 1; E \times A \times \{2\})| \\
&= \left| \int \bar{P}(x, x, 1; dx_1 \times dx_2 \times \{0, 1, 2\}) \cdots \int (1 - \alpha) Q(x_{n-1}, A \cap E) \right. \\
&\quad \left. - \int \bar{P}(x, x, 1; dx_1 \times dx_2 \times \{0, 1, 2\}) \cdots \int (1 - \alpha) Q(x_{n-1}, E \cap A) \right| \\
&= 0.
\end{aligned}$$

Thus

$$\begin{aligned}
|P^n(x, A) - P'^n(x, A)| &= |\bar{P}^n(x, x, 1; A \times E \times \{0\}) - \bar{P}^n(x, x, 1; E \times A \times \{0\})| \\
&\leq \max(\bar{P}^n(x, x, 1; A \times E \times \{0\}), \bar{P}^n(x, x, 1; E \times A \times \{0\})) \\
&\leq \bar{P}^n(x, x, 1; E \times E \times \{0\}),
\end{aligned}$$

$$\begin{aligned}
& \int |P^n(x, dy) - P'^n(x, dy)| v(y) \\
&= \left| \int \bar{P}^n(x, x, 1; dy \times E \times \{0\}) v(y) - \int \bar{P}^n(x, x, 1; E \times dy \times \{0\}) v(y) \right| \\
&\leq \max \left(\int \bar{P}^n(x, x, 1; dy \times E \times \{0\}) v(y), \int \bar{P}^n(x, x, 1; E \times dy \times \{0\}) v(y) \right) \\
&\leq \int_{E \times E} \bar{P}^n(x, x, 1; dy \times dy' \times \{0\}) (v(y) + v(y')).
\end{aligned}$$

Now

$$\begin{aligned}
\bar{P}^n(x, x, 1; E \times E \times \{0\}) &= \bar{P}_{x,x,1}(\tau > n) + \bar{P}_{x,x,1}(\tau \leq n, \tau_1 > n) + \bar{P}_{x,x,1}(\tau_1 \leq n), \\
\bar{P}_{x,x,1}(\tau > n) &= h_n(x), \\
\bar{P}_{x,x,1}(\tau \leq n, \tau_1 > n) &= (a(x) \star h)_n, \\
\bar{P}_{x,x,1}(\tau_1 \leq n) &= (a(x) \star u \star h)_n \leq \sup_{n \geq 1} u_n \sum_{k \geq 0} h_k \sum_{k \geq 0} a_k(x) \leq \sup_{n \geq 1} u_n \sum_{k \geq 0} h_k.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \int \bar{P}^n(x, x, 1; dy \times dy' \times \{0\}) (v(y) + v(y')) \\
&= \int \left[\bar{P}_{x,x,1}(\tau > n, (X_n, X'_n) \in (dy, dy')) \right. \\
&\quad \left. + \bar{P}_{x,x,1}(\tau \leq n, \tau_1 > n, (X_n, X'_n) \in (dy, dy')) \right. \\
&\quad \left. + \bar{P}_{x,x,1}(\tau_1 \leq n, (X_n, X'_n) \in (dy, dy')) \right] (v(y) + v(y')), \\
& \int \bar{P}_{x,x,1}(\tau > n, (X_n, X'_n) \in (dy, dy')) (v(y) + v(y')) = \int h_n(x, dy, dy') (v(y) + v(y')), \\
& \int \bar{P}_{x,x,1}(\tau \leq n, \tau_1 > n, (X_n, X'_n) \in (dy, dy')) = \left(a(x) \star \int h(dy, dy') (v(y) + v(y')) \right)_n, \\
& \int \bar{P}_{x,x,1}(\tau_1 \leq n, (X_n, X'_n) \in (dy, dy')) = \left(a(x) \star u \star \int h(dy, dy') [v(y) + v(y')] \right)_n \\
&\leq \sup_{n \geq 1} u_n \sum_{k \geq 0} \int h_k(dy, dy') [v(y) + v(y')] \sum_{k \geq 0} a_k(x) \\
&\leq \sup_{n \geq 1} u_n \sum_{k \geq 0} \int h_k(dy, dy') [v(y) + v(y')].
\end{aligned}$$

The lemma is proved. $\square$

Lemma 3.2.

$$(25) \quad \sup_{n \geq 1} u_n \leq \varepsilon \times \frac{2K + \frac{3}{1-\varepsilon m}}{1 - \varepsilon K \left(2 + \frac{3\varepsilon}{1-\varepsilon m} \right)}.$$

Proof. We have

$$\begin{aligned}
a_n &= \sum_{k=1}^{n-1} (1-\varepsilon)^{k-1} \varepsilon \int \pi(dy) R(y, dz) R'(y, dz') T^{n-k}(z, z') = \varepsilon \sum_{k=1}^n (1-\varepsilon)^{k-1} s_{n-k}, \\
e_n &= \varepsilon \sum_{k=1}^n (1-\varepsilon)^{k-1} (\nu Q^{k-1} - \pi) \int R \times R'(dy, dy') T^{n-k}(y, y', E, E).
\end{aligned}$$

Adding and subtracting π in the definition (19) of g_n , we get $g_n = a_n + e_n$. Then

$$u_n = \delta_0(n) + (g \star u)_n = \delta_0(n) + (a \star u)_n + (e \star u)_n.$$

Put

$$y_n = \delta_0(n) + (e \star u)_n.$$

Hence

$$u_n = y_n + (a \star u)_n.$$

Therefore, for an arbitrary $n \geq 0$,

$$u_n = (y \star b)_n = b_n + (e \star b \star u)_n,$$

where

$$b_n = \delta_0(n) + (a \star b)_n = \sum_{t \geq 0} (a^{\star t})_n.$$

The moment generating function $b(z)$ of the sequence b_n is given by

$$b(z) = \frac{1}{1 - a(z)},$$

where $a(z)$ is the moment generating function of the sequence a_n :

$$\begin{aligned} a(z) &= 0 + z\varepsilon s_0 + z^2\varepsilon \sum_{k=1}^2 (1-\varepsilon)^k s_{2-k} + \cdots = \varepsilon z \sum_{n \geq 0} z^n \sum_{k=0}^n (1-\varepsilon)^k s_{n-k} \\ &= \varepsilon z \frac{s(z)}{1-z(1-\varepsilon)}. \end{aligned}$$

Then the function $b(z)$ is such that

$$\begin{aligned} b(z) &= \frac{1}{1 - \frac{\varepsilon z s(z)}{1-z(1-\varepsilon)}} = \frac{1-z(1-\varepsilon)}{1-z(1-\varepsilon) - \varepsilon z s(z)} = \frac{1-z(1-\varepsilon)}{1-z+z\varepsilon - \varepsilon z s(z)} \\ &= \frac{1-z+\varepsilon z}{1-z} \frac{1}{1 - \varepsilon z \frac{s(z)-1}{1-z}} = \left(1 + \frac{\varepsilon z}{1-z}\right) \left(\frac{1}{1 - \varepsilon z \frac{s(z)-1}{1-z}}\right) \\ &= \left(1 + \frac{\varepsilon z}{1-z}\right) \left(\frac{1}{1 + \varepsilon z S(z)}\right), \end{aligned}$$

where S_n is defined by (23).

Set

$$\begin{aligned} d(z) &= \sum_{n \geq 0} z^n d_n = \frac{1}{1 + \varepsilon z S(z)} = \sum_{n \geq 0} z^n (-\varepsilon)^n S(z)^n = \sum_{n \geq 0} z^n (-\varepsilon)^n \sum_{k \geq 0} z^k S_k^{\star n} \\ &= \sum_{n \geq 0} z^n \sum_{k=0}^n (-\varepsilon)^k S_{n-k}^{\star k}. \end{aligned}$$

Thus

$$(26) \quad d_0 = 1, \quad d_n = \sum_{k=0}^n (-\varepsilon)^k S_{n-k}^{\star k}$$

and

$$\begin{aligned} b(z) &= d(z) + \frac{\varepsilon z}{1-z} d(z) = \sum_{k \geq 0} z^k d_k + \varepsilon z \sum_{k \geq 0} z^k \sum_{i=0}^k d_i \\ &= d_0 + \sum_{k \geq 1} z^k d_k + \varepsilon \sum_{k \geq 0} z^{k+1} \sum_{i=0}^k d_i = d_0 + \sum_{k \geq 1} z^k d_k + \varepsilon \sum_{k \geq 1} z^k \sum_{i=0}^{k-1} d_i \\ &= d_0 + \sum_{k \geq 1} z^k \left(d_k + \varepsilon \sum_{i=0}^{k-1} d_i \right), \end{aligned}$$

whence we obtain a representation for b_n :

$$\begin{aligned} b_0 &= d_0 = 1, \\ b_n &= d_n + \varepsilon \sum_{k=0}^{n-1} d_k. \end{aligned}$$

Consider the following auxiliary sequence c_n :

$$c_0 = 1, \quad c_n = b_n - b_{n-1}.$$

Note that

$$\begin{aligned} b_n &= \sum_{k=0}^n c_k \leq \sum_{k=0}^n |c_k|, \quad \sup_{n \geq 0} b_n \leq \sum_{k=0}^{\infty} |c_k|, \\ \sup_{n \geq 1} b_n &\leq \sum_{k \geq 2} |b_k - b_{k-1}| + |b_1| = \sum_{k \geq 0} |b_k - b_{k-1}| - 2 = \sum_{k \geq 0} |c_k| - 2. \end{aligned}$$

Here we have used the property that $b_1 = 0$.

Our aim is to show that

$$\sum_{n \geq 0} |b_n - b_{n-1}| = \sum_{n \geq 0} |c_n| < \infty.$$

For all $n \geq 1$, we have

$$b_n - b_{n-1} = d_n - d_{n-1} + \varepsilon d_{n-1}.$$

Therefore,

$$\sum_{n \geq 0} |b_n - b_{n-1}| \leq \sum_{n \geq 0} |d_n - d_{n-1}| + \varepsilon \sum_{n \geq 0} |d_n|,$$

where $b_{-1} = d_{-1} = 0$.

Now we evaluate $\sum_{n \geq 0} |d_n - d_{n-1}|$. Note that

$$(27) \quad S_n^{\star k} - S_{n-1}^{\star k} = S_n^{\star k-1} - (S_n^{\star k-1} \star s)_{n-1}$$

for $k \geq 1$ and $n \geq 1$, where S_n is defined by (23). Indeed,

$$\begin{aligned} S_n^{\star k} - S_{n-1}^{\star k} &= \sum_{i=0}^n S_i^{\star k-1} S_{n-i} - \sum_{i=0}^{n-1} S_i^{\star k-1} S_{n-1-i} \\ &= S_n^{\star k-1} S_0 + \sum_{i=0}^{n-1} S_i^{\star k-1} (S_{n-i} - S_{n-i-1}) \\ &= S_n^{\star k-1} 1 + \sum_{i=0}^{n-1} S_i^{\star k-1} s_{n-i-1} = S_n^{\star k-1} - (S_n^{\star k-1} \star s)_{n-1}. \end{aligned}$$

Note also that

$$S_0^{\star 0} = 1, \quad S_i^{\star 0} = 0$$

for all $i > 0$. Putting

$$D_1 = \sum_{n \geq 1} \sum_{k=0}^{n-1} \varepsilon^k |S_{n-k}^{\star k} - S_{n-k-1}^{\star k}|$$

and taking into account that

$$\begin{aligned} S_0^{\star n} &= S_0^n = 1^n = 1, \\ d_0 &= 1, \end{aligned}$$

we get

$$\begin{aligned} \sum_{n \geq 0} |d_n - d_{n-1}| &= d_0 + \sum_{n \geq 1} |d_n - d_{n-1}| = d_0 + \sum_{n \geq 1} \left[\sum_{k=0}^{n-1} \varepsilon^k |S_{n-k}^{\star k} - S_{n-k-1}^{\star k}| + \varepsilon^n S_0^n \right] \\ &= 1 + \frac{\varepsilon}{1 - \varepsilon} + D_1. \end{aligned}$$

Now we evaluate D_1 :

$$\begin{aligned}
D_1 &= \sum_{n \geq 1} \sum_{k=0}^{n-1} \varepsilon^k |S_{n-k}^{\star k} - S_{n-k-1}^{\star k}| \\
&= \sum_{n \geq 2} \sum_{k=0}^{n-1} \varepsilon^k |S_{n-k}^{\star k} - S_{n-k-1}^{\star k}| + \sum_{k=0}^0 \varepsilon^k |S_{1-k}^{\star k} - S_{1-k-1}^{\star k}| \\
&= \sum_{n \geq 2} \sum_{k=0}^{n-1} \varepsilon^k |S_{n-k}^{\star k} - S_{n-k-1}^{\star k}| + \varepsilon^0 |S_1^{\star 0} - S_0^{\star 0}| = 1 + D_2,
\end{aligned}$$

where

$$D_2 = \sum_{n \geq 2} \sum_{k=0}^{n-1} \varepsilon^k |S_{n-k}^{\star k} - S_{n-k-1}^{\star k}|.$$

Using equality (27) we get

$$D_2 \leq \sum_{n \geq 2} \sum_{k=0}^{n-1} \varepsilon^k S_{n-k}^{\star k-1} + \sum_{n \geq 2} \sum_{k=0}^{n-1} \varepsilon^k (S^{\star k-1} \star s)_{n-k-1} = D_{21} + D_{22},$$

where

$$\begin{aligned}
D_{21} &= \sum_{n \geq 2} \sum_{k=0}^{n-1} \varepsilon^k S_{n-k}^{\star k-1}, \\
D_{22} &= \sum_{n \geq 2} \sum_{k=0}^{n-1} \varepsilon^k (S^{\star k-1} \star s)_{n-k-1}.
\end{aligned}$$

First we deal with D_{21} :

$$\begin{aligned}
D_{21} &= \varepsilon \sum_{n \geq 0} \varepsilon^n \sum_{k \geq 1} S_k^{\star n} = \varepsilon \sum_{n \geq 0} \varepsilon^n \sum_{k \geq 0} S_k^{\star n} - \varepsilon \sum_{n \geq 0} \varepsilon^n S_0^{\star n} \\
&= \varepsilon \sum_{n \geq 0} \varepsilon^n \left(\sum_{k \geq 0} S_k \right)^n - \frac{\varepsilon}{1-\varepsilon} = \varepsilon \sum_{n \geq 0} (\varepsilon m)^n - \frac{\varepsilon}{1-\varepsilon} = \frac{\varepsilon}{1-\varepsilon m} - \frac{\varepsilon}{1-\varepsilon}.
\end{aligned}$$

Then we evaluate D_{22} :

$$D_{22} = \varepsilon \sum_{n \geq 0} \varepsilon^n \sum_{k \geq 0} (S^{\star k} \star s)_k = \varepsilon \sum_{n \geq 0} \varepsilon^n m^n 1 = \frac{\varepsilon}{1-\varepsilon m}.$$

Thus

$$\begin{aligned}
D_2 &= D_{21} + D_{22} = \frac{\varepsilon}{1-\varepsilon m} - \frac{\varepsilon}{1-\varepsilon} + \frac{\varepsilon}{1-\varepsilon m} = \frac{2\varepsilon}{1-\varepsilon m} - \frac{\varepsilon}{1-\varepsilon}, \\
D_1 &= 1 + D_2 \leq 1 + \frac{2\varepsilon}{1-\varepsilon m} - \frac{\varepsilon}{1-\varepsilon}, \\
\sum_{n \geq 0} |d_n - d_{n-1}| &= 1 + \frac{\varepsilon}{1-\varepsilon} + D_1 \leq 1 + \frac{\varepsilon}{1-\varepsilon} + 1 + \frac{2\varepsilon}{1-\varepsilon m} - \frac{\varepsilon}{1-\varepsilon} = 2 + \frac{2\varepsilon}{1-\varepsilon m}.
\end{aligned}$$

Note that

$$\sum_{n \geq 0} |d_n| = \frac{1}{1-\varepsilon S(1)} = \frac{1}{1-\varepsilon m}.$$

Hence we write

$$\begin{aligned}
 \sum_{n \geq 0} |c_n| &= \sum_{n \geq 0} |b_n - b_{n-1}| \leq \sum_{n \geq 0} |d_n - d_{n-1}| + \varepsilon \sum_{n \geq 0} |d_n| \\
 &\leq 2 + \frac{2\varepsilon}{1 - \varepsilon m} + \frac{\varepsilon}{1 - \varepsilon m} = 2 + \frac{3\varepsilon}{1 - \varepsilon m}, \\
 (28) \quad \sup_{n \geq 1} b_n &\leq \sum_{n \geq 2} |b_n - b_{n-1}| \leq \frac{3\varepsilon}{1 - \varepsilon m}.
 \end{aligned}$$

Now we turn to the sequence $\{u_n\}$:

$$\begin{aligned}
 u_n &= (y \star b)_n = (\delta_0 \star b)_n + (e \star b \star u)_n = b_n + (e \star b \star u)_n, \\
 (e \star b \star u)_n &= \sum_{k=0}^n u_k (e \star b)_{n-k} \leq (e \star b)_n + \sum_{k=1}^n (e \star b)_{n-k} \sup_{n \geq 1} u_n \\
 &= (e \star b)_n + (e \star b \star 1)_{n-1} \sup_{n \geq 1} u_n.
 \end{aligned}$$

Note that

$$b = 1 \star c.$$

Since e is the difference of two probability distributions, we have

$$\begin{aligned}
 \sum_{k \geq 0} e_k &= 0, \\
 E_n &= (e \star 1)_n = \sum_{k=0}^n e_k = - \sum_{k > n} e_k.
 \end{aligned}$$

Consider the series $\sum_{n \geq 0} |E_n|$:

$$\begin{aligned}
 \sum_{n \geq 0} |E_n| &= \sum_{n \geq 0} \sum_{k > n} |e_k| = \sum_{n \geq 0} k |e_k| \leq \varepsilon \sum_{n \geq 1} n \sum_{k=1}^n (1 - \varepsilon)^{k-1} \int (\nu Q^{k-1} - \pi)(dy) s_{n-k}(y) \\
 &\leq \varepsilon \int \sum_{n \geq 1} n (1 - \varepsilon)^{n-1} (\nu Q^{n-1} - \pi)(dy) \times \sum_{n \geq 1} s_n(y) \\
 &= \varepsilon \sum_{n \geq 1} n (1 - \varepsilon)^{n-1} \|\nu Q^{n-1} - \pi\| \leq \varepsilon K.
 \end{aligned}$$

We have

$$(e \star b \star 1)_n = (e \star 1 \star c \star 1)_n \leq \sum_{n \geq 0} |E_n| \sum_{n \geq 0} |c_n| \leq \varepsilon K \times \left(2 + \frac{3\varepsilon}{1 - \varepsilon m}\right).$$

Inequality (28) implies that

$$(29) \quad (e \star b)_n \leq \sum_{n \geq 0} |E_n| \sup |c_n| \leq \varepsilon K.$$

Thus

$$\begin{aligned}
 \sup_{n \geq 1} u_n &\leq \sup_{n \geq 1} b_n + \varepsilon K + \varepsilon K \times \left(2 + \frac{3\varepsilon}{1 - \varepsilon m}\right) \sup_{n \geq 1} u_n, \\
 \sup_{n \geq 1} u_n &\leq \frac{\varepsilon K + \frac{3\varepsilon}{1 - \varepsilon m}}{1 - \varepsilon K \times \left(2 + \frac{3\varepsilon}{1 - \varepsilon m}\right)}.
 \end{aligned}$$

The lemma is proved. $\square$

4. ADDITIONAL LEMMAS

Lemma 4.1.

$$\sum_{k \geq 0} h_k \leq \varepsilon K + \bar{L}.$$

Proof. We have

$$\begin{aligned} \sum_{n \geq 0} h_n &= \varepsilon \sum_{n \geq 0} \sum_{k=1}^n (1-\varepsilon)^{k-1} \nu Q^{k-1} R \times R' T^{n-k} \\ &= \varepsilon \sum_{n \geq 0} \sum_{k=1}^n (1-\varepsilon)^{k-1} (\nu Q^{k-1} - \pi) R \times R' T^{n-k} \\ &\quad + \varepsilon \sum_{n \geq 0} \sum_{k=1}^n (1-\varepsilon)^{k-1} \pi R \times R' T^{n-k} \\ &\leq \varepsilon K + \varepsilon \sum_{n \geq 0} (1-\varepsilon)^n \sum_{k \geq 0} \lambda T^k \leq \varepsilon K + \bar{L}. \end{aligned}$$

The lemma is proved. $\square$

Lemma 4.2.

$$\begin{aligned} \sup_{n \geq 0} h_n(x) &\leq \varepsilon [K(x) + \bar{L}], \\ \sup_{n \geq 0} \int h_n(x, dy, dy') (v(x) + v(y)) &\leq \varepsilon \left[K(x) + \int \bar{L}(dy, dy') (v(y) + v(y')) \right]. \end{aligned}$$

Proof. We have

$$\begin{aligned} h_n(x) &= \varepsilon \sum_{k=1}^n (1-\varepsilon)^{k-1} \delta_x Q^{k-1} R \times R' T^{n-k} \\ &= \varepsilon \sum_{k=1}^n (1-\varepsilon)^{k-1} (\delta_x Q^{k-1} - \pi) R \times R' T^{n-k} + \varepsilon \sum_{k=1}^n (1-\varepsilon)^{k-1} \pi R \times R' T^{n-k} \\ &\leq \varepsilon \sum_{k \geq 0} |\delta_x Q^k - \pi| + \varepsilon \sum_{k \geq 0} \lambda T^k = \varepsilon [K(x) + \bar{L}]. \end{aligned}$$

Substituting T^k for $T^k(dy, dy')$ and reasoning in the same way as before we prove the second equality.

The lemma is proved. $\square$

Lemma 4.3.

$$\sup_{n \geq 0} a_n(x) \leq \varepsilon [K(x) + 1].$$

Proof. We have

$$\begin{aligned} a_n(x) &= \varepsilon \sum_{k=1}^n (1-\varepsilon)^{k-1} \delta_x Q^{k-1} R \times R' T^{n-k} \\ &\leq \varepsilon \sum_{k=1}^n (1-\varepsilon)^{k-1} (\delta_x Q^{k-1} - \pi) R \times R' T^{n-k} + \varepsilon \sum_{k=1}^n (1-\varepsilon)^{k-1} \lambda T^{n-k} \\ &\leq \varepsilon \left[K(x) + \sum_{k \geq 0} \lambda s_k \right] = \varepsilon [K(x) + 1]. \end{aligned}$$

The lemma is proved. $\square$

5. THE PROOF OF THE MAIN RESULTS

Proof of Theorem 2.1. Lemma 3.1 implies that

$$\sup_{n \geq 0} \|P^n(x, \cdot) - P'^n(x, \cdot)\| \leq \sup_{n \geq 0} h_n(x) + \left(\sup_{n \geq 0} a_n(x) + \sup_{n \geq 1} u_n \right) \sum_{k \geq 0} h_k.$$

Lemmas 4.1–4.3 yield

$$\sup_{n \geq 0} \|P^n(x, \cdot) - P'^n(x, \cdot)\| \leq \varepsilon[K(x) + \bar{L}] + \left(\varepsilon[K(x) + 1] + \sup_{n \geq 1} u_n \right) (\varepsilon K + \bar{L}).$$

Substituting the bound (25) for u_n , we obtain

$$\begin{aligned} & \sup_{n \geq 0} \|P^n(x, \cdot) - P'^n(x, \cdot)\| \\ & \leq \varepsilon[K(x) + \bar{L}] + \varepsilon \left(K(x) + 1 + \frac{2K + \frac{3}{1-\varepsilon m}}{1 - \varepsilon K \left(2 + \frac{3\varepsilon}{1-\varepsilon m} \right)} \right) (\varepsilon K + \bar{L}) \\ & = \varepsilon[K(x) + \bar{L}] + \varepsilon \left(K(x) + 1 + \frac{2K - 2K\varepsilon m + 3}{1 - \varepsilon m - 2\varepsilon K + 2\varepsilon^2 K m - 3\varepsilon^2 K} \right) (\varepsilon K + \bar{L}) \\ & = \varepsilon[K(x) + \bar{L}] + \varepsilon \left(K(x) + 1 + \frac{2K(1 - \varepsilon m) + 3}{1 - \varepsilon(m + 2K - 2\varepsilon K m + 3\varepsilon K)} \right) (\varepsilon K + \bar{L}) \\ & \leq \varepsilon[K(x) + \bar{L}] + \varepsilon \left(K(x) + 1 + \frac{2K(1 - \varepsilon m) + 3}{1 - \varepsilon(m + 5K)} \right) (\varepsilon K + \bar{L}) = \varepsilon M(x, \varepsilon), \\ & \lim_{\varepsilon \rightarrow \infty} M(x, \varepsilon) = K(x) + \bar{L} + K(x)\bar{L} + \bar{L} + 2K\bar{L} + 3\bar{L} = K(x) + \bar{L}(5 + K(x) + 2K). \end{aligned}$$

It is clear that $M(x, \varepsilon)$ does not exceed

$$K(x) + 5 + 2K + \bar{L}(5 + K(x) + 2K) = (\bar{L} + 1)(K(x) + 2K + 5)$$

starting with some small number ε_1 . $\square$

Proof of Theorem 2.2. It is sufficient to prove that $\int \bar{L}(dy, dy')(v(y) + v(y')) = \bar{L}_g$. The rest of the proof is the same as that of Theorem 2.1. We have

$$\begin{aligned} \int \lambda T^n(dx, dy)(g(x) + g(y)) & \leq \bar{E}_\lambda [(g(X_n) + g(X'_n))(1 - \alpha)^{N_{n-1}}] \\ & \leq \bar{E}_\lambda [\beta(V(X_n, X'_n))(1 - \alpha)^{N_{n-1}}]. \end{aligned}$$

Here N_{n-1} is the number of visits to the set $C \times C$ over the time $n - 1$. Further,

$$\begin{aligned} & \alpha(R(n) + M_U) \int \lambda T^n(dx, dy)(g(x) + g(y)) \\ & \leq \bar{E}_\lambda [\alpha(R(n) + M_U)\beta(V(X_n, X'_n))(1 - \alpha)^{N_{n-1}}] \\ & \leq \rho(R(n) + M_U)\bar{E}_\lambda [(1 - \alpha)^{N_{n-1}}] + (1 - \rho)\bar{E}_\lambda [V(X_n, X'_n)(1 - \alpha)^{N_{n-1}}]. \end{aligned}$$

Given $n \geq 0$, we put

$$U_n(x, x') = \bar{E}_\lambda \left[\sum_{k=0}^{\sigma_{C \times C}} r(n + k) \right].$$

Then

$$\bar{P}U_{n+1} \leq U_n - r(n) + b_U r(n) I_{C \times C}.$$

Similarly,

$$\bar{P}V \leq V - v + b_V I_{C \times C}.$$

For $n \geq 0$, put

$$W_n^0 = U_n(X_n, X'_n) + \sum_{k=0}^{n-1} r(k) + M_U,$$

$$W_n^1 = V_n(X_n, X'_n) + \sum_{k=0}^{n-1} v(X_k, X'_k) + M_V.$$

By construction,

$$W_n^0 \geq R(n) + M_U,$$

$$W_n^1 \geq V(X_n, X'_n),$$

whence

$$\alpha(R(n) + M_U) \int \lambda T^n(dx, dy)(g(x) + g(y))$$

$$\leq \bar{\rho} E_\lambda [W_n^0(1 - \alpha)_{n-1}^N] + (1 - \rho) \bar{E}_\lambda [W_n^1(1 - \alpha)_{n-1}^N].$$

Let

$$T_n^0 = \prod_{i=0}^{n-1} \frac{W_i^0 + b_U r(i) I_{C \times C}(X_i, X'_i)}{W_i^0},$$

$$T_n^1 = \prod_{i=0}^{n-1} \frac{W_i^1 + b_V U I_{C \times C}(X_i, X'_i)}{W_i^1},$$

$$\sigma_0 = \sigma_{C \times C}, \quad \sigma_{j+1} = \inf \{n \geq \sigma_j : (X_n, X'_n) \in C \times C\}.$$

If $\alpha = 1$, then $T_n^i(1 - \alpha)_{n-1}^N = T_n^i I_{\sigma_0 \geq n} = I_{\sigma_0 \geq n} \leq 1$. If $\alpha < 1$, then by definition $N_{n-1} = 0$, $T_n^i = 1$, and, for $N_{n-1} > 0$,

$$T_n^0 = \prod_{i=0}^{N_{n-1}-1} \frac{W_{\sigma_i}^0 + b_U r(\sigma_i)}{W_{\sigma_i}^0},$$

$$T_n^1 = \prod_{i=0}^{N_{n-1}-1} \frac{W_{\sigma_i}^1 + b_V}{W_{\sigma_i}^1}.$$

Since

$$W_n^0 \geq R(n+1) + M_U$$

and

$$\frac{1 + b_U r(n)}{R(n+1) + M_U} \leq \frac{1}{1 - \alpha}, \quad N_{n-1} > 0,$$

we obtain

$$T_n^0(1 - \alpha)_{n-1}^N \leq \prod_{i=0}^{N_{n-1}-1} \left(1 + \frac{b_U r(\sigma_i)}{R(\sigma_i + 1) + M_U} (1 - \alpha) \right) \leq 1.$$

Taking into account that $W_n^1 \geq M_V$ and $1 + b_V/M_V \leq (1 - \alpha)^{-1}$, we get

$$T_n^1(1 - \alpha)_{n-1}^N \leq 1.$$

Therefore,

$$\bar{E}_\lambda [W_n^0(1 - \alpha)_{n-1}^N] \leq \bar{E}_\lambda \left[\frac{W_n^0}{T_n^0} \right],$$

$$\bar{E}_\lambda [W_n^1(1 - \alpha)_{n-1}^N] \leq \bar{E}_\lambda \left[\frac{W_n^1}{T_n^1} \right].$$

Now we evaluate $\bar{E}_\lambda[W_n^i/T_n^i]$. By construction,

$$\bar{E}_\lambda \left[\frac{W_n^0}{T_n^0} \mid F_{n-1} \right] = \bar{E}_\lambda [W_n^0 \mid F_{n-1}] \frac{W_{n-1}^0}{W_{n-1}^0 + b_U r(n-1) I_{C \times C}(X_n, X'_n)} \frac{1}{T_{n-1}^0},$$

where $F_n = \sigma[(X_k, X'_k)]_{k \leq n}$. The shift condition implies that

$$\bar{E}_\lambda [W_n^0 \mid F_{n-1}] \leq W_{n-1}^0 + b_U r(n-1) I_{C \times C}(X_{n-1}, X'_{n-1}).$$

Thus $\bar{E}_\lambda[W_n^0/T_n^0]$ is a supermartingale, whence we deduce that

$$\bar{E}_\lambda [W_n^0(1 - \alpha)_{n-1}^N] \leq \bar{E}_\lambda [W_n^0] = \lambda U + M_U.$$

Similarly,

$$\bar{E}_\lambda [W_n^1(1 - \alpha)_{n-1}^N] \leq \bar{E}_\lambda [W_n^1] = \lambda V + M_V,$$

whence

$$\alpha(R(n) + M_U) \int \lambda T^n(dx, dy)(g(x) + g(y)) \leq \rho(\lambda U + M_U) + (1 - \rho)(\lambda V + M_V),$$

and this completes the proof. $\square$

6. CONCLUDING REMARKS

The stability of time-homogeneous Markov chains is considered in this paper. The assumption imposed on transition probabilities is

$$P = (1 - \varepsilon)Q + \varepsilon R, \quad P' = (1 - \varepsilon)Q + \varepsilon R'.$$

Other assumptions are minorization condition (A1) and technical conditions (A2) and (A3). Under these conditions, we prove that the difference of transition probabilities after n steps is of order ε . The results are obtained for the uniform metric and for $\|\cdot\|_v$ as well.

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