SOME LIMIT THEOREMS FOR CONTROLLED BRANCHING PROCESSES

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ABSTRACT. Limit theorems are obtained for branching processes depending on the size of the population. These results contain the case of convergence of such processes to the deterministic limit. Functional central limit theorems for fluctuations are proved.

Let $\{\xi_{k,j}(l), k, j, l \in \mathbb{N}\}$ be a family of nonnegative integer-valued independent random variables. Assume that the random variables $\xi_{k,j}(l)$, $k, j \in \mathbb{N}$, are identically distributed. We define the stochastic process X_k , $k \geq 0$, by the following recurrence relation:

$$(1) \quad X_0 = 1, \quad X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,j}(X_{k-1}), \quad k \geq 1.$$

This process is called a branching process dependent on the size of the population or a controlled branching process.

The branching processes dependent on the population size are studied, for example, in the papers [1, 2], [5]–[10], [14]–[16], [18]–[21]. Conditions for the extinction of processes (1) as well as those for the convergence of normalized random variables X_n to a finite random variable are studied in [2]–[5] and [6, 7, 18]. The central limit theorem for controlled branching processes (1) is proved in [8]. Sufficient conditions for the weak convergence as $n \rightarrow \infty$ of the distribution of X_n/n to the gamma distribution are found in [16, 21].

In [19] the authors consider a model for a population of cells in which every cell lives for a unit of time and then divides into two or dies, with probability of division depending on the size of the population. Also the asymptotic distribution function for the time when the size of the population exceeds given limits is found there. The results of [19] are extended in [20] to a more general class of stochastic processes. Some statistical problems for the processes of type (1) are studied in [9, 10].

The so-called φ -controlled branching processes are studied in [14]–[15]. In the current paper, we obtain some sufficient conditions for the convergence as $n \rightarrow \infty$ of the processes $\{X_{[nt]}, t \geq 0\}$, $n \geq 1$, to a deterministic process (the symbol $[a]$ stands for the integer part of a number a), as well as functional central limit theorems for the deviation of $X_{[nt]}$, $t \geq 0$, from the limit process.

In what follows, we always assume that $E \xi_{1,1}^2(x) < \infty$ for all $x \in \mathbb{N}$. We introduce the notation

$$m(x) = E \xi_{1,1}(x), \quad \sigma^2(x) = \text{Var} \xi_{1,1}(x).$$

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Let $\mathcal{F}_k = \sigma\{X_0, X_1, \dots, X_{k-1}\}$ be the σ -algebra generated by random variables

$$\{X_0, X_1, \dots, X_{k-1}\}.$$

A process of the form (1) can be represented as

$$(2) \quad X_k = X_{k-1} + (m(X_{k-1}) - 1) X_{k-1} + M_k,$$

where

$$M_k = \sum_{j=1}^{X_{k-1}} (\xi_{k,j}(X_{k-1}) - m(X_{k-1})).$$

It is clear that M_k , $k \geq 1$, is a martingale difference with respect to the filtration $\mathcal{F}_k$, $k \geq 1$. Consider the random event $\varepsilon_\infty = \{X_n \rightarrow \infty\}$ and put $q = 1 - \mathbb{P}(\varepsilon_\infty)$. In what follows $T > 0$ denotes an arbitrary fixed number and $I(A)$ stands for the indicator of a random event A .

Theorem 1. *Let $m(x) = 1 + \alpha/x$ for $x > 0$ and some $\alpha > 0$. Suppose $x\sigma^2(x) \leq Cx^\beta$ for some $0 \leq \beta < 1$. Then*

$$\sup_{0 \leq t \leq T} \left| \frac{X_{[nt]}}{n} - \alpha t \right| \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \rightarrow \infty$$

on the event ε_∞ .

Theorem 1 implies, in particular, that $\mathbb{P}(\varepsilon_\infty) > 0$. This can be shown by using some results of the paper [4].

Proof of Theorem 1. We have

$$X_{k+1} = X_k + (m(X_k) - 1)X_k I(X_k > 0) + M_k.$$

Passing to the expectation and then summing in k we obtain

$$\mathbb{E} X_{n+1} = 1 + \alpha \sum_{k=0}^n \mathbb{P}(X_k > 0).$$

Since $\lim_{n \rightarrow \infty} \mathbb{P}(X_n > 0) = 1 - q$, the latter relation together with the Toeplitz lemma implies that

$$(3) \quad \mathbb{E} \frac{X_n}{n} \rightarrow \alpha(1 - q) \quad \text{as } n \rightarrow \infty.$$

Further, relation (2) yields

$$X_{k+1}^2 = X_k^2 + 2(m(X_k) - 1)X_k^2 + (m(X_k) - 1)^2 X_k^2 + M_k^2 + 2m(X_k)X_k M_k,$$

whence

$$\mathbb{E} X_{k+1}^2 = \mathbb{E} X_k^2 + 2\alpha \mathbb{E} X_k + \alpha^2 \mathbb{P}(X_k > 0) + \mathbb{E} X_k \sigma^2(X_k).$$

Summing these equalities we get

$$(4) \quad \mathbb{E} X_{n+1}^2 = 1 + 2\alpha \sum_{k=0}^n \mathbb{E} X_k + \alpha^2 \sum_{k=0}^n \mathbb{P}(X_k > 0) + \sum_{k=0}^n \mathbb{E} X_k \sigma^2(X_k).$$

It is obvious that

$$(5) \quad \frac{1}{(n+1)^2} \sum_{k=0}^n \mathbb{P}(X_k > 0) \leq \frac{n}{(n+1)^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We obtain from (3) that

$$(6) \quad \frac{2\alpha}{(n+1)^2} \sum_{k=0}^n \mathbb{E} X_k \sim \frac{2\alpha^2(1-q)}{(n+1)^2} \sum_{k=1}^n k \rightarrow \alpha^2(1-q) \quad \text{as } n \rightarrow \infty$$

by the Toeplitz lemma.

The assumptions of the theorem and the well-known inequality

$$(\mathbb{E} |\xi|^r)^{1/r} \leq (\mathbb{E} |\xi|^l)^{1/l}, \quad r \leq l,$$

imply that

$$\mathbb{E} X_k \sigma^2(X_k) \leq C \mathbb{E} X_k^\beta \leq C (\mathbb{E} X_k)^\beta.$$

Taking into account asymptotic relation (3) and applying the Toeplitz lemma again, we prove that

$$(7) \quad \begin{aligned} \sum_{k=0}^n \mathbb{E} X_k \sigma^2(X_k) &\leq C \sum_{k=1}^n (\mathbb{E} X_k)^\beta \sim C(\alpha(1-q))^\beta \sum_{k=1}^n k^\beta \\ &\sim C(\alpha(1-q))^\beta \int_0^1 x^\beta dx \cdot n^{1+\beta} = C \frac{(\alpha(1-q))^\beta}{1+\beta} n^{1+\beta} \end{aligned}$$

as $n \rightarrow \infty$. Therefore

$$\frac{1}{(n+1)^2} \sum_{k=1}^n \mathbb{E} X_k \sigma^2(X_k) \leq C \frac{(\alpha(1-q))^\beta}{1+\beta} \cdot \frac{n^{1+\beta}}{(n+1)^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

since $\beta < 1$. Thus relations (4)–(6) imply that

$$\mathbb{E} \left(\frac{X_n}{n} \right)^2 \rightarrow \alpha^2(1-q) \quad \text{as } n \rightarrow \infty.$$

We derive from the latter result and (3) that

$$(8) \quad \frac{X_{[nt]}}{n} \xrightarrow{P} \alpha t \quad \text{as } n \rightarrow \infty$$

for all $t \geq 0$ on the event ε_∞ . Applying Cramér's and Wald's techniques, we conclude that the finite-dimensional distributions of $n^{-1}X_{[nt]}$, $t \geq 0$, weakly converge to the corresponding distributions of αt , $t \geq 0$, on the event ε_∞ .

Now we prove the density for the distributions of $\{n^{-1}X_{[nt]}, t \in [0, T]\}$, $n \geq 1$. Let $s, t \in [0, T]$ and $0 \leq s < t \leq T$. Using the elementary inequality $(a+b)^2 \leq 2(a^2 + b^2)$, we obtain

$$\begin{aligned} \mathbb{E} \left(\frac{X_{[nt]}}{n} - \frac{X_{[ns]}}{n} \right)^2 &= \frac{1}{n^2} \mathbb{E} \left(\alpha \sum_{k=[ns]+1}^{[nt]} I(X_k > 0) + \sum_{k=[ns]+1}^{[nt]} M_k \right)^2 \\ &\leq \frac{2}{n^2} \left(\alpha^2([nt] - [ns])^2 + \sum_{k=[ns]+1}^{[nt]} \mathbb{E} M_k^2 \right) \\ &\leq 3 \left(\alpha^2(t-s)^2 + \frac{1}{n^2} \sum_{k=[ns]+1}^{[nt]} \mathbb{E} X_k \sigma^2(X_k) \right). \end{aligned}$$

An argument similar to that used to prove (7) shows that

$$\begin{aligned} \mathbb{E} \left(n^{-1} X_{[nt]} - n^{-1} X_{[ns]} \right)^2 &\leq 3 \left((\alpha^2(t-s)^2 + C \frac{\alpha^\beta}{1+\beta} \frac{n^{1+\beta}}{n^2} (t^{1+\beta} - s^{1+\beta})) \right) \\ &\leq 3 \left(\alpha^2(t-s)^2 + \frac{C(\alpha T)^\beta}{n^{1-\beta}}(t-s) \right) \end{aligned}$$

for sufficiently large n . Thus Theorem 12.3 of [13] implies that the distributions of $X_{[nt]}/n$, $t \in [0, T]$, $n \geq 1$, are dense, since $X_0/n \xrightarrow{P} 0$ as $n \rightarrow \infty$. By (8) and by the convergence criteria proposed in [13] we obtain the weak convergence in $D[0, T]$:

$$(9) \quad \frac{X_{[nt]}}{n} \rightarrow \alpha t \quad \text{as } n \rightarrow \infty$$

on the event ε_∞ . Since the limit process is continuous, convergence (9) holds also in the uniform topology, and this completes the proof of Theorem 1. $\square$

Throughout the rest of the paper we consider the branching process X_k , $k \geq 0$, on the event ε_∞ only. By the restriction method, one always can assume that ε_∞ is the sample space. It is obvious that $X_k \geq 1$ for all $k \geq 0$ on this event.

We denote by $\xrightarrow{D}$ the weak convergence of stochastic processes in the space $D[0, T]$; the symbol W stands for a standard Wiener process.

Theorem 2. *Let $m(x) = 1 + \alpha/x$ for some $\alpha > 0$ and let $x\sigma^2(x) = 1$. Suppose that, for all $\varepsilon > 0$,*

$$\gamma_n(\varepsilon) = n \sup_{x \geq 1} \mathbb{E} \left((\xi_{1,1}(x) - m(x))^2 I(|\xi_{1,1}(x) - m(x)| > \varepsilon \sqrt{n}) \right) \rightarrow 0$$

as $n \rightarrow \infty$. Then

$$\frac{X_{[nt]} - n\alpha t}{\sqrt{n}}, \quad t \in [0, T] \quad \xrightarrow{D} \quad W(t), \quad t \in [0, T] \quad \text{as } n \rightarrow \infty$$

in the space $D[0, T]$ on the event ε_∞ .

Theorem 3. *Let $m(x) = 1 + \alpha/x$ for some $\alpha > 0$ and let $x\sigma^2(x) = 1$. Suppose there exists a nonrandom number $A > 0$ such that*

$$\xi_{k,j}(x) \leq A \quad \text{a.s.}$$

for all $k, j, x \in \mathbb{N}$. Then Theorem 2 holds.

Theorem 4. *Let $m(x) = 1 + \alpha/x$ and $x\sigma^2(x) = x^{-2(1-\gamma)}$ for some $\alpha > 0$ and $0 < \gamma < 1$. Suppose*

$$n^{2(1-\gamma)} \sup_{x \geq 1} \mathbb{E} \left((\xi_{1,1}(x) - m(x))^2 \cdot I(|\xi_{1,1}(x) - m(x)| > \varepsilon n^\gamma) \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for every $\varepsilon > 0$. Then

$$n^{-\gamma} (X_{[nt]} - n\alpha t), \quad t \in [0, T] \quad \xrightarrow{D} \quad W(\rho(t)), \quad t \in [0, T] \quad \text{as } n \rightarrow \infty$$

in the space $D[0, T]$ on the event ε_∞ , where

$$\rho(t) = \frac{\alpha^{2\gamma-1}}{2\gamma} t^{2\gamma}.$$

Note that $P(\varepsilon_\infty) > 0$ under the assumptions imposed on the functions $m(x)$ and $\sigma^2(x)$ in Theorems 2-4 (see [4]).

It is easy to see that the condition of uniform boundedness for the random variables $\xi_{k,j}(x)$ in Theorem 3 implies that $\gamma_n(\varepsilon) \rightarrow 0$ as $n \rightarrow \infty$ in Theorem 2. Since Theorems 2 and 4 are proved similarly, we restrict ourselves to the proof of Theorem 2.

Remark. Theorems 2 and 3 remain true if we assume that $x\sigma^2(x) \rightarrow 1$ as $x \rightarrow \infty$ instead of $x\sigma^2(x) = 1$.

Proof of Theorem 2. It is easy to see that

$$(10) \quad \frac{X_{[nt]} - n\alpha t}{\sqrt{n}} = \frac{1 + [nt]\alpha - n\alpha t}{\sqrt{n}} + \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} M_k.$$

Since

$$(11) \quad \sup_{0 \leq t \leq T} \frac{|1 + [nt]\alpha - n\alpha t|}{\sqrt{n}} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

Theorem 4.1 of [13] implies that Theorem 2 follows from

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} M_k, \quad t \in [0, T] \xrightarrow{D} W(t), \quad t \in [0, T] \quad \text{as } n \rightarrow \infty.$$

The latter relation holds if

$$(12) \quad \frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{E} (M_k^2 / \mathcal{F}_k) \rightarrow t \quad \text{as } n \rightarrow \infty$$

and

$$(13) \quad L_n(t) = \frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{E} (M_k^2 I(|M_k| > \varepsilon\sqrt{n}) / \mathcal{F}_k) \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty$$

for all $\varepsilon > 0$ (see Theorem 7.1.11 in [12]).

Consider (12). It is clear that

$$\frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{E} (M_k^2 / \mathcal{F}_k) = \frac{1}{n} \sum_{k=1}^{[nt]} X_{k-1} \sigma^2(X_{k-1}) = \frac{[nt]}{n} \rightarrow t \quad \text{as } n \rightarrow \infty,$$

whence (12) follows.

Now we prove (13). We have

$$(14) \quad \begin{aligned} L_n(t) &= \frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{E} \left(\sum_{i=1}^{X_{k-1}} (\xi_{k,i}(X_{k-1}) - m(X_{k-1}))^2 I(|M_k| > \varepsilon\sqrt{n}) / \mathcal{F}_k \right) \\ &\quad + \frac{2}{n} \sum_{k=1}^{[nt]} \mathbb{E} \sum_{i=2}^{X_{k-1}} \sum_{j=1}^{i-1} \left((\xi_{k,i}(X_{k-1}) - m(X_{k-1})) (\xi_{k,j}(X_{k-1}) - m(X_{k-1})) \right. \\ &\quad \left. \times I(|M_k| > \varepsilon\sqrt{n}) / \mathcal{F}_k \right) \\ &= L_{n1}(t) + L_{n2}(t). \end{aligned}$$

Put

$$S_j^k = \sum_{i=1}^{X_{k-1}} (\xi_{k,i}(X_{k-1}) - m(X_{k-1})) - (\xi_{k,j}(X_{k-1}) - m(X_{k-1})), \quad j \leq X_{k-1}.$$

Using the inequality

$$I(|X + Y| > 2\varepsilon) \leq I(|X| > \varepsilon) + I(|Y| > \varepsilon),$$

which is true for arbitrary random variables X, Y and for all $\varepsilon > 0$, we get

$$\begin{aligned}
 L_{n1}(t) &\leq \frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{E} \left(\sum_{j=1}^{X_{k-1}} (\xi_{k,j}(X_{k-1}) - m(X_{k-1}))^2 I \left(|S_j^k| > \frac{\varepsilon\sqrt{n}}{2} \right) / \mathcal{F}_k \right) \\
 &+ \frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{E} \left(\sum_{j=1}^{X_{k-1}} (\xi_{k,j}(X_{k-1}) - m(X_{k-1}))^2 \right. \\
 &\quad \left. \times I \left(|\xi_{k,j}(X_{k-1}) - m(X_{k-1})| > \frac{\varepsilon\sqrt{n}}{2} \right) / \mathcal{F}_k \right) \\
 &= L_{n1}^{(1)}(t) + L_{n1}^{(2)}(t).
 \end{aligned}
 \tag{15}$$

Since $\xi_{k,j}(X_{k-1}) - m(X_{k-1})$ and S_j^k are conditionally independent with respect to $\mathcal{F}_k$, we apply the second Chebyshev inequality (see [17]) and obtain

$$\begin{aligned}
 L_{n1}^{(1)}(t) &\leq \frac{1}{n} \sum_{k=1}^{[nt]} \sum_{j=1}^{X_{k-1}} \sigma^2(X_{k-1}) \cdot \mathbb{P} \left(|S_j^k| > \frac{\varepsilon\sqrt{n}}{2} / \mathcal{F}_k \right) \\
 &\leq \frac{4}{\varepsilon^2 n^2} \sum_{k=1}^{[nt]} X_{k-1} (X_{k-1} - 1) \sigma^4(X_{k-1}) \leq \frac{4[nt]}{\varepsilon^2 n^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned}
 \tag{16}$$

Theorem 1 and continuous mapping theorem (Theorem 5.1 of [13]) imply that

$$\frac{1}{n^2} \sum_{k=1}^{[nt]} X_k \rightarrow \frac{1}{n} \int_0^{\frac{[nt]}{n}} X_{[ns]} ds \xrightarrow{\mathbb{P}} \frac{\alpha t^2}{2} \quad \text{as } n \rightarrow \infty.$$

Therefore,

$$L_{n1}^{(2)}(t) \leq \gamma_n \left(\frac{\varepsilon}{2} \right) \frac{1}{n^2} \sum_{k=1}^{[nt]} X_{k-1} = \frac{1}{n} \int_0^{\frac{[nt]}{n}} X_{[ns]} ds \cdot \gamma_n \left(\frac{\varepsilon}{2} \right) \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \rightarrow \infty.
 \tag{17}$$

We derive from (15)–(17) that

$$L_{n1}(t) \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \rightarrow \infty.
 \tag{18}$$

We now pass to the estimation of $L_{n2}(t)$. Put

$$\theta_k = 2 \sum_{i=2}^{X_{k-1}} \sum_{j=1}^{i-1} (\xi_{k,i}(X_{k-1}) - m(X_{k-1})) (\xi_{k,j}(X_{k-1}) - m(X_{k-1})).$$

Then

$$L_{n2}(t) = \frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{E} (\theta_k \cdot I(|M_k| > \varepsilon\sqrt{n}) / \mathcal{F}_k).$$

It is not hard to see that

$$\mathbb{E} (\theta_k^2 / \mathcal{F}_k) = 4 \sum_{i=2}^{X_{k-1}} (i-1) \sigma^4(X_{k-1}) = 2X_{k-1}(X_{k-1} - 1) \sigma^4(X_{k-1}) \leq 2.$$

Applying the Cauchy–Bunyakovskii inequality for conditional expectations [17], we deduce from the latter relation that

$$\begin{aligned} L_{n2}(t) &\leq \frac{1}{n} \sum_{k=1}^{[nt]} (\mathbb{E}(\theta_k^2 / \mathcal{F}_k))^{1/2} \cdot (\mathbb{P}(|M_k| > \varepsilon \sqrt{n} / \mathcal{F}_k))^{1/2} \\ &\leq \frac{\sqrt{2}}{n} \sum_{k=1}^{[nt]} \left(\frac{\mathbb{E}(M_k^2 / \mathcal{F}_k)}{\varepsilon^2 n} \right)^{1/2} = \frac{\sqrt{2}}{\varepsilon n^{3/2}} \sum_{k=1}^{[nt]} (X_{k-1} \cdot \sigma^2(X_{k-1}))^{1/2} = \frac{\sqrt{2}[nt]}{\varepsilon n^{3/2}} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. This together with (14) and (18) implies (13). The proof of Theorem 2 is complete. $\square$

Example. Suppose a random variable $\xi_{k,j}(x)$ takes values 0, 1, and 2 with probabilities $\frac{1}{2}x^{-2}$, $1 - x^{-1} - x^{-2}$, and $x^{-1} + \frac{1}{2}x^{-2}$, respectively. In this case, $m(x) = 1 + x^{-1}$ and $\sigma^2(x) = x^{-1}$. Thus, the assumptions of Theorem 1 are satisfied with $\alpha = 1$ and $\beta = 0$. Hence Theorem 1 implies that the process $n^{-1}X_{[nt]}$, $t \in [0, T]$, weakly converges as $n \rightarrow \infty$ to the process t , $t \in [0, T]$, on the event ε_∞ . Moreover, $q < 1$ by Theorem 1.4 in [3].

All the assumptions of Theorem 3 hold for this process and therefore the process $n^{-1/2}(X_n(t) - nt)$, $t \geq 0$, weakly converges in $D[0, T]$ to the Wiener process $W(t)$, $t \geq 0$, as $n \rightarrow \infty$.

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