LIGHT TAILED ASYMPTOTICS IN AN UNRELIABLE $M/G/1$ RETRIAL QUEUE

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ABSTRACT. We consider the standard unreliable $M/G/1$ retrial queuing system with active and passive breakdowns. The explicit expressions of the probability generating functions of distribution of server state and orbit size are well known from early works. However, some problems particularly related to Cybernetic and Artificial Intellect need to save computational effort. So, we give here another look to solve this problem more simply, but under some light tailed assumptions.

1. INTRODUCTION

Queueing Systems with breakdowns have been the subject of many investigations since they are interesting modeling tools in many modern systems: production, and telecommunication, computer science, finance and so on. Such studies are interesting since they create a gap between Queueing Theory and Reliability theory in the context of probabilistic framework, which is one of the logics for uncertainty problems [7], [6].

In recent years, particular attention has been devoted to Retrial Queueing Systems for their interest in many modern systems (call centers, mobile systems, CSMA/CD protocol with star topology, random access protocols. . .); see, for example, [11] and the references in the didactical book [9], or bibliographical surveys of [3, 4].

Early works were concerned with analytical or algorithmic resolution of the underlying problems; see, for example, [6, 5]. Asymptotic methods are interesting in the sense that we can save computational effort for the resolution of the problem under study; see [2]. We refer also to some alternative approaches to study more qualitative properties such as stability (see [10]) or monotonicity and comparability methods; see [12]. The last paper gives also a short survey about recent advances in Unreliable Retrial Queues.

In this paper, we consider the approximation of the queue size of the unreliable $M/G/1$ retrial queue under light tailed assumptions. In the next section we describe the mathematical model and in section 3 we provide some preliminary results. The light tailed approximation is given in section 4.

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2. THE MATHEMATICAL MODEL

We consider the standard $M/G/1$ retrial queuing system with active and passive breakdowns, with the following parameters and assumptions:

- λ is the arrival rate of primary customers.
- η (resp. μ) is the Poisson rate of passive (resp. active) breakdowns.
- $F(x)$ is the probability distribution (p.d.) of the service time.
- $G(x)$ (resp. $H(x)$) is the p.d. of duration of passive D_i (resp. active D_b) interruption.
- η is the retrial rate of each secondary call, i.e., the retrial protocol is such that the probability of a retrial during $(t, t+dt)$ is proportional to the number of units in orbit.
- If a customer whose service is interrupted by an active breakdown needs to leave the service zone, then the displaced customer must decide either to join the retrial group (with probability c), or leave the system after interruption (with probability $1 - c$).

Now, if $A(x)$ is one of the above p.d.'s, we denote by $\tilde{A}(s)$, $\text{Re}(s) \geq 0$ its Laplace–Stieltjes transform. Let

- $X(t) = 0$, if the server is operative and idle at time t ,
- $X(t) = 1$, if the server is operative and busy at time t ,
- $X(t) = 2$, if the server is down due to an independent breakdown at time t ,
- $X(t) = 3$, if the server is down due to an active breakdown at time t .

Let $Q(t)$ be the number of customers in orbit at time t . We are interested in the limiting probability

$$P_{in} = \lim_{t \rightarrow \infty} \mathbf{P}\{X(t) = i, Q(t) = n\},$$

$$i = 0, \dots, 3, n \geq 0.$$

The explicit expressions of the probability generating functions of distribution of server state and orbit size are well known from [1] and we recall them in the next section.

3. PRELIMINARY RESULTS

Consider the Standard $M/G/1$ Retrial Queue as described in section 1. We assume that the following ergodicity condition holds:

$$(1) \quad \rho = \lambda \frac{1 - \tilde{F}(\mu)}{\mu} \left(1 + \mu \left(\mathbf{E}(D_b) + \frac{c}{\mu} \right) \right) < 1.$$

Denote by

$$P_{in} = \lim_{t \rightarrow \infty} \mathbf{P}\{X(t) = i, Q(t) = n\}, \quad i = 0, \dots, 3, n \geq 0$$

and

$$P_i(z) = \sum_{n=0}^{\infty} z^n P_{in}, \quad i = 0, \dots, 3, |z| < 1.$$

From theorem 4.2. of [1] we know the explicit expression of these generating functions:

$$(2) \quad P_0(z) = \frac{K}{A} \exp\left(\frac{\lambda}{\theta} \int_1^z \frac{1 - N(\lambda - \lambda u)}{N(\lambda - \lambda u) - u} du + \frac{\eta}{\theta} \int_1^z \frac{1 - \tilde{G}(\lambda - \lambda u)}{N(\lambda - \lambda u) - u} du\right),$$

$$(3) \quad P_1(z) = \frac{1 - \tilde{F}(\lambda - \lambda z + \mu)}{\lambda - \lambda z + \mu} \times \frac{\lambda - \lambda z + \eta - \eta \tilde{G}(\lambda - \lambda z)}{N(\lambda - \lambda z) - z} P_0(z),$$

$$(4) \quad P_2(z) = \eta \frac{1 - \tilde{G}(\lambda - \lambda z)}{\lambda - \lambda z} P_0(z),$$

$$(5) \quad P_3(z) = (1 - c + cz) \mu \frac{1 - \tilde{H}(\lambda - \lambda z)}{\lambda - \lambda z} P_1(z),$$

where

$$K = \frac{1 - \rho}{\lambda(1 + \eta \mathbf{E}(D_i)) + \eta(1 - \rho)}$$

and

$$(6) \quad A = K(1 + \eta \mathbf{E}(D_i)) + (1 - \tilde{F}(\mu))(1 - \eta K) \left(\mathbf{E}(D_b) + \frac{1}{\mu} \right).$$

These relations are obtained in different ways (method of imbedded Markov chain, method of supplementary variable, algorithmic methods...); see [1] and its references.

Recall that the function $N(\lambda - \lambda z)$ has the following interpretation. Consider the random variable Φ representing the fundamental period, i.e., the duration determined by the competition between the service and the arrival of an active interruption. Thus the Laplace-Stieltjes of the distribution of Φ is given by

$$f(s) = \tilde{F}(s + \mu) + \frac{\mu \tilde{H}(s)(1 - \tilde{F}(s + \mu))}{s + \mu}.$$

Let N be the number of customers that joins the retrial group during the period $[0, \Phi]$. Then the distribution of N is given by [1]

$$N(\lambda - \lambda z) = \mathbf{E}(z^N) = \tilde{F}(\lambda - \lambda z + \mu) + (1 - c + cz) \frac{\mu \tilde{H}(\lambda - \lambda z)(1 - \tilde{F}(\lambda - \lambda z + \mu))}{\lambda - \lambda z + \mu}.$$

4. LIGHT TAILED APPROXIMATION

The behavior of the orbit size distribution is based on the analytical properties of probability generating functions (2)–(5).

The light tailed asymptotic of the queue size distribution in the standard Unreliable $M/G/1$ Retrial Queue is based on the assumption that the generalized service time distribution Φ has a finite exponential moment, i.e.,

$$(7) \quad \gamma = \sup \{t \in \mathbb{C}: \mathbf{E}(e^{t\Phi}) < \infty, \mathbf{E}(e^{tD_i}) < \infty\} > 0.$$

Using this assumption, we can locate the zeros of $N(\lambda - \lambda z) - z$, $|z| \leq \sigma$. More precisely, the analytic function $N(\lambda - \lambda z) - z$, $|z| \leq 1 + \gamma/\lambda$ has simple zero at 1 and σ . Furthermore, it has no other zeros on $\{z \in \mathbb{C}: |z| \leq \sigma\}$, where $\sigma = \sigma(\lambda, \mu, c, F(\cdot), H(\cdot))$ is the unique root of the equation

$$(8) \quad N(\lambda - \lambda z) - z = \sigma, \quad 1 < \sigma < 1 + \frac{\gamma}{\lambda}.$$

The proof is similar to that of Lemma 1 of [8]. The function $f(z) = N(\lambda - \lambda z)$, $|z| < 1 + \gamma/\lambda$ has a simple zero at $z = 1$ since $f'(1) < 0$. Since $f(z)$ is strictly convex in $z \in (0, 1 + \gamma/\lambda)$ and $f(1) = f(\sigma) = 0$, then $f(z) < 0$ for $1 < z < \sigma$. The assertion is a consequence of the Rouché's theorem.

Now, the function

$$g(z) = \frac{1 - N(\lambda - \lambda z)}{N(\lambda - \lambda z) - z}$$

is analytic at $z = 1$, since it has a removable singularity at this point. On the other hand, this function has a simple pole at $z = \sigma$ and the residue at this point $\text{Re } s$,

$$g(\sigma) = \frac{-(\sigma - 1)}{-\lambda N'(\lambda - \lambda\sigma) - 1}.$$

So, the function

$$h(z) = \frac{1 - N(\lambda - \lambda z)}{N(\lambda - \lambda z) - z} + \frac{-(\sigma - 1)}{-\lambda N'(\lambda - \lambda\sigma) - 1} \frac{1}{z - \sigma}$$

has a removable singularity at $z = \sigma$. Therefore, the function $g(z)$ can be decomposed as a sum of the principal part at $z = \sigma$ and the analytical part. More precisely, we can find $\delta' \in (0, 1 + \gamma/\lambda - \sigma)$ and an analytical function $P(z)$ on $\Theta_{\delta'} = \{z \in \mathbb{C} : |z| < \sigma + \delta'\}$ such that

$$(9) \quad \frac{1 - N(\lambda - \lambda z)}{N(\lambda - \lambda z) - z} = -\frac{\sigma - 1}{-\lambda N'(\lambda - \lambda\sigma) - 1} \frac{1}{z - \sigma} + P(z), \quad |z| < \sigma + \delta', \quad z \neq \sigma,$$

where

$$P(z) = \begin{cases} \frac{1 - N(\lambda - \lambda z)}{N(\lambda - \lambda z) - z} + \frac{\sigma - 1}{-\lambda N'(\lambda - \lambda\sigma) - 1} \frac{1}{z - \sigma}, & |z| < \sigma + \delta', \quad z \neq \sigma, \\ \lim_{u \rightarrow \sigma} \frac{1 - N(\lambda - \lambda u)}{N(\lambda - \lambda u) - u}, & z = \sigma. \end{cases}$$

In a similar way, we can find δ'' and an analytical function $Q(z)$ on $\Theta_{\delta''}$ such that

$$(10) \quad \frac{1 - \tilde{G}(\lambda - \lambda z)}{N(\lambda - \lambda z) - z} = -\frac{\lambda \tilde{G}'(\lambda - \lambda\sigma)}{-\lambda N'(\lambda - \lambda\sigma) - 1} \frac{1}{z - \sigma} + Q(z), \quad |z| < \sigma + \delta'', \quad z \neq \sigma,$$

where

$$Q(z) = \begin{cases} \frac{1 - \tilde{G}(\lambda - \lambda z)}{N(\lambda - \lambda z) - z} + \frac{\lambda \tilde{G}'(\lambda - \lambda\sigma)}{-\lambda N'(\lambda - \lambda\sigma) - 1} \frac{1}{z - \sigma}, & |z| < \sigma + \delta'', \quad z \neq \sigma, \\ \lim_{u \rightarrow \sigma} \frac{1 - \tilde{G}(\lambda - \lambda u)}{N(\lambda - \lambda u) - u}, & z = \sigma. \end{cases}$$

Next, we take the domain $|z| < \sigma + \delta$, where $\delta = \min(\delta', \delta'')$ gives the range of validity of the two above decompositions. Substituting (9) and (10) into (2), we have

$$(11) \quad \begin{aligned} P_0(z) &= E(z^Q; X = 0) \\ &= \frac{K}{A} \exp \left(\frac{\lambda}{\theta} \int_1^z \frac{1 - N(\lambda - \lambda u)}{N(\lambda - \lambda u) - u} du + \frac{\eta}{\theta} \int_1^z \frac{1 - \tilde{G}(\lambda - \lambda u)}{N(\lambda - \lambda u) - u} du \right). \end{aligned}$$

Thus,

$$\begin{aligned} P_0(z) &= \frac{K}{A} \exp \left(\frac{\lambda}{\theta} \int_1^z \frac{a}{u - \sigma} du \right) \exp \left(\frac{\lambda}{\theta} \int_1^z P(u) du \right) \\ &\quad \times \exp \left(\frac{\eta}{\theta} \int_1^z \frac{b}{u - \sigma} du \right) \exp \left(\frac{\eta}{\theta} \int_1^z Q(u) du \right), \end{aligned}$$

where

$$a = \frac{\lambda}{\theta} \frac{\sigma - 1}{-\lambda N'(\lambda - \lambda\sigma)}$$

and

$$b = \frac{\eta}{\theta} \frac{\tilde{G}(\lambda - \lambda\sigma) - 1}{-\lambda N'(\lambda - \lambda\sigma)}.$$

Finally, we get

$$(12) \quad P_0(z) = \frac{K}{A} \left(\frac{\sigma - z}{\sigma - 1} \right)^{-a-b} \Lambda(z), \quad |z| < \sigma,$$

where

$$(13) \quad \Lambda(z) = \exp \left(\frac{\lambda}{\theta} \int_1^z P(u) du + \frac{\eta}{\theta} \int_1^z Q(u) du \right).$$

Since $P(z)$ and $Q(z)$ are analytic on $\{z \in \mathbb{C}: |z| < \sigma + \delta\}$, so is $\Lambda(z)$.

Consider the power series expansion of the function $\Lambda(z)$ at $z = 0$, i.e.,

$$(14) \quad \Lambda(z) = \sum_{n=0}^{\infty} \varphi_n z^n, \quad |z| < \sigma + \delta.$$

On the other hand, we note that we have the following power series expansion for the first factor

$$(15) \quad \frac{K}{A} \left(\frac{\sigma - z}{\sigma - 1} \right)^{-a-b} = \frac{K}{A} \left(\frac{\sigma - 1}{\sigma} \right)^{a+b} \sum_{n=0}^{\infty} \frac{\Gamma(a+b+n)}{\Gamma(a+b)\Gamma(n+1)} \sigma^{-n} z^n.$$

By using verbatim the computation [8] we show

$$(16) \quad \begin{aligned} \lim_{n \rightarrow \infty} \frac{P(X=0, Q=n)}{(\Gamma(a+b+n)/\Gamma(a+b)\Gamma(n+1))\sigma^{-n}} &= \frac{K}{A} \left(\frac{\sigma - 1}{\sigma} \right)^{a+b} \sum_{n=0}^{\infty} \varphi_n \sigma^n \\ &= \frac{K}{A} \left(\frac{\sigma - 1}{\sigma} \right)^{a+b} \Lambda(z) = \Gamma(a+b)c_0, \end{aligned}$$

where

$$(17) \quad \begin{aligned} c_0 &= \frac{K}{A\Gamma(a+b)} \left(\frac{\sigma - 1}{\sigma} \right)^{a+b} \\ &\times \exp \left(\int_1^{\sigma} \left(\frac{\frac{\lambda}{\theta}\{1 - N(\lambda - \lambda u)\} + \frac{\eta}{\theta}\{1 - \tilde{G}(\lambda - \lambda u)\}}{\{1 - N(\lambda - \lambda u)\}} + \frac{a+b}{u - \sigma} \right) du \right). \end{aligned}$$

Now, taking into account that $\Gamma(a+b+n)/\Gamma(n+1) \propto n^{a+b-1}$ as $n \rightarrow \infty$, the relation (16) is equivalent to

$$(18) \quad P(X=0, Q=n) \propto c_0 n^{a+b-1} \sigma^{-n}$$

as $n \rightarrow \infty$. Here $g_n \propto h_n$ means that

$$\lim_{n \rightarrow \infty} \frac{g_n}{h_n} = 1.$$

Similar approximations can be obtained for the probability generating functions (3)–(5). Consider, for example, the case of the distribution of the number of customers in orbit when the server is operative and busy $P_1(z)$. Denote by

$$(19) \quad A(z) = \frac{1 - \tilde{F}(\lambda - \lambda z + \mu)}{\lambda - \lambda z + \mu} \frac{\lambda - \lambda z + \eta - \eta \tilde{G}(\lambda - \lambda z)}{N(\lambda - \lambda z) - z} P_0(z).$$

Similarly to (10), we can find some $\beta \in (0, 1 + \gamma/\lambda - \sigma)$ and an analytical function $S(z)$ on Θ_δ such that

$$(20) \quad \frac{A(z)}{N(\lambda - \lambda z) - z} = - \frac{A'(\sigma)}{-\lambda N'(\lambda - \lambda \sigma) - 1} \frac{1}{z - \sigma} + S(z), \quad |z| < \sigma + \beta, \quad z \neq \sigma,$$

where

$$S(z) = \begin{cases} \frac{A(z)}{N(\lambda-\lambda z)-z} + \frac{A'(\sigma)}{-\lambda N'(\lambda-\lambda\sigma)-1} \frac{1}{z-\sigma}, & |z| < \sigma + \beta, \ z \neq \sigma, \\ \lim_{u \rightarrow \sigma} \frac{A(u)}{N(\lambda-\lambda u)-u}, & z = \sigma. \end{cases}$$

Now substituting (19) into (3), we have the following representation:

$$(21) \quad \begin{aligned} P_1(z) &= \frac{K}{A} \frac{A(\sigma)(\sigma-1)}{-\lambda N'(\lambda-\lambda\sigma)-1} \left(\frac{\sigma-z}{\sigma-1} \right)^{-a-b-1} \Lambda(z) \\ &\quad + \frac{K}{A} A(\sigma) \left(\frac{\sigma-z}{\sigma-1} \right)^{-a-b} S(z) \Lambda(z). \end{aligned}$$

Denote, respectively, by $\sum_{n=0}^{\infty} p_n z^n$ and $\sum_{n=0}^{\infty} q_n z^n$ the series expansion of the two terms in (21). We obtain by the same method that

$$(22) \quad p_n = P(X=1, Q=n) \propto c'_0 n^{a+b-1} \sigma^{-n}, \quad c'_0 = A(\sigma)(\sigma-1)c_0,$$

$$(23) \quad q_n \propto c_1 S(\sigma) n^{a+b} \sigma^{-n}.$$

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