SEMIGROUPS OF OPERATORS THAT DESCRIBE A FELLER PROCESS ON THE LINE, WHICH IS THE RESULT OF PASTING TOGETHER TWO DIFFUSION PROCESSES

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ABSTRACT. We use the method of the classical potential theory to construct the semigroup of operators that describe a Feller process on the line by pasting together two diffusion processes that satisfy a nonlocal Feller–Wentzell type condition for the pasting.

1. MAIN NOTATION AND SETTING OF THE PROBLEM

Let $D_i = \{x: (-1)^i x > 0\}$, $i = 1, 2$, be two domains on the real line, let $\partial D = \{0\}$ be the joint boundary for these domains, and let $\overline{D}_i = D_i \cup \partial D$ be the closure of D_i . If D is a domain in $\mathbb{R}$ (in particular, $D = \mathbb{R}$) and $\overline{D}$ is its closure, then $C_b(\overline{D})$ denotes the Banach space of bounded continuous functions $\varphi(x)$ on $\overline{D}$ equipped with the norm $\|\varphi\| = \sup_{x \in \overline{D}} |\varphi(x)|$ and $C_0(\overline{D})$ denotes the space of bounded uniformly continuous functions on $\overline{D}$.

In what follows, if $\varphi \in C_b(\mathbb{R})$, then the restriction of φ to $\overline{D}_i$ is denoted by φ_i . Let $C_2(\overline{D}_i)$ be the subset of $C_b(\overline{D}_i)$ containing all the functions φ such that $\varphi_i, \varphi'_i, \varphi''_i \in C_0(\overline{D}_i)$, $i = 1, 2$.

Assume that a differential operator L_i is defined in the domain D_i , $i = 1, 2$, and that it acts on functions φ of $C_2(\overline{D}_i)$,

$$(1) \quad L_i \varphi(x) = \frac{1}{2} b_i(x) \frac{d^2 \varphi_i}{dx^2}(x) + a_i(x) \frac{d \varphi_i}{dx}(x), \quad i = 1, 2,$$

where $b_i(x)$ and $a_i(x)$ are bounded continuous functions on $\overline{D}_i$ and, moreover, $b_i(x) \geq 0$. We further assume that each operator L_i generates a homogeneous diffusion process on D_i . Denote by $C_2(\mathbb{R})$ the subset of $C_b(\mathbb{R})$ consisting of the functions $\varphi(x)$ such that $\varphi_i \in C_2(\overline{D}_i)$, $i = 1, 2$, and $L_1 \varphi_1(0) = L_2 \varphi_2(0)$. Now we introduce the operator L acting on functions φ of $C_2(\mathbb{R})$ according to the rule

$$(2) \quad L \varphi(x) = \begin{cases} L_1 \varphi(x) & \text{if } x \in \overline{D}_1, \\ L_2 \varphi(x) & \text{if } x \in \overline{D}_2. \end{cases}$$

Then we introduce the additional pasting condition at the point $x = 0$ that reduces the operator L (its closure, more precisely) to the infinitesimal operator of the Feller semigroup on $C_0(\mathbb{R})$. This condition is written as

$$(3) \quad L_0 \varphi(0) \equiv q_1 \varphi'(0-) - q_2 \varphi'(0+) + \int_{D_1 \cup D_2} (\varphi(0) - \varphi(y)) \mu(dy) = 0.$$

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Here q_i , $i = 1, 2$, are nonnegative numbers and $\mu(\cdot)$ is a nonnegative measure on $D_1 \cup D_2$ such that

$$(4) \quad \lambda(\delta) = \int_{(D_1 \cup D_2) \setminus D_\delta} |y| \mu(dy) < \infty, \quad \mu(D_\delta) < \infty,$$

where $D_\delta = \{x \in \mathbb{R}: |x| > \delta > 0\}$. It is assumed that not all numbers among q_1 , q_2 , and $m = \mu(D_1 \cup D_2)$ equal 0.

Note that (see [1]) condition (3) is a particular case of a more general Feller–Wentzell type condition for pasting ([2, 3]). Conditions of this type are used to describe the behavior of a diffusion process after it reaches the joint boundary of the domains D_i , $i = 1, 2$. In the case under consideration, the left hand side of (3) contains the terms corresponding to the partial reflection of the process at the point $x = 0$ and those corresponding to its jumps to one of the domains D_1 or D_2 . Besides the two types of behavior mentioned above, the diffusion process may have a delay or may disappear at the boundary. Thus two extra terms appear in the general Feller–Wentzell pasting condition.

We study the question of the existence of a semigroup of operators $\{\mathcal{T}_t\}_{t \geq 0}$ describing a Feller process (not necessarily continuous) that is defined on $\mathbb{R}$ and coincides with a diffusion process generated by the operator L_i in D_i , $i = 1, 2$. The behavior of the process at the point $x = 0$ is determined by the pasting condition (3). We consider the case where the parameters q_1 and q_2 in (3) satisfy the condition

$$(5) \quad q_1 + q_2 > 0.$$

The problem stated above is often called the problem on pasting together two diffusion processes on the real line (see [1, 4, 5, 6]).

We use analytical methods to solve this problem. When applying this approach (see [4, 5, 6]), the problem reduces to the corresponding problem of conjugation for a linear parabolic second order equation with discontinuous coefficients. The latter problem is to find a function $u(t, x)$ defined for $t > 0$ and $x \in \mathbb{R}$ such that

$$(6) \quad \frac{\partial u}{\partial t}(t, x) = L_i u(t, x), \quad t > 0, \quad x \in D_i, \quad i = 1, 2,$$

$$(7) \quad u(0, x) = \varphi(x), \quad x \in \mathbb{R},$$

$$(8) \quad u(t, 0-) = u(t, 0+), \quad t \geq 0,$$

$$(9) \quad L_0 u(t, 0) = 0, \quad t > 0,$$

where $\varphi \in C_b(\mathbb{R})$ is a given function.

Conjugation condition (8) in the problem (6)–(9) reflects the Feller property of the semigroup to be found, while equality (9) corresponds to the pasting condition (3).

We prove that problem (6)–(9) has a solution under some additional assumptions imposed on the coefficients of the operators L_i , $i = 1, 2$. The proof applies the method of limit integral equations and uses an ordinary fundamental solution of a parabolic equation and heat potentials generated by this equation. A similar problem is studied in [4] by using the potential method if $m = 0$. The case of $q_1 = q_2 = 0$ is considered in [6] for a finite measure $\mu(\cdot)$. In the paper [7], the Markov process constructed by pasting together two Brownian motions is obtained in a different way if condition (3) does not contain the integral term and if $q_1 = q_2$ (this is the so-called classical case of pasting together two diffusion processes).

In what follows let T be a fixed positive number, $\mathbb{R}_T^2 \equiv (0, T] \times \mathbb{R}$,

$$\mathbb{R}_\infty^2 \equiv (0, \infty) \times \mathbb{R};$$

let D_t^r (D_x^p) be the symbol of the derivative (partial derivative) of order r (of order p) with respect to the variable t (x), where r and p are nonnegative integers; let $C^{m,l}(\Omega)$ ($C^{m,l}(\overline{\Omega})$), $m = 0, 1$, $l = 0, 1, 2$, be the family of continuous functions in Ω (in $\overline{\Omega}$) whose derivatives D_t^r , D_x^p , $r \leq m$, $p \leq l$, are continuous in Ω ($\overline{\Omega}$), where Ω is a subset of $\mathbb{R}_\infty^2$; let $H^\alpha(\mathbb{R})$, $\alpha \in (0, 1)$, denote the Hölder space (see [8]). Here

$$C^{0,0}(\Omega) \equiv C(\Omega) \quad \text{and} \quad C^{0,0}(\overline{\Omega}) \equiv C(\overline{\Omega}).$$

Throughout the paper C and c denote those constants whose precise values do not matter for our reasoning. These constants may be different in different places but do not depend on (t, x) . Other notation will be introduced in appropriate places.

2. FUNDAMENTAL SOLUTION OF A SECOND ORDER PARABOLIC EQUATION, AND RELATED POTENTIALS

Consider parabolic operators $D_t - L_i$, $i = 1, 2$, in the domain $\mathbb{R}_\infty^2$. Without loss of generality, assume that the coefficients of these operators are defined in $\mathbb{R}$ and satisfy the following conditions:

- a) $b_i, a_i \in H^\alpha(\mathbb{R})$, $i = 1, 2$;
- b) there are positive constants b_{i0} , $i = 1, 2$, such that $b_i(x) \geq b_{i0}$ for all $x \in \mathbb{R}$.

Let $g_i(t, x, y)$, $t > 0$, $x, y \in \mathbb{R}$, be the fundamental solutions of the operators $D_t - L_i$, $i = 1, 2$ (see [5, Chapter II, §2] or [8, Chapter IV, §11]):

$$(10) \quad g_i(t, x, y) = g_{i0}(t, x, y) + g_{i1}(t, x, y), \quad i = 1, 2,$$

where

$$g_{i0}(t, x, y) = (2\pi b_i(y)t)^{-1/2} \exp\left(-\frac{(y-x)^2}{2b_i(y)t}\right),$$

and the g_{i1} are integral terms that have a weaker singularity as $t \rightarrow 0$ than the g_{0i} do; moreover $g_i(t, x, y) = 0$ for $t \leq 0$. Recall that the functions $g_i(t, x, y)$, $i = 1, 2$, are continuous and nonnegative with respect to all arguments and satisfy equation (6) in the domain $(t, x) \in \mathbb{R}_\infty^2$ if y is fixed. Further,

$$\lim_{t \downarrow 0} \int_{\mathbb{R}} g_i(t, x, y) \varphi(y) dy = \varphi(x), \quad i = 1, 2,$$

for $x \in \mathbb{R}$ and $\varphi \in C_b(\mathbb{R})$.

If $2r + p \leq 2$, then

$$(11) \quad |D_t^r D_x^p g_i(t, x, y)| \leq Ct^{-(1+2r+p)/2} \exp\left(-c\frac{(y-x)^2}{t}\right), \quad t \in (0, T],$$

and

$$(12) \quad |D_t^r D_x^p g_{i1}(t, x, y)| \leq Ct^{-(1+2r+p-\alpha)/2} \exp\left(-c\frac{(y-x)^2}{t}\right), \quad t \in (0, T],$$

for $i = 1, 2$ and $x, y \in \mathbb{R}$. We have

$$\begin{aligned}
 & \int_{\mathbb{R}} g_i(t, x, y) dy = 1, \\
 & \int_{\mathbb{R}} g_i(t, x, y)(y - x) dy = \int_0^t d\tau \int_{\mathbb{R}} g_i(\tau, x, y) a_i(y) dy, \\
 & \int_{\mathbb{R}} g_i(t, x, y)(y - x)^2 dy \\
 & = \int_0^t d\tau \int_{\mathbb{R}} g_i(\tau, x, y) b_i(y) dy + 2 \int_0^t d\tau \int_{\mathbb{R}} g_i(\tau, x, y) a_i(y)(y - x) dy
 \end{aligned}
 \tag{13}$$

for $i = 1, 2$, $t > 0$, and $x \in \mathbb{R}$.

Consider the following integrals:

$$u_{i0}(t, x) = \int_{\mathbb{R}} g_i(t, x, y) \varphi(y) dy, \quad i = 1, 2, \tag{14}$$

$$u_{i1}(t, x) = \int_0^t g_i(t - \tau, x, 0) V_i(\tau) d\tau, \quad i = 1, 2. \tag{15}$$

Here $\varphi(x)$ and $V_i(t)$, $i = 1, 2$, are given functions. The functions u_{i0} and u_{i1} are called the Poisson potential and a simple layer potential, respectively, in the theory of parabolic equations. The definition and main properties of the fundamental solution $\overline{g_i}$, $i = 1, 2$, imply that if $\varphi \in C_b(\mathbb{R})$, then the functions $u_{i0}(t, x)$ are continuous in $\overline{\mathbb{R}_{\infty}^2}$, bounded with respect to x , and satisfy equation (6) in the domain $(t, x) \in \mathbb{R}_{\infty}^2$ and the initial condition (7). Moreover,

$$|D_t^r D_x^p u_{i0}(t, x)| \leq C \|\varphi\| t^{-(2r+p)/2}, \quad 2r + p \leq 2, \quad i = 1, 2, \tag{16}$$

in $\mathbb{R}_T^2$.

If the functions $V_i(t)$, $i = 1, 2$, in (15) are bounded and continuous in $[0, \infty)$, then the functions $u_{i1}(t, x)$ are continuous in $\overline{\mathbb{R}_{\infty}^2}$, bounded with respect to the argument x , satisfy equation (6) in the domains $(t, x) \in (0, \infty) \times D_1$ and $(t, x) \in (0, \infty) \times D_2$, and the initial condition $u_{i1}(0, x) = 0$, $x \in \mathbb{R}$, $i = 1, 2$, holds.

Note another important property of the simple layer potential $u_{i1}(t, x)$ concerning the behavior of its derivatives with respect to the argument x in a neighborhood of the point $x = 0$. The so-called jump formula is known in this case (see [5, Chapter II, §5], [8, Chapter IV, §15]). For the case under consideration, this result is written as

$$D_x u_{i1}(t, 0\pm) = \int_0^t D_x g_{i1}(t - \tau, 0, 0) V_i(\tau) d\tau \mp \frac{V_i(t)}{b_i(0)}, \quad t > 0, \quad i = 1, 2. \tag{17}$$

The existence of the integral in (17) follows from inequality (12) with $r = 0$, $p = 1$, and $x = y = 0$.

The properties of the simple layer potential mentioned above hold under even more general assumptions concerning the functions V_i , $i = 1, 2$, in the integrals on the right hand side of (15).

3. SOLUTION OF THE PARABOLIC PROBLEM OF CONJUGATION

Consider the problem (6)–(9). First we prove that a solution $u(t, x)$ exists. We search for a solution represented as a sum of two potentials,

$$u(t, x) = u_{i0}(t, x) + u_{i1}(t, x), \quad (t, x) \in [0, \infty) \times \overline{D_i}, \quad i = 1, 2, \tag{18}$$

where the functions u_{i0} and u_{i1} are defined by (14) and (15), respectively, and where the densities V_i , $i = 1, 2$, are unknown functions. To find the densities, we use the

conjugation conditions (8) and (9). Substituting the right hand sides of relations (18) into equalities (8) and (9) and taking (17) into account, we obtain the following system of integral equations with respect to V_i , $i = 1, 2$:

$$(19) \quad \begin{aligned} \sum_{i=1}^2 \int_0^t (-1)^{i-1} g_i(t-\tau, 0, 0) V_i(\tau) d\tau &= \Phi(t), \quad t > 0, \\ \sum_{i=1}^2 \left(\frac{q_i}{b_i(0)} V_i(t) - \int_0^t K_i(t-\tau) V_i(\tau) d\tau \right) &= \Psi(t), \quad t > 0, \end{aligned}$$

where

$$\begin{aligned} \Phi(t) &= u_{20}(t, 0) - u_{10}(t, 0), \\ \Psi(t) &= \sum_{i=1}^2 \left((-1)^i q_i D_x u_{i0}(t, 0) + \int_{D_i} (u_{i0}(t, y) - u_{i0}(t, 0)) \mu(dy) \right), \\ K_i(t-\tau) &= (-1)^i q_i D_x g_{i1}(t-\tau, 0, 0) + \int_{D_i} (g_i(t-\tau, y, 0) - g_i(t-\tau, 0, 0)) \mu(dy), \\ &\quad i = 1, 2. \end{aligned}$$

The first equation in the system of integral equations (19) is the Volterra equation of the first kind, while the second is the Volterra equation of the second kind. Following Holmgren's idea [9], we reduce the Volterra equation of the first kind to the Volterra equation of the second kind. First we introduce the integro-differential operator $\mathcal{E}$ whose action is given by

$$(20) \quad \mathcal{E}(t)\Phi = \sqrt{\frac{2}{\pi}} \frac{d}{dt} \int_0^t (t-s)^{-1/2} \Phi(s) ds, \quad t > 0.$$

Denote the function on the right hand side of (20) by $\Phi_0(t)$. Since the function $\Phi(t)$ in (19) is continuously differentiable for $t > 0$, we easily check that the right hand side of (20) can be written as

$$(21) \quad \Phi_0(t) \equiv \mathcal{E}(t)\Phi = \frac{1}{\sqrt{2\pi}} \int_0^t (t-s)^{-3/2} (\Phi(t) - \Phi(s)) ds + \sqrt{\frac{2}{\pi}} \Phi(t) t^{-1/2}, \quad t > 0.$$

Moreover,

$$(22) \quad |\Phi_0(t)| \leq C \|\varphi\| t^{-1/2}$$

for $t \in (0, T]$, where T is an arbitrary number.

Indeed, inequality (22) for the second term on the right hand side of (21) follows from the bound (16) with $r = p = 0$. To estimate the integral on the right hand side of (21), we split it into two parts, namely

$$\int_0^t = \int_0^{t/2} + \int_{t/2}^t.$$

Applying estimate (16) for integrals involving the function Φ and the Lagrange mean value theorem for the difference $\Phi(t) - \Phi(s)$ in the second integral, we prove inequality (22) for the integral term in (21), whence this bound follows for $\Phi_0(t)$. Note that the function $\Psi(t)$ in (19) satisfies inequality (22), too. This follows from the mean value theorem for $u_{i0}(t, y) - u_{i0}(t, 0)$, $i = 1, 2$, bounds (16), and condition (4). However, the constant C in (22) depends on T and on δ in this case.

A simple algebra checks that the operator $\mathcal{E}$ applied to both sides of the first equation in (19) transforms it to the Volterra integral equation of the second kind. This changes

the system of equations (19) to the equivalent system of two integral equations with respect to V_i , $i = 1, 2$, namely

$$(23) \quad V_i(t) = \sum_{j=1}^2 \int_0^t K_{ij}(t-\tau) V_j(\tau) d\tau + \Psi_i(t), \quad t > 0, \quad i = 1, 2,$$

where

$$\begin{aligned} \Psi_i(t) &= d_i \left(\Psi(t) + \frac{(-1)^{i+1} q_{3-i}}{\sqrt{b_{3-i}(0)}} \Phi_0(t) \right), \quad i = 1, 2, \\ K_{ij}(t-\tau) &= d_i \left(K_j(t-\tau) + \frac{(-1)^{i-1} q_{3-i}}{\sqrt{b_{3-i}(0)}} R_j(t-\tau) \right), \quad i, j = 1, 2, \\ d_i &= \frac{\sqrt{b_i(0)}}{\frac{q_1}{\sqrt{b_1(0)}} + \frac{q_2}{\sqrt{b_2(0)}}}, \quad i = 1, 2, \\ R_j(t-\tau) &= \frac{(-1)^j}{\sqrt{2\pi}} \left(\int_\tau^t (t-s)^{-3/2} (g_{j1}(t-\tau, 0, 0) - g_{j1}(s-\tau, 0, 0)) ds \right. \\ &\quad \left. + 2(t-\tau)^{-1/2} g_{j1}(t-\tau, 0, 0) \right), \quad j = 1, 2. \end{aligned}$$

The properties of $\Psi(t)$ and $\Phi_0(t)$ imply that $\Psi_i(t)$, $i = 1, 2$, are continuous functions for $t > 0$ and satisfy the following bound:

$$(24) \quad |\Psi_i(t)| \leq M(\delta) \|\varphi\| t^{-1/2}, \quad i = 1, 2,$$

where $t \in (0, T]$ and $M(\delta)$ is a constant.

Now we study the kernels in the integral equations (23). Let $D_{j,\delta} = \{y \in D_j : |y| > \delta\}$, $j = 1, 2$. Using (10) we represent K_{ij} as follows:

$$\begin{aligned} (25) \quad K_{ij}(t-\tau) &= d_i \left(K_j^{(1)}(t-\tau) + \frac{(-1)^{i+1} q_{3-i}}{\sqrt{b_{3-i}(0)}} R_j(t-\tau) \right) + d_i K_j^{(2)}(t-\tau) \\ &= K_{ij}^{(1)}(t-\tau) + K_{ij}^{(2)}(t-\tau), \quad i, j = 1, 2, \end{aligned}$$

where

$$\begin{aligned} K_j^{(1)}(t-\tau) &= (-1)^j q_j D_x g_{j1}(t-\tau, 0, 0) + \int_{D_{j,\delta}} (g_j(t-\tau, y, 0) - g_j(t-\tau, 0, 0)) \mu(dy) \\ &\quad + \int_{D_j \setminus D_{j,\delta}} (g_{j1}(t-\tau, y, 0) - g_{j1}(t-\tau, 0, 0)) \mu(dy), \\ K_j^{(2)}(t-\tau) &= \int_{D_j \setminus D_{j,\delta}} (g_{j0}(t-\tau, y, 0) - g_{j0}(t-\tau, 0, 0)) \mu(dy). \end{aligned}$$

Using the mean value theorem for $g_{j1}(t-\tau, y, 0) - g_{j1}(t-\tau, 0, 0)$, the bounds (11) and (12), and condition (4), we obtain the inequality

$$(26) \quad |K_{ij}^{(1)}(t-\tau)| \leq N(\delta) (t-\tau)^{-1+\alpha/2}, \quad i, j = 1, 2,$$

for $0 \leq \tau < t \leq T$ and some constant $N(\delta)$.

Following the same method for the integral $K_j^{(2)}(t-\tau)$, we get

$$|K_{ij}^{(2)}(t-\tau)| \leq C(t-\tau)^{-1}.$$

This inequality means that the corresponding part of every kernel K_{ij} has a nonintegrable singularity. Nevertheless, we prove that the method of sequential approximations applies to the system (23).

We find a solution of the system (23) such that

$$(27) \quad V_i(t) = \sum_{k=0}^{\infty} V_i^{(k)}(t), \quad i = 1, 2,$$

where

$$\begin{aligned} V_i^{(0)}(t) &= \Psi_i(t), \quad i = 1, 2, \\ V_i^{(k)}(t) &= \sum_{j=1}^2 \int_0^t K_{ij}(t-\tau) V_j^{(k-1)}(\tau) d\tau, \quad i = 1, 2, \quad k \in \mathbb{N}. \end{aligned}$$

First we estimate $V_i^{(1)}(t)$. Applying (25), we obtain

$$V_i^{(1)} = \sum_{j=1}^2 \int_0^t K_{ij}^{(1)}(t-\tau) \Psi_j(\tau) d\tau + \sum_{j=1}^2 \int_0^t K_{ij}^{(2)}(t-\tau) \Psi_j(\tau) d\tau = I_{i1} + I_{i2}, \quad i = 1, 2.$$

The first integral on the right hand side is estimated with the help of inequalities (26) and (27). Indeed,

$$\begin{aligned} (28) \quad |I_{i1}| &\leq M(\delta) \|\varphi\| 2N(\delta) \int_0^t (t-\tau)^{-1+\alpha/2} \tau^{-1/2} d\tau \\ &= M(\delta) \|\varphi\| \frac{2N(\delta) \Gamma(\frac{\alpha}{2}) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + \frac{\alpha}{2})} t^{-1/2+\alpha/2}, \quad t \in (0, T], \quad i = 1, 2. \end{aligned}$$

To estimate the second term I_{i2} , we use the equalities

$$\begin{aligned} g_{j0}(t-\tau, y, 0) - g_{j0}(t-\tau, 0, 0) &= \frac{1}{\sqrt{2\pi b_j(0)(t-\tau)}} \int_0^1 \frac{d}{d\Theta} \exp\left(-\frac{y^2 \Theta}{2b_j(0)(t-\tau)}\right) d\theta, \\ \int_0^t (t-\tau)^{-3/2} \tau^{-1/2} \exp\left(-\frac{y^2 \Theta}{2b_j(0)(t-\tau)}\right) d\Theta &= \frac{\sqrt{2\pi b_j(0)}}{|y|\sqrt{\theta t}} \exp\left(-\frac{y^2 \Theta}{2b_j(0)t}\right) \end{aligned}$$

for $j = 1, 2$. Then we use estimate (24) and the inequality

$$\frac{d_i}{b_j(0)} \leq d = \frac{b^+(0)}{b^-(0)(q_1 + q_2)}, \quad i, j = 1, 2,$$

where $b^+(0) = \max(b_1(0), b_2(0))$ and $b^-(0) = \min(b_1(0), b_2(0))$. As a result, we get

$$\begin{aligned} (29) \quad |I_{i2}| &\leq M(\delta) \|\varphi\| d_i \sum_{j=1}^2 \frac{1}{2\sqrt{2\pi}(b_j(0))^{3/2}} \int_{D_j \setminus D_{j,\delta}} y^2 \mu(dy) \\ &\quad \times \int_0^1 d\Theta \int_0^t (t-\tau)^{-3/2} \tau^{-1/2} \exp\left(-\frac{y^2 \Theta}{2b_j(0)(t-\tau)}\right) d\tau \\ &\leq M(\delta) \|\varphi\| t^{-1/2} d \sum_{j=1}^2 \int_{D_j \setminus D_{j,\delta}} |y| \mu(dy) \\ &= M(\delta) \|\varphi\| t^{-1/2} d\lambda(\delta), \quad t \in (0, T], \quad i = 1, 2. \end{aligned}$$

Now consider $d\lambda(\delta)$. Condition (4) implies that $\lambda(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. This allows one to choose a number $\delta = \delta_0$ such that $d_0 = d\lambda(\delta_0) < 1$. In what follows, all the constants

depending on δ are considered for $\delta = \delta_0$. Combining the estimates (28) and (29), we get

$$(30) \quad \left| V_i^{(1)}(t) \right| \leq M(\delta_0) \|\varphi\| t^{-1/2} \left(\frac{2N(\delta_0) \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right)} t^{\alpha/2} + d_0 \right), \quad i = 1, 2,$$

for $t \in (0, T]$. Let $M = M(\delta_0)$, $N = N(\delta_0)$, and

$$h_t = \frac{2N\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right)} t^{\alpha/2}$$

in the right hand side of (30). Using the induction in k for $V_i^{(k)}(t)$, $t \in (0, T]$, $i = 1, 2$, we prove the estimate

$$\left| V_i^{(k)}(t) \right| \leq M \|\varphi\| t^{-1/2} \sum_{m=0}^k \binom{n}{k} h_t^{(k-n)} d_0^n, \quad k = 0, 1, \dots,$$

where

$$h_t^{(k)} = \frac{(2N\Gamma\left(\frac{\alpha}{2}\right))^k \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+k\alpha}{2}\right)} t^{k\alpha/2}.$$

This implies that

$$\begin{aligned} \sum_{k=0}^{\infty} \left| V_i^{(k)}(t) \right| &\leq M \|\varphi\| t^{-1/2} \sum_{k=0}^{\infty} \sum_{n=0}^k \binom{n}{k} h_t^{(k-n)} d_0^n \\ &= M \|\varphi\| t^{-1/2} \sum_{k=0}^{\infty} h_t^{(k)} \sum_{n=0}^{\infty} \binom{k+n}{n} d_0^n \\ (31) \quad &= M \|\varphi\| t^{-1/2} \sum_{k=0}^{\infty} \frac{h_t^{(k)}}{(1-d_0)^{k+1}} \\ &= M \|\varphi\| t^{-1/2} \sum_{k=0}^{\infty} \frac{(2N\Gamma\left(\frac{\alpha}{2}\right))^k \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+k\alpha}{2}\right) (1-d_0)^{k+1}} t^{k\alpha/2} \end{aligned}$$

for $t \in (0, T]$ and $i = 1, 2$. Inequality (31) implies the convergence of series (27) for $t > 0$ and the bound

$$(32) \quad |V_i(t)| \leq c \|\varphi\| t^{-1/2}, \quad i = 1, 2,$$

where $t \in (0, T]$ and C is a constant.

Estimates (11) for $r = p = 0$ and (32) prove that the integrals on the right hand side of (15) exist and that

$$(33) \quad |u_{i1}(t, x)| \leq C \|\varphi\|, \quad i = 1, 2,$$

in the domain $\mathbb{R}_T^2$.

A further analysis of series (27) and integrals in (15) shows that the functions $u_{i1}(t, x)$ are continuous for $t = 0$ and $x \in \mathbb{R}$ as well, and that $u_{i1}(0, x) = 0$, $i = 1, 2$. Thus we conclude that the function $u(t, x)$ defined by relations (18) and (27) is a solution of the problem (6)–(9).

The proof of the uniqueness of a solution of the problem (6)–(9) follows the lines of the proof of an analogous result in [6].

The results proved above are stated below.

Theorem 3.1. *Let the coefficients of the operators L_i , $i = 1, 2$, in (1) satisfy assumptions a) and b) (in §2), while the parameters q_1 , q_2 , and the measure $\mu(\cdot)$ in (3) satisfy conditions (5) and (4), respectively. Then the problem (6)–(9) has a unique solution*

$$(34) \quad u \in C(\overline{\mathbb{R}_\infty^2}) \cap C^{1,2}((0, \infty) \times D_i), \quad i = 1, 2,$$

for all functions $\varphi \in C_b(\mathbb{R})$. Moreover

$$(35) \quad |u(t, x)| \leq C \|\varphi\|, \quad (t, x) \in \overline{\mathbb{R}_T^2}.$$

Theorem 3.2. *A solution of the problem (6)–(9) that belongs to the class (34) is the sum of potentials (14) and (15), where V_i , $i = 1, 2$, is a solution of the system of integral equations (23).*

4. CONSTRUCTION OF THE PROCESS

Theorems 3.1 and 3.2 imply that a family of operators $(\mathcal{T}_t)_{t \geq 0}$ can be defined with the help of a solution of the problem (6)–(9) (the operators act in the space $C_b(\mathbb{R})$). For $t > 0$, $x \in \mathbb{R}$, and $\varphi \in C_b(\mathbb{R})$ put

$$(36) \quad \mathcal{T}_t \varphi(x) = \int_{\mathbb{R}} g_i(t, x, xy) \varphi(y) dy + \int_0^t g_i(t - \tau, x, 0) V_i(\tau, \varphi) d\tau$$

for $t > 0$, $x \in \overline{D_i}$, and $i = 1, 2$, where $V_i(t, \varphi) \equiv V_i(t)$, $i = 1, 2$, is a solution of the system of integral equations (23) defined by relation (27). Note that $\mathcal{T}_0 = I$, where I is the unit operator, and that $\mathcal{T}_t \varphi(x)$ admits bound (35) in the domain $(t, x) \in \overline{\mathbb{R}_T^2}$.

The above integral representation for the family of operators $(\mathcal{T}_t)_{t \geq 0}$ implies the following properties:

- 1) if $\varphi_n \in C_b(\mathbb{R})$, $n \in \mathbb{N}$, $\sup_n \|\varphi_n\| < \infty$, and if $\lim_{n \rightarrow \infty} \varphi_n(x) = \varphi(x)$ for all $x \in \mathbb{R}$, where $\varphi \in C_b(\mathbb{R})$, then for all $t \geq 0$ and $x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathcal{T}_t \varphi_n(x) = \mathcal{T}_t \varphi(x);$$

- 2) for all $t_1 \geq 0$ and $t_2 \geq 0$,

$$\mathcal{T}_{t_1+t_2} = \mathcal{T}_{t_1} \cdot \mathcal{T}_{t_2};$$

- 3) $\mathcal{T}_t \varphi(x) \geq 0$ for all $t \geq 0$ and $x \in \mathbb{R}$ if $\varphi \in C_b(\mathbb{R})$ and $\varphi(x) \geq 0$;
- 4) $\|\mathcal{T}_t\| \leq 1$ for all $t \geq 0$.

We outline the proof of these properties. Property 1) follows from the equality

$$\lim_{n \rightarrow \infty} V_i(t, \varphi_n) = V_i(t, \varphi), \quad t > 0, \quad i = 1, 2,$$

and from the Lebesgue dominated convergence theorem. Property 2), called the semigroup property of operators $\mathcal{T}_t$, follows from the uniqueness of a solution of the problem (6)–(9) proved in Theorem 3.1. Property 3) means that the cone of nonnegative functions is invariant under the semigroup $\mathcal{T}_t$. The proof of this property is the same as that of a lemma in [4] (see also [5, p. 82]) and uses the maximum principle for parabolic equations. Finally, property 4) means that $\mathcal{T}_t$ is a contraction operator for all $t \geq 0$. To prove this property, we note that $\mathcal{T}_t \varphi_0(x) \equiv 1$ for all $t \geq 0$ and $x \in \mathbb{R}$ if $\varphi_0(x) \equiv 1$; then we use property 3).

Properties 1)–4) imply that the semigroup of operators $\mathcal{T}_t$, $t \geq 0$, constructed by using equalities (36) and (27), defines a homogeneous Feller process. The transition probabilities of this process are denoted by $P(t, x, dy)$. Thus

$$(37) \quad \mathcal{T}_t \varphi(x) = \int_{\mathbb{R}} P(t, x, dy) \varphi(y).$$

Therefore we proved the following result.

Theorem 4.1. *The semigroup of operators defined by (36) and (27) generates a homogeneous Feller process in $\mathbb{R}$ that coincides at inner points of the domains D_1 and*

D_2 with the two given diffusion processes generated by the operators L_1 and L_2 , respectively. The behavior of this Feller process at the point $x = 0$ is determined by the pasting condition (3).

Remark 4.1. Conditions (4) and (5) imply that the construction of the Feller process presented in Theorem 4.1 also works for the case where $m = \mu(D_1 \cup D_2) = 0$. As noted above, this case is studied in the paper [4]. In particular, it is shown in [4] that the trajectories of the constructed process are continuous and that it can be treated as a generalized diffusion process in the sense of Portenko [5]. This means that the diffusion coefficients for such a process are generalized functions of the Dirack δ -function type concentrated at the point $x = 0$ (more precisely, only the drift coefficient is a generalized function in this case).

We now turn back to the process constructed above. The presence of the integral term in the pasting condition (3) means that the trajectories of the process are, in general, discontinuous. On the other hand, condition (5) guarantees the positive probability that the original diffusion processes, after they pass through the point $x = 0$, may continue without breaks in their trajectories, that is, continuously. A natural question arises about the influence of the jump measure of the process $\mu(\cdot)$ on the local behavior at the point $x = 0$ of a diffusion particle moving along continuous trajectories. A partial answer to this question can be obtained by defining the generalized diffusion coefficients of the process constructed above. The existence of these characteristics can be proved under the condition that the measure $\mu(\cdot)$ in (3) satisfies the following additional conditions:

$$(38) \quad \int_{D_1 \cup D_2} |y| \mu(dy) < \infty, \quad \int_{D_1 \cup D_2} y^2 \mu(dy) < \infty.$$

If conditions (38) hold, then equality (13) implies that

$$(39) \quad \begin{aligned} \lim_{t \downarrow 0} \int_{\mathbb{R}} \varphi(x) \left(\frac{1}{t} \int_{\mathbb{R}} (y - x) P(t, x, dy) \right) dx &= \int_{\mathbb{R}} a(x) \varphi(x) dx + a_0 \varphi(0), \\ \lim_{t \downarrow 0} \int_{\mathbb{R}} \varphi(x) \left(\frac{1}{t} \int_{\mathbb{R}} (y - x)^2 P(t, x, dy) \right) dx &= \int_{\mathbb{R}} b(x) \varphi(x) dx + b_0 \varphi(0) \end{aligned}$$

for all continuous real-valued compactly supported functions $\varphi(x)$, $x \in \mathbb{R}$, where

$$\begin{aligned} a(x) &= \begin{cases} a_i(x), & x \in D_i, \ i = 1, 2, \\ \sum_{i=1}^2 l_i a_i(0), & x = 0, \end{cases} \\ b(x) &= \begin{cases} b_i(x), & x \in D_i, \ i = 1, 2, \\ \sum_{i=1}^2 l_i b_i(0), & x = 0, \end{cases} \\ l_i &= \frac{q_i \sqrt{b_{3-i}(0)}}{q_1 \sqrt{b_2(0)} + q_2 \sqrt{b_1(0)}}, \quad i = 1, 2, \quad l_1 + l_2 = 1, \\ a_0 &= \frac{1}{2}(d_1 + d_2)(q_2 - q_1 + m_1), \quad b_0 = \frac{1}{2}(d_1 + d_2)m_2, \\ m_1 &= \int_{D_1 \cup D_2} y \mu(dy), \quad m_2 = \int_{D_1 \cup D_2} y^2 \mu(dy). \end{aligned}$$

Equalities (39) mean that the diffusion coefficients exist in a generalized sense for the process constructed above; namely, there exists a diffusion coefficient $b(x) + b_0 \delta(x)$ and drift coefficient $a(x) + a_0 \delta(x)$, where $\delta(x)$ is the Dirack δ -function.

The following result complements Theorem 4.1.

Theorem 4.2. *Suppose the coefficients of the differential operators L_i , $i = 1, 2$, in (1) satisfy conditions a) and b). Assume further that the parameters q_i , $i = 1, 2$, and measure $\mu(\cdot)$ in (3) satisfy conditions (5) and (38), respectively. Then the semigroup of operators $\mathcal{T}_t$, $t \geq 0$, constructed by using equalities (36) and (27), generates a generalized diffusion process in $\mathbb{R}$ whose transition probability satisfies relation (39).*

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