STABILITY OF A TANDEM OF QUEUEING SYSTEMS WITH BERNOULLI NONINSTANTANEOUS TRANSFER OF CUSTOMERS

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ABSTRACT. A tandem of queueing systems with cyclic control algorithm and with independent conflict input flows is considered. The customers served in the first queueing system are transferred to the second system with random speeds. A cybernetic approach is used to construct a mathematical model in terms of a multidimensional denumerable Markov chain. The classification of the states of this Markov chain is carried out, and a sufficient condition for the existence of the stationary distribution is obtained.

1. THE SETTING OF THE PROBLEM

Consider two queueing systems serving the customers under the following assumptions. There are four independent conflict input flows, two flows for each of the systems. Each of these flows is a nonordinary Poisson process. The flows are conflicting in the sense that customers arriving from different sources cannot be served simultaneously in the same queueing system. This means that the problem cannot be reduced to a corresponding problem with fewer input flows by merging the flows together. The customers coming from each flow are placed in a special buffer with an unbounded capacity. The service time of an arbitrary customer is random and service times of different customers are independent random variables with unknown probability distributions. Each system may serve several customers during the same working period. The service algorithm of conflict flows in each of the two systems belongs to the class of cyclic algorithms. We assume that the durations of working periods are commensurable constants.

At the moment 0, the service starts in both systems. After a service is completed in the first system for a customer coming from the first flow of the first system, he is transferred to the second system. We assume that the transfer is not instant. We further assume that the speed of the transfer for each of the customers is random and its probability distribution is unknown. Moreover, the speeds of transfer of different customers are varying with time and have different probability distributions. We agree that the transfer of every customer from the first system to the second system is completed with a known probability during a working period. Respectively, the transfer is not completed during a working period and will be continued during the next one with the complementary probability.

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The customers from the first system are placed in the queue together with the first input flow of the second system. This means that the input flow for the second queueing system has a complicated probability structure.

Finally, the customers coming from the second input flow of the first system as well as all the customers in the second system leave the tandem of queueing systems after the service.

An example of a tandem of queueing systems with a cyclic control algorithm in each of the two systems is represented by a chain of two crossroads (see Figure 1) with input traffic flows $\Pi_1, \Pi_2, \dots, \Pi_5$. The flows Π_1 and Π_4 are conflicting as well as the superposition of the flows Π_2 and Π_3 is conflicting with the flow Π_5 . The flow Π_3 is the output for the first crossroad.

FIGURE 1. Tandem of crossroads

Note that the output flow Π_3 comes from the first system that cyclically switches from the service of the flow Π_1 to blocking it and vice versa. The times when the customers leave the first system depend on the service times that are dependent and have unknown and different probability distributions.

This model does not fit the classical description of the output flow in terms of a counting process $\{\eta(t); t \geq 0\}$ with continuous time (the same concerns any equivalent classical description). Instead, we use a nonlocal description of the flows and a cybernetic approach to construct and analyze the mathematical models of controlled queueing systems [1]–[4]. The nonlocal description of a flow of customers Π means that a marked point process $\{(\tau'_i, \eta'_i, \nu'_i); i = 1, 2, \dots\}$ is given, where τ'_i is the moment of observation, $\tau'_0 = 0$, η_i is the number of customers arriving during the interval $(\tau_{i-1}, \tau_i]$, and where ν'_i is the mark of these customers.

The following are the three main principles of the cybernetic approach:

- a) the controlled queueing system operates at discrete moments τ_i , $i = 0, 1, \dots$;
- b) the nonlocal description of the structure of a controlled queueing system is given blockwise;
- c) the block structure of a controlled queueing system and its functioning in time are studied jointly.

Every controlled queueing system has its *scheme*, *information*, *coordinates*, and *function*. The scheme of a queueing system consists of the following seven blocks:

- 1) external environment,
- 2) input poles,
- 3) external memory,

- 4) a device processing the information in the external memory,
- 5) internal memory,
- 6) a device processing the information in the internal memory,
- 7) output poles.

The information in the queueing system is the set of all possible states of its blocks. A coordinate of the queueing system defines the state of a corresponding block. The function of the queueing system describes the way in which the controlled queueing system passes from one discrete time moment to another one.

In the rest of the paper we consider the input flows Π_1, Π_2, Π_3 only. An analogous approach can be applied to study all five input flows $\Pi_1, \Pi_2, \dots, \Pi_5$.

Denote by O_1 and O_2 the buffers for the customers arriving from the flow Π_1 and Π_2 , respectively. By O_3 , we denote the buffer for the flow of customers transferred from the first system to the second one.

The following notation is needed to define the input processes and service discipline. Let λ_j denote the intensity for the bulk arrival for the flow Π_j , $j = 1, 2$. Further, let $p_x^{(j)}$ be the probability that the size of a group of customers arriving from the flow Π_j is equal to x , $x = 1, 2, \dots$. We assume that the following two series

$$\sum_{x=1}^{\infty} z^x p_x^{(j)}, \quad j = 1, 2,$$

converge in the circle $|z| < 1 + \varepsilon$ for some $\varepsilon > 0$.

The duration of the working period for customers from the queue O_1 is equal to T'_1 . Similarly, T'_2 is the duration of the working period for the flow Π_4 , T'_3 is the duration of the working period for the flows Π_2 and Π_3 , and T'_4 is the duration of the working period for the flow Π_5 .

To fix a discrete time scale consider the moments $\tau_1, \tau_2, \dots$ when at least one of the servers changes its state; we also put $\tau_0 = 0$. We treat two servers as a new server with $n < \infty$ states $\Gamma^{(1)}, \Gamma^{(2)}, \dots, \Gamma^{(n)}$ and with sojourn times $T_1, T_2, \dots, T_n$. The number n and sojourn times are chosen according to T'_1, T'_2, T'_3, T'_4 and also to the initial states of servers for both queueing systems. In the state $\Gamma^{(r)}$, either none of the customers arriving from the queues O_1, O_2 is served (the state of type I), or only the customers from the queue O_1 are served (the state of type II), or only the customers from the queue O_2 are served (the state of type III), or customers from both queues O_1 and O_2 are served (the state of type IV). For example, let $T'_1 = 5, T'_2 = 4, T'_3 = 3, T'_4 = 3$ and let the service start at a moment τ_0 for flows Π_1 and Π_5 . Then $n = 8$ and

$$\begin{aligned} \tau_1 = 3, \tau_2 = 5, \tau_3 = 6, \tau_4 = 9, \tau_5 = 12, \tau_6 = 14, \tau_7 = 15, \tau_8 = 18, \tau_9 = 21, \dots; \\ T_1 = 3, T_2 = 2, T_3 = 1, T_4 = 3, T_5 = 3, T_6 = 2, T_7 = 1, T_8 = 3. \end{aligned}$$

Thus the states $\Gamma^{(4)}$ and $\Gamma^{(7)}$ are of type I, $\Gamma^{(1)}$ and $\Gamma^{(6)}$ are of type II, $\Gamma^{(3)}$ and $\Gamma^{(8)}$ are of type III, and $\Gamma^{(2)}$ and $\Gamma^{(5)}$ are of type IV.

Finally, we assume that each customer from the queue O_3 is transferred to the queue O_2 with probability p_r independently of all other customers in the queue O_3 provided the server is in the state $\Gamma^{(r)}$ during the time T_r .

Since the durations of working periods for different customers are dependent and have different probability distributions, it is convenient to describe the service process in terms of the saturation flows Π_1^{sat} and Π_2^{sat} instead of in terms of the distribution of the duration of the working period for an arbitrary customer [1]. The saturation flow is a virtual output flow corresponding to the maximal workload of the servers and highest queue loads. The saturation flow Π_1^{sat} contains 0 customers if the state of the server $\Gamma^{(r)}$ is of type I or III, while Π_1^{sat} contains a nonrandom number $\ell_{r,1} > 0$ of customers if the

state of the server $\Gamma^{(r)}$ is of type II or IV. Similarly, the saturation flow Π_2^{sat} contains 0 customers if the state $\Gamma^{(r)}$ is of type I or II and contains a nonrandom number $\ell_{r,2} > 0$ of customers if $\Gamma^{(r)}$ is of type III or IV.

The scheme of a controlled queueing system is depicted in Figure 2. The scheme contains the following blocks:

- 1) external environment,
- 2) input flows Π_1 , Π_2 , and Π_3 (input flows of the first type) and saturation flows Π_1^{sat} and Π_2^{sat} (input flows of the second type),
- 3) buffers O_1 , O_2 , and O_3 (external memory),
- 4) devices β_1 , β_2 , and β_3 processing the information (the discipline of service in queues) in the external memory,
- 5) server (internal memory),
- 6) graph of changes of states of the server (a device processing the information in the internal memory),
- 7) output flows Π_1^{out} and Π_2^{out} (output poles).

The set of all possible states of the random environment, input flows, buffers, server, saturation flows, and output flows describes the information in the controlled queueing system. The numbers corresponding to the states of the external environment, input flows, buffers, devices realizing the discipline in the queues, server, saturation flows, and output flows are the coordinates of the queueing system. The function of the system is to serve customers according to a cyclic algorithm when the order and the working period are fixed.

FIGURE 2. The scheme of the tandem of queueing systems (with cyclic service algorithm when the order and the working period are fixed) as a controlled queueing system

2. MATHEMATICAL MODEL

We assume that all random variables and other random elements are defined on a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. The sequence $\tau_0 = 0, \tau_1, \dots, \tau_i, \dots$ of observation moments is defined above. By Γ_i , we denote the state of the server at the moment τ_i and let $\kappa_{s,i}$ be the number of customers in the buffer O_s for $s = 1, 2$, and 3 at the moment τ_i . Further let $\eta_{s,i}$ and $\xi_{j,i}$ be the number of customers arriving from the flows Π_s and Π_j^{sat} ,

respectively, during the interval $(\tau_i, \tau_{i+1}]$, and let $\bar{\xi}_{1,i}$ be the number of customers from the flow Π_1^{out} whose service is completed during the interval $(\tau_i, \tau_{i+1}]$.

For r , $1 \leq r < n$, let $r \oplus 1 = r + 1$ and $n \oplus 1 = 1$. Set $\Gamma = \{\Gamma^{(1)}, \Gamma^{(2)}, \dots, \Gamma^{(n)}\}$ and introduce the mapping $u(\cdot): \Gamma \rightarrow \Gamma$ by $u(\Gamma^{(r)}) = \Gamma^{(r \oplus 1)}$. The controlled queueing system is such that for $i = 0, 1, \dots$,

$$\begin{aligned} \Gamma_{i+1} &= u(\Gamma_i), & \kappa_{1,i+1} &= \max\{0, \kappa_{1,i} + \eta_{1,i} - \xi_{1,i}\}, \\ \kappa_{2,i+1} &= \max\{0, \kappa_{2,i} + \eta_{2,i} + \eta_{3,i} - \xi_{2,i}\}, & \kappa_{3,i+1} &= \kappa_{3,i} + \bar{\xi}_{1,i} - \eta_{3,i}, \\ \bar{\xi}_{1,i+1} &= \min\{\xi_{1,i}, \kappa_{1,i} + \eta_{1,i}\}. \end{aligned}$$

The nonlocal description of the input flows as well as that of the saturation flows is given with the help of conditional distributions of the discrete components

$$\{(\eta_{1,i}, \eta_{2,i}); i = 0, 1, \dots\}, \quad \{(\xi_{1,i}, \xi_{2,i}); i = 0, 1, \dots\}, \quad \{\eta_{3,i}; i = 0, 1, \dots\}$$

of the marked point processes

$$\begin{aligned} \{(\tau_i, \eta_{1,i}, \eta_{2,i}, \nu_i); i = 0, 1, \dots\}, & \quad \{(\tau_i, \xi_{1,i}, \xi_{2,i}, \nu_i); i = 0, 1, \dots\}, \\ \{(\tau_i, \eta_{3,i}, \bar{\nu}_i); i = 0, 1, \dots\}, \end{aligned}$$

respectively. Here $\nu_i = \Gamma_i$ and $\bar{\nu}_i = (\Gamma_i, \kappa_{3,i})$ are the marks of customers from the corresponding flows arriving to the system during the interval $(\tau_i, \tau_{i+1}]$.

For $x \geq 0$, $0 \leq k \leq x$, and $0 < \alpha < 1$ let

$$\psi(k; x, \alpha) = \binom{x}{k} \alpha^k (1 - \alpha)^{x-k}, \quad \Gamma = \Gamma^{\text{I}} \cup \Gamma^{\text{II}} \cup \Gamma^{\text{III}} \cup \Gamma^{\text{IV}},$$

where Γ^{I} , Γ^{II} , Γ^{III} , and Γ^{IV} are the sets of states of types I, II, III, and IV, respectively. Now we introduce $\varphi_j(x, T)$, $T > 0$, $x = 0, 1, \dots$, $j = 1, 2$, as follows:

$$\sum_{x=0}^{\infty} z^x \varphi_j(x, T) = \exp \left\{ \lambda_j T \left(\sum_{x=1}^{\infty} z^x p_x^{(j)} - 1 \right) \right\}, \quad |z| < 1.$$

Then

$$\begin{aligned} \mathbb{P}(\{\omega: \eta_{j,i} = b\} \mid \{\omega: \Gamma_i = \Gamma^{(r)}\}) &= \varphi_j(b; T_{r \oplus 1}) \quad \text{if } b = 0, 1, \dots, \\ \mathbb{P}(\{\omega: \xi_{1,i} = 0\} \mid \{\omega: \Gamma_i = \Gamma^{(r)}\}) &= 1 \quad \text{if } \Gamma^{(r \oplus 1)} \in \Gamma^{\text{I}} \cup \Gamma^{\text{III}}, \\ \mathbb{P}(\{\omega: \xi_{1,i} = \ell_{r \oplus 1, 1}\} \mid \{\omega: \Gamma_i = \Gamma^{(r)}\}) &= 1 \quad \text{if } \Gamma^{(r \oplus 1)} \in \Gamma^{\text{II}} \cup \Gamma^{\text{IV}}, \\ \mathbb{P}(\{\omega: \xi_{2,i} = 0\} \mid \{\omega: \Gamma_i = \Gamma^{(r)}\}) &= 1 \quad \text{if } \Gamma^{(r \oplus 1)} \in \Gamma^{\text{I}} \cup \Gamma^{\text{II}}, \\ \mathbb{P}(\{\omega: \xi_{2,i} = \ell_{r \oplus 1, 2}\} \mid \{\omega: \Gamma_i = \Gamma^{(r)}\}) &= 1 \quad \text{if } \Gamma^{(r \oplus 1)} \in \Gamma^{\text{III}} \cup \Gamma^{\text{IV}}, \\ \mathbb{P}(\{\omega: \eta_{3,i} = b\} \mid \{\omega: \Gamma_i = \Gamma^{(r)}, \kappa_{3,i} = x\}) &= \psi(b; x, p_{r \oplus 1}) \quad \text{if } 0 \leq b \leq x. \end{aligned}$$

For nonnegative integers $i, x_1, x_2, x_3, b_1, b_2, b_3, y_1, y_2$ and $r = 1, 2, \dots, n$, we introduce the random events

$$\begin{aligned} A_i(r, x_1, x_2, x_3) &= \{\omega: \Gamma_i = \Gamma^{(r)}, \kappa_{1,i} = x_1, \kappa_{2,i} = x_2, \kappa_{3,i} = x_3\}, \\ B_i(b_1, b_2, b_3, y_1, y_2) &= \{\omega: \eta_{1,i} = b_1, \eta_{2,i} = b_2, \eta_{3,i} = b_3, \xi_{1,i} = y_1, \xi_{2,i} = y_2\}. \end{aligned}$$

Finally let

$$\begin{aligned}
 (1) \quad & \mathbb{P} \left(B_i(b_1, b_2, b_3, y_1, y_2) \left| \bigcap_{\bar{i}=0}^i A_{\bar{i}}(r_{\bar{i}}, x_{1,\bar{i}}, x_{2,\bar{i}}, x_{3,\bar{i}}) \right. \right) \\
 &= \mathbb{P}(\{\omega: \eta_{1,i} = b_1\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}\}) \mathbb{P}(\{\omega: \eta_{2,i} = b_2\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}\}) \\
 &\quad \times \mathbb{P}(\{\omega: \eta_{3,i} = b_3\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}, \kappa_{3,i} = x_i\}) \\
 &\quad \times \mathbb{P}(\{\omega: \xi_{1,i} = y_1\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}\}) \mathbb{P}(\{\omega: \xi_{2,i} = y_2\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}\})
 \end{aligned}$$

for all nonnegative integers

$$i, b_1, b_2, b_3, y_1, y_2, x_{1,0}, x_{2,0}, x_{3,0}, x_{1,1}, x_{2,1}, x_{3,1}, \dots, x_{1,i}, x_{2,i}, x_{3,i}$$

and arbitrary states $\Gamma^{(r_0)} \in \Gamma, \Gamma^{(r_1)} \in \Gamma, \dots, \Gamma^{(r_i)} \in \Gamma$.

Theorem 1. *Given the probability distribution of the random vector*

$$(\Gamma_0, \kappa_{1,0}, \kappa_{2,0}, \kappa_{3,0}),$$

the sequence

$$(2) \quad \{(\Gamma_i, \kappa_{1,i}, \kappa_{2,i}, \kappa_{3,i}); i = 0, 1, \dots\}$$

is a Markov chain.

Proof. The law of total probability and assumption (1) imply that

$$\begin{aligned}
 & \mathbb{P}(A_{i+1}(r_i \oplus 1, x_{1,i+1}, x_{2,i+1}, x_{3,i+1}) \mid W(i)) \\
 &= \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \sum_{b_3=0}^{\infty} \sum_{y_1=0}^{\infty} \sum_{y_2=0}^{\infty} \mathbb{P}(B_i(b_1, b_2, b_3, y_1, y_2) \mid W(i)) \\
 &\quad \times \mathbb{P}(A_{i+1}(r_i \oplus 1, x_{1,i+1}, x_{2,i+1}, x_{3,i+1}) \mid W(i) \cap B_i(b_1, b_2, b_3, y_1, y_2)) \\
 &= \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \sum_{b_3=0}^{\infty} \sum_{y_1=0}^{\infty} \sum_{y_2=0}^{\infty} \mathbb{P}(\{\omega: \eta_{1,i} = b_1\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}\}) \\
 &\quad \times \mathbb{P}(\{\omega: \eta_{2,i} = b_2\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}\}) \\
 &\quad \times \mathbb{P}(\{\omega: \eta_{3,i} = b_3\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}, \kappa_{3,i} = x_i\}) \\
 &\quad \times \mathbb{P}(\{\omega: \xi_{1,i} = y_1\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}\}) \mathbb{P}(\{\omega: \xi_{2,i} = y_2\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}\}) \\
 &\quad \times \mathbb{P}(\{\omega: x_{1,i+1} = \max\{0, x_{1,i} + b_1 - y_1\}, x_{2,i+1} = \max\{0, x_{2,i} + b_2 + b_3 - y_2\}, \\
 &\quad \quad \quad x_{3,i+1} = x_{3,i} + \min\{y_1, x_{1,i} + b_1 - b_3\}\})
 \end{aligned}$$

for all $\Gamma^{(r_0)} \in \Gamma, \Gamma^{(r_1)} \in \Gamma, \dots, \Gamma^{(r_i)} \in \Gamma$ and all nonnegative numbers

$$x_{1,0}, x_{2,0}, x_{3,0}, x_{1,1}, x_{2,1}, x_{3,1}, \dots, x_{1,i+1}, x_{2,i+1}, x_{3,i+1},$$

where

$$W(i) = \bigcap_{\bar{i}=0}^i A_{\bar{i}}(r_{\bar{i}}, x_{1,\bar{i}}, x_{2,\bar{i}}, x_{3,\bar{i}}).$$

Similarly

$$\begin{aligned}
 & P(A_{i+1}(r_i \oplus 1, x_{1,i+1}, x_{2,i+1}, x_{3,i+1}) \mid A_i(r_i, x_{1,i}, x_{2,i}, x_{3,i})) \\
 &= \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \sum_{b_3=0}^{\infty} \sum_{y_1=0}^{\infty} \sum_{y_2=0}^{\infty} P(\{\omega: \eta_{1,i} = b_1\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}\}) \\
 &\quad \times P(\{\omega: \eta_{2,i} = b_2\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}\}) \\
 (3) \quad &\quad \times P(\{\omega: \eta_{3,i} = b_3\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}, \kappa_{3,i} = x_i\}) \\
 &\quad \times P(\{\omega: \xi_{1,i} = y_1\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}\}) P(\{\omega: \xi_{2,i} = y_2\} \mid \{\omega: \Gamma_i = \Gamma^{(r_i)}\}) \\
 &\quad \times P(\{\omega: x_{1,i+1} = \max\{0, x_{1,i} + b_1 - y_1\}, \\
 &\quad \quad x_{2,i+1} = \max\{0, x_{2,i} + b_2 + b_3 - y_2\}, \\
 &\quad \quad x_{3,i+1} = x_{3,i} + \min\{y_1, x_{1,i} + b_1\} - b_3\}).
 \end{aligned}$$

The right hand sides of the latter two equations coincide, whence we conclude that the conditional probabilities on the left hand sides coincide, too. $\square$

Put $Q_i(r; x_1, x_2, x_3) = P(A_i(r, x_1, x_2, x_3))$. The symbols $w_1, w_2, w_3, x_1, x_2, x_3, b_1, b_2$, and c denote nonnegative integers. From (3), we obtain

$$Q_{i+1}(r \oplus 1; w_1, w_2, w_3) = \sum Q_i(r; x_1, x_2, x_3) \varphi_1(b_1; T_{r \oplus 1}) \varphi_2(b_2; T_{r \oplus 1}) \psi(c; x_3, p_{r \oplus 1}),$$

where the summation is taken over all nonnegative integers x_1, x_2, x_3, b_1, b_2 , and c such that

$$\begin{aligned}
 w_1 &= \max\{0, x_1 + b_1 - y_1\}, \quad w_2 = \max\{0, x_2 + b_2 + c - y_2\}, \\
 (4) \quad w_3 &= x_3 + \min\{y_1, x_1 + b_1\} - c,
 \end{aligned}$$

and $c \leq x_3$. The number y_j above is equal to 0 if the customers from the queue O_j are not served in the state $\Gamma^{(r \oplus 1)}$, while $y_j = \ell_{r \oplus 1, j}$ otherwise.

Now let $\Gamma^{(r \oplus 1)} \in \Gamma^I$. Then $y_1 = y_2 = 0$ and the system of equations (4) reduces to

$$w_1 = x_1 + b_1, \quad w_2 = x_2 + b_2 + c, \quad w_3 = x_3 - c$$

for $w_1 \geq 0, w_2 \geq 0$, and $w_3 \geq 0$. The solution of this system is given by

$$b_1 = w_1 - x_1, \quad b_2 = w_2 - c - x_2, \quad x_3 = w_3 + c$$

with $0 \leq x_1 \leq w_1, 0 \leq c \leq w_2$, and $0 \leq x_2 \leq w_2 - c$. Therefore

$$\begin{aligned}
 & Q_{i+1}(r \oplus 1; w_1, w_2, w_3) \\
 (5) \quad &= \sum_{x_1=0}^{w_1} \sum_{c=0}^{w_2} \sum_{x_2=0}^{w_2-c} Q_i(r; x_1, x_2, w_3 + c) \\
 &\quad \times \varphi_1(w_1 - x_1; T_{r \oplus 1}) \varphi_2(w_2 - c - x_2; T_{r \oplus 1}) \psi(c; w_3 + c, p_{r \oplus 1}).
 \end{aligned}$$

Similar reasoning for $\Gamma^{(r \oplus 1)} \in \Gamma^{II}$ proves that if $w_3 \geq 0$,

$$\begin{aligned}
 & Q_{i+1}(r \oplus 1; 0, 0, w_3) \\
 (6) \quad &= \sum_{x_1=0}^{\min\{w_3, \ell_{r \oplus 1, 1}\}} \sum_{b_1=0}^{\min\{w_3, \ell_{r \oplus 1, 1}\} - x_1} Q_i(r; x_1, 0, w_3 - x_1 - b_1) \\
 &\quad \times \varphi_1(b_1; T_{r \oplus 1}) \varphi_2(0; T_{r \oplus 1}) \psi(0; w_3 - x_1 - b_1, p_{r \oplus 1});
 \end{aligned}$$

if $w_2 \geq 1$,

$$(7) \quad \begin{aligned} & Q_{i+1}(r \oplus 1; 0, w_2, 0) \\ &= \sum_{c=0}^{w_2} \sum_{x_2=0}^{w_2-c} \varphi_1(0; T_{r \oplus 1}) \varphi_2(w_2 - x_2 - c; T_{r \oplus 1}) \psi(c; c, p_{r \oplus 1}) Q_i(r; 0, x_2, c); \end{aligned}$$

if $w_2 \geq 1$ and $w_3 \geq 1$,

$$(8) \quad \begin{aligned} & Q_{i+1}(r \oplus 1; 0, w_2, w_3) \\ &= \sum_{c=0}^{w_2} \sum_{x_2=0}^{w_2-c} \sum_{x_1=0}^{\min\{w_3+c, \ell_{r \oplus 1, 1}\}} \sum_{b_1=0}^{\min\{w_3+c, \ell_{r \oplus 1, 1}\}-x_1} \varphi_1(b_1; T_{r \oplus 1}) \\ & \quad \times Q_i(r; x_1, x_2, w_3 + c - x_1 - b_1) \varphi_2(w_2 - c - x_2; T_{r \oplus 1}) \\ & \quad \times \psi(c; w_3 + c - x_1 - b_1, p_{r \oplus 1}); \end{aligned}$$

and if $w_1 \geq 1$, $w_2 \geq 0$, and $w_3 \geq \ell_{r \oplus 1, 1}$,

$$(9) \quad \begin{aligned} & Q_{i+1}(r \oplus 1; w_1, w_2, w_3) \\ &= \sum_{x_1=0}^{w_1+\ell_{r \oplus 1, 1}} \sum_{c=0}^{w_2} \sum_{x_2=0}^{w_2-c} Q_i(r; x_1, x_2, c + w_3 - \ell_{r \oplus 1, 1}) \\ & \quad \times \varphi_1(w_1 + \ell_{r \oplus 1, 1} - x_1; T_{r \oplus 1}) \varphi_2(w_2 - c - x_2; T_{r \oplus 1}) \\ & \quad \times \psi(c; c + w_3 - \ell_{r \oplus 1, 1}, p_{r \oplus 1}). \end{aligned}$$

In the case $\Gamma^{(r \oplus 1)} \in \Gamma^{\text{III}}$,

$$(10) \quad \begin{aligned} & Q_{i+1}(r \oplus 1; w_1, 0, w_3) \\ &= \sum_{x_1=0}^{w_1} \sum_{x_2=0}^{\ell_{r \oplus 1, 2}} \sum_{c=0}^{\ell_{r \oplus 1, 2}-x_2} \sum_{b_2=0}^{\ell_{r \oplus 1, 2}-x_2-c} \varphi_1(w_1 - x_1; T_{r \oplus 1}) \\ & \quad \times Q_i(r; x_1, x_2, w_3 + c) \varphi_2(b_2; T_{r \oplus 1}) \psi(c; w_3 + c, p_{r \oplus 1}) \end{aligned}$$

if $w_1 \geq 0$ and $w_3 \geq 0$, and

$$(11) \quad \begin{aligned} & Q_{i+1}(r \oplus 1; w_1, w_2, w_3) \\ &= \sum_{x_1=0}^{w_1} \sum_{c=0}^{w_2+\ell_{r \oplus 1, 2}} \sum_{x_2=0}^{w_2+\ell_{r \oplus 1, 2}-c} Q_i(r; x_1, x_2, w_3 + c) \varphi_1(w_1 - x_1; T_{r \oplus 1}) \\ & \quad \times \varphi_2(w_2 + \ell_{r \oplus 1, 2} - x_2 - c; T_{r \oplus 1}) \psi(c; w_3 + x, p_{r \oplus 1}) \end{aligned}$$

if $w_1 \geq 0$, $w_2 \geq 1$, and $w_3 \geq 0$.

In the case $\Gamma^{(r \oplus 1)} \in \Gamma^{\text{IV}}$, for $w_3 \geq 0$,

$$(12) \quad \begin{aligned} & Q_{i+1}(r \oplus 1; 0, 0, w_3) \\ &= \sum_{x_1=0}^{\min\{w_3, \ell_{r \oplus 1, 1}\}} \sum_{b_1=0}^{\min\{w_3, \ell_{r \oplus 1, 1}\}-x_1} \sum_{c=0}^{\ell_{r \oplus 1, 2}} \sum_{x_2=0}^{\ell_{r \oplus 1, 2}-c} \sum_{b_2=0}^{\ell_{r \oplus 1, 2}-c-x_2} \varphi_1(b_1; T_{r \oplus 1}) \\ & \quad \times Q_i(r; x_1, x_2, w_3 + c - x_1 - b_1) \varphi_2(b_2; T_{r \oplus 1}) \psi(c; w_3 + c - x_1 - b_1, p_{r \oplus 1}); \end{aligned}$$

for $w_2 \geq 1$ and $w_3 \geq 0$,

$$\begin{aligned}
 & Q_{i+1}(\Gamma^{(r \oplus 1)}; 0, w_2, w_3) \\
 (13) \quad &= \sum_{x_1=0}^{\min\{\ell_{r \oplus 1,1}, w_3\}} \sum_{b_1=0}^{\min\{\ell_{r \oplus 1,1}, w_3\} - x_1} \sum_{c=0}^{w_2 + \ell_{r \oplus 1,2} - x_1 - b_1} \sum_{x_2=0}^{\ell_{r \oplus 1,2} - c} \varphi_1(b_1; T_{r \oplus 1}) \\
 & \quad \times Q_i(r; x_1, x_2, w_3 + c - x_1 - b_1) \varphi_2(w_2 + \ell_{r \oplus 1,2} - x_2 - c; T_{r \oplus 1}) \\
 & \quad \times \psi(c; w_3 + c - x_1 - b_1, p_{r \oplus 1});
 \end{aligned}$$

for $w_1 \geq 1$ and $w_3 \geq \ell_{r \oplus 1,1}$,

$$\begin{aligned}
 & Q_{i+1}(\Gamma^{(r \oplus 1)}; w_1, 0, w_3) \\
 (14) \quad &= \sum_{x_1=0}^{w_1 + \ell_{r \oplus 1,1}} \sum_{c=0}^{\ell_{r \oplus 1,2}} \sum_{x_2=0}^{\ell_{r \oplus 1,2} - c} \sum_{b_2=0}^{\ell_{r \oplus 1,2} - c - x_2} Q_i(\Gamma^{(r)}; x_1, x_2, w_3 + c - \ell_{r \oplus 1,1}) \\
 & \quad \times \varphi_1(w_1 + \ell_{r \oplus 1,1} - x_1; T_{r \oplus 1}) \psi(c; w_3 + c - \ell_{r \oplus 1,1}, p_{r \oplus 1}) \\
 & \quad \times \varphi_2(b_2; T_{r \oplus 1});
 \end{aligned}$$

and for $w_1 \geq 1$, $w_2 \geq 1$, and $w_3 \geq \ell_{r \oplus 1,1}$,

$$\begin{aligned}
 & Q_{i+1}(\Gamma^{(r \oplus 1)}; w_1, w_2, w_3) \\
 (15) \quad &= \sum_{x_1=0}^{w_1 + \ell_{r \oplus 1,1}} \sum_{c=0}^{\ell_{r \oplus 1,2}} \sum_{x_2=0}^{\ell_{r \oplus 1,2} - c} Q_i(\Gamma^{(r)}; x_1, x_2, w_3 + c - \ell_{r \oplus 1,1}) \\
 & \quad \times \varphi_1(w_1 + \ell_{r \oplus 1,1} - x_1; T_{r \oplus 1}) \varphi_2(w_2 + \ell_{r \oplus 1,2} - x_2 - c; T_{r \oplus 1}) \\
 & \quad \times \psi(c; w_3 + c - \ell_{r \oplus 1,1}, p_{r \oplus 1}).
 \end{aligned}$$

Finally,

$$(16) \quad Q_{i+1}(r \oplus 1; w_1, w_2, w_3) = 0$$

if $\Gamma^{(r \oplus 1)} \in \Gamma^{\text{II}} \cup \Gamma^{\text{IV}}$ and $w_1 \geq 1$, $w_2 \geq 0$, $0 \leq w_3 \leq \ell_{r \oplus 1,1} - 1$. Thus we have proved the theorem for the marginal probability distributions of the Markov chain (2).

Theorem 2. *The probabilities $Q_i(r; x_1, x_2, x_3)$ satisfy recurrence equations (5)–(16) with respect to $i = 0, 1, \dots$.*

An arbitrary element of the set X is denoted by $w = (w_1, w_2, w_3)$. Let

$$\begin{aligned}
 E'_1 &= \{(\Gamma^{(r)}, w) : \Gamma^{(r)} \in \Gamma^{\text{II}} \cup \Gamma^{\text{IV}}, w_1 > 0, w_3 < \ell_{r,1}\}, \\
 E_1 &= \{(\gamma, w) : \gamma \in \Gamma^{\text{I}} \cup \Gamma^{\text{III}}, w \in X\}, \\
 E_2 &= \{(\gamma, w) : \gamma \in \Gamma^{\text{II}} \cup \Gamma^{\text{IV}}, w_1 = 0\}, \\
 E_3 &= \{(\Gamma^{(r)}, w) : \Gamma^{(r)} \in \Gamma^{\text{II}} \cup \Gamma^{\text{IV}}, w_1 > 0, w_3 \geq \ell_{r,1}\}.
 \end{aligned}$$

Corollary 1. *The set of states of the Markov chain (2) is the union of the nonclosed set E'_1 of inessential states and the closed set $E_1 \cup E_2 \cup E_3$ of essential periodic states with period n .*

Consider the probability generating functions

$$\begin{aligned}
 \Psi_i(r; z_1, z_2, z_3) &= \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\infty} \sum_{x_3=0}^{\infty} z_1^{x_1} z_2^{x_2} z_3^{x_3} Q_i(r; x_1, x_2, x_3), \\
 q_{r,j}(z_j) &= \exp \left\{ \lambda_j T_r \left(\sum_{x=1}^{\infty} p_x^{(j)} z_j^x - 1 \right) \right\}
 \end{aligned}$$

for $r = 1, 2, \dots, n$, $j = 1, 2$, and $|z_1| \leq 1$, $|z_2| \leq 1$, $|z_3| \leq 1$.

Theorem 3. *The following recurrence relations for the probability generating functions hold with respect to $i = 0, 1, \dots$:*

$$\Psi_{i+1}(r \oplus 1; z_1, z_2, z_3) = q_{r \oplus 1, 1}(z_1)q_{r \oplus 1, 2}(z_2)\Psi_i(r; z_1, z_2, z_2p_{r \oplus 1} + z_3(1 - p_{r \oplus 1}))$$

if $\Gamma^{(r \oplus 1)} \in \Gamma^I$;

$$\begin{aligned} & \Psi_{i+1}(r \oplus 1; z_1, z_2, z_3) \\ &= \left(\frac{z_3}{z_1}\right)^{\ell_{r \oplus 1, 1}} q_{r \oplus 1, 1}(z_1)q_{r \oplus 1, 2}(z_2)\Psi_i(r; z_1, z_2, z_2p_{r \oplus 1} + z_3(1 - p_{r \oplus 1})) \\ &+ q_{r \oplus 1, 2}(z_2) \sum_{x_1=0}^{\ell_{r \oplus 1, 1}} \sum_{x_2=0}^{\infty} \sum_{x_3=0}^{\infty} Q_i(r; x_1, x_2, x_3) z_2^{x_2} (z_2p_{r \oplus 1} + z_3(1 - p_{r \oplus 1}))^{x_3} \\ &\quad \times \sum_{b_1=0}^{\ell_{r \oplus 1, 1} - x_1} (z_3^{x_1+b_1} - z_1^{x_1+b_1-\ell_{r \oplus 1, 1}} z_3^{\ell_{r \oplus 1, 1}}) \varphi_1(b_1; T_{r \oplus 1}) \end{aligned}$$

if $\Gamma^{(r \oplus 1)} \in \Gamma^{II}$;

$$\begin{aligned} & \Psi_{i+1}(r \oplus 1; z_1, z_2, z_3) \\ &= z_2^{-\ell_{r \oplus 1, 2}} q_{r \oplus 1, 1}(z_1)q_{r \oplus 1, 2}(z_2)\Psi_i(r; z_1, z_2, z_2p_{r \oplus 1} + z_3(1 - p_{r \oplus 1})) \\ &+ q_{r \oplus 1, 1}(z_1) \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\ell_{r \oplus 1, 2}} \sum_{x_3=0}^{\infty} Q_i(r; x_1, x_2, x_3) \\ &\quad \times \sum_{c=0}^{\min\{x_3, \ell_{r \oplus 1, 2} - x_2\}} \sum_{b_2=0}^{\ell_{r \oplus 1, 2} - x_2 - c} r(x_1, x_2, x_3, c, b_2) \end{aligned}$$

if $\Gamma^{(r \oplus 1)} \in \Gamma^{III}$, where

$$r(x_1, x_2, x_3, c, b_2) = z_1^{x_1} \left(1 - z_2^{x_2+b_2+c-\ell_{r \oplus 1, 2}}\right) z_3^{x_3-c} \varphi_2(b_2; T_{r \oplus 1}) \psi(c; x_3, p_{r \oplus 1});$$

and

$$\begin{aligned} & \Psi_{i+1}(r \oplus 1; z_1, z_2, z_3) \\ &= \left(\frac{z_3}{z_1}\right)^{\ell_{r \oplus 1, 1}} z_2^{-\ell_{r \oplus 1, 2}} q_{r \oplus 1, 1}(z_1)q_{r \oplus 1, 2}(z_2)\Psi_i(r; z_1, z_2, z_2p_{r \oplus 1} + z_3(1 - p_{r \oplus 1})) \\ &+ \sum_{x_1=0}^{\ell_{r \oplus 1, 1}} \sum_{x_2=0}^{\infty} \sum_{x_3=0}^{\infty} Q_i(r; x_1, x_2, x_3) g_1(x_1, x_2, x_3, r) \\ &+ \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\ell_{r \oplus 1, 2}} \sum_{x_3=0}^{\infty} Q_i(r; x_1, x_2, x_3) g_2(x_1, x_2, x_3, r) \\ &+ \sum_{x_1=0}^{\ell_{r \oplus 1, 1}} \sum_{x_2=0}^{\ell_{r \oplus 1, 2}} \sum_{x_3=0}^{\infty} Q_i(r; x_1, x_2, x_3) g_3(x_1, x_2, x_3, r) \end{aligned}$$

if $\Gamma^{(r \oplus 1)} \in \Gamma^{\text{IV}}$, where

$$\begin{aligned}
 g_1(x_1, x_2, x_3, r) &= \sum_{b_1=0}^{\ell_{r \oplus 1, 1} - x_1} \sum_{c=m_1}^{x_3} \sum_{b_2=m_2}^{\infty} z_2^{x_2 + b_2 + c - \ell_{r \oplus 1, 2}} \\
 &\quad \times \left(z_3^{x_1 + x_3 + b_1 - c} - z_1^{x_1 + b_1 - \ell_{r \oplus 1, 1}} z_3^{x_3 + \ell_{r \oplus 1, 1} - c} \right) \\
 &\quad \times \varphi_1(b_1; T_{r \oplus 1, 1}) \varphi_2(b_2; T_{r \oplus 1}) \psi(c; x_3, p_{r \oplus 1}), \\
 m_1 &= \max\{0, \ell_{r \oplus 1, 2} - x_2\}, \quad m_2 = \max\{0, l(r, x_2)\}, \quad l(r, x_2) = \ell_{r \oplus 1, 2} - x_2 - c, \\
 g_2(x_1, x_2, x_3, r) &= \sum_{b_1=m_3}^{\infty} \sum_{c=0}^{m_4} \sum_{b_2=0}^{l(r, x_2)} \varphi_1(b_1; T_{r \oplus 1}) \varphi_2(b_2; T_{r \oplus 1}) \psi(c; x_3, p_{r \oplus 1}) z_1^{x_1 + b_1 - \ell_{r \oplus 1, 1}} \\
 &\quad \times \left(1 - z_2^{x_2 + b_2 + c - \ell_{r \oplus 1, 2}} \right) z_3^{x_3 + \ell_{r \oplus 1, 1} - c}, \\
 m_3 &= \max\{0, \ell_{r \oplus 1, 1} - x_1\}, \quad m_4 = \min\{x_3, \ell_{r \oplus 1, 2} - x_2\}, \\
 g_3(x_1, x_2, x_3, r) &= \sum_{b_1=0}^{\ell_{r \oplus 1, 1} - x_1} \sum_{c=0}^{m_5} \sum_{b_2=0}^{l(r, x_2)} \varphi_1(b_1; T_{r \oplus 1}) \varphi_2(b_2; T_{r \oplus 1}) \psi(c; x_3, p_{r \oplus 1}) \\
 &\quad \times \left(z_3^{x_1 + x_3 + b_1 - c} - z_1^{x_1 + b_1 - \ell_{r \oplus 1, 1}} z_2^{x_2 + b_2 + c - \ell_{r \oplus 1, 2}} z_3^{x_3 + \ell_{r \oplus 1, 1} - c} \right), \\
 m_5 &= \min\{x_3, \ell_{r \oplus 1, 2} - x_2\}.
 \end{aligned}$$

Proof. Denote by $E(\cdot)$ the operator of the mathematical expectation. Let $I_B(\omega)$ be the indicator of an event $B \in \mathfrak{F}$ and let $E(\theta; B) = E(\theta I_B)$ for a random variable θ . Then the probability generating functions $\Psi_i(r; z_1, z_2, z_3)$, $r = 1, 2, \dots, n$, can be rewritten as follows:

$$\Psi_i(r; z_1, z_2, z_3) = E \left(z_1^{\kappa_{1,i}} z_2^{\kappa_{2,i}} z_3^{\kappa_{3,i}}; \{\omega: \Gamma_i = \Gamma^{(r)}\} \right).$$

Let $x = (x_1, x_2, x_3)$ be an arbitrary element of the set X . We exhibit the general proof for the most complicated case, namely for $\Gamma^{(r \oplus 1)} \in \Gamma^{\text{IV}}$. The other cases can be treated in an analogous way.

If $\Gamma^{(r \oplus 1)} \in \Gamma^{\text{IV}}$, then

$$\begin{aligned}
 \Psi_{i+1}(r \oplus 1; z_1, z_2, z_3) &= \sum_{x \in X} Q_i(r; x_1, x_2, x_3) E \left(z_1^{m_6} z_2^{m_7} z_3^{x_3 + m_8 - \eta_{3,i}} \mid A_i(r; x_1, x_2, x_3) \right) \\
 &= \sum_{x \in X} E(Z(x_1, x_2, x_3) \mid A_i(r; x_1, x_2, x_3)) Q_i(r; x_1, x_2, x_3) \\
 &\quad + \sum_{x \in X} E \left(z_1^{m_6} z_2^{m_7} z_3^{x_3 + m_8 - \eta_{3,i}} - Z(x_1, x_2, x_3) \mid A_i(r; x_1, x_2, x_3) \right) Q_i(r; x_1, x_2, x_3),
 \end{aligned}$$

where

$$\begin{aligned}
 m_6 &= \max\{0, x_1 + \eta_{1,i} - \ell_{r \oplus 1, 1}\}, \quad m_7 = \max\{0, x_2 + \eta_{2,i} + \eta_{3,i} - \ell_{r \oplus 1, 2}\}, \\
 m_8 &= \min\{x_1 + \eta_{1,i}, \ell_{r \oplus 1, 1}\}, \\
 Z(x_1, x_2, x_3) &= z_1^{x_1 + \eta_{1,i} - \ell_{r \oplus 1, 1}} z_2^{x_2 + \eta_{2,i} + \eta_{3,i} - \ell_{r \oplus 1, 2}} z_3^{x_3 + \ell_{r \oplus 1, 1} - \eta_{3,i}}.
 \end{aligned}$$

The first term on the right hand side is equal to

$$\left(\frac{z_3}{z_1} \right)^{\ell_{r \oplus 1, 1}} z_2^{-\ell_{r \oplus 1, 2}} q_{r \oplus 1, 1}(z_1) q_{r \oplus 1, 2}(z_2) \Psi_i(r; z_1, z_2, z_2 p_{r \oplus 1} + z_3(1 - p_{r \oplus 1})).$$

Since

$$I_{\{\eta_{1,i} \leq \ell_{r \oplus 1,1} - x_1, \eta_{2,i} + \eta_{3,i} > \ell_{r \oplus 1,2} - x_2\}} + I_{\{\eta_{1,i} > \ell_{r \oplus 1,1} - x_1, \eta_{2,i} + \eta_{3,i} \leq \ell_{r \oplus 1,2} - x_2\}} \\ + I_{\{\eta_{1,i} \leq \ell_{r \oplus 1,1} - x_1, \eta_{2,i} + \eta_{3,i} \leq \ell_{r \oplus 1,2} - x_2\}} + I_{\{\eta_{1,i} > \ell_{r \oplus 1,1} - x_1, \eta_{2,i} + \eta_{3,i} > \ell_{r \oplus 1,2} - x_2\}} = 1$$

and

$$z_1^{m_6} z_2^{m_7} z_3^{x_3 + m_8 - \eta_{3,i}} = z_1^{x_1 + \eta_{1,i} - \ell_{r \oplus 1,1}} z_2^{x_2 + \eta_{2,i} + \eta_{3,i} - \ell_{r \oplus 1,2}} z_3^{x_3 + \ell_{r \oplus 1,1} - \eta_{3,i}}$$

for $\eta_{1,i} > \ell_{r \oplus 1,1} - x_1$ and $\eta_{2,i} + \eta_{3,i} > \ell_{r \oplus 1,2} - x_2$, we obtain the following equation:

$$\begin{aligned} & \mathbb{E} \left(z_1^{m_6} z_2^{m_7} z_3^{x_3 + m_8 - \eta_{3,i}} - Z(x_1, x_2, x_3) \right) \\ &= \mathbb{E} \left(\left(z_2^{x_2 + \eta_{2,i} + \eta_{3,i} - \ell_{r \oplus 1,2}} z_3^{x_3 + x_1 + \eta_{1,i} - \eta_{3,i}} - Z(x_1, x_2, x_3) \right) \right. \\ & \quad \times I_{\{\eta_{1,i} \leq \ell_{r \oplus 1,1} - x_1, \eta_{2,i} + \eta_{3,i} > \ell_{r \oplus 1,2} - x_2\}} \left. \middle| A_i(r; x_1, x_2, x_3) \right) \\ &+ \mathbb{E} \left(\left(z_1^{x_1 + \eta_{1,i} - \ell_{r \oplus 1,1}} z_3^{x_3 + \ell_{r \oplus 1,1} - \eta_{3,i}} - Z(x_1, x_2, x_3) \right) \right. \\ & \quad \times I_{\{\eta_{1,i} > \ell_{r \oplus 1,1} - x_1, \eta_{2,i} + \eta_{3,i} \leq \ell_{r \oplus 1,2} - x_2\}} \left. \middle| A_i(r; x_1, x_2, x_3) \right) \\ &+ \mathbb{E} \left(\left(z_3^{x_3 + x_1 + \eta_{1,i} - \eta_{3,i}} - Z(x_1, x_2, x_3) \right) \right. \\ & \quad \times I_{\{\eta_{1,i} \leq \ell_{r \oplus 1,1} - x_1, \eta_{2,i} + \eta_{3,i} \leq \ell_{r \oplus 1,2} - x_2\}} \left. \middle| A_i(r; x_1, x_2, x_3) \right). \end{aligned}$$

Note that some terms above are vanishing for some combinations of values of x_1 , x_2 , and x_3 . For example, the event $\eta_{1,i} \leq \ell_{r \oplus 1,1} - x_1$ is impossible for $x_1 > \ell_{r \oplus 1,1}$. This implies that

$$\begin{aligned} & \sum_{x \in X} Q_i(r; x_1, x_2, x_3) \mathbb{E} \left(\left(z_2^{x_2 + \eta_{2,i} + \eta_{3,i} - \ell_{r \oplus 1,2}} z_3^{x_3 + x_1 + \eta_{1,i} - \eta_{3,i}} - Z(x_1, x_2, x_3) \right) \right. \\ & \quad \times I_{\{\eta_{1,i} \leq \ell_{r \oplus 1,1} - x_1, \eta_{2,i} + \eta_{3,i} > \ell_{r \oplus 1,2} - x_2\}} \left. \middle| A_i(r; x_1, x_2, x_3) \right) \\ &= \sum_{x_1=0}^{\ell_{r \oplus 1,1}} \sum_{x_2=0}^{\infty} \sum_{x_3=0}^{\infty} Q_i(r; x_1, x_2, x_3) \\ & \quad \times \sum_{b_1=0}^{\ell_{r \oplus 1,1} - x_1} \sum_{c=m_1}^{\infty} \sum_{b_2=m_2}^{\infty} \varphi_1(b_1; T_{r \oplus 1}) \psi(c; x_3, T_{r \oplus 1}) z_2^{x_2 + b_2 + c - \ell_{r \oplus 1,2}} \\ & \quad \times \left(z_3^{x_1 + x_3 + b_1 - c} - z_1^{x_1 + b_1 - \ell_{r \oplus 1,1}} z_3^{\ell_{r \oplus 1,1} + x_3 - c} \right) \end{aligned}$$

and

$$\begin{aligned} & \sum_{x \in X} Q_i(r; x_1, x_2, x_3) \mathbb{E} \left(\left(z_1^{x_1 + \eta_{1,i} - \ell_{r \oplus 1,1}} z_3^{x_3 + \ell_{r \oplus 1,1} - \eta_{3,i}} - Z(x_1, x_2, x_3) \right) \right. \\ & \quad \times I_{\{\eta_{1,i} > \ell_{r \oplus 1,1} - x_1, \eta_{2,i} + \eta_{3,i} \leq \ell_{r \oplus 1,2} - x_2\}} \left. \middle| A_i(r; x_1, x_2, x_3) \right) \\ &= \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\ell_{r \oplus 1,2}} \sum_{x_3=0}^{\infty} Q_i(r; x_1, x_2, x_3) z_1^{x_1 - \ell_{r \oplus 1,1}} z_3^{x_3 + \ell_{r \oplus 1,2}} \\ & \quad \times \sum_{b_1=m_3+1}^{\infty} \sum_{c=0}^{\min\{x_3, \ell_{r \oplus 1,2}\}} \sum_{b_2=0}^{\ell_{r \oplus 1,2} - x_2 - c} \varphi_1(b_1; T_{r \oplus 1}) \varphi_2(b_2; T_{r \oplus 1}) \\ & \quad \times \psi(c; x_3, T_{r \oplus 1}) z_1^{b_1} \left(1 - z_2^{x_2 + b_2 + c - \ell_{r \oplus 1,2}} \right) z_3^{-c}. \end{aligned}$$

Also

$$\begin{aligned}
& \sum_{x \in X} Q_i(r; x_1, x_2, x_3) \mathbb{E} \left[\left(z_3^{x_3+x_1+\eta_{1,i}-\eta_{3,i}} - Z(x_1, x_2, x_3) \right) \right. \\
& \quad \left. \times I_{\{\eta_{1,i} \leq \ell_{r \oplus 1,1} - x_1, \eta_{2,i} + \eta_{3,i} \leq \ell_{r \oplus 1,2} - x_2\}} \mid A_i(r; x_1, x_2, x_3) \right] \\
& = \sum_{x_1=0}^{\ell_{r \oplus 1,1}} \sum_{x_2=0}^{\ell_{r \oplus 1,2}} \sum_{x_3=0}^{\infty} Q_i(r; x_1, x_2, x_3) \\
& \quad \times \sum_{b_1=0}^{\ell_{r \oplus 1,1} - x_1} \sum_{c=0}^{m_4} \sum_{b_2=0}^{\ell_{r \oplus 1,2} - x_2 - c} \varphi_1(b_1; T_{r \oplus 1}) \varphi_2(b_2; T_{r \oplus 1}) \psi(c; x_3, T_{r \oplus 1}) \\
& \quad \times \left(z_3^{x_1+x_3+b_1-c} - z_1^{x_1+b_1-\ell_{r \oplus 1,1}} z_2^{x_2+b_2+c-\ell_{r \oplus 1,2}} z_3^{x_3+\ell_{r \oplus 1,1}-c} \right).
\end{aligned}$$

The theorem is proved for $\Gamma^{(r \oplus 1)} \in \Gamma^{\text{IV}}$. $\square$

The recurrence relations of Theorem 3 allow one to evaluate the probability generating functions $\Psi_{i+1}(r; z_1, z_2, z_3)$, $r = 1, 2, \dots, n$, $i = 0, 1, \dots$, from a given probability distribution of the vector $(\Gamma_0, \kappa_{1,0}, \kappa_{2,0}, \kappa_{3,0})$. Note that the probability generating functions converge in the polydisk $D_0 = \{(z_1, z_2, z_3) : |z_i| \leq 1\}$.

The following result shows that the probability generating functions converge in the polydisk $D_\varepsilon = \{(z_1, z_2, z_3) : |z_i| \leq 1 + \varepsilon\}$ provided that a point (z_1, z_2, z_3) does not belong to D_ε if z_1 is a singular point of the function $q_{r,1}(z_1)$ for some r or if z_2 is a singular point of the function $q_{r,2}(z_2)$ for some r if $\Psi_0(r; z_1, z_2, z_3)$ is chosen appropriately.

Lemma 1. *The function $(z_1, z_2, z_3) \rightarrow (z_1, z_2, pz_2 + (1-p)z_3)$ maps the polydisk D_ε into itself if $0 < p < 1$.*

Proof. Consider the polar coordinates $z_j = \rho_j e^{iu_j}$, $j = 1, 2, 3$. Then $\rho_j \leq 1 + \varepsilon$ in the polydisk D_ε . Hence

$$\begin{aligned}
|pz_2 + (1-p)z_3|^2 &= p^2 \rho_2^2 + (1-p)^2 \rho_3^2 + 2p(1-p) \rho_2 \rho_3 \cos(u_2 - u_3) \\
&\leq (p \rho_2 + (1-p) \rho_3)^2 \leq (1 + \varepsilon)^2.
\end{aligned}$$

Thus $(z_1, z_2, pz_2 + (1-p)z_3) \in D_\varepsilon$. The lemma is proved. $\square$

3. A SUFFICIENT CONDITION FOR THE EXISTENCE OF THE STATIONARY DISTRIBUTION

Let

$$\ell_1 = \sum_{r: \Gamma^{(r)} \in \Gamma^{\text{III}} \cup \Gamma^{\text{IV}}} \ell_{r,1}, \quad \ell_2 = \sum_{r: \Gamma^{(r)} \in \Gamma^{\text{III}} \cup \Gamma^{\text{IV}}} \ell_{r,2},$$

$T = T_1 + T_2 + \dots + T_n$, and

$$\bar{\lambda}_j = \lambda_j \sum_{x=1}^{\infty} x p_x^{(j)}.$$

Here ℓ_j means the number of customers coming from the saturation flow Π_j^{sat} during the server working period of duration T , while λ_j is the intensity of the flow Π_j , since $\sum_{x=1}^{\infty} x p_x^{(j)}$ equals the mathematical expectation of the bulk size in the flow Π_j .

The following result contains some restrictions on intensities of the input flows, mean bulk sizes, saturation flows, and sojourn times of the server that are sufficient for the boundedness in time of the expected number of customers in the system.

Theorem 4. *If*

$$\bar{\lambda}_1 T - \ell_1 < 0, \quad (\bar{\lambda}_1 + \bar{\lambda}_2) T - \ell_2 < 0,$$

then the stationary distribution of the Markov chain (2) exists.

Proof. Assume that the stationary distribution does not exist. Then the sequence

$$\{\mathbf{E}(\kappa_{1,i} + \kappa_{2,i} + \kappa_{3,i}); i = 0, 1, \dots\}$$

increases unboundedly for any given initial distribution. Choose a polydisk D_ε that does not contain the points (z_1, z_2, z_3) such that z_1 is a singular point of the function $q_{r',1}(z_1)$ or z_2 is a singular point of the function $q_{r',2}(z_2)$ for some $r' = 1, 2, \dots, n$. Then we choose the initial probability distribution such that the functions $\Psi_0(r; z_1, z_2, z_3)$, $r = 1, 2, \dots, n$, converge in the polydisk D_ε . Now we show that the sequence $\{\mathbf{E}(\kappa_{1,i} + \kappa_{2,i} + \kappa_{3,i}); i = 0, 1, \dots\}$ is bounded for this initial distribution.

Let $(z^*, 1, 1) \in D_\varepsilon$ be a point such that $1 < z^* < 1 + \varepsilon$. We prove that the sequence $\Psi_i(r; z^*, 1, 1)$, $i = 0, 1, \dots$, is bounded for each r . Theorem 3 implies that

$$(17) \quad \Psi_{i+1}(r \oplus 1; z^*, 1, 1) = q_{r \oplus 1, 1}(z^*) \Psi_i(r; z^*, 1, 1)$$

for $\Gamma^{(r \oplus 1)} \in \Gamma^I \cup \Gamma^{III}$, while

$$\begin{aligned} \Psi_{i+1}(r \oplus 1; z^*, 1, 1) &= (z^*)^{-\ell_{r \oplus 1, 1}} q_{r \oplus 1, 1}(z^*) \Psi_i(r; z^*, 1, 1) \\ &+ \sum_{x_1=0}^{\ell_{r \oplus 1, 1}} \sum_{x_2=0}^{\infty} \sum_{x_3=0}^{\infty} Q_i(r; x_1, x_2, x_3) \\ &\times \sum_{b_1=0}^{\ell_{r \oplus 1, 1} - x_1} (1 - (z^*)^{x_1 + b_1 - \ell_{r \oplus 1, 1}}) \varphi_1(b_1; T_{r \oplus 1}) \end{aligned}$$

for $\Gamma^{(r \oplus 1)} \in \Gamma^{II} \cup \Gamma^{IV}$. Since $Q_i(r; x_1, x_2, x_3)$ is a probability distribution,

$$\sum_{x_2=0}^{\infty} \sum_{x_3=0}^{\infty} Q_i(r; x_1, x_2, x_3) \leq 1,$$

whence

$$(18) \quad \begin{aligned} \Psi_{i+1}(r \oplus 1; z^*, 1, 1) &\leq (z^*)^{-\ell_{r \oplus 1, 1}} q_{r \oplus 1, 1}(z^*) \Psi_i(r; z^*, 1, 1) \\ &+ \sum_{x_1=0}^{\ell_{r \oplus 1, 1}} \sum_{b_1=0}^{\ell_{r \oplus 1, 1} - x_1} (1 - (z^*)^{x_1 + b_1 - \ell_{r \oplus 1, 1}}) \varphi_1(b_1; T_{r \oplus 1}). \end{aligned}$$

We find from relations (17) and (18) that

$$\sum_{r=1}^n \Psi_{i+n}(r; z^*, 1, 1) \leq \left(\prod_{r=1}^n q_{r,1}(z^*) \right) (z^*)^{-\ell_1} \left(\sum_{r=1}^n \Psi_i(r; z^*, 1, 1) \right) + B(z^*).$$

The inequality $\bar{\lambda}_1 T - \ell_1 < 0$ yields

$$\prod_{r=1}^n q_{r,1}(z^*) (z^*)^{-\ell_1} < 1.$$

The sequence

$$M_0 = \sum_{r=1}^n \Psi_0(r; z^*, 1, 1), \dots, M_{n-1} = \sum_{r=1}^n \Psi_{n-1}(r; z^*, 1, 1),$$

$$M_{i+n} = M_i \prod_{r=1}^n q_{r,1}(z^*) (z^*)^{-\ell_1} + B(z^*), \quad i = 0, 1, \dots,$$

converges and thus is bounded. Moreover,

$$\sum_{r=1}^n \Psi_i(r; z^*, 1, 1) \leq M_i, \quad i = 0, 1, \dots$$

Hence $\Psi_i(r; z^*, 1, 1)$, $i = 0, 1, \dots$, are bounded by a constant M^* for all r .

Substituting $z_1 = z_2 = z_3 = z^*$ in the recurrence relations of Theorem 3, we get

$$(19) \quad \Psi_{i+1}(r \oplus 1; z^*, z^*, z^*) = q_{r \oplus 1, 1}(z^*) q_{r \oplus 1, 2}(z^*) \Psi_i(r; z^*, z^*, z^*)$$

for $\Gamma^{(r \oplus 1)} \in \Gamma^I \cup \Gamma^{II}$, while for $\Gamma^{(r \oplus 1)} \in \Gamma^{III} \cup \Gamma^{IV}$,

$$(20) \quad \begin{aligned} & \Psi_{i+1}(r \oplus 1; z^*, 1, 1) \\ &= (z^*)^{-\ell_{r \oplus 1, 2}} q_{r \oplus 1, 1}(z^*) q_{r \oplus 1, 2}(z^*) \Psi_i(r; z^*, z^*, z^*) \\ &+ (z^*)^{-\ell_{r \oplus 1, 2}} q_{r \oplus 1, 1}(z^*) q_{r \oplus 1, 2}(z^*) \\ &\times \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\ell_{r \oplus 1, 2}} \sum_{x_3=0}^{\ell_{r \oplus 1, 2} - x_2} Q_i(r; x_1, x_2, x_3) \\ &\times \sum_{c=0}^{m_4} \sum_{b_2=0}^{\ell_{r \oplus 1, 2} - x_2 - c} (z^*)^{x_1 + x_3 - c} (1 - (z^*)^{x_2 + b_2 + c - \ell_{r \oplus 1, 2}}) \\ &\times \varphi_2(b_2; T_{r \oplus 1}) \psi(c; x_3, T_{r \oplus 1}). \end{aligned}$$

It is easy to see that

$$\begin{aligned} & \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\ell_{r \oplus 1, 2}} \sum_{x_3=0}^{\ell_{r \oplus 1, 2} - x_2} Q_i(r; x_1, x_2, x_3) \\ & \times \sum_{c=0}^{m_4} \sum_{b_2=0}^{\ell_{r \oplus 1, 2} - x_2 - c} (z^*)^{x_1 + x_3 - c} (1 - (z^*)^{x_2 + b_2 + c - \ell_{r \oplus 1, 2}}) \varphi_2(b_2; T_{r \oplus 1}) \psi(c; x_3, T_{r \oplus 1}) \\ & \leq \left(\max_{0 \leq x_2 + x_3 \leq \ell_{r \oplus 1, 2}} \left\{ \sum_{c=0}^{m_4} \sum_{b_2=0}^{\ell_{r \oplus 1, 2} - x_2 - c} (z^*)^{x_3 - c} \right. \right. \\ & \quad \left. \left. \times (1 - (z^*)^{x_2 + b_2 + c - \ell_{r \oplus 1, 2}}) \varphi_2(b_2; T_{r \oplus 1}) \psi(c; x_3, T_{r \oplus 1}) \right\} \right) \\ & \times \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\ell_{r \oplus 1, 2}} \sum_{x_3=0}^{\ell_{r \oplus 1, 2} - x_2} (z^*)^{x_1} Q_i(r; x_1, x_2, x_3). \end{aligned}$$

At the same time,

$$\begin{aligned} & \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\ell_{r \oplus 1, 2}} \sum_{x_3=0}^{\ell_{r \oplus 1, 2} - x_2} (z^*)^{x_1} Q_i(r; x_1, x_2, x_3) \leq \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\infty} \sum_{x_3=0}^{\infty} (z^*)^{x_1} Q_i(r; x_1, x_2, x_3) \\ &= \Psi_i(r; z^*, 1, 1). \end{aligned}$$

Since $\Psi_i(r; z^*, 1, 1) < M^*$, the second term on the right hand side of equation (20) is bounded from above by another constant that does not depend on i . Thus equations (19) and (20) imply

$$\sum_{r=1}^n \Psi_{i+n}(r; z^*, z^*, z^*) \leq \left(\prod_{r=1}^n q_{r, 1}(z^*) q_{r, 2}(z^*) \right) (z^*)^{-\ell_2} \left(\sum_{r=1}^n \Psi_i(r; z^*, z^*, z^*) \right) + B_1(z^*).$$

By inequality $(\bar{\lambda}_1 + \bar{\lambda}_2)T - \ell_2 < 0$, we have

$$\prod_{r=1}^n q_{r,1}(z^*)q_{r,2}(z^*)(z^*)^{-\ell_2} < 1.$$

The sequence

$$M'_0 = \sum_{r=1}^n \Psi_0(r; z^*, z^*, z^*), \dots, M'_{n-1} = \sum_{r=1}^n \Psi_{n-1}(r; z^*, z^*, z^*),$$

$$M'_{i+n} = M'_i \left(\prod_{r=1}^n q_{r,1}(z^*)q_{r,2}(z^*) \right) (z^*)^{-\ell_2} + B_1(z^*), \quad i = 0, 1, \dots,$$

converges and thus is bounded. Moreover,

$$\sum_{r=1}^n \Psi_i(r; z^*, z^*, z^*) \leq M'_i, \quad i = 0, 1, \dots$$

Therefore the sequence $\Psi_i(r; z^*, z^*, z^*)$, $i = 0, 1, \dots$, is bounded by a constant M^{**} for all r . According to Cauchy's theorem, the expectation

$$\mathbf{E}(\kappa_{1,i} + \kappa_{2,i} + \kappa_{3,i}) = \frac{1}{2\pi\sqrt{-1}} \int_{|z-1|=\rho} (z-1)^{-2} \sum_{r=1}^n \Psi_i(r; z, z, z) dz$$

is bounded from above and the bound does not depend on ρ and M^{**} . This proves that the mathematical expectation of the total number of customers in the system is bounded. The theorem is proved. $\square$

It would be instructive to analyze the sufficient condition in Theorem 4. The first inequality ensures that the queue O_1 remains bounded in mean as the time increases. The intensity of the output flow from the queue O_1 is equal to the intensity of the input flow Π_1 . The second inequality means that the cumulative input intensity for the queue O_2 is equal to $\bar{\lambda}_1 + \bar{\lambda}_2$ and does not depend on the probabilities p_r , $r = 1, 2, \dots, n$. The interpretation of the second inequality is that the system is capable of serving "in time" all the customers coming to the queue O_2 .

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