CONDITIONS FOR THE UNIFORM CONVERGENCE IN PROBABILITY OF WAVELET DECOMPOSITIONS FOR STOCHASTIC PROCESSES FROM THE SPACE $\text{Exp}_\varphi(\Omega)$

UDC 519.21

YU. V. KOZACHENKO AND O. V. POLOS'MAK

ABSTRACT. Conditions for the uniform convergence in probability on the interval $[0, T]$ of wavelet decompositions of Orlicz stochastic processes of exponential type are found in the paper.

1. INTRODUCTION

In this paper we present the conditions for the uniform convergence in probability of wavelet decompositions of Orlicz stochastic processes of exponential type. The uniform convergence of wavelet decompositions of nonrandom functions is considered in the book [6]. Some problems related to the uniform convergence with probability one and in probability of wavelet decompositions of stochastic processes are studied in the papers [7, 8, 11, 12] for various spaces of random variables.

We consider a strictly Orlicz stochastic process $X = \{X(t), t \in \mathbb{R}\}$ of exponential type. We derive its wavelet decomposition in terms of continuous wavelet functions:

$$(1) \quad X(t) = \sum_{k \in \mathbb{Z}} \xi_{0k} \phi_{0k}(t) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \eta_{jk} \psi_{jk}(t).$$

We find sufficient conditions for the uniform convergence of these decompositions in probability to the original stochastic process $X(t)$. Namely, we find conditions under which

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq T} |X(t) - X_{n,k_j}(t)| > \varepsilon \right\} \rightarrow 0$$

as $n \rightarrow \infty$ and $k_j \rightarrow \infty$ for all $j \in \mathbb{N}_0 := \{0, 1, \dots\}$, where

$$(2) \quad X_{n,k_j}(t) = \sum_{|k| \leq k_0} \xi_{0k} \phi_{0k}(t) + \sum_{j=0}^{n-1} \sum_{|k| \leq k_j} \eta_{jk} \psi_{jk}(t).$$

The paper consists of four sections. Section 2 contains necessary definitions of wavelet analysis and a theorem on the convergence of the wavelet decomposition to the original process in the norm of the space $L_2(\mathbb{R})$. In Section 3, we provide some notions of the theory of Orlicz stochastic processes. Section 4 contains conditions for the uniform

2000 *Mathematics Subject Classification.* Primary 60G12, 42C40.

Key words and phrases. Stochastic processes, wavelet decomposition, convergence in probability, Orlicz spaces.

This research was supported by the La Trobe University Research Grant #501821.

convergence in probability of wavelet decompositions of Orlicz stochastic processes of exponential type.

2. WAVELET DECOMPOSITIONS OF STOCHASTIC PROCESSES

Let $\phi = \{\phi(x), x \in \mathbb{R}\}$ and $\psi = \{\psi(x), x \in \mathbb{R}\}$ be two functions of the space $L_2(\mathbb{R})$ and let $\widehat{\phi}(y)$ be the Fourier transform of ϕ :

$$(3) \quad \widehat{\phi}(y) = \int_{\mathbb{R}} e^{-iyx} \phi(x) dx.$$

Assume that

$$(4) \quad \sum_{k \in \mathbb{Z}} |\widehat{\phi}(y + 2\pi k)|^2 = 1$$

almost everywhere.

Suppose there exists a periodic function $m_0(x) \in L_2([0, 2\pi])$ with period 2π and such that $\widehat{\phi}(0) \neq 0$,

$$(5) \quad \widehat{\phi}(y) = m_0[y/2] \widehat{\phi}[y/2],$$

and $\widehat{\phi}(y)$ is continuous at the origin. In this case, the function $\phi(x)$ is called an f -wavelet.

Let $\psi(x)$ be the inverse Fourier transform of the function $\widehat{\psi}(y)$ satisfying the following functional equality:

$$\widehat{\psi}(y) = \overline{m_0\left(\frac{y}{2} + \pi\right)} \cdot \exp\left\{-i\frac{y}{2}\right\} \cdot \widehat{\psi}\left(\frac{y}{2}\right).$$

Then the function

$$\psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iyx} \widehat{\psi}(y) dy$$

is called an m -wavelet.

Let

$$(6) \quad \phi_{jk}(x) = 2^{j/2} \phi(2^j x - k), \quad \psi_{jk}(x) = 2^{j/2} \psi(2^j x - k), \quad j, k \in \mathbb{Z}.$$

It is known that the family of functions $\{\phi_{0k}; \psi_{jk}, j = 0, 1, 2, \dots; k \in \mathbb{Z}\}$ is an orthonormal basis in the space $L_2(\mathbb{R})$ (see, for example, [3, 4]).

Any function $f(x) \in L_2(\mathbb{R})$ can be represented in the form

$$(7) \quad f(x) = \sum_{k \in \mathbb{Z}} \alpha_{0k} \phi_{0k}(x) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \beta_{jk} \psi_{jk}(x),$$

$$(8) \quad \alpha_{0k} = \int_{\mathbb{R}} f(x) \overline{\phi_{0k}(x)} dx, \quad \beta_{jk} = \int_{\mathbb{R}} f(x) \overline{\psi_{jk}(x)} dx.$$

Representation (7) is called the wavelet decomposition.

The series (7) converge in the norm of the space $L_2(\mathbb{R})$, that is,

$$\sum_{k \in \mathbb{Z}} |\alpha_{0k}|^2 + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} |\beta_{jk}|^2 < \infty.$$

The integrals determining α_{0k} and β_{jk} are well defined for functions in $L_1(\mathbb{R})$ or for those in some other spaces. Thus representation (7) may, in fact, hold for a larger class of functions than the space $L_2(\mathbb{R})$.

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a standard probability space. Let $X = \{X(t), t \in \mathbb{R}\}$ be a stochastic process and let $\mathbf{E} X(t) = 0$.

Representation (7) can be proved for stochastic processes whose trajectories belong to the space $L_2(\mathbb{R})$ with probability one. However, many stochastic processes do not

possess this property. For example, the trajectories of stationary processes do not belong to $L_2(\mathbb{R})$.

We study a representation for $X(t)$ similar to (7), namely

$$(9) \quad X(t) = \sum_{k \in \mathbb{Z}} \xi_{0k} \phi_{0k}(t) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \eta_{jk} \psi_{jk}(t),$$

where the integrals

$$\xi_{0k} = \int_{\mathbb{R}} X(t) \overline{\phi_{0k}(t)} dt, \quad \eta_{jk} = \int_{\mathbb{R}} X(t) \overline{\psi_{jk}(t)} dt$$

are understood in the mean square sense.

We prove the uniform convergence of the wavelet decomposition (9) for the following approximation:

$$(10) \quad X_{n,k_j}(t) = \sum_{|k| \leq k_0} \xi_{0k} \phi_{0k}(t) + \sum_{j=0}^{n-1} \sum_{|k| \leq k_j} \eta_{jk} \psi_{jk}(t).$$

Below we state Theorem 2.1 containing some sufficient conditions for the convergence in the norm of the space $L_2(\Omega)$ of wavelet decompositions. The proof of this result can be found in [7].

Definition 2.1 ([6]). Let ϕ be an f -wavelet. We say that ϕ satisfies condition S if there exists a function $\Phi = \{\Phi(x), x \geq 0\}$ such that $\Phi(0) < \infty$, $\Phi(x)$ decreases for $x \geq 0$, $|\phi(x)| \leq \Phi(|x|)$ almost everywhere, and

$$\int_{\mathbb{R}} \Phi(|x|) dx < \infty.$$

Theorem 2.1. Let $X = \{X(t), t \in \mathbb{R}\}$ be a stochastic process such that $\mathbb{E} X(t) = 0$, $\mathbb{E} |X(t)|^2 < \infty$ for all $t \in \mathbb{R}$, and the covariance function $R_X(t, s) = R(t, s)$ is continuous. Let wavelets ϕ and ψ be continuous functions and let condition S hold for both of them.

Assume that $c = \{c(x), x \in \mathbb{R}\}$ is an even function such that $c(x) > 1$ for all $x \in \mathbb{R}$. Assume further that the function c is nondecreasing in the domain $x > 0$ and there exists a function $0 < A(a) < \infty$ defined for $a > 0$ and such that

$$c(ax) \leq c(x) \cdot A(a)$$

for sufficiently large x .

If

$$\int_{\mathbb{R}} c(x) \Phi(|x|) dx < \infty$$

and $|R(t, t)|^{1/2} \leq c(t)$, then

- 1) $X_{n,k_j}(t) \in L_2(\Omega)$;
- 2) $X_{n,k_j}(t) \rightarrow X(t)$ as $n \rightarrow \infty$ and $k_j \rightarrow \infty$ for all $j = 0, 1, \dots$ in the mean square sense, that is,

$$\mathbb{E} |X_{n,k_j}(t) - X(t)|^2 \rightarrow 0.$$

3. EXPONENTIAL TYPE STOCHASTIC PROCESSES FROM ORLICZ SPACES

Definition 3.1 ([1]). A continuous even convex function $U = \{U(x), x \in \mathbb{R}\}$ is called a C -function if $U(0) = 0$ and $U(x)$ increases for $x > 0$.

Let $\{\Omega, \mathfrak{F}, \mathbb{P}\}$ be a standard probability space.

Definition 3.2 ([1]). Let $U(x)$ be an arbitrary C -function. A family of random variables $L_U(\Omega)$ is called an Orlicz space if, for every $\xi \in L_U(\Omega)$, there exists a constant $r_\xi > 0$ such that $\mathbf{E} U(\xi/r_\xi) < \infty$.

Theorem 3.1 ([1]). An Orlicz space $L_U(\Omega)$ is a Banach space with respect to the Luxemburg norm

$$\|\xi\|_U = \inf\{r > 0: \mathbf{E} U(\xi/r) \leq 1\}.$$

Definition 3.3 ([1]). The space $L_U(\Omega)$ is called an Orlicz space of exponential type if it is generated by a function

$$U(x) = \exp\{\varphi(x)\} - 1,$$

where $\varphi(x)$ is a C -function. Following [1], Orlicz spaces of exponential type are denoted by $\text{Exp}_\varphi(\Omega)$, while the norms in these space are denoted by $\|\cdot\|_{\text{Exp}_\varphi}$.

Definition 3.4 ([1]). Let $\mathbb{T}$ be a set of parameters. We say that a stochastic process

$$X = \{X(t), t \in \mathbb{T}\}$$

belongs to a space $L_U(\Omega)$ if, for all $t \in \mathbb{T}$, the random variable $X(t)$ belongs to the space $L_U(\Omega)$.

Definition 3.5 ([14]). We say that $X(t) \in L_U(\Omega)$ is a strictly Orlicz stochastic process if there is a constant C_T such that, for all $t_1, t_2, \dots, t_n \in \mathbb{T}$ and for all $c_1, c_2, \dots, c_n \in \mathbb{R}$,

$$\left\| \sum_{k=1}^n c_k X(t_k) \right\|_U < C_T \sqrt{\mathbf{E} \sum_{k=1}^n c_k X(t_k)}.$$

Theorem 3.2. Let $\mathbb{T} = [0, T]$ and let

$$X_n = \{X_n(t), t \in \mathbb{T}\}$$

be a sequence of separable stochastic processes from an Orlicz space $L_U(\Omega)$. Assume that

1) there exists an increasing function σ such that $\sigma(h) \rightarrow 0$ as $h \rightarrow 0$ and

$$(11) \quad \sup_{n \geq 1} \rho(t, s) = \sup_{\substack{|t-s| \leq h \\ t, s \in \mathbb{T}}} \|X_n(t) - X_n(s)\|_{\text{Exp}_\varphi} \leq \sigma(h);$$

2) the processes $X_n(t)$ converge in probability as $n \rightarrow \infty$ to a separable process $X(t)$ for all points t ;

3) for all $\varepsilon > 0$, the integral

$$(12) \quad \int_0^\varepsilon U^{(-1)} \left(\frac{1}{2\sigma^{(-1)}(t)} + 1 \right) dt < \infty$$

converges.

Then the processes $X_n(t)$ and $X(t)$ are continuous and $X_n(t) \rightarrow X(t)$ as $n \rightarrow \infty$ uniformly in probability on $\mathbb{T}$.

This result follows from the bound

$$N_\rho(t) \leq \frac{1}{2\sigma^{(-1)}(t)} + 1$$

(see Theorem 3.2 in [10]).

Lemma 3.1. *Let the function $\sigma(u)$ in (11) be given by*

$$\sigma(u) = \frac{c}{(\ln(e^\alpha + 1/u))^\alpha}, \quad 0 < \alpha < 1,$$

and let $U(x) = \exp\{\varphi(x)\} - 1$. Then condition (12) holds if

$$(13) \quad \int_0^\varepsilon \varphi^{(-1)} \left(\left(\frac{c}{t} \right)^{1/\alpha} \right) dt < \infty.$$

Proof. Indeed,

$$\sigma^{(-1)}(t) = \frac{1}{e^{(c/t)^{1/\alpha}} - e^\alpha}$$

and

$$U^{-1}(t) = \varphi^{-1}(\ln(t+1)).$$

Then the integral in (12) is estimated as follows:

$$\begin{aligned} \int_0^\varepsilon U^{(-1)} \left(\frac{1}{2\sigma^{(-1)}(t)} + 1 \right) dt &= \int_0^\varepsilon \varphi^{(-1)} \left(\ln \left(\frac{1}{2\sigma^{(-1)}(t)} + 2 \right) \right) dt \\ &= \int_0^\varepsilon \varphi^{(-1)} \left(\ln \left(\frac{1}{2} \left(e^{(c/t)^{1/\alpha}} - e^\alpha \right) + 2 \right) \right) dt \\ &\leq \int_0^\varepsilon \varphi^{(-1)} \left(\left(\frac{c}{t} \right)^{1/\alpha} \right) dt < \infty. \quad \square \end{aligned}$$

4. UNIFORM CONVERGENCE OF WAVELET DECOMPOSITIONS FOR STOCHASTIC PROCESSES BELONGING TO ORLICZ SPACES OF EXPONENTIAL TYPE

Theorem 4.1. *Let $X = \{X(t), t \in \mathbb{R}\}$ be a separable strictly Orlicz stochastic process in the space $\text{Exp}_\varphi(\Omega)$ whose covariance function $R(t, s)$ is continuous. Suppose the assumptions of Theorem 2.1 as well as condition (13) hold. We further assume that, for an f -wavelet ϕ and for the corresponding m -wavelet ψ , the following conditions hold:*

- 1) *the derivatives $\widehat{\psi}'(u)$ and $\widehat{\psi}''(u)$ exist for all u and $\widehat{\psi}(0) = 0$ and $\widehat{\psi}'(0) = 0$;*
- 2) *the constants c_ϕ , $c_{\phi'}$, and $c_{\psi''}$ are finite, where*

$$c_\phi = \sup_u |\widehat{\phi}(u)|, \quad c_{\phi'} = \sup_u |\widehat{\phi}'(u)|, \quad \text{and} \quad c_{\psi''} = \sup_u |\widehat{\psi}''(u)|;$$

- 3) *$\widehat{\phi}(u) \rightarrow 0$ as $u \rightarrow \pm\infty$ and $\widehat{\psi}(u) \rightarrow 0$ as $u \rightarrow \pm\infty$;*
- 4) *the following two integrals are well defined:*

$$\begin{aligned} \int_{\mathbb{R}} (\ln(1+|u|))^\alpha |\widehat{\psi}(u)|^\beta du &< \infty, \\ \int_{\mathbb{R}} (\ln(1+|u|))^\alpha |\widehat{\phi}(u)|^\beta du &< \infty, \end{aligned}$$

where $0 < \beta \leq 1$ and $0 \leq \alpha \leq 1$;

- 5) *the following integrals exist:*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\partial^{k+l} \widehat{R}_2(z, w)}{\partial w \partial z} \right| |w|^t |z|^s dw dz < \infty, \quad k, l = 0, 1, \quad t, s = 0, 1, 2;$$

- 6) *$|\widehat{R}_2(t, s)| < c < \infty$.*

Then, for all $j = 0, 1, \dots$, as $n \rightarrow \infty$ and $k_j \rightarrow \infty$,

$$X_{n, k_j}(t) \rightarrow X(t)$$

uniformly in probability in every interval $[0, T]$.

Proof. Assumption 2) of Theorem 3.2 follows from Theorem 2.1. In view of Lemma 3.1, assumption (13) implies condition 3) of Theorem 3.2. Now we check that condition (11) holds for the representation $X_{n,k_j}(t)$ and the function

$$\sigma(h) = \frac{B}{(\ln(e^\alpha + 1/|h|))^\alpha}, \quad 0 < \alpha < 1.$$

Since $X(t)$ is a strictly Orlicz stochastic process, $X_{n,k_j}(t)$ is also a strictly Orlicz process (see [14]). Thus we need to check the inequality

$$\begin{aligned} \|X_{n,k_j}(t) - X_{n,k_j}(s)\|_{\text{Exp}_\varphi} &\leq C_T \left(\mathbb{E} |X_{n,k_j}(t) - X_{n,k_j}(s)|^2 \right)^{1/2} \\ &\leq \frac{\tilde{B}}{(\ln(e^\alpha + 1/|t-s|))^\alpha}, \end{aligned}$$

where $\tilde{B}$ is a certain constant. We have

$$\begin{aligned} X_{n,k_j}(t) - X_{n,k_j}(s) &= \sum_{|k| \leq k_0} \xi_{0k}(\phi_{0k}(t) - \phi_{0k}(s)) + \sum_{j=0}^{n-1} \sum_{|k| \leq k_j} \eta_{jk}(\psi_{jk}(t) - \psi_{jk}(s)), \\ \left(\mathbb{E} |X_{n,k_j}(t) - X_{n,k_j}(s)|^2 \right)^{1/2} &\leq \left(\mathbb{E} \left| \sum_{|k| \leq k_0} \xi_{0k}(\phi_{0k}(t) - \phi_{0k}(s)) \right|^2 \right)^{1/2} \\ &\quad + \sum_{j=0}^{n-1} \left(\mathbb{E} \left| \sum_{|k| \leq k_j} \eta_{jk}(\psi_{jk}(t) - \psi_{jk}(s)) \right|^2 \right)^{1/2} \\ &= (S_1)^{1/2} + \sum_{j=0}^{n-1} (S_2)^{1/2}. \end{aligned}$$

Now we estimate S_2 . A similar method applies to S_1 . Indeed,

$$S_2 \leq \sum_{|k| \leq k_j} \sum_{|l| \leq k_j} |\mathbb{E} \eta_{jk} \eta_{jl}| \cdot |\psi_{jk}(t) - \psi_{jk}(s)| \cdot |\psi_{jl}(t) - \psi_{jl}(s)|.$$

First we consider $\mathbb{E} \eta_{jk} \eta_{jl}$. Applying the Parseval equality, we get

$$\begin{aligned} \mathbb{E} \eta_{jk} \eta_{jl} &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \xi(u) \xi(v) \psi_{jk}(u) \psi_{jl}(v) du dv \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{R}_2(z, w) \widehat{\psi}_{jk}(z) \widehat{\psi}_{jl}(w) dz dw, \end{aligned}$$

where

$$\widehat{R}_2(z, w) = \int_{\mathbb{R}} \int_{\mathbb{R}} R(u, v) e^{-izu} e^{-i w v} du dv.$$

Note that

$$(14) \quad \widehat{\psi}_{jk}(z) = \frac{\exp\{-i(k/2^j)z\}}{2^{j/2}} \cdot \widehat{\psi}\left(\frac{z}{2^j}\right).$$

In the next transformation we use integration by parts, assumptions 1), 2), and 5) of the theorem, and the inequality

$$\left| \widehat{\psi}\left(\frac{z}{2^j}\right) \right| = \left| \frac{\widehat{\psi}''(\tilde{z})}{2} \left(\frac{z}{2^j}\right)^2 \right| \leq c_{\psi''} \frac{|z|^2}{2^{2j}}.$$

Thus

$$\begin{aligned}
|\mathbb{E} \eta_{jk} \eta_{jl}| &= \left| \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{R}_2(z, w) \frac{\exp \left\{ -i \left(\frac{k}{2^j} \right) w \right\}}{2^{j/2}} \widehat{\psi} \left(\frac{w}{2^j} \right) \right. \\
&\quad \left. \times \frac{\exp \left\{ -i \left(\frac{l}{2^j} \right) z \right\}}{2^{j/2}} \widehat{\psi} \left(\frac{z}{2^j} \right) dw dz \right| \\
&= \frac{2^j}{(2\pi)^2 |k| |l|} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{\partial^2 \widehat{R}_2(z, w)}{\partial w \partial z} \widehat{\psi} \left(\frac{z}{2^j} \right) \widehat{\psi} \left(\frac{w}{2^j} \right) \right. \right. \\
&\quad + \frac{\partial \widehat{R}_2(z, w)}{\partial z} \frac{1}{2^j} \widehat{\psi}' \left(\frac{w}{2^j} \right) \widehat{\psi} \left(\frac{z}{2^j} \right) \\
&\quad + \frac{\partial \widehat{R}_2(z, w)}{\partial w} \widehat{\psi} \left(\frac{w}{2^j} \right) \frac{1}{2^j} \widehat{\psi}' \left(\frac{z}{2^j} \right) \\
&\quad \left. \left. + \frac{\widehat{R}_2(z, w)}{2^j} \widehat{\psi}' \left(\frac{w}{2^j} \right) \frac{1}{2^j} \widehat{\psi}' \left(\frac{z}{2^j} \right) \right) \right. \\
&\quad \left. \times \exp \left\{ -i \frac{k}{2^j} w \right\} dw \exp \left\{ -i \frac{l}{2^j} z \right\} dz \right| \\
&\leq \frac{1}{|k| |l| 2^{3j}} A^\psi,
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
A^\psi &= \frac{c_{\psi''}^2}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\left| \frac{\partial^2 \widehat{R}_2(z, w)}{\partial w \partial z} \right| |w|^2 |z|^2 + \left| \frac{\partial \widehat{R}_2(z, w)}{\partial z} \right| |w| \cdot |z|^2 \right. \\
&\quad \left. + \left| \frac{\partial \widehat{R}_2(z, w)}{\partial w} \right| |w|^2 |z| + |\widehat{R}_2(z, w)| \cdot |w| \cdot |z| \right) dw dz \\
&< \infty.
\end{aligned}$$

To estimate the difference $|\psi_{jl}(t) - \psi_{jl}(s)|$, we use equality (14). Then

$$\begin{aligned}
(16) \quad \psi_{jl}(t) &= \frac{1}{(2\pi)} \int_{\mathbb{R}} e^{itz} \frac{1}{2^{j/2}} \exp \left\{ -i \frac{l}{2^j} z \right\} \widehat{\psi} \left(\frac{z}{2^j} \right) dz, \\
\psi_{jl}(t) &= \frac{1}{(2\pi) 2^{j/2}} \int_{\mathbb{R}} \exp \left\{ it \left(z + \frac{2^j}{l} \pi \right) \right\} \exp \left\{ -i \frac{l}{2^j} z \right\} e^{-i\pi} \widehat{\psi} \left(\frac{z}{2^j} + \frac{\pi}{l} \right) dz.
\end{aligned}$$

Hence

$$\psi_{jl}(t) = \frac{1}{(4\pi) 2^{j/2}} \int_{\mathbb{R}} \exp \left\{ -i \frac{l}{2^j} z \right\} \left(e^{itz} \widehat{\psi} \left(\frac{z}{2^j} \right) - \exp \left\{ it \left(z + \frac{2^j}{l} \pi \right) \right\} \widehat{\psi} \left(\frac{z}{2^j} + \frac{\pi}{l} \right) \right) dz,$$

whence

$$\begin{aligned}
& |\psi_{jl}(t) - \psi_{jl}(s)| \\
&= \frac{1}{(4\pi)} \frac{1}{2^{j/2}} \left| \int_{\mathbb{R}} \exp \left\{ -i \frac{l}{2^j} z \right\} \right. \\
&\quad \times \left[\left(e^{itz} \widehat{\psi} \left(\frac{z}{2^j} \right) - \exp \left\{ it \left(z + \frac{2^j}{l} \pi \right) \right\} \widehat{\psi} \left(\frac{z}{2^j} + \frac{\pi}{l} \right) \right) \right. \\
&\quad \left. \left. - \left(e^{isz} \widehat{\psi} \left(\frac{z}{2^j} \right) - \exp \left\{ is \left(z + \frac{2^j}{l} \pi \right) \right\} \widehat{\psi} \left(\frac{z}{2^j} + \frac{\pi}{l} \right) \right) \right] dz \right| \\
&\leq \frac{1}{(4\pi)} \frac{1}{2^{j/2}} \int_{\mathbb{R}} \left(\left| e^{itz} - e^{isz} - \exp \left\{ it \left(z + \frac{2^j}{l} \pi \right) \right\} + \exp \left\{ is \left(z + \frac{2^j}{l} \pi \right) \right\} \right| \right. \\
&\quad \times \left| \widehat{\psi} \left(\frac{z}{2^j} \right) \right| \\
&\quad \left. + \left| \exp \left\{ it \left(z + \frac{2^j}{l} \pi \right) \right\} - \exp \left\{ is \left(z + \frac{2^j}{l} \pi \right) \right\} \right| \right. \\
&\quad \left. \times \left| \widehat{\psi} \left(\frac{z}{2^j} \right) - \widehat{\psi} \left(\frac{z}{2^j} + \frac{\pi}{l} \right) \right| \right) dz \\
&= \left| \frac{z}{2^j} = u \right| \\
&= \frac{2^{j/2}}{(4\pi)} \int_{\mathbb{R}} \left(\left| \exp \{ it 2^j u \} - \exp \{ is 2^j u \} - \exp \left\{ it 2^j \left(u + \frac{\pi}{l} \right) \right\} \right. \right. \\
&\quad \left. \left. + \exp \left\{ is 2^j \left(u + \frac{\pi}{l} \right) \right\} \right| \left| \widehat{\psi}(u) \right| \right. \\
&\quad \left. + \left| \exp \left\{ it 2^j \left(u + \frac{\pi}{l} \right) \right\} - \exp \left\{ is 2^j \left(u + \frac{\pi}{l} \right) \right\} \right| \right. \\
&\quad \left. \times \left| \widehat{\psi}(u) - \widehat{\psi} \left(u + \frac{\pi}{l} \right) \right| \right) du.
\end{aligned}$$

To estimate the latter integral, we use the inequality

$$\begin{aligned}
(17) \quad & |e^{itz} - e^{isz}| \leq 2 \left| \sin \frac{z(t-s)}{2} \right| \leq 2 \left(\frac{\ln(e^\alpha + z/2)}{\ln(e^\alpha + 1/|t-s|)} \right)^\alpha, \\
& 0 < \alpha \leq 1
\end{aligned}$$

(see, for example, [13]). We estimate the second term of the latter integral as follows:

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}} \left| \exp \left\{ it 2^j \left(u + \frac{\pi}{l} \right) \right\} - \exp \left\{ is 2^j \left(u + \frac{\pi}{l} \right) \right\} \right| \left| \widehat{\psi}(u) - \widehat{\psi} \left(u + \frac{\pi}{l} \right) \right| du \\
&= \left| u + \frac{\pi}{l} = v \right| \\
&= \int_{\mathbb{R}} \left| \exp \{ it 2^j v \} - \exp \{ is 2^j v \} \right| \left| \widehat{\psi} \left(v - \frac{\pi}{l} \right) - \widehat{\psi}(v) \right| dv \\
&\leq \frac{2}{(\ln(e^\alpha + \frac{1}{|t-s|}))^\alpha} \int_{\mathbb{R}} \left(\ln \left(e^\alpha + \frac{|v| 2^j}{2} \right) \right)^\alpha \left| \widehat{\psi} \left(v - \frac{\pi}{l} \right) - \widehat{\psi}(v) \right|^\beta \left| \widehat{\psi} \left(v - \frac{\pi}{l} \right) - \widehat{\psi}(v) \right|^{1-\beta} dv.
\end{aligned}$$

In the next transformation, we apply assumptions 1) and 4) of Theorem 4.1 and the inequalities

$$\begin{aligned} \left| \widehat{\psi}\left(v - \frac{\pi}{l}\right) - \widehat{\psi}(v) \right|^\beta &= \frac{b}{|l|^\beta}, \quad b = a\pi^\beta, \\ \left| \widehat{\psi}\left(v - \frac{\pi}{l}\right) - \widehat{\psi}(v) \right|^{1-\beta} &\leq \left(\left| \widehat{\psi}\left(v - \frac{\pi}{l}\right) \right|^{1-\beta} + \left| \widehat{\psi}(v) \right|^{1-\beta} \right), \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathbb{R}} b \left(\ln \left(e^\alpha + \frac{|v|2^j}{2} \right) \right)^\alpha \left| \widehat{\psi}\left(v - \frac{\pi}{l}\right) \right|^{1-\beta} dv \\ &\leq b \int_{\mathbb{R}} (\ln(2e^\alpha + |v| + \pi))^j \left| \widehat{\psi}(v) \right|^{1-\beta} dv = j^\alpha c_1 < \infty, \\ c_1 &= b \int_{\mathbb{R}} (\ln(2e^\alpha + \pi + |v|))^\alpha \left| \widehat{\psi}(v) \right|^{1-\beta} dv < \infty. \end{aligned}$$

From these results we have

$$(18) \quad I_2 \leq \frac{2}{(\ln(e^\alpha + 1/|t-s|))^\alpha} \frac{1}{|l|^\beta} j^\alpha 2c_1.$$

The first term of the same integral (we call it I_1) can be estimated similarly. First we use the inequality

$$\begin{aligned} &\left| \exp\{it2^j u\} - \exp\{is2^j u\} - \exp\left\{it2^j \left(u + \frac{\pi}{l}\right)\right\} + \exp\left\{is2^j \left(u + \frac{\pi}{l}\right)\right\} \right| \\ &\leq \left| \exp\{it2^j u\} - \exp\{is2^j u\} \right| \left| 1 - \exp\left\{it2^j \frac{\pi}{l}\right\} \right| + \left| \exp\left\{is2^j \frac{\pi}{l}\right\} - \exp\left\{it2^j \frac{\pi}{l}\right\} \right| \\ &=: |\Delta| \\ &\leq 2 \left| \sin \frac{2^j u(t-s)}{2} \right| \cdot \left| \sin \frac{\pi}{l} 2^j \frac{t}{2} \right| + 2 \left| \sin \frac{(t-s)}{2} \frac{\pi}{l} 2^j \right|. \end{aligned}$$

Note that

$$\begin{aligned} \left| \sin \frac{2^j u(t-s)}{2} \right| &\leq \left(\frac{\ln(e^\alpha + \frac{2^j |u|}{2})}{\ln(e^\alpha + 1/|t-s|)} \right)^\alpha, \quad 0 < \alpha \leq 1, \\ \left| \sin \frac{\pi}{l} 2^j \frac{t}{2} \right| &\leq \left(\pi \frac{t}{2} \right)^\beta \cdot \left(\frac{2^j}{|l|} \right)^\beta \leq \left(\pi \frac{T}{2} \right)^\beta \cdot \frac{2^{j\beta}}{|l|^\beta}, \\ \left| \sin \frac{(t-s)}{2} \frac{\pi}{l} 2^j \right| &\leq \left(\frac{\pi 2^j}{|l|} \right)^\beta \cdot \left(\frac{|s-t|}{2} \right)^\beta \leq \left(\frac{2^j}{|l|} \right)^\beta \frac{c_{\alpha,\beta,T}}{(\ln(e^\alpha + 1/|t-s|))^\alpha}. \end{aligned}$$

Then

$$|\Delta| \leq \frac{2^{j\beta}}{|l|^\beta} \cdot \frac{j^\alpha}{(\ln(e^\alpha + 1/|t-s|))^\alpha} \left(2 \left(\pi \frac{T}{2} \right)^\beta (\ln(1+|u|))^\alpha + 2c_{\alpha,\beta,T} \right).$$

Now the first part of the integral is estimated as follows:

$$(19) \quad I_1 \leq \frac{2^{j\beta}}{|l|^\beta} \cdot \frac{j^\alpha}{(\ln(e^\alpha + 1/|t-s|))^\alpha} \cdot c_2,$$

where

$$c_2 = \int_{\mathbb{R}} \left(2 \left(\pi \frac{T}{2} \right)^\beta (\ln(1+|u|))^\alpha + 2c_{\alpha,\beta,T} \right) \left| \widehat{\psi}(u) \right| du < \infty.$$

Taking into account the bounds (18) and (19), we obtain

$$\begin{aligned}
 & |\psi_{jk}(t) - \psi_{jk}(s)| \cdot |\psi_{jl}(t) - \psi_{jl}(s)| \\
 (20) \quad & \leq \frac{2^j}{(4\pi)^2} \frac{j^{2\alpha}}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}} \left(\frac{2^{j\beta}}{|l|^\beta} \cdot c_2 + \frac{4c_1}{|l|^\beta} \right) \left(\frac{2^{j\beta}}{|k|^\beta} \cdot c_2 + \frac{4c_1}{|k|^\beta} \right) \\
 & \leq \frac{2^{j(1+2\beta)} j^{2\alpha}}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}} \frac{1}{|l|^\beta} \frac{1}{|k|^\beta} \cdot (c^\psi)^2,
 \end{aligned}$$

where $c^\psi = (c_2 + 4c_1)/(4\pi)$. Therefore, for the case where $l \neq 0$ and $k \neq 0$, we have that

$$\begin{aligned}
 & \sum_{\substack{|k| \leq k_j, \\ k \neq 0}} \sum_{\substack{|l| \leq k_j, \\ l \neq 0}} |\mathbb{E} \eta_{jk} \eta_{jl}| |\psi_{jk}(t) - \psi_{jk}(s)| |\psi_{jl}(t) - \psi_{jl}(s)| \\
 (21) \quad & \leq \sum_{\substack{|k| \leq k_j, \\ k \neq 0}} \sum_{\substack{|l| \leq k_j, \\ l \neq 0}} \frac{1}{2^{3j}} \frac{1}{|k||l|} \cdot A^\psi \frac{2^{(2\beta+1)j}}{|l|^\beta |k|^\beta} (c^\psi)^2 \frac{j^{2\alpha}}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}} \\
 & = \frac{j^{2\alpha}}{2^{(2-2\beta)j}} \frac{A^\psi (c^\psi)^2 \hat{q}^2}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}},
 \end{aligned}$$

where

$$\hat{q}^2 = \sum_{\substack{|k| \leq k_j, \\ k \neq 0}} \sum_{\substack{|l| \leq k_j, \\ l \neq 0}} \frac{1}{|l||k|} \cdot \frac{1}{|l|^\beta |k|^\beta} \leq \sum_{0 < k \leq k_j} \sum_{0 < l \leq k_j} \frac{2}{k^{1+\beta}} \cdot \frac{2}{l^{1+\beta}} < \infty.$$

The series on the right hand side converges if $\beta > 0$.

The case of $l \neq 0$, $k = 0$ can be treated similarly. Thus

$$\begin{aligned}
 |\mathbb{E} \eta_{j0} \eta_{jl}| &= \frac{1}{2\pi} \left| \int_{\mathbb{R}} \widehat{R}_2(z, w) \frac{1}{2^{j/2}} \widehat{\psi}\left(\frac{w}{2^j}\right) \cdot \frac{\exp\{-i\frac{l}{2^j}z\}}{2^{j/2}} \widehat{\psi}\left(\frac{z}{2^j}\right) dz \right| \\
 (22) \quad &= \frac{1}{(2\pi)^2 |l|} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{\partial \widehat{R}_2(z, w)}{\partial z} \widehat{\psi}\left(\frac{z}{2^j}\right) + \frac{\widehat{R}_2(z, w)}{2^j} \widehat{\psi}'\left(\frac{z}{2^j}\right) \right) \right. \\
 &\quad \left. \times \exp\left\{-i\frac{l}{2^j}z\right\} \widehat{\psi}\left(\frac{w}{2^j}\right) dz dw \right| \\
 &\leq \frac{1}{2^{4j}} \cdot \frac{1}{|l|} \cdot A_1^\psi,
 \end{aligned}$$

where

$$A_1^\psi = \frac{(c_{\psi''})^2}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\left| \frac{\partial \widehat{R}_2(z, w)}{\partial z} \right| |z|^2 |w|^2 + |\widehat{R}_2(z, w)| |z| |w|^2 \right) dz dw < \infty.$$

Similar reasoning is applied in the case of $k \neq 0$, $l = 0$. Hence

$$|\mathbb{E} \eta_{jk} \eta_{j0}| \leq \frac{A_1^\psi}{2^{4j} |k|}.$$

Note that

$$\begin{aligned}
|\psi_{j0}(t) - \psi_{j0}(s)| &= \frac{1}{(2\pi)} \frac{1}{2^{j/2}} \left| \int_{\mathbb{R}} (e^{itz} - e^{isz}) \widehat{\psi}\left(\frac{z}{2^j}\right) dz \right| \\
&= \frac{2^{j/2}}{(2\pi)} \left| \int_{\mathbb{R}} (e^{it2^j u} - e^{is2^j u}) \widehat{\psi}(u) du \right| \\
&\leq \frac{2^{j/2}}{(2\pi)} \int_{\mathbb{R}} \left(\frac{\ln(e^\alpha + \frac{2^j |u|}{2})}{\ln(e^\alpha + 1/|t-s|)} \right)^\alpha |\widehat{\psi}(u)| du \\
&\leq \frac{2^{j/2}}{(2\pi)} \frac{j^\alpha}{(\ln(e^\alpha + 1/|t-s|))^\alpha} \int_{\mathbb{R}} (\ln(2e^\alpha + |u|))^\alpha |\widehat{\psi}(u)| du \\
&= \frac{2^{j/2} j^\alpha \cdot c_1^\psi}{(\ln(e^\alpha + 1/|t-s|))^\alpha},
\end{aligned}$$

where

$$c_1^\psi = \frac{1}{2\pi} \int_{\mathbb{R}} (\ln(2e^\alpha + |u|))^\alpha |\widehat{\psi}(u)| du < \infty.$$

Then

$$\begin{aligned}
(23) \quad & \sum_{|l| \leq k_j, l \neq 0} |\mathbb{E} \eta_{j0} \eta_{jl}| |\psi_{j0}(t) - \psi_{j0}(s)| |\psi_{jl}(t) - \psi_{jl}(s)| \\
& \leq \sum_{|l| \leq k_j, l \neq 0} \frac{1}{2^{4j}} \frac{1}{|l|} \cdot A_1^\psi \frac{2^{j(1/2+\beta)} j^\alpha}{(\ln(e^\alpha + 1/|t-s|))^\alpha} \frac{1}{|l|^\beta} \cdot c^\psi \frac{2^{j/2} j^\alpha \cdot c_1^\psi}{(\ln(e^\alpha + 1/|t-s|))^\alpha} \\
& \leq \frac{j^{2\alpha}}{2^{(3-\beta)j}} \cdot \frac{A_1^\psi \cdot c^\psi \cdot c_1^\psi \cdot q}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}},
\end{aligned}$$

where

$$q = \sum_{|l| \leq k_j, l \neq 0} \frac{1}{|l|^{1+\beta}} < \infty$$

and

$$\begin{aligned}
(24) \quad & \sum_{|k| \leq k_j, k \neq 0} |\mathbb{E} \eta_{j0} \eta_{jk}| |\psi_{j0}(t) - \psi_{j0}(s)| |\psi_{jk}(t) - \psi_{jk}(s)| \\
& \leq \frac{j^{2\alpha}}{2^{(3-\beta)j}} \cdot \frac{A_1^\psi \cdot c^\psi \cdot c_1^\psi \cdot q}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}}.
\end{aligned}$$

Using the bounds (21), (23), and (24), we get

$$\begin{aligned}
(25) \quad & \sum_{j=0}^{n-1} \left(\mathbb{E} \left| \sum_{|k| \leq k_j} \eta_{jk} (\psi_{jk}(t) - \psi_{jk}(s)) \right|^2 \right)^{1/2} \\
& \leq \sum_{j=0}^{n-1} \left(\frac{j^{2\alpha}}{2^{(2-2\beta)j}} \frac{A^\psi (c^\psi)^2 \hat{q}^2}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}} + 2 \frac{j^{2\alpha}}{2^{(3-\beta)j}} \cdot \frac{A_1^\psi \cdot c^\psi \cdot c_1^\psi \cdot q}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}} \right)^{1/2} \\
& \leq q_1 \left[c^\psi (c^\psi A^\psi \hat{q}^2 + 2c_1^\psi A_1^\psi q) \right]^{1/2} \frac{1}{(\ln(e^\alpha + 1/|t-s|))^\alpha},
\end{aligned}$$

where

$$q_1 = \sum_{j=0}^{n-1} \frac{j^\alpha}{2^{(1-\beta)j}} < \infty, \quad \beta < 1.$$

An analogous method is applied to

$$S_1 = \left(\mathbb{E} \left| \sum_{|k| \leq k_0} \xi_{0k} (\phi_{0k}(t) - \phi_{0k}(s)) \right|^2 \right)^{1/2}.$$

Namely, we estimate $|\mathbb{E} \xi_{0k} \xi_{0l}|$ in the case of $k \neq 0, l \neq 0$:

$$\begin{aligned} |\mathbb{E} \xi_{0k} \xi_{0l}| &= \left| \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{R}_2(z, w) e^{-ikw} \widehat{\phi}(z) e^{-ilz} \widehat{\phi}(w) dz dw \right| \\ &= \frac{1}{(2\pi)^2} \frac{1}{|k|} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{\partial \widehat{R}_2(z, w)}{\partial w} \widehat{\phi}(w) + \widehat{R}_2(z, w) \widehat{\phi}'(w) \right) e^{-ikw} dw e^{-ilz} \widehat{\phi}(z) dz \right| \\ &= \frac{1}{(2\pi)^2} \frac{1}{|k||l|} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{\partial^2 \widehat{R}_2(z, w)}{\partial w \partial z} \widehat{\phi}(w) \widehat{\phi}(z) + \frac{\partial \widehat{R}_2(z, w)}{\partial z} \widehat{\phi}'(w) \widehat{\phi}(z) \right. \right. \\ &\quad \left. \left. + \frac{\partial \widehat{R}_2(z, w)}{\partial w} \widehat{\phi}(w) \widehat{\phi}'(z) + \widehat{R}_2(z, w) \widehat{\phi}'(w) \widehat{\phi}'(z) \right) \right. \\ &\quad \left. \times e^{-ilz} dz e^{-ikw} dw \right| \\ &\leq \frac{1}{|k| \cdot |l|} \cdot A^\phi, \end{aligned} \tag{26}$$

where

$$\begin{aligned} A^\phi &= \frac{1}{(2\pi)^2} \left(c_\phi^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\partial^2 \widehat{R}_2(z, w)}{\partial w \partial z} \right| dz dw \right. \\ &\quad \left. + 2c_\phi c'_\phi \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\partial \widehat{R}_2(z, w)}{\partial z} \right| dz dw + (c'_\phi)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} |\widehat{R}_2(z, w)| dz dw \right) < \infty, \\ c_\phi &= \sup_u |\widehat{\phi}(u)| < \infty, \quad c'_\phi = \sup_u |\widehat{\phi}'(u)| < \infty. \end{aligned}$$

Similarly to (20) we derive the bound

$$|\phi_{0k}(t) - \phi_{0k}(s)| \cdot |\phi_{0l}(t) - \phi_{0l}(s)| \leq \frac{(c^\phi)^2}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}} \frac{1}{|l|^\beta} \frac{1}{|k|^\beta}, \tag{27}$$

where

$$\begin{aligned} c^\phi &= \frac{1}{(4\pi)} \left(4 \int_{\mathbb{R}} \left(2 \left(\pi \frac{T}{2} \right)^\beta (\ln(1+|u|))^\alpha + 2c_\alpha \right) |\widehat{\phi}(u)| du \right. \\ &\quad \left. + \int_{\mathbb{R}} (\ln(1+|u|))^\alpha |\widehat{\phi}(u)|^{1-\beta} du \right) < \infty. \end{aligned}$$

Thus

$$\begin{aligned}
 (28) \quad & \sum_{|k| \leq k_0, k \neq 0} \sum_{|l| \leq k_0, l \neq 0} |\mathbb{E} \xi_{0k} \xi_{0l}| |\phi_{0k}(t) - \phi_{0k}(s)| |\phi_{0l}(t) - \phi_{0l}(s)| \\
 & \leq \sum_{|k| \leq k_0, k \neq 0} \sum_{|l| \leq k_0, l \neq 0} \frac{1}{|k||l|} \cdot A^\phi \frac{(c^\phi)^2}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}} \frac{1}{|l|^\beta} \frac{1}{|k|^\beta} \\
 & \leq \frac{A^\phi (c^\phi)^2 \hat{q}^2}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}}.
 \end{aligned}$$

Consider the case of $k = 0, l \neq 0$:

$$\begin{aligned}
 |\mathbb{E} \xi_{00} \xi_{0l}| &= \left| \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{R}_2(z, w) \widehat{\phi}(w) \cdot e^{-ilz} \widehat{\phi}(z) dz dw \right| \\
 &= \frac{1}{(2\pi)^2} \frac{1}{|l|} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{\partial \widehat{R}_2(z, w)}{\partial z} \widehat{\phi}(z) + \widehat{R}_2(z, w) \widehat{\phi}'(z) \right) e^{-ilz} dz \widehat{\phi}(w) dw \right| \\
 &\leq \frac{1}{|l|} \cdot A_1^\phi,
 \end{aligned}$$

where

$$A_1^\phi = \frac{c_\phi}{(2\pi)^2} \left(c_\phi \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\partial \widehat{R}_2(z, w)}{\partial z} \right| dz dw + c'_\phi \int_{\mathbb{R}} \int_{\mathbb{R}} |\widehat{R}_2(z, w)| dz dw \right) < \infty.$$

In the case of $l = 0, k \neq 0$, we have $|\mathbb{E} \xi_{00} \xi_{0k}| \leq A_1^\phi / |k|$.

From the inequality (17) we get

$$\begin{aligned}
 |\phi_{00}(t) - \phi_{00}(s)| &= \frac{1}{(2\pi)} \left| \int_{\mathbb{R}} (e^{itz} - e^{isz}) \widehat{\phi}(z) dz \right| \\
 &\leq \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{\ln(e^\alpha + |z|/2)}{\ln(e^\alpha + 1/|t-s|)} \right)^\alpha |\widehat{\phi}(z)| dz \leq \frac{c_1^\phi}{(\ln(e^\alpha + 1/|t-s|))^\alpha},
 \end{aligned}$$

where

$$c_1^\phi = \pi^{-1} \int_{\mathbb{R}} (\ln(2e^\alpha + |z|))^\alpha |\widehat{\phi}(z)| dz < \infty,$$

whence we deduce that

$$\begin{aligned}
 (29) \quad & \sum_{|l| \leq k_0, l \neq 0} |\mathbb{E} \xi_{00} \xi_{0l}| \cdot |\phi_{00}(t) - \phi_{00}(s)| \cdot |\phi_{0l}(t) - \phi_{0l}(s)| \\
 & \leq \sum_{|l| \leq k_0, l \neq 0} \frac{1}{|l|} \cdot A_1^\phi \frac{c_1^\phi}{(\ln(e^\alpha + 1/|t-s|))^\alpha} \frac{1}{|l|^\beta} \frac{c^\phi}{(\ln(e^\alpha + 1/|t-s|))^\alpha} \\
 & = \frac{A_1^\phi \cdot c^\phi \cdot c_1^\phi \cdot q}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}}
 \end{aligned}$$

and

$$(30) \quad \sum_{|k| \leq k_0, k \neq 0} |\mathbb{E} \xi_{00} \xi_{0l}| \cdot |\phi_{00}(t) - \phi_{00}(s)| \cdot |\phi_{0k}(t) - \phi_{0k}(s)| \leq \frac{A_1^\phi \cdot c^\phi \cdot c_1^\phi \cdot q}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}}.$$

Using the bounds (28), (29), and (30), we obtain

$$\begin{aligned}
 (31) \quad & \left(\mathbb{E} \left| \sum_{|k| \leq k_0} \xi_{0k} (\phi_{0k}(t) - \phi_{0k}(s)) \right|^2 \right)^{1/2} \\
 & \leq \left(\frac{A^\phi (c^\phi)^2 \hat{q}^2}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}} + 2 \frac{A_1^\phi \cdot c^\phi \cdot c_1^\phi \cdot q}{(\ln(e^\alpha + 1/|t-s|))^{2\alpha}} \right)^{1/2} \\
 & = \frac{[c^\phi (c^\phi A^\phi \hat{q}^2 + 2qc_1^\phi A_1^\phi)]^{1/2}}{(\ln(e^\alpha + 1/|t-s|))^\alpha}, \quad 0 < \alpha \leq 1.
 \end{aligned}$$

Finally, (25) and (31) imply that

$$\begin{aligned}
 (32) \quad & \left(\mathbb{E} |X_{n,k_j}(t) - X_{n,k_j}(s)|^2 \right)^{1/2} \\
 & \leq \frac{q_1 [c^\psi (\hat{q}^2 A^\psi c^\psi + 2qc_1^\psi A_1^\psi)]^{1/2}}{(\ln(e^\alpha + 1/|t-s|))^\alpha} + \frac{[c^\phi (c^\phi A^\phi \hat{q}^2 + 2qc_1^\phi A_1^\phi)]^{1/2}}{((e^\alpha + 1/|t-s|))^\alpha} \\
 & = \frac{B}{(\ln(e^\alpha + 1/|t-s|))^\alpha}, \quad 0 < \alpha \leq 1,
 \end{aligned}$$

where

$$B = q_1 \left[c^\psi (\hat{q}^2 A^\psi c^\psi + 2qA_1^\psi c_1^\psi) \right]^{1/2} + \left[c^\phi (\hat{q}^2 A^\phi c^\phi + 2qc_1^\phi A_1^\phi) \right]^{1/2}.$$

To complete the proof, we use Theorem 2.1, where $X_{n,k_j}(t) \rightarrow X(t)$ in the mean square sense as $n \rightarrow \infty$ and $k_j \rightarrow \infty$ for $j = 0, 1, \dots$. Then relation (32), Theorem 3.2, and Lemma 3.1 imply that $X_{n,k_j}(t) \rightarrow X(t)$ uniformly in probability in the interval $[0, T]$ as $n \rightarrow \infty$ and $k_j \rightarrow \infty$ for $j = 0, 1, \dots$. $\square$

5. CONCLUDING REMARKS

We have obtained sufficient conditions for the uniform convergence in probability in an interval $[0, T]$ of wavelet decompositions of strictly Orlicz stochastic processes of exponential type. We plan to apply these results for studying the rate of convergence of wavelet decompositions of strictly Orlicz stochastic processes.

BIBLIOGRAPHY

1. V. V. Buldygin and Yu. V. Kozachenko, *Metric Characterization of Random Variables and Random Processes*, TViMS, Kiev, 1998; English transl., American Mathematical Society, Providence, RI, 2000. MR1743716 (2001g:60089)
2. V. V. Buldygin and Yu. V. Kozachenko, *Sub-Gaussian random variables*, Ukrain. Mat. Zh. **32** (1980), no. 6, 723–730; English transl. in Ukrainian Math. J. **32** (1981), no. 6, 483–489. MR598605 (82d:60023)
3. C. K. Chui, *An Introduction to Wavelets*, Academic Press, New York, 1992. MR1150048 (93f:42055)
4. I. Daubechies, *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, Philadelphia, 1992. MR1162107 (93e:42045)
5. P. Flandrin, *Time-Frequency/Time-Scale Analysis*, Academic Press, New York, 1999. MR1681043 (2000e:94014)
6. W. Härdle, G. Kerkycharian, D. Picard, and A. Tsybakov, *Wavelet. Approximation and Statistical Applications*, Springer, New York, 1998. MR1618204 (99f:42065)
7. Yu. V. Kozachenko and O. V. Polosmak, *Uniform convergence in probability of wavelet expansions of random processes from $L_2(\Omega)$* , Random Oper. Stoch. Equ. **16** (2008), no. 4, 12–37. MR2494935 (2010c:42081)

8. Yu. V. Kozachenko, M. M. Perestyuk, and O. I. Vasylyk, *On uniform convergence of wavelet expansion of ϕ -sub-Gaussian random processes*, Random Oper. Stoch. Equ. **14** (2006), no. 3, 209–233. MR2264363 (2008e:60092)
9. Yu. V. Kozachenko, *Lectures on Wavelet Analysis*, TViMS, Kyiv, 2004. (Ukrainian)
10. Yu. V. Kozachenko and G. I. Slivka, *Justification of the Fourier method for hyperbolic equations with random initial conditions*, Teor. ĭmovir. Mat. Stat. **69** (2003), 63–78; English transl. in Theory Probab. Math. Statist. **69** (2004), 67–83. MR2110906 (2005k:60127)
11. Yu. V. Kozachenko and M. M. Perestyuk, *On the uniform convergence of wavelet expansions of random processes from Orlicz spaces of random variables. I*, Ukrain. Mat. Zh. **59** (2007), no. 12, 1647–1660; English transl. in Ukrainian Math. J. **59** (2007), no. 12, 1850–1869. MR2411593 (2009b:60058)
12. Yu. V. Kozachenko and M. M. Perestyuk, *On the uniform convergence of wavelet expansions of random processes from Orlicz spaces of random variables. II*, Ukrain. Mat. Zh. **60** (2008), no. 6, 759–775; English transl. in Ukrainian Math. J. **60** (2008), no. 6, 876–900. MR2485176 (2009m:60095)
13. Yu. V. Kozachenko and E. V. Turchin, *Conditions for the uniform convergence of distributions of ϕ -sub-Gaussian random processes in systems of functions generated by wavelets*, Teor. ĭmovir. Mat. Stat. **78** (2008), 74–85; English transl. in Theory Probab. Math. Statist. **78** (2009), 83–95. MR2446851 (2009m:60085)
14. E. Barrasa de la Krus and Yu. V. Kozachenko, *Boundary-value problems for equations of mathematical physics with strictly Orlicz random initial conditions*, Random Oper. Stoch. Equ. **3** (1995), no. 3, 201–220. MR1354813 (96h:60009)

DEPARTMENT OF PROBABILITY THEORY, STATISTICS, AND ACTUARIAL MATHEMATICS, FACULTY FOR MECHANICS AND MATHEMATICS, NATIONAL TARAS SHEVCHENKO UNIVERSITY, ACADEMICIAN GLUSHKOV AVENUE 2, KIEV 03127, UKRAINE

E-mail address: yvk@univ.kiev.ua

DEPARTMENT OF PROBABILITY THEORY, STATISTICS, AND ACTUARIAL MATHEMATICS, FACULTY FOR MECHANICS AND MATHEMATICS, NATIONAL TARAS SHEVCHENKO UNIVERSITY, ACADEMICIAN GLUSHKOV AVENUE 2, KIEV 03127, UKRAINE

E-mail address: olgapolosmak@yandex.ru

Received 11/MAR/2009

Translated by O. KLESOV