AN INEQUALITY FOR THE LÉVY DISTANCE BETWEEN TWO DISTRIBUTION FUNCTIONS AND ITS APPLICATIONS

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ABSTRACT. We prove a nonuniform bound for the deviation between two distribution functions expressed in terms of the Lévy distance. Applications of this bound to the global version of the central limit theorem are given and complete convergence is shown.

1. INTRODUCTION

Let F and G be two distribution functions. Then the Lévy distance $\mathcal{L}(F, G)$ between F and G is defined as

$$(1.1) \quad \mathcal{L}(F, G) = \inf \mathbb{H},$$

where

$$\mathbb{H} = \{h: G(x-h) - h \leq F(x) \leq G(x+h) + h \text{ for all } x\}.$$

The Lévy distance is much less popular in probability theory than the uniform distance

$$(1.2) \quad \Delta(F, G) = \sup_{x \in \mathbf{R}} |F(x) - G(x)|.$$

The advantage of the Lévy distance becomes evident if one considers the weak convergence

$$F_n \xrightarrow{w} G, \quad n \rightarrow \infty,$$

which is equivalent to $\mathcal{L}(F_n, G) \rightarrow 0$ ($n \rightarrow \infty$) (see, for example, [7]). If G is continuous, then the weak convergence $F_n \xrightarrow{w} G$ ($n \rightarrow \infty$) is also equivalent to $\Delta(F_n, G) \rightarrow 0$ ($n \rightarrow \infty$); however, this property may fail if G has discontinuities. We also recall that always

$$\mathcal{L}(F, G) \leq \Delta(F, G).$$

Having weak convergence in mind, we compare the deviation $|F(x) - G(x)|$ between two distribution functions. The bound we obtain in Section 2 is nonuniform in x and is expressed in terms of the Lévy distance $\mathcal{L}(F, G)$. Furthermore, we discuss some examples of applications of our bound. In Section 3, we obtain a generalized global version of the central limit theorem. Finally, Section 4 contains some results for the generalized complete convergence.

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2. A BOUND IN TERMS OF THE LÉVY DISTANCE

In this section, we concentrate on the case of $G = \Phi$ where Φ is the standard $N(0, 1)$ Gaussian law.

Bounds for $|F(x) - \Phi(x)|$ expressed in terms of the uniform distance have been studied in many papers. The most popular is the case where F corresponds to a sum of independent random variables. Research in this direction originated from the work of Esseen [5].

Kolodyazhnyi [11] extends the results of [5] by proving the following theorem.

Theorem A (Kolodyazhnyi). *Let F be an arbitrary distribution function and let $\Delta = \Delta(F, \Phi)$. Let $p > 0$ and assume that the moment of order p exists for the distribution function F . Denote*

$$(2.1) \quad \lambda_p = \left| \int_{-\infty}^{\infty} |x|^p dF(x) - \int_{-\infty}^{\infty} |x|^p d\Phi(x) \right|.$$

If

$$(2.2) \quad 0 < \Delta \leq \frac{1}{\sqrt{e}},$$

then there exists a universal constant c_Δ , depending only on p , such that

$$(2.3) \quad |F(x) - \Phi(x)| \leq \frac{\lambda_p + c_\Delta \Delta \left(\ln \frac{1}{\Delta}\right)^{p/2}}{1 + |x|^p}$$

for all $x \in \mathbf{R}$.

A similar result is obtained in [10] in terms of the Lévy distance instead of the uniform distance.

Theorem B (Indlekofer–Klesov). *Let F be an arbitrary distribution function and let $L = \mathcal{L}(F, \Phi)$. Let $p > 0$ and assume that the moment of order p exists for the distribution function F . If*

$$(2.4) \quad 0 < L \leq \frac{1}{\sqrt{e}},$$

then there exists a universal constant c_L , depending only on p , such that

$$(2.5) \quad |F(x) - \Phi(x)| \leq \frac{\lambda_p + c_L L \left(\ln \frac{1}{L}\right)^{p/2}}{1 + |x|^p}$$

for all $x \in \mathbf{R}$, where the constant λ_p is defined by (2.1).

Remark 2.1. Theorems A and B look very similar. Note, however, that in the case of

$$L \leq \frac{1}{\sqrt{e}} < \Delta$$

only Theorem B is applicable.

Remark 2.2. Since we provide a better result below, we do not go into a deep comparison of these theorems. Nevertheless a couple of comments are in order.

There are some cases in which one of the two Theorems A or B implies the other. Since the function $x(\ln x^{-1})^{p/2}$ increases on the interval $(0, e^{-p/2})$ and decreases on $(e^{-p/2}, 1)$, assertion (2.5) implies (2.3) if

$$(2.6) \quad \Delta \leq \min \left\{ \frac{1}{\sqrt{e^p}}, \frac{1}{\sqrt{e}} \right\}.$$

Moreover, the constant c_Δ inherits the value of the constant c_L in this case. On the other hand, (2.3) implies (2.5) if $p \geq 1$ and

$$(2.7) \quad \frac{1}{\sqrt{e^p}} \leq L \leq \Delta \leq \frac{1}{\sqrt{e}}.$$

Now the constant c_L inherits the value of the constant c_Δ . The case of (2.6) fits better in the context of the central limit theorem compared to (2.7) (because $\Delta \rightarrow 0$ for the central limit theorem).

Since

$$\Delta \leq \left(1 + \frac{1}{\sqrt{2\pi}}\right) L$$

if $G = \Phi$, there are several other interrelations between Theorems A and B. In what follows we use (2.5) rather than (2.3) even in the cases where both of them are applicable. This can be explained by the fact that other functions G (even discontinuous ones) can be used in (2.5) in place of Φ , while (2.3) has no nice generalizations in the discontinuous case if we are concerned with weak convergence. Theorem B for other G instead of Φ will be discussed elsewhere.

In the following result we avoid the restriction (2.4).

Theorem 1. *Let F be an arbitrary distribution function. Denote by $L = \mathcal{L}(F, \Phi)$ the Lévy distance between F and Φ . Let $p > 0$ and suppose the moment of order p exists for F . Then there exists a function g defined on $(0, 1)$, depending only on p , and such that*

$$\lim_{s \downarrow 0} g(s) = 0$$

and

$$(2.8) \quad |F(x) - \Phi(x)| \leq \frac{\lambda_p + g(L)}{1 + |x|^p}$$

for all $x \in \mathbf{R}$, provided $0 \leq L < 1$.

Remark 2.3. The case of $L = 0$ is equivalent to $F = \Phi$. Indeed, for any $h \in \mathbb{H}$,

$$\Phi(x) \geq F(x - h) - h, \quad x \in \mathbf{R}.$$

If $L = 0$, then $\Phi(x) \geq F(x)$, $x \in \mathbf{R}$, in view of the left continuity of F . Exchanging Φ and F in the definition of L , we prove the converse inequality, whence $F = \Phi$.

Remark 2.4. The case of $L = 1$ is not possible for any distribution function F . This follows from the estimate

$$L \leq \Delta < 1, \quad \text{since } 0 < \Phi(x) < 1 \text{ for all } x.$$

Remark 2.5. Inequality (2.8) improves the Markov inequality in some cases. Indeed, let $F^*(x)$ be the tail of F , that is,

$$(2.9) \quad F^*(x) = 1 - F(x) + F(-x), \quad x \geq 0.$$

For simplicity we assume that $p = 2$ and that the first two moments of F and Φ coincide. Then the Markov inequality yields

$$F^*(x) \leq \frac{1}{x^2},$$

while (2.8) implies that

$$F^*(x) \leq |F^*(x) - \Phi^*(x)| + \Phi^*(x) \leq \frac{2g(L)}{1 + x^2} + \Phi^*(x).$$

If L is small and x is large, then the latter bound is better than the one from the Markov inequality.

An explicit form of the function g is available under a minor additional restriction.

Theorem 2. *Let F be an arbitrary distribution function and let $L = \mathcal{L}(F, \Phi)$. Let $p > 0$ and assume that the moment of order p exists for F . Finally, let $L_0 \in (0, 1)$ be fixed and $0 < L \leq L_0$. Then there exists a constant c_p , depending only on p and L_0 , such that*

$$(2.10) \quad |F(x) - \Phi(x)| \leq \frac{\lambda_p + c_p L \left(\ln \frac{1}{L}\right)^{p/2}}{1 + |x|^p}$$

for all $x \in \mathbf{R}$.

Theorem 2 coincides with Theorem B if $L_0 = e^{-1/2}$. Note also that the constant c_p decreases with L_0 if p is fixed.

Proof of Theorem 1. In view of Remarks 2.3 and 2.4, we restrict our consideration to the case of $0 < L < 1$. The proof below uses the main ideas of the proof in [11].

Denote by $\mathcal{C}(F)$ the set of continuity points of F . For all $a > 0$ such that $\pm a \in \mathcal{C}(F)$ we have

$$(2.11) \quad \begin{aligned} & \int_{(-a, a)} |x|^p dF(x) \\ &= \int_{(-a, a)} |x|^p d[F(x) - \Phi(x)] + \int_{(-a, a)} |x|^p d\Phi(x) \\ &= a^p[F(a) - \Phi(a)] - a^p[F(-a) - \Phi(-a)] \\ &\quad - p \int_{(0, a)} x^{p-1}[F(x) - \Phi(x)] dx \\ &\quad + p \int_{(-a, 0)} |x|^{p-1}[F(x) - \Phi(x)] dx + \int_{(-a, a)} |x|^p d\Phi(x). \end{aligned}$$

For all $h \in \mathbb{H}(F, \Phi)$ and all $x \in \mathbf{R}$,

$$\begin{aligned} F(x) - \Phi(x) &= F(x) - \Phi(x - h) + h - h + \Phi(x - h) - \Phi(x) \\ &\geq -h - [\Phi(x) - \Phi(x - h)]. \end{aligned}$$

By the mean value theorem and the fact that $\Phi'(\xi) = \varphi(\xi) \leq 1/\sqrt{2\pi}$,

$$\Phi(x) - \Phi(x - h) \leq \frac{h}{\sqrt{2\pi}},$$

whence

$$(2.12) \quad F(x) - \Phi(x) \geq -h \left(1 + \frac{1}{\sqrt{2\pi}}\right), \quad h \in \mathbb{H}(F, \Phi),$$

for all $x \in \mathbf{R}$. In particular,

$$(2.13) \quad F(a) - \Phi(a) \geq -h \left(1 + \frac{1}{\sqrt{2\pi}}\right), \quad h \in \mathbb{H}(F, \Phi).$$

Similarly,

$$\begin{aligned} F(x) - \Phi(x) &= F(x) - \Phi(x + h) - h + h + \Phi(x + h) - \Phi(x) \\ &\leq h + [\Phi(x + h) - \Phi(x)] \end{aligned}$$

and thus

$$(2.14) \quad F(x) - \Phi(x) \leq h \left(1 + \frac{1}{\sqrt{2\pi}}\right), \quad h \in \mathbb{H}(F, \Phi),$$

for all $x \in \mathbf{R}$. In particular,

$$(2.15) \quad F(-a) - \Phi(-a) \leq h \left(1 + \frac{1}{\sqrt{2\pi}} \right), \quad h \in \mathbb{H}(F, \Phi).$$

Applying (2.14) we obtain, for every $h \in \mathbb{H}(F, \Phi)$, that

$$\begin{aligned} \int_{(0,a)} |x|^{p-1} [F(x) - \Phi(x)] dx &\leq \int_{(0,a)} |x|^{p-1} h \left(1 + \frac{1}{\sqrt{2\pi}} \right) dx \\ &= \frac{1}{p} a^p h \left(1 + \frac{1}{\sqrt{2\pi}} \right). \end{aligned}$$

Similarly, from (2.12), we derive that

$$\int_{(-a,0)} |x|^{p-1} [F(x) - \Phi(x)] dx \geq -\frac{1}{p} a^p h \left(1 + \frac{1}{\sqrt{2\pi}} \right).$$

Combining the latter estimates with (2.13), (2.15) and inserting them into (2.11), we obtain

$$\int_{(-a,a)} |x|^p dF(x) \geq -hB_p + \int_{(-a,a)} |x|^p d\Phi(x),$$

where

$$(2.16) \quad B_p = 4 \left(1 + \frac{1}{\sqrt{2\pi}} \right) a^p.$$

Recalling the definition of λ_p , we have

$$\begin{aligned} \lambda_p &\geq \int_{|x|<a} |x|^p dF(x) - \int_{|x|<a} |x|^p d\Phi(x) \\ &\quad + \int_{|x|\geq a} |x|^p dF(x) - \int_{|x|\geq a} |x|^p d\Phi(x) \\ &\geq -hB_p + \int_{|x|\geq a} |x|^p dF(x) - \int_{x\geq a} |x|^p d\Phi(x), \end{aligned}$$

whence

$$\int_{|x|\geq a} |x|^p dF(x) \leq \lambda_p + hB_p + \int_{|x|\geq a} |x|^p d\Phi(x).$$

Further, if $x \geq a$, then

$$\int_{|y|\geq a} |y|^p dF(y) \geq \int_{y\geq x} y^p dF(y) \geq x^p(1 - F(x)) \geq x^p[\Phi(x) - F(x)].$$

Therefore, for every $h \in \mathbb{H}(F, \Phi)$,

$$(2.17) \quad x^p[\Phi(x) - F(x)] \leq \lambda_p + hB_p + \int_{|y|\geq a} |y|^p d\Phi(y).$$

Now,

$$(2.18) \quad \begin{aligned} F(x) - \Phi(x) &\leq 1 - \Phi(x) \\ &\leq \int_{|y|\geq x} d\Phi(y) \leq \frac{1}{x^p} \int_{|y|\geq x} |y|^p d\Phi(y), \quad x \in \mathbf{R}, \end{aligned}$$

whence, for all $x \geq a$,

$$x^p[F(x) - \Phi(x)] \leq \int_{|y|\geq a} |y|^p d\Phi(y).$$

Combining this bound with (2.17), we get, for $x \geq a$ and $h \in \mathbb{H}(F, \Phi)$, that

$$(2.19) \quad |x|^p |F(x) - \Phi(x)| \leq \lambda_p + hB_p + \int_{|y| \geq a} |y|^p d\Phi(y).$$

A similar bound holds for $x \leq -a$. Finally, in view of (2.12) and (2.14), it also holds for $|x| < a$. Therefore (2.19) holds for all $x \in \mathbf{R}$.

The same reasoning applies for $p = 0$. Note that $\lambda_0 = 0$. Thus

$$(2.20) \quad \begin{aligned} & (1 + |x|^p) |F(x) - \Phi(x)| \\ & \leq \lambda_p + hB_p + hB_0 + \int_{|y| \geq a} |y|^p d\Phi(y) + \int_{|y| \geq a} d\Phi(y). \end{aligned}$$

The right-hand side of this estimate is a continuous function of a (see (2.16)); therefore, one can remove the assumption that $\pm a \in \mathcal{C}(F)$. Thus (2.20) holds for all $a > 0$. Moreover, on taking the infimum over $h \in \mathbb{H}(F, \Phi)$, we have, for all $x \in \mathbf{R}$ and all $a > 0$, that

$$(2.21) \quad \begin{aligned} & (1 + |x|^p) |F(x) - \Phi(x)| \\ & \leq \lambda_p + LB_p + LB_0 + \int_{|y| \geq a} |y|^p d\Phi(y) + \int_{|y| \geq a} d\Phi(y), \end{aligned}$$

where L is the Lévy distance between F and Φ . Thus (2.8) follows with

$$g(L) = \inf_{a>0} f(L, a),$$

where, for $0 \leq s < 1$ and $a > 0$,

$$(2.22) \quad f(s, a) = s\kappa a^p + s\kappa + \int_{|y| \geq a} |y|^p \varphi(y) dy + \int_{|y| \geq a} \varphi(y) dy,$$

with $\kappa = 4(1 + 1/\sqrt{2\pi})$ (see (2.16)).

If α is an increasing function on $(1, \infty)$ such that

$$\lim_{z \rightarrow \infty} \alpha(z) = \infty, \quad \lim_{z \rightarrow \infty} \frac{\alpha(z)}{z^{1/p}} = 0,$$

then

$$g(s) \leq f(s, \alpha(1/s)) \rightarrow 0, \quad s \downarrow 0.$$

This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. First we proceed as in the proof of Theorem 1 and arrive at (2.21). We choose $\alpha(z) = \sqrt{2 \ln z}$ and consider the function in (2.22) for $a \geq a_0$, where

$$a = \sqrt{2 \ln \frac{1}{L}} \quad \text{and} \quad a_0 = \sqrt{2 \ln \frac{1}{L_0}}.$$

Then

$$L \leq c_1 L \left(\ln \frac{1}{L} \right)^{p/2}, \quad \text{with} \quad c_1 = \left(\ln \frac{1}{L_0} \right)^{-p/2},$$

and

$$\int_a^\infty \varphi(x) dx \leq c_2 \int_a^\infty x^p \varphi(x) dx, \quad \text{with} \quad c_2 = a_0^{-p}.$$

Note also that, for some constant c_3 depending on a_0 ,

$$\int_a^\infty x^p \varphi(x) dx \leq c_3 a^{p-1} \varphi(a), \quad a \geq a_0.$$

Thus

$$\begin{aligned}
 f(L, a) &\leq c_3 \left(La^p + \int_a^\infty x^p \varphi(x) dx \right) \\
 &\leq c_4 \left(L \left(\ln \frac{1}{L} \right)^{p/2} + L \left(\ln \frac{1}{L} \right)^{(p-1)/2} \right) \\
 &\leq c_5 L \left(\ln \frac{1}{L} \right)^{p/2}. \quad \square
 \end{aligned}$$

3. GENERALIZED GLOBAL VERSION OF THE CENTRAL LIMIT THEOREM

Agnew [2] was apparently the first to study the relationship between weak convergence and the convergence to zero of the integral in (3.3) below. We should mention, however, that in Agnew's setting the limit distribution function can be arbitrary and not just Φ . Nevertheless, we restrict our considerations here to Φ as a weak limit in order to highlight the main features of our approach. Moreover, for the sake of comparison, we state the results of other authors also with Φ only, even if they have been proved for more general limiting distribution functions.

Similar results can also be obtained via Theorem A (see, e.g., [15] and [16]). Another method of proof is due to Studnev and Ignat [18] (see also [12]).

For example, from Agnew [2] we have the following result.

Theorem C (Agnew [2]). *Let $\{F_n\}$ be a sequence of distribution functions such that*

$$(3.1) \quad \int_{-\infty}^{\infty} x dF_n(x) = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} x^2 dF_n(x) = 1.$$

Suppose

$$(3.2) \quad F_n \xrightarrow{w} \Phi, \quad n \rightarrow \infty.$$

Then, for all $r > \frac{1}{2}$,

$$(3.3) \quad \int_{-\infty}^{\infty} |F_n(x) - \Phi(x)|^r dx \rightarrow 0, \quad n \rightarrow \infty.$$

In fact, Agnew [2] proved that the weak convergence is equivalent to (3.3) in a certain sense.

We can complete this result by considering also the case of $r = \frac{1}{2}$.

Theorem 3. *Let $\{F_n\}$ be a sequence of distribution functions satisfying (3.1). If (3.2) holds, then*

$$(3.4) \quad \int_{-\infty}^{\infty} \frac{|F_n(x) - \Phi(x)|^{1/2}}{(\ln(1 + |x|))^{1+\delta}} dx \rightarrow 0, \quad n \rightarrow \infty,$$

for all $\delta > 0$.

Note that the distribution functions $\{F_n\}$ in Theorem C are arbitrary and need not correspond to partial sums of independent random variables. The latter more specific and more typical case will be considered next.

Corollary 1. *Let $\{X_n\}$ be a sequence of independent, identically distributed random variables and let $\{S_n\}$ denote their partial sums. Assume that*

$$\mathbb{E} X_1 = 0 \quad \text{and} \quad \mathbb{E} X_1^2 = 1.$$

If $r > \frac{1}{2}$, then (3.3) holds, where F_n denotes the distribution function of $n^{-1/2}S_n$.

This result was generalized by Esseen [6] to the case of independent, but not necessarily identically distributed random variables $\{X_n\}$, provided they satisfy the central limit theorem. A further extension is due to de Acosta and Giné [1].

Of course, the smaller the value of $r > 0$, the stronger is the convergence in (3.3). So, it is natural to ask whether the restriction $r > \frac{1}{2}$ in the above results can be weakened to $r > 0$. A first positive answer has been given by Nishimura [14]. However, as pointed out in Rosalsky [17], the argument in [14] is incomplete and even the assumptions can be improved.

Theorem D (Laube [13] and Rosalsky [17]). *Let $\{F_n\}$ be a sequence of distribution functions such that*

$$(3.5) \quad \sup_{n \geq 1} \int_{-\infty}^{\infty} |x|^p dF_n(x) < \infty$$

for some $p > 1$. Assume that F_n converges weakly to Φ as $n \rightarrow \infty$. Then (3.3) holds for all $r > 1/p$.

We extend Theorem D to the case $r = 1/p$.

Theorem 4. *Suppose all the assumptions of Theorem D hold. Then*

$$(3.6) \quad \int_{-\infty}^{\infty} \frac{|F_n(x) - \Phi(x)|^{1/p}}{(\ln(1 + |x|))^{1+\delta}} dx \rightarrow 0, \quad n \rightarrow \infty,$$

for all $\delta > 0$.

The proof of Theorem 4 is the same as that of Theorem 3.

Theorem D can be generalized as follows.

Theorem 5. *Let $p > 0$ and $r > 0$. Let $\{F_n\}$ be a sequence of distribution functions such that $F_n \xrightarrow{w} \Phi$ ($n \rightarrow \infty$) and condition (3.5) holds. Let g be a positive function such that*

$$(3.7) \quad \int_{-\infty}^{\infty} \frac{g(x)}{(1 + |x|)^{rp_1}} dx < \infty \quad \text{for some } 0 < p_1 < p.$$

Then

$$\int_{-\infty}^{\infty} g(x) |F_n(x) - \Phi(x)|^r dx \rightarrow 0, \quad n \rightarrow \infty.$$

Remark 3.1. The case of $g(x) = 1 + |x|^q$, $q \geq 0$, in Theorem 5 has been proved by V. V. Petrov (see Theorem 14 in Chapter 1 of [16]). For this case condition (3.7) reduces to $pr > 1 + q$.

The case of $q = 0$ in Petrov's theorem corresponds to Theorem D. For the particular case of $q = 0$, $p = 2$, Petrov shows that the result can be proved via Theorem A (see Theorem 15, §5, Chapter V of [15] or §5, Chapter V of [16]).

Excluding the weak convergence from the set of assumptions in Theorem 5, we obtain the following general result.

Theorem 6. *Let $p > 0$ and $r > 0$. Let $\{F_n\}$ be a sequence of distribution functions such that condition (3.5) holds. Let g be a positive function satisfying (3.7). Then*

$$\limsup_{n \rightarrow \infty} \int_{-\infty}^{\infty} g(x) |F_n(x) - \Phi(x)|^r dx < \infty.$$

Remark 3.2. In the proofs below, we apply Theorem 2. One of the assumptions of this theorem concerns the upper bound L_0 for the Lévy distances. Thus we must have

$$(3.8) \quad \limsup_{n \rightarrow \infty} L_n < 1$$

in order to apply Theorem 2. We derive relation (3.8) from the boundedness of moments required in (3.5). Indeed, denote $L_n = \mathcal{L}(F_n, \Phi)$ and $\Delta_n = \Delta(F_n, \Phi)$. If (3.8) does not hold, then, for some subsequence $\{n_k\}$,

$$\lim_{k \rightarrow \infty} L_{n_k} = 1,$$

whence

$$\lim_{k \rightarrow \infty} \Delta_{n_k} = 1.$$

Without loss of generality we assume that $\{n_k\}$ coincides with the whole sequence of integers. Then, there is a subsequence $\{x_n\}$ such that

$$|F_n(x_n) - \Phi(x_n)| \rightarrow 1.$$

Choose a convergent subsequence of $\{\Phi(x_n)\}$ and assume again without loss of generality that it coincides with the whole sequence of integers. Obviously it converges either to 0 or to 1. In the first case, $x_n \rightarrow -\infty$ and $F_n(x_n) \rightarrow 1$, while, in the second case, $x_n \rightarrow \infty$ and $F_n(x_n) \rightarrow 0$. In either case, $F_n^*(c) \rightarrow 1$ for all $c > 0$ (see (2.9) for the definition of the tail of a distribution function). Thus, for any $c > 0$,

$$\int_{-\infty}^{\infty} |x|^p dF_n \geq \int_{|x| \geq c} |x|^p dF_n \geq c^p F_n^*(c),$$

which clearly contradicts (3.5), since c is arbitrary.

Proof of Theorem 3. In view of Remark 3.2, there is an $L_0 < 1$ such that $L_n \leq L_0$ for all $n \geq 1$. Then, by Theorem 2, for $p = 2$ and with $\lambda_p = 0$,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{|F_n(x) - \Phi(x)|^{1/2}}{(\ln(1 + |x|))^{1+\delta}} dx &\leq \int_{-\infty}^{\infty} \left(\frac{c_2 L_n \ln \frac{1}{L_n}}{1 + x^2} \right)^{1/2} \frac{1}{(\ln(1 + |x|))^{1+\delta}} dx \\ &= o(1) \int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^{1/2} (\ln(1 + |x|))^{1+\delta}} dx. \end{aligned}$$

This estimate proves Theorem 3. $\square$

Proof of Theorem 5. In view of Remark 3.2, there is an $L_0 < 1$ such that $L_n \leq L_0$ for all $n \geq 1$. Then, by Theorem 2,

$$\begin{aligned} \int_{-\infty}^{\infty} g(x) |F_n(x) - \Phi(x)|^r dx &\leq \int_{-\infty}^{\infty} g(x) \left(\frac{\lambda_{p_1, n} + c_{p_1} L_n (\ln \frac{1}{L_n})^{p_1/2}}{(1 + |x|)^{p_1}} \right)^r dx \\ &= o(1) \int_{-\infty}^{\infty} \frac{g(x)}{(1 + |x|)^{rp_1}} dx, \end{aligned}$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof of Theorem 5. $\square$

Proof of Theorem 6. Here one can proceed in the same way as in the proof of Theorem 5. The only difference is that $O(1)$ replaces $o(1)$. $\square$

4. GENERALIZED COMPLETE CONVERGENCE

The concept of *complete convergence* has been introduced by Hsu and Robbins in [9]. By definition, a sequence of random variables $\{\xi_n\}$ is said to converge completely to zero (denoted by $\xi_n \xrightarrow{\text{c.c.}} 0$, $n \rightarrow \infty$) if not only $\xi_n \xrightarrow{\text{a.s.}} 0$ but also $\xi'_n \xrightarrow{\text{a.s.}} 0$ as $n \rightarrow \infty$, for any sequence of random variables $\{\xi'_n\}$ such that $\xi_n \stackrel{d}{=} \xi'_n$ for all $n \geq 1$, i.e., for ξ_n and ξ'_n having the same distribution function for all $n \geq 1$. Note that other limit values are easily obtained from this case.

Remark 4.1. Hsu and Robbins [9] proved that the complete convergence $\xi_n \xrightarrow{\text{c.c.}} 0$ as $n \rightarrow \infty$ is equivalent to

$$\sum_{n=1}^{\infty} \mathbb{P}(|\xi_n| \geq \varepsilon) < \infty \quad \text{for all } \varepsilon > 0.$$

Using the notation $F^*(x)$ for the tail of a distribution function F (cf. (2.9)), we see that the latter condition is equivalent to

$$\sum_{n=1}^{\infty} F_n^*(\varepsilon) < \infty \quad \text{for all } \varepsilon > 0,$$

where F_n denotes the distribution function of ξ_n .

The convergence $n^{-1}S_n \xrightarrow{\text{c.c.}} 0$ as $n \rightarrow \infty$ has been studied by Hsu and Robbins [9] for the case where $\{S_n\}$ are partial sums of independent, identically distributed random variables $\{X_n\}$.

Theorem E (Hsu and Robbins [9]). *Let $\{X_n\}$ be a sequence of independent, identically distributed random variables and let $\{S_n\}$ denote their partial sums. Then $n^{-1}S_n \xrightarrow{\text{c.c.}} 0$ as $n \rightarrow \infty$ if and only if condition (3.1) holds.*

If F_n denotes the distribution function of $n^{-1/2}S_n$ (with corresponding F_n^*), Theorem E can be stated as follows:

$$\frac{S_n}{n} \xrightarrow{\text{c.c.}} 0, n \rightarrow \infty \iff (3.1) \text{ holds} \iff \sum_{n=1}^{\infty} F_n^*(\varepsilon\sqrt{n}) < \infty \text{ for all } \varepsilon > 0.$$

The following result provides conditions for the convergence of the latter series in the general case.

Theorem 7. *Let $\{F_n\}$ be an arbitrary sequence of distribution functions. Assume that condition (3.5) holds with some $p > 2$. Then*

$$(4.1) \quad \sum_{n=1}^{\infty} F_n^*(\varepsilon\sqrt{n}) < \infty \quad \text{for all } \varepsilon > 0.$$

We do not give the proof of Theorem 7, since it follows the lines of the proof of more general Theorem 8 below.

Heyde [8] considered the asymptotics as $\varepsilon \downarrow 0$ of the series (4.1) in the Hsu–Robbins case of sums of independent, identically distributed random variables.

Theorem F (Heyde [8]). *Let $\{X_n\}$ be a sequence of independent, identically distributed random variables and let $\{S_n\}$ denote their partial sums. Let F_n be the distribution function of $n^{-1/2}S_n$ and let F_n^* be the tail of F_n . If condition (3.1) holds, then*

$$(4.2) \quad \lim_{\varepsilon \downarrow 0} \varepsilon^2 \sum_{n=1}^{\infty} F_n^*(\varepsilon\sqrt{n}) = 1.$$

We complete this result as follows.

Theorem 8. *Let $p > 2$. Let $\{F_n\}$ be a sequence of distribution functions such that $F_n \xrightarrow{w} \mathcal{N}(0, \sigma^2)$, $\sigma^2 > 0$, and condition (3.5) holds. Then*

$$(4.3) \quad \lim_{\varepsilon \downarrow 0} \varepsilon^2 \sum_{n=1}^{\infty} F_n^*(\varepsilon\sqrt{n}) = \sigma^2.$$

Proof. We prove the theorem for the case of $\sigma^2 = 1$. The general case of σ^2 immediately follows from this special case. Therefore, we assume that $F_n \xrightarrow{w} \Phi$.

First we note that

$$(4.4) \quad \lim_{\varepsilon \downarrow 0} \varepsilon^2 \sum_{n=1}^{\infty} \Phi^*(\varepsilon\sqrt{n}) = 1$$

by Theorem F for Gaussian random variables X_n . Further let $c > 0$ be fixed. Then

$$\sum_{n < c\varepsilon^{-2}} |F_n^*(\varepsilon\sqrt{n}) - \Phi^*(\varepsilon\sqrt{n})| \leq \frac{c}{\varepsilon^2},$$

whence

$$(4.5) \quad \limsup_{\varepsilon \downarrow 0} \varepsilon^2 \sum_{n < c\varepsilon^{-2}} |F_n^*(\varepsilon\sqrt{n}) - \Phi^*(\varepsilon\sqrt{n})| \leq c.$$

Now let $2 < p_1 < p$. In view of Remark 3.2, there is an $L_0 < 1$ such that $L_n \leq L_0$ for all $n \geq 1$. Then, by Theorem 2,

$$\sum_{n \geq c\varepsilon^{-2}} |F_n^*(\varepsilon\sqrt{n}) - \Phi^*(\varepsilon\sqrt{n})| \leq 2 \sum_{n \geq c\varepsilon^{-2}} \frac{\lambda_{p_1, n} + c_{p_1} L_n \left(\ln \frac{1}{L_n}\right)^{p_1/2}}{(1 + \varepsilon\sqrt{n})^{p_1}},$$

whence

$$\sum_{n \geq c\varepsilon^{-2}} |F_n^*(\varepsilon\sqrt{n}) - \Phi^*(\varepsilon\sqrt{n})| = o(1) \sum_{n \geq c\varepsilon^{-2}} \frac{1}{(1 + \varepsilon\sqrt{n})^{p_1}},$$

where $o(1) \rightarrow 0$ as $\varepsilon \downarrow 0$. Therefore

$$\varepsilon^2 \sum_{n \geq c\varepsilon^{-2}} |F_n^*(\varepsilon\sqrt{n}) - \Phi^*(\varepsilon\sqrt{n})| = o(1).$$

This together with (4.5) gives

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^2 \sum_{n=1}^{\infty} |F_n^*(\varepsilon\sqrt{n}) - \Phi^*(\varepsilon\sqrt{n})| \leq c.$$

Since $c > 0$ is arbitrary, (4.3) can be derived from (4.4). $\square$

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