LAW OF THE ITERATED LOGARITHM
FOR SOLUTIONS OF STOCHASTIC EQUATIONS

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ABSTRACT. Strassen's law of the iterated logarithm for a solution $x(t)$ of Itô's stochastic equation is considered in the paper. We obtain a result for small times in the uniform metric and for a more general normalizing function than the classical $\sqrt{h \ln \ln \frac{1}{h}}$.

1. INTRODUCTION

We consider Strassen's law of the iterated logarithm for solutions of stochastic equations for small times in the uniform metric. We use a general normalizing function $\sqrt{h} \varphi_h$ instead of the classical one $\sqrt{h \ln \ln \frac{1}{h}}$. A. Bulinskiĭ developed in [1] an approach to study such general normalizing functions for the Wiener process as the time goes to infinity. The result obtained below allows one to study the asymptotic behavior of integral functionals of solutions of stochastic equations. In particular, Theorem 2.2 of [5] for a Wiener process $w(t)$,

$$\varlimsup_{h \to 0} \frac{\frac{1}{h} \int_t^{t+h} w(s)\,ds - w(t)}{\sqrt{2h \ln \ln \frac{1}{h}}} = \frac{1}{\sqrt{3}},$$

follows from our Theorem 2 stated below.

2. MAIN RESULTS

This section contains the definitions and statements of the main results of the paper. The proofs are given in Section 4 below on the basis of some auxiliary results obtained in Section 3.

Let $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$ be a probability space equipped with a filtration $\mathfrak{F}_t$, $t \in [0,1]$. Denote by E_d the d-dimensional Euclidean space with the scalar product $(\cdot\,,\cdot)$. Let $C([0,1],d)$ denote the space of all continuous functions on $[0,1]$ assuming values in E_d and let $\rho(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)|$ and $\mathcal{B}$ be the metric and Borel σ-algebra in this space, respectively. The symbol $I(A)$ stands for the indicator of a random event A and $\bar{A}$ denotes the complement of the event A.

Let $x(t)$ be a solution of the stochastic equation

$$(1) \qquad x(t) = x_0 + \int_0^t b(s, x(s))\,ds + \int_0^t \sigma(s, x(s))\,dw(s),$$

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where $b(t,x) \in E_d$, $\sigma(t,x)$ is a $d \times n$ matrix, and w is an n-dimensional and $\mathfrak{F}_t$-adapted Wiener process. Put $a(t,x) = \sigma(t,x)\sigma^*(t,x)$, $\sigma_0 = \sigma(0,x_0)$, and $a_0 = \sigma_0\sigma_0^*$, where * is the transposition symbol.

We assume the following conditions.

(i) There exists a constant K such that

$$|b_i(t,x)| + |\sigma_{ij}(t,x)| \leq K$$

for all $i = 1, 2, \ldots, d$ and $j = 1, 2, \ldots, n$, where the functions $\sigma_{ij}(t,x)$ are continuous at the point $(0, x_0)$.

(ii) There exists $\lambda > 0$ such that $(a_0\theta, \theta) \geq \lambda|\theta|^2$ for all $\theta \in E_d$.

By Φ we denote the class of increasing functions $\varphi(x)$, $x \geq 0$, such that

$$\lim_{x\to\infty} \varphi(x) = \infty, \qquad \lim_{x\to\infty} \frac{\varphi(x)}{\sqrt{x}} = 0.$$

Let $c > 1$ be a number and let $[c]$ denote its integer part. For every $\varphi \in \Phi$, we define $R^2(\varphi)$ as follows:

$$(2) \qquad R^2(\varphi) = \inf\left\{ r > 0 \colon \sum_{n=1}^{\infty} \exp\left\{ -r\frac{\varphi^2([c^n])}{2} \right\} < \infty \right\}.$$

We set $R^2(\varphi) = \infty$ if the series in (2) diverges for all finite r. Note that if the series in (2) converges for some $c > 1$, then it converges for all $c > 1$. It is proved in [1, Theorem 3] that $R = Q^{-1}$, where

$$Q = \lim_{x\to\infty} \frac{\varphi(x)}{\sqrt{2 \ln \ln x}}.$$

For an absolutely continuous function $f(t)$, put

$$I_0(f) = \frac{1}{2} \int_0^1 |\dot{f}(t)|^2 \, dt$$

and introduce the following classes of functions:

$$\mathcal{K}_R = \left\{ f \in (C[0,1], n) \colon f(0) = 0, I_0(f) \leq R^2/2 \right\},$$
$$\mathcal{H}_R = \{ g(t) \colon g(t) = \sigma_0 f(t), f \in \mathcal{K}_R, t \in [0,1] \}.$$

If $R = \infty$, then $\mathcal{K}_R = \{ f \in (C[0,1], n) \colon f(0) = 0 \}$. Put $\varphi_h = \varphi(1/h)$ and let $x_h(t) = (x(th) - x_0)/(\sqrt{h}\varphi_h)$. We derive from (1) that

$$(3) \qquad x(th) = x_0 + h \int_0^t b(sh, x(sh)) \, ds + \sqrt{h} \int_0^t \sigma(sh, x(sh)) \, dw_h(s)$$

and

$$(4) \qquad x_h(t) = \frac{\sqrt{h}}{\varphi_h} \int_0^t b(sh, x(sh)) \, ds + \frac{1}{\varphi_h} \int_0^t \sigma(sh, x(sh)) \, dw_h(s),$$

where $w_h(t) = w(th)/\sqrt{h}$ is a Wiener process.

Theorem 2.1 provides the functional law of the iterated logarithm for a solution of the stochastic equation (1) considered for small times. This result is used in Theorem 2.3 to study the limit behavior of integral functionals of a solution.

Below are two main results of the paper.

Theorem 2.1. *Let conditions (i) and (ii) hold for the coefficients of equation (1). Then, with probability one, the family of random functions $\{x_h(t)\}_{h>0}$ is relatively compact in $C([0,1], d)$ and the set of limit points of this family coincides with $\mathcal{H}_R$.*

The properties of continuous mappings together with Theorem 2.1 imply the following result concerning the limit points of $F(x_h)$.

Corollary 2.2. *Let $F(\cdot)$ be a continuous on $C([0,1], d)$ functional. Then, with probability one, the set of limit points of $F(x_h)$ coincides with $F(\mathcal{H}_R)$.*

Consider the following family of stochastic processes:

$$\xi_h(t) = \frac{\frac{1}{h} \int_0^{th} x(s)\, ds - t\, x_0}{\sqrt{h}\varphi_h}, \qquad t \in [0,1].$$

Theorem 2.3. *With probability one, the family of random functions $\{\xi_h(t)\}_{h>0}$ is relatively compact in $C([0,1], d)$ and the set of its limit points coincides with*

$$\widetilde{\mathcal{H}}_R = \left\{ \tilde{g}(t) = \int_0^t g(s)\, ds \colon g(s) = \sigma_0 f(s), f \in \mathcal{K}_R, t \in [0,1] \right\}.$$

Corollary 2.4. *For $d = 1$,*

$$\overline{\lim_{h \to 0}} \, \frac{\frac{1}{h} \int_0^h x(s)\, ds - t\, x_0}{\sqrt{h}\varphi_h} = \frac{|\sigma_0|R}{\sqrt{3}},$$

$$\underline{\lim_{h \to 0}} \, \frac{\frac{1}{h} \int_0^h x(s)\, ds - t\, x_0}{\sqrt{h}\varphi_h} = -\frac{|\sigma_0|R}{\sqrt{3}}.$$

3. Auxiliary results

In this section, we prove some auxiliary results needed in the proof of the main results of this paper.

We say that the large deviations principle holds for a family of probability measures $\{\mathsf{P}_h\}_{h>0}$ defined in $(C([0,1], d), \mathcal{B})$ with a normalizing function φ_h^2 and a rate functional $I(f)$ if

1) for all open sets $G \subset \mathcal{B}$,

$$\underline{\lim_{h \to 0}} \, \varphi_h^{-2} \ln \mathsf{P}_h(G) \geq -\inf\{I(f); f \in G\};$$

2) for all closed sets $U \subset \mathcal{B}$,

$$\overline{\lim_{h \to 0}} \, \varphi_h^{-2} \ln \mathsf{P}_h(U) \leq -\inf\{I(f); f \in U\};$$

3) for all $a < \infty$, the set $\{\psi \colon I(\psi) \leq a\}$ is compact;

see [3, Chapter 3].

Lemma 3.1. *Let conditions (i) and (ii) hold for the coefficients of equation (1). Then the large deviations principle holds for a family of probability measures $\{\mathsf{P}_h\}_{h>0}$ as $h \downarrow 0$ with the normalizing function $\varphi_h \in \Phi$ and the following rate functional:*

$$I(f) = \begin{cases} \frac{1}{2} \int_0^1 \left(a_0^{-1} \dot{f}(t), \dot{f}(t) \right) dt, & \text{if } f(t) \text{ is absolutely continuous and } f(0) = 0; \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\mathsf{P}_h(A) = \mathsf{P}\{x_h(\cdot) \in A\}$ for $A \in \mathcal{B}$.

Proof. We rewrite representation (4) for the process $x_h(t)$ and obtain the following equation:

$$x_h(t) = \int_0^t b_h(s, x_h(s))\, ds + \frac{1}{\varphi_h} \int_0^t \sigma_h(h, x_h(s))\, dw_h(s),$$

where

$$b_h(s, x) = \sqrt{h}\varphi_h^{-1} b\left(sh, x\sqrt{h}\varphi_h + x_0\right) \quad \text{and} \quad \sigma_h(s, x) = \sigma\left(sh, x\sqrt{h}\varphi_h + x_0\right).$$

Put $a_h(s, x) = \sigma_h(s, x)\sigma_h^*(s, x)$. Under condition (i),

$$\lim_{h \to 0} b_h(s, x) = 0 \quad \text{and} \quad \lim_{h \to 0} a_h(s, x) = a_0$$

uniformly in (s, x) belonging to an arbitrary compact set. Thus the proof is completed by [2, Theorem 3.2.1 and Corollary 2]. $\qquad\square$

The following result is a multidimensional analogue of a corollary to Theorem 5 in Section 2 of Chapter 3 in [7]. The changes in the proof are obvious.

Lemma 3.2. *Let $w(t)$ be an n-dimensional Wiener process, $f(t)$ be a $d \times n$ matrix function whose entries are bounded by a constant K. We also assume that $f(t)$ is $\mathfrak{F}_t$-adapted. Then*

$$(5) \qquad \mathsf{P}\left\{ \sup_{t \in [0,1]} \left| \int_0^t f(s) \, dw(s) \right| \geq \delta \right\} \leq 2d \exp\left\{ -\frac{\delta^2}{2nK^2} \right\}$$

for all $\delta > 0$.

Lemma 3.3. *Assume that conditions (i) and (ii) hold. Then, for every $\delta > 0$, there exists h_0 such that*

$$(6) \qquad \mathsf{P}\left\{ \sup_{t \in [0,1]} |x(th) - x_0| \geq \delta \right\} \leq 2d \exp\left\{ -\frac{\delta^2}{8dnK^2h} \right\}$$

for all $h < h_0$.

Proof. It follows from (3) that

$$(7) \qquad \begin{aligned} \mathsf{P}\left\{ \sup_{t \in [0,1]} |x(th) - x_0| \geq \delta \right\} &\leq \mathsf{P}\left\{ h \sup_{t \in [0,1]} \left| \int_0^t b(sh, x(sh)) \, ds \right| \geq \frac{\delta}{2} \right\} \\ &+ \mathsf{P}\left\{ \sqrt{h} \sup_{t \in [0,1]} \left| \int_0^t \sigma(sh, x(sh)) \, dw_h(s) \right| \geq \frac{\delta}{2} \right\}. \end{aligned}$$

Since the coefficient $b(t, x)$ is bounded, the first term on the right hand side of (7) vanishes if h is sufficiently small. The second term on the right hand side of (7) is estimated by Lemma 3.2. The proof is completed. $\qquad\square$

Lemma 3.4. *Assume that conditions (i) and (ii) hold. For all $L > 0$ and $\delta > 0$, there exists h_0 such that*

$$(8) \qquad \mathsf{P}\left\{ \sup_{t \in [0,1]} \left| \int_0^t (\sigma(sh, x(sh)) - \sigma(0, x_0)) \, dw(s) \right| \geq \delta\varphi_h \right\} \leq 4d \exp\left\{ -L\varphi_h^2 \right\}$$

for all $h < h_0$.

Proof. According to condition (i), the functions $\sigma_{ij}(t, x)$ are continuous at the point $(0, x_0)$. Thus, given $\varepsilon > 0$, there exists $\gamma > 0$ such that

$$(9) \qquad |\sigma(s, x) - \sigma(0, x_0)| < \varepsilon$$

for $0 \leq s < \gamma$ and $|x - x_0| < \gamma$.

Further,

$$(10) \qquad \mathsf{P}\left\{ \sup_{t \in [0,1]} \left| \int_0^t (\sigma(sh, x(sh)) - \sigma(0, x_0)) \, dw(s) \right| \geq \delta\varphi_h \right\} = I_1 + I_2,$$

where

$$I_1 = \mathsf{P}\left\{\sup_{t\in[0,1]}\left|\int_0^t (\sigma(sh, x(sh)) - \sigma(0, x_0))\,dw(s)\right| \geq \delta\varphi_h,\ \sup_{t\in[0,1]}|x(th) - x_0| \geq \gamma\right\},$$

$$I_2 = \mathsf{P}\left\{\sup_{t\in[0,1]}\left|\int_0^t (\sigma(sh, x(sh)) - \sigma(0, x_0))\,dw(s)\right| \geq \delta\varphi_h,\ \sup_{t\in[0,1]}|x(th) - x_0| < \gamma\right\}.$$

The term I_1 in (10) is estimated by inequality (6) as follows:

$$(11) \qquad I_1 \leq \mathsf{P}\left\{\sup_{t\in[0,1]}|x(th) - x_0| \geq \gamma\right\} \leq 2d\exp\left\{-\frac{\gamma^2}{8dnK^2h}\right\}.$$

Since $h\varphi_h^2 \to 0$, one can find h_1 for a given $L > 0$ such that

$$\frac{\gamma^2}{8ndK^2h\varphi_h^2} \geq L$$

for $h < h_1$. This together with (11) implies that

$$(12) \qquad I_1 \leq 2d\exp\left\{-L\varphi_h^2\right\}.$$

To estimate the term I_2, we use (9) and bound (5). Then

$$(13) \qquad I_2 \leq 2d\exp\left\{-\varphi_h^2\left(\frac{\delta^2}{8nd\varepsilon^2 h}\right)\right\}.$$

For all $L > 0$, there exists h_2 such that the expression in the parentheses on the right hand side of (13) is greater than or equal to L if $h < h_2$. In this case, (14) implies that

$$(14) \qquad I_2 \leq 2d\exp\left\{-L\varphi_h^2\right\}.$$

Putting $h_0 = h_1 \wedge h_2$ in (12) and (14) we prove (8). The proof is completed. $\qquad\square$

Lemma 3.5. *For all functions $f \in \mathcal{K}_R$ and all $\delta > 0$, there exist $\gamma > 0$ and $h_0 > 0$ such that*

$$(15) \qquad \mathsf{P}\left\{\sup_{t\in[0,1]}\left|\int_0^t (\sigma(sh, x(sh)) - \sigma_0)\dot{f}(s)\,ds\right| \geq \delta,\ \sup_{t\in[0,1]}|x(th) - x_0| < \gamma\right\} = 0$$

for all $h < h_0$.

Proof. Using the γ specified for (9) we obtain

$$\delta^2 \leq \sup_{t\in[0,1]}\left|\int_0^t (\sigma(sh, x(sh)) - \sigma_0)\dot{f}(s)\,ds\right|^2 \leq \sup_{t\in[0,1]}\varepsilon^2\int_0^t |\dot{f}(s)|^2\,ds \leq \varepsilon^2 R^2.$$

The random event in the braces on the left hand side of (15) has the probability zero if $\varepsilon < \delta/R$ in (9). The lemma is proved. $\qquad\square$

Denote by $H_1([0,1], n)$ the space of absolutely continuous functions $f(t) \in E_n$ such that

$$\int_0^1 |\dot{f}(s)|^2\,ds < \infty.$$

Lemma 3.6. *The functional $I(f)$ defined in Lemma 3.1 admits the following representation:*

$$(16) \qquad I(f) = \inf_{g\in H_1([0,1],n)}\{I_0(g)\colon f(t) = \sigma_0 g(t),\ g(0) = 0\}$$

(we agree that the infimum of an empty set is equal to $+\infty$).

Proof. Consider the following stochastic processes $\xi_\varepsilon(t) = \varepsilon w(t)$ and $\eta_\varepsilon(t) = \varepsilon\sigma_0 w(t)$. It is well known that $I_0(f)$ and $I(f)$ are rate functionals for the processes $\xi_\varepsilon(t)$ and $\eta_\varepsilon(t)$, respectively (see [3, Theorem 5.3.1]). On the other hand, $\eta_\varepsilon(t) = F(\xi_\varepsilon(t))$, where

$$F(x) = \sigma_0 x.$$

Representation (16) follows from the contraction principle; see [3, Theorem 3.3.1]. The lemma is proved. $\qquad\square$

The following result for $\varphi_h = \sqrt{2\ln\ln\frac{1}{h}}$ is Theorem 1 of [4]. The proof for the case considered in the current paper is given in [6, Theorem 2].

Lemma 3.7. *With probability one, the set of limit points, as $h \to 0$, of the family*

$$\left\{ \frac{w(th)}{\sqrt{h}\varphi_h} \right\}$$

coincides with the set $\mathcal{K}_R$.

4. Proof of the main results

In this section, we prove the main results of the paper.

Proof of Theorem 2.1. We split the proof of this result into the three standard steps. Put

$$\mathcal{L}_R = \left\{ f \colon f(0) = 0, I(f) \leq \frac{R^2}{2} \right\}.$$

Lemma 3.6 implies that if $f \in \mathcal{L}_R$, then a function $g \in \mathcal{K}_R$ exists such that $f(t) = \sigma_0 g(t)$, that is, $f \in \mathcal{H}_R$, whence

$$\mathcal{L}_R \subseteq \mathcal{H}_R. \tag{17}$$

Set $h_k = [c^k]^{-1}$ and $z_k(t) = x_{h_k}(t)$, where $c > 1$ is a given number.

Step 1. We show that, for all $c > 1$, $\delta > 0$, and $R^2 < \infty$, an integer number $k_0 > 0$ exists such that

$$\rho(z_k, \mathcal{H}_R) < \delta \tag{18}$$

with probability one for all $k > k_0$. Put $B_\delta = \{f \colon \rho(f, \mathcal{L}_R) \geq \delta\}$. Note that a number $\delta > 0$ exists such that

$$\inf_{f \in B_\delta} I(f) \geq \frac{R^2(\varphi)}{2} + \delta.$$

Since the large deviation principle holds for $z_h(t)$,

$$\ln \mathsf{P}\{z_k \in B_\delta\} \leq -\varphi^2\left([c^k]\right) \inf_{f \in B_\delta} I(f)$$

for a closed set B_δ and for sufficiently large k. Then

$$\mathsf{P}\{z_k \in B_\delta\} \leq \exp\left\{ -\varphi^2\left([c^k]\right)\left(\frac{R^2(\varphi)}{2} + \delta \right) \right\}.$$

Recalling the definition of the number $R^2(\varphi)$ we deduce from the Borel–Cantelli lemma that, with probability one,

$$\rho(z_k, \mathcal{L}_R) < \delta \tag{19}$$

for sufficiently large k. Now (17) and (19) imply (18).

Step 2. Now we show that each limit point of the sequence x_h belongs to $\mathcal{H}_R$. This is already proved for $h_k = [c^k]^{-1}$ in Step 1. Let $h \in [h_{k+1}, h_k]$.

Since $\sqrt{h}\varphi(h)$ is an increasing function,

$$(20) \qquad \frac{1}{\sqrt{h}\varphi(h)} = \frac{\alpha(h,k)}{\sqrt{h_k}\varphi_{h_k}} + \frac{\beta(h,k)}{\sqrt{h_{k+1}}\varphi_{h_{k+1}}},$$

where $\alpha(h,k) > 0$ and $\beta(h,k) > 0$ are such that $\alpha(h,k) + \beta(h,k) = 1$. Let

$$\hat{x}_{h,k}(t) = \alpha(h,k)z_k(t) + \beta(h,k)z_{k+1}(t).$$

Then $\hat{x}_{h,k} \in \{f\colon \rho(f, \mathcal{H}_R) < \delta\}$ for sufficiently large k, since $\alpha f(t) + \beta g(t) \in \mathcal{L}_R$ for $\alpha > 0$ and $\beta > 0$ such that $\alpha + \beta = 1$, provided that $f(t), g(t) \in \mathcal{L}_R$.

The goal of Step 2 is achieved if we show that, for all $\delta > 0$, there exist a number $c_\delta > 1$ and an integer k_0 such that

$$(21) \qquad \sup_{t\in[0,1],h\in[h_{k+1},h_k]} |x_h(t) - \hat{x}_{h,k}(t)| \le \delta$$

for all $k > k_0$ and $c > c_\delta$. Recalling the definition of $z_k(t)$ and accounting equality (20) we obtain

$$x_h(t) = z_k\left(t\frac{h}{h_k}\right)\frac{\sqrt{h_k}\varphi_{h_k}}{\sqrt{h}\varphi_h} = z_k\left(t\frac{h}{h_k}\right)\alpha(h,k) + z_{k+1}\left(t\frac{h}{h_{k+1}}\right)\beta(h,k).$$

Then

$$\sup_{t\in[0,1],h\in[h_{k+1},h_k]} |x_h(t) - \hat{x}_{h,k}(t)| \le \sup_{t\in[0,1],s\in[t/c,t]} |z_k(t) - z_k(s)|$$
$$+ \sup_{t\in[0,1],\ s\in[t,tc\wedge 1]} |z_{k+1}(t) - z_{k+1}(s)|.$$

Now we derive the inequality

$$\mathsf{P}\left\{\sup_{t\in[0,1],\ h\in[h_{k+1},h_k]} |x_h(t) - \hat{x}_{h,k}(t)| \ge \delta\right\}$$
$$\le \mathsf{P}\left\{\sup_{t\in[0,1],\ s\in[t/c,t]} |z_k(t) - z_k(s)| \ge \frac{\delta}{2}\right\}$$
$$+ \mathsf{P}\left\{\sup_{t\in[0,1],\ s\in[t,tc\wedge 1]} |z_{k+1}(t) - z_{k+1}(s)| \ge \frac{\delta}{2}\right\}.$$

We apply the large deviation principle to estimate the terms on the right hand side of the latter relation. Let

$$Q_\delta = \left\{ f \in C([0,1], n)\colon f(t) \text{ is absolutely continuous} \right.$$

$$\left. \text{and such that} \sup_{t\in[0,1],s\in[t/c,t]} |f(t) - f(s)| \ge \frac{\delta}{2} \right\}.$$

Note that the set Q_δ, $\delta > 0$, is closed. Applying the Cauchy–Schwarz inequality, we prove that

$$\frac{\delta^2}{4} \le |f(t) - f(s)|^2 \le \left|\int_s^t \dot{f}(v)\,dv\right|^2 \le (t-s)\int_s^t |\dot{f}(v)|^2 dv \le \frac{2(c-1)}{\lambda}I(f),$$

whence

$$I(f) \ge \frac{\lambda\delta^2}{8(c-1)}$$

for $f \in Q_\delta$. If $c_\delta = 1 + \lambda \delta^2 / (4R^2)$, then

$$(22) \qquad \mathsf{P}\left\{\sup_{t \in [0,1],\, s \in [t/c, t]} |z_k(t) - z_k(s)| \geq \frac{\delta}{2}\right\} = \mathsf{P}\{z_k \in Q_\delta\} \leq \exp\left\{-\frac{R^2 \varphi_{h_k}^2}{2}\right\}$$

for $c > c_\delta$.

Reasoning similarly we prove that

$$(23) \qquad \mathsf{P}\left\{\sup_{t \in [0,1],\, s \in [t, tc \wedge 1]} |z_{k+1}(t) - z_{k+1}(s)| \geq \frac{\delta}{2}\right\} \leq \exp\left\{-\frac{R^2 \varphi_{h_k}^2}{2}\right\}$$

for the same c. Now we obtain inequality (21) from relations (22) and (23) and from the Borel–Cantelli lemma.

Step 3. We show, for an arbitrary function $g(t) = \sigma_0 f(t) \in \mathcal{H}_R$, that a subsequence exists such that $x_{h_k} \to g$ almost surely. Consider the following random events:

$$C_h = \left\{\sup_{t \in [0,1]} \left|\frac{w_h(t)}{\varphi_h} - f(t)\right| \leq \varepsilon\right\}, \qquad D_h = \left\{\sup_{t \in [0,1]} |x_h(t) - g(t)| < \delta\right\}.$$

It is easy to see that

$$C_h = (C_h \cap D_h) \cup (C_h \cap \overline{D}_h) \subset D_h \cup (C_h \cap \overline{D}_h).$$

By Lemma 3.7, the events C_h occur infinitely often (i.o.) for whatever $\varepsilon > 0$. Thus

$$(24) \qquad 1 = \mathsf{P}\{C_h \text{ i.o.}\} \leq \mathsf{P}\{D_h \text{ i.o.}\} + \mathsf{P}\{C_h \cap \overline{D}_h \text{ i.o.}\}.$$

Now we show that a sequence h_k exists such that the second term on the right hand side of inequality (24) equals zero. We have

$$\mathsf{P}\{C_h \cap \overline{D}_h\} = \mathsf{P}\left\{\sup_{t \in [0,1]} \left|\frac{\sqrt{h}}{\varphi_h} \int_0^t b(sh, x(sh))\, ds \right.\right.$$

$$+ \int_0^t \sigma(sh, x(sh)) d\left(\frac{w_h(s)}{\varphi_h} - f(s)\right)$$

$$\left.+ \int_0^t (\sigma(sh, x(sh) - \sigma_0)) \dot{f}(s)\, ds\right| \geq \delta;$$

$$\left.\sup_{t \in [0,1]} \left|\frac{w_h(t)}{\varphi_h} - f(t)\right| \leq \varepsilon\right\}$$

$$(25) \qquad \leq \mathsf{P}\left\{\sup_{t \in [0,1]} \left|\frac{\sqrt{h}}{\varphi_h} \int_0^t b(sh, x(sh))\, ds\right| \geq \frac{\delta}{3}\right\}$$

$$+ \mathsf{P}\left\{\sup_{t \in [0,1]} \left|\int_0^t (\sigma(sh, x(sh)) - \sigma_0) \dot{f}(s)\, ds\right| \geq \frac{\delta}{3}\right\}$$

$$+ \mathsf{P}\left\{\sup_{t \in [0,1]} \left|\int_0^t \sigma(sh, x(sh)) d\left(\frac{w_h(s)}{\varphi_h} - f(s)\right)\right| \geq \frac{\delta}{3};\right.$$

$$\left.\sup_{t \in [0,1]} \left|\frac{w_h}{\varphi_h} - f\right| \leq \varepsilon\right\}$$

$$= P_1 + P_2 + P_3.$$

Since $b(t,x)$ is a bounded function, there exists h_0 such that

$$(26) \qquad P_1 = \mathsf{P}\left\{\sup_{t\in[0,1]}\left|\frac{\sqrt{h}}{\varphi_h}\int_0^t b(sh, x(sh))\,ds\right| \geq \frac{\delta}{3}\right\} = 0$$

for all $0 < h < h_0$. Further,

$$P_2 = \mathsf{P}\left\{\sup_{t\in[0,1]}\left|\int_0^t (\sigma(sh, x(sh)) - \sigma(0, x_0))\dot{f}(s)\,ds\right| \geq \frac{\delta}{3}\right\}$$

$$= \mathsf{P}\left\{\sup_{t\in[0,1]}\left|\int_0^t (\sigma(sh, x(sh)) - \sigma(0, x_0))\dot{f}(s)\,ds\right| \geq \frac{\delta}{3}; \sup_{t\in[0,1]}|x(ht) - x_0| \geq \gamma\right\}$$

$$+ \mathsf{P}\left\{\sup_{t\in[0,1]}\left|\int_0^t (\sigma(sh, x(sh)) - \sigma(0, x_0))\dot{f}(s)\,ds\right| \geq \frac{\delta}{3}; \sup_{t\in[0,1]}|x(ht) - x_0| < \gamma\right\}$$

$$= P_4 + P_5.$$

We pick up a number γ such that $P_5 = 0$ by Lemma 3.5. Now Lemma 3.3 with such a number γ yields

$$P_4 \leq \mathsf{P}\left\{\sup_{t\in[0,1]}|x(ht) - x_0| \geq \gamma\right\} \leq 2d\exp\left\{-\frac{\gamma^2}{8nK^2dh}\right\}.$$

Choosing h_0 in such a way that

$$-\frac{\gamma^2}{8nK^2dh} \leq -R^2\varphi_h^2$$

for $h < h_0$ we get

$$(27) \qquad P_2 \leq 2d\exp\left\{-\frac{R^2\varphi_h^2}{2}\right\}.$$

Now we estimate the probability P_3 in (25). Let

$$G_h = \exp\left\{\varphi_h\int_0^1 \dot{f}(s)\,dw_h(s) - \frac{\varphi_h^2}{2}\int_0^1 |\dot{f}(s)|^2\,ds\right\}$$

and let $\widetilde{\mathsf{P}}_h$ be a probability measure on (Ω, F) possessing a density which we denote by G_h. By Girsanov's theorem, $\widetilde{w}_h(t) = w_h(t) - \varphi_h f(t)$ is a Wiener process on $(\Omega, \mathcal{F}, \widetilde{\mathsf{P}}_h)$.

Put

$$S_h = \left\{\sup_{t\in[0,1]}\left|\int_0^t \sigma(sh, x(sh))\,d\widetilde{w}_h(s)\right| \geq \frac{\delta\varphi_h}{3}; \sup_{t\in[0,1]}|\widetilde{w}_h(t)| \leq \varepsilon\varphi_h\right\}.$$

By $\widetilde{\mathsf{E}}_h$ we denote the expectation with respect to the measure $\widetilde{\mathsf{P}}_h$. Then

$$(28) \qquad P_3 = \widetilde{\mathsf{E}}_h I(S_h)G_h^{-1} \leq \left(\widetilde{\mathsf{P}}_h(S_h)\right)^{\frac{1}{2}}\left(\widetilde{\mathsf{E}}_h G_h^{-2}\right)^{\frac{1}{2}}.$$

Further,

$$(29) \qquad \begin{aligned} \widetilde{\mathsf{P}}_h(S_h) &\leq \widetilde{\mathsf{P}}_h\left\{\sup_{t\in[0,1]}\left|\int_0^t (\sigma(sh, x(sh) - \sigma_0)\,d\widetilde{w}_h(s)\right| \geq \frac{\delta\varphi_h}{6}\right\} \\ &+ \widetilde{\mathsf{P}}_h\left\{\sup_{t\in[0,1]}|\sigma_0\widetilde{w}_h(s)| \geq \frac{\delta\varphi_h}{6}; \sup_{t\in[0,1]}|\widetilde{w}_h(t)| \leq \varepsilon\varphi_h\right\}. \end{aligned}$$

Let $\alpha = \sum_{ij}(\sigma_0)^2_{ij}$. We choose $\varepsilon < \delta/(6\alpha)$. Then the second term on the right hand side of (29) equals zero. We estimate the first term in (29) by Lemma 3.4. Therefore, for all $L > 0$ and $\delta > 0$, there exists h_0 such that

$$\widetilde{\mathsf{P}}_h(S_h) \le 4d \exp\left\{-L\varphi_h^2\right\} \tag{30}$$

for $h < h_0$. Since $f \in \mathcal{K}_R$, we obtain

$$\widetilde{\mathsf{E}}_h\left(G_h^{-1}\right)^2 = \widetilde{\mathsf{E}}_h\left(\exp\left\{-2\varphi_h \int_0^1 \dot{f}(s)\, dw_h(s) + \varphi_h^2 \int_0^1 |\dot{f}(s)|^2\, ds\right\}\right)$$

$$= \widetilde{\mathsf{E}}_h\left(\exp\left\{-2\varphi_h \int_0^1 \dot{f}(s)\, dw_h(s) - 2\varphi_h^2 \int_0^1 |\dot{f}(s)|^2\, ds\right\}\right) \tag{31}$$

$$\times \exp\left\{3\varphi_h^2 \int_0^1 |\dot{f}(s)|^2\, ds\right\}$$

$$\le \exp\left\{3R^2\varphi_h^2\right\}.$$

Relations (30) and (31) yield

$$P_3 \le \sqrt{4d}\exp\left\{-\frac{\varphi_h^2}{2}\left(L - 3R^2\right)\right\},$$

where the probability P_3 is defined by (25). Choosing $L = 4R^2$ in this inequality, we get

$$P_3 \le \exp\left\{-\frac{R^2\varphi_h^2}{2}\right\}. \tag{32}$$

Turning back to relations (25), we note by (26), (27), and (32) that a number h_0 exists such that

$$\mathsf{P}\{C_h \cap \overline{D}_h\} \le (2d+1)\exp\left\{-\frac{R^2\varphi_h^2}{2}\right\}$$

for $h < h_0$. Thus a number $c > 1$ exists such that

$$\sum_k P\{C_{h_k} \cap \overline{D}_{h_k}\} \le (2d+1)\sum_k \exp\left\{-\frac{R^2\varphi^2([c^k])}{2}\right\}$$

for the sequence $h_k = [c^k]^{-1}$.

Recalling the definition of R, we conclude that the latter series converges. Using the Borel–Cantelli lemma, we conclude that $\mathsf{P}\{C_{h_k} \cap \overline{D}_{h_k}$ i.o.$\} = 0$ for all $\delta > 0$. Inequality (24) implies that

$$\mathsf{P}\{D_{h_k}\} = \mathsf{P}\left\{\sup_{t\in[0,1]}|x_{h_k}(t) - g(t)| < \delta \text{ i.o.}\right\} = 1$$

for all $\delta > 0$. The theorem is proved. $\qquad\square$

Proof of Theorem 2.3. It remains to apply Corollary 2.2 and choose the functional

$$F(x(\cdot)) = \int_0^t x(s)\, ds. \qquad\square$$

Proof of Corollary 2.4. If $\tilde{g} \in \widetilde{\mathcal{H}}_R$, then

$$\tilde{g}^2(1) = \sigma_0^2\left(\int_0^1\int_0^t \dot{f}(s)\,ds\,dt\right)^2 = \sigma_0^2\left(\int_0^1(1-s)\dot{f}(s)\,ds\right)^2$$

$$\le \sigma_0^2\int_0^1(1-s)^2\,ds\int_0^1(\dot{f}(s))^2\,ds = \frac{\sigma_0^2}{3}\int_0^1(\dot{f}(s))^2\,ds \le \frac{\sigma_0^2 R^2}{3}.$$

This implies that

$$-\frac{|\sigma_0|R}{\sqrt{3}} \le \tilde{g}(1) \le \frac{|\sigma_0|R}{\sqrt{3}}.$$

Note that the upper and lower bounds are attained at the functions

$$f_1(t) = \frac{|\sigma_0|R\sqrt{3}}{\sigma_0}\left(t - \frac{t^2}{2}\right)$$

and

$$f_2(t) = -f_1(t),$$

respectively. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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