LIMIT BEHAVIOR OF SYMMETRIC RANDOM WALKS WITH A MEMBRANE

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ABSTRACT. Let $\{X(k), k \in \mathbb{Z}_+\}$ be a random walk in $\mathbb{Z}$. Assume that its transition probabilities coincide with those of a symmetric random walk with unit steps throughout except for a fixed neighborhood of zero. The weak convergence of the sequence of normalized walks $\{X_n(k) = n^{-1/2}X(nk), k \geq 0\}_{n \geq 1}$ is proved. The main result generalizes a Harrison and Shepp theorem on the weak convergence to a skew Brownian motion in the case where the symmetry of the random walk fails at a single point. All possible limits for the corresponding random walks are described.

1. INTRODUCTION

Let $\{S(k), k \in \mathbb{Z}_+\}$ be a symmetric random walk in $\mathbb{Z}$, that is, $S(0) = 0$ and

$$p_{i,i+1} = p_{i,i-1} = \frac{1}{2}, \quad i \in \mathbb{Z}.$$

We construct a continuous process $\{S(t), t \geq 0\}$ from the sequence $\{S(k), k \geq 0\}$ by using the linear interpolation between the values at integer points, and we study the stochastic processes

$$S_n(t) = \frac{1}{\sqrt{n}}S(nt), \quad n \in \mathbb{N}.$$

According to the well known Donsker theorem (see, for example, the book [2]), the processes $\{S_n(t), t \in [0, 1]\}_{n \geq 1}$ weakly converge in the space $C[0, 1]$ to the Wiener process as $n \rightarrow \infty$.

We consider the weak convergence of normalized random walks $\{X(k), k \in \mathbb{Z}_+\}$ whose transition probabilities differ from the transition probabilities of $\{S(k), k \in \mathbb{Z}_+\}$ in a neighborhood $[-m, m]$ of the origin. This neighborhood is called a membrane.

The case where the membrane consists of a single point, that is $m = 0$, is considered by Harrison and Shepp in [3]. They proved that if

$$p_{0,1} = p, \quad p_{0,-1} = q = 1 - p, \quad \text{and} \quad p_{i,i+1} = p_{i,i-1} = \frac{1}{2} \quad \text{for } i \neq 0,$$

then the sequence of appropriately normalized random walks $\{X_n\}$ weakly converges to the skew Brownian motion $W_\gamma(\cdot)$, $\gamma = p - q$, defined as a continuous Markov process with the transition density

$$p_t(x, y) = \varphi_t(x - y) + \gamma \operatorname{sign}(y) \varphi_t(|x| + |y|), \quad x, y \in \mathbb{R},$$

where $\varphi_t(x) = (2\pi t)^{-1/2} e^{-x^2/2t}$ is the density of the normal distribution $N(0, t)$.

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The parameter $\gamma \in [-1, 1]$ is called the penetrability coefficient. Note that if $\gamma = +1$ (or $\gamma = -1$), then W_γ is the Brownian motion with upward (downward) reflection, while if $\gamma = 0$, then W_γ is the usual Brownian motion.

More detail on the diffusion with membranes can be found in [4].

Minlos and Zhizhina [5] generalize the result of [3] to the case of an arbitrary bounded membrane. The proof in [5] is based on some methods of the theory of semigroups.

In the present paper, we prove a similar result by using direct probabilistic methods that are simpler as compared to those in [5] and can be used to treat more general cases. We also provide a probabilistic meaning of the coefficient γ in terms of some characteristics of the behavior of the walk in the interior of the membrane. Finally we describe all possible processes that may occur as a limit depending on the properties of a membrane.

We should like to mention the papers [6]–[9] where a similar topic is considered (also see the references therein).

2. SETTING OF THE PROBLEM AND MAIN RESULTS

Consider a homogeneous Markov chain $\{X(k) = X(x_0, k), k \in \mathbb{Z}_+\}$ in $\mathbb{Z}$ that starts from a point $x_0 \in \mathbb{Z}$ and denote its transition probabilities by $p_{i,j}$. We assume that the probabilities $p_{i,j}$ may differ from the corresponding transition probabilities of a symmetric random walk $\{S(k), k \in \mathbb{Z}_+\}$ only if $|i| \leq m$, namely

$$p_{i,i+1} = p_{i,i-1} = \frac{1}{2}, \quad i \notin \{-m, \dots, m\}.$$

We further assume that the chain X jumps from a point of the set $\{-m, \dots, m\}$ to an arbitrary point of the set $\{-m-1, \dots, m+1\}$:

$$\sum_{j=-m-1}^{m+1} p_{i,j} = 1, \quad i \in \{-m, \dots, m\}.$$

The chain $\{X(k), k \in \mathbb{Z}_+\}$ can be viewed as a symmetric random walk with a (non-symmetric) membrane in the set $\{-m, \dots, m\}$.

Now we define $X(x_0, t)$ for all $t \geq 0$ by using the linear interpolation

$$X(x_0, t) := X(x_0, [t]) + (t - [t]) (X(x_0, [t] + 1) - X(x_0, [t])).$$

Let $x \in \mathbb{R}$. Consider the following sequence of processes

$$X_n(t) = X_n(x, t) := \frac{1}{\sqrt{n}} X([\sqrt{n}x], nt), \quad t \geq 0.$$

To state the main result of the paper, we need the following probability measures in the space $C[0, 1]$. Denote by P_{x, W_γ} the distribution of the skew Brownian motion $W_\gamma(\cdot)$ that starts from a point x , and by $P_{x,0}$ we denote the distribution of the Brownian motion that starts from a point x and has a sticky point at the origin.

2.1. Main result.

Theorem 1. *Given an arbitrary $x \in \mathbb{R}$, the sequence of processes $\{X_n(x, t), t \in [0, 1]\}_{n \geq 1}$ weakly converges in the space $C[0, 1]$ to a continuous process $\{X_\infty(x, t), t \in [0, 1]\}$ as $n \rightarrow \infty$. In particular,*

- A.** *If at least one of the states of the chain $\{X(k), k \in \mathbb{Z}_+\}$, either $-m-1$ or $m+1$,*
 - 1) *is essential, and*
 - 2) *can be reached with probability one,*

then the limit process X_∞ is the skew Brownian motion W_γ whose parameter is defined as follows.

A1. If assumptions 1) and 2) are satisfied for both states $-m-1$ and $m+1$, then

$$\gamma = \frac{\alpha - \beta}{\alpha + \beta},$$

where α is the probability that the walk X reaches the point $m+1$ from $-m$ omitting the point $-m-1$ and where β is the probability that the walk X reaches the point $-m-1$ from m omitting the point $m+1$.

A2. If both assumptions 1) and 2) are satisfied for the state $-m-1$ only, then $\gamma = -1$, while if both assumptions 1) and 2) are satisfied for the state $m+1$ only, then $\gamma = 1$.

B. Let $x > 0$, and let the state $-m-1$ be essential and it can be reached by the chain $\{X(k), k \in \mathbb{Z}_+\}$ with probability q , $0 < q < 1$. Then the distribution of the limit process X_∞ equals

$$qP_{x, W_{-1}} + (1 - q)P_{x, 0}.$$

An analogous result holds if $x < 0$ and the state $m+1$ is essential.

C. Let $x = 0$. Assume that the states $-m-1$ and $m+1$ can be reached by the chain $\{X(k), k \in \mathbb{Z}_+\}$ with probabilities q and p , respectively, and let both states be essential.¹

C1. If these states are not communicating, then the distribution of the limit process X_∞ equals

$$qP_{0, W_{-1}} + pP_{0, W_{+1}} + (1 - q - p)P_{0, 0}.$$

C2. If the states $-m-1$ and $m+1$ are communicating, then the distribution of the limit process X_∞ equals

$$(q + p)P_{0, W_\gamma} + (1 - q - p)P_{0, 0},$$

where the number γ is defined in the same way as in **A**.

D. If a state $-m-1$ or state $m+1$ is accessible¹ and inessential, then the limit process X_∞ is the Brownian motion with sticky point at the origin.

Remark 1. If $m = 0$, that is if the symmetricity of the random walk fails only at a single point, then Theorem 1 coincides with the Harrison and Shepp result [3].

Remark 2. The result of Theorem 1 remains true if the condition

$$\sum_{j=-m-1}^{m+1} p_{i,j} = 1$$

in the definition of X is changed by

$$\sum_{j=-m-N}^{m+N} p_{i,j} = 1$$

for a given number N . Indeed, X in this case can be viewed as a walk with the membrane $[-m-N, m+N]$ and with the corresponding transition probabilities.

¹If a state is inaccessible, then it does not matter for the result of the theorem whether that state is essential or not. For simplicity, we assume that inaccessible states are essential in statement **C** or inessential in statement **D**.

2.2. **Examples.** a) Let the symmetricity of the random walk fail at two points -1 and 1 . Thus $m = 1$ and

$$\begin{aligned} p_{i,i\pm 1} &= \frac{1}{2}, & i \notin \{-1, 1\}, \\ p_{-1,-2} &= q', & p_{-1,0} = p', \\ p_{+1,0} &= q'', & p_{+1,+2} = p'', \end{aligned}$$

where q' , p' , q'' , and p'' are positive numbers such that $q' + p' = q'' + p'' = 1$. One can check that $\alpha = p'p''/(q' + p'')$ and $\beta = q'q''/(q' + p'')$. By Theorem 1, the sequence of such walks normalized appropriately weakly converges to the skew Brownian motion with parameter

$$\gamma = \frac{p'p'' - q'q''}{p'p'' + q'q''}.$$

b) Let the transition probabilities $p_{i,j}$ of a chain X differ from the corresponding transition probabilities of a symmetric random walk at the origin only. We further assume that the jump of X from the origin is bounded. More precisely, the transition probabilities are such that

$$p_{i,i-1} = p_{i,i+1} = \frac{1}{2} \quad \text{for } i \neq 0, \quad \sum_{j=-N}^N p_{0,j} = 1, \quad p_{0,0} \neq 1.$$

Now we apply Theorem 1 with $m = N$. It is clear that the probabilities ρ_i that a walk reaches the state $m+1$ starting from i and omitting the point $-m-1$ satisfy the following system of equations:

$$\begin{cases} \rho_{-m-1} = 0, \\ \rho_{-m} = \frac{1}{2}\rho_{-m+1} + \frac{1}{2}\rho_{-m-1}, \\ \dots \\ \rho_{-1} = \frac{1}{2}\rho_{-2} + \frac{1}{2}\rho_0, \\ \rho_0 = \sum_{j=-m}^m p_{0,j}\rho_j, \\ \rho_1 = \frac{1}{2}\rho_0 + \frac{1}{2}\rho_2, \\ \dots \\ \rho_m = \frac{1}{2}\rho_{m-1} + \frac{1}{2}\rho_{m+1}, \\ \rho_{m+1} = 1. \end{cases}$$

Note that all the points $(-m-1, 0)$, $(-m, \rho_{-m})$, $\dots$, $(-1, \rho_{-1})$, and $(0, \rho_0)$ lie on the same straight line. Using the coordinates of the first and last points, we determine the coefficients of this line, namely

$$y = \frac{\rho_0 - 0}{0 - (-m-1)}x + \rho_0,$$

that is

$$\rho_k = \left(1 + \frac{k}{m+1}\right)\rho_0, \quad k \leq 0.$$

Similarly,

$$\rho_k = \left(1 - \frac{k}{m+1}\right)\rho_0 + \frac{k}{m+1}, \quad k \geq 0.$$

Substituting these expressions for ρ_k to the equation corresponding to $k = 0$ in the above system, we evaluate ρ_0 as follows:

$$\begin{aligned}\rho_0 &= \sum_{j=-m}^m p_{0,j} \rho_j = \sum_{j=-m}^m p_{0,j} \left(1 - \frac{|j|}{m+1}\right) \rho_0 + \sum_{j=1}^m p_{0,j} \frac{j}{m+1} \\ &= \rho_0 - \sum_{j=-m}^m p_{0,j} \frac{|j|}{m+1} \rho_0 + \sum_{j=1}^m p_{0,j} \frac{j}{m+1},\end{aligned}$$

whence

$$\rho_0 = \frac{\sum_{j=1}^m j p_{0,j}}{\sum_{j=-m}^m |j| p_{0,j}}.$$

Therefore we have that, with $\alpha = \rho_{-m}$ and $\beta = 1 - \rho_m$,

$$\gamma = \frac{\sum_{j=1}^m |j| p_{0,j} - \sum_{j=-m}^{-1} |j| p_{0,j}}{\sum_{j=1}^m |j| p_{0,j} + \sum_{j=-m}^{-1} |j| p_{0,j}} = \frac{\sum_{j=-m}^m j p_{0,j}}{\sum_{j=-m}^m |j| p_{0,j}}.$$

If the jump from the origin is a bounded random variable ξ with the distribution $P(\xi = j) = p_{0,j}$, then the limit process X_∞ for the sequence $\{X_n\}$ is the skew Brownian motion with parameter

$$\gamma = \frac{E \xi_+ - E \xi_-}{E |\xi|} = \frac{E \xi}{E |\xi|}.$$

Note also that Harrison and Shepp mentioned without a proof in [3] that such a result holds for an arbitrary integrable random variable ξ .

3. PROOF OF THEOREM 1

We prove Theorem 1 for the case where $x_0 > 0$, the state $m+1$ is essential, and if the chain X reaches it with probability 1. Other cases can be considered analogously.

3.1. Construction of an auxiliary sequence. To study the limit behavior of the Markov chain $\{X(k), k \in \mathbb{Z}_+\}$, it is convenient to represent its trajectory by pasted parts of the trajectories of two independent Markov chains $\{Y(k), k \in \mathbb{Z}_+\}$ and $\{Z(k), k \in \mathbb{Z}_+\}$ whose structures are simpler. More precisely, let Y be the absolute value of an usual random walk and let Z be a Markov chain in $\{-m-1, \dots, m+1\}$ describing the evolution of the process X in the interior of the membrane. Below we describe the main idea of the construction.

Consider a trajectory $\{X(x_0, k), k \in \mathbb{Z}_+\}$ starting from a point $x_0 \in \mathbb{Z}$. Without loss of generality we assume that $x_0 > m$. Then we define the following sequence of stopping times $\{\tau_k, \sigma_k, k \geq 1\}$.

Let $\tau_1 := \inf\{j > 0: |X(x_0, j)| = m\}$ be the time when the process X reaches the point m for the first time. Further, let

$$\sigma_k := \inf\{j > \tau_k: |X(x_0, j)| = m+1\}, \quad k \geq 1,$$

$$\tau_{k+1} := \inf\{j > \sigma_k: |X(x_0, j)| = m\}, \quad k \geq 1,$$

be the sequential moments when the walk enters the sets $\{-m-1, m+1\}$ and $\{-m, m\}$, respectively.

Then the process $\{Y(k), k \geq 0\}$ is constructed in the following way.

Put $Y(k) := |X(x_0, k)| - m$, $k = 0, \dots, \tau_1$. Thus we define a part of the trajectory of Y up to the moment τ_1 when Y reaches the origin. Then we put

$$\begin{aligned} Y(\tau_1 + 1) &:= 1, \\ Y(\tau_1 + 1 + k) &:= |X(x_0, \sigma_1 + k)| - m, \quad k = 1, \dots, \tau_2 - \sigma_1, \\ Y(\tau_1 + 1 + \tau_2 - \sigma_1 + 1) &:= 1, \\ Y(\tau_1 + 1 + \tau_2 - \sigma_1 + 1 + k) &:= |X(x_0, \sigma_2 + k)| - m, \quad k = 1, \dots, \tau_3 - \sigma_2, \\ Y(\tau_1 + 1 + \tau_2 - \sigma_1 + 1 + \tau_3 - \sigma_2 + 1) &:= 1, \end{aligned}$$

and so forth.

It is easy to see that $\{Y(k), k \in \mathbb{Z}_+\}$ is a symmetric random walk in $\mathbb{Z}_+$ with reflection at the origin, that is

$$P(Y(k+1) = i \pm 1 | Y(k) = i) = \frac{1}{2}, \quad i \neq 0; \quad P(Y(k+1) = 1 | Y(k) = 0) = 1.$$

Note that the sequence $\{Y(k)\}$ is constructed from the part of the trajectory of $\{X(k)\}$ that omits the membrane $[-m, m]$. Another part of the trajectory of X is used to construct the sequence $\{Z(k), k \in \mathbb{Z}_+\}$:

$$\begin{aligned} Z(0) &:= m, \\ Z(k) &:= X(x_0, \tau_1 + k), \quad k = 1, \dots, \sigma_1 - \tau_1, \\ Z(\sigma_1 - \tau_1 + 1) &:= m \operatorname{sign}(Z(\sigma_1 - \tau_1)), \\ Z(\sigma_1 - \tau_1 + 1 + k) &:= X(x_0, \tau_2 + k), \quad k = 1, \dots, \sigma_2 - \tau_2, \\ Z(\sigma_1 - \tau_1 + 1 + \sigma_2 - \tau_2 + 1) &:= m \operatorname{sign}(Z(\sigma_1 - \tau_1 + 1 + \sigma_2 - \tau_2)), \end{aligned}$$

and so forth.

It is easy to see that Z is a Markov chain in $\{-m-1, \dots, m+1\}$ with transition probabilities $p_{i,j}$ if $i = -m, \dots, m$ and $p_{-m-1, -m} = p_{m+1, m} = 1$.

The trajectories of the processes Y and Z are constructed from different parts of the trajectory of the Markov chain X . One can check that the processes Y and Z are independent. Therefore we constructed two independent processes Y and Z from the trajectories of the process X .

On the other hand, if Y is the absolute value of a symmetric random walk and Z is a Markov chain assuming values in $\{-m-1, \dots, m+1\}$ with the corresponding transition probabilities and being independent of Y , then one can uniquely construct a sequence $\tilde{X}$ (whose distribution is the same as that of X) by pasting together the corresponding excursions of the processes Y and Z as described below.

Denote by η_k the moment when the process Y visits the origin for the k th time and by ζ_k the moment when the process Z visits the set $\{-m-1, m+1\}$ for the k th time. As the first part of the trajectory of a new process $\tilde{X}$, we take the excursion of the process Y up to the moment η_1 and shift it by m units upward (recall that $x_0 > m > 0$). As the next part of the trajectory of $\tilde{X}$, we take the excursion of Z from 0 to ζ_1 . Then we use the excursion of the process Y from $\eta_k + 1$ to η_{k+1} and shift it by m units upward to continue the trajectory of $\tilde{X}$. If $\operatorname{sign}(Z(\zeta_k)) = -1$, then this part of the trajectory is reflected with respect to the axis Ot . Then we use the trajectory of Z again, namely we use its excursion from $\zeta_1 + 1$ to ζ_2 , and so on. Since we shift and reflect the excursions of Y and Z appropriately, the trajectory of $\tilde{X}$ is continuous.

It is easy to see that the process $\tilde{X}$ constructed in this way is equivalent to the original process X . Therefore we constructed a one-to-one correspondence between the distributions of X and a pair (Y, Z) .

Next we introduce the sequences

$$Y'(k) := \text{sign}(Z(\zeta_j))Y(k), \quad k = \eta_j, \dots, \eta_{j+1}, \quad j \geq 1.$$

We linearly extend the definition of all the processes for $t \geq 0$ in the same as in the case of X . Finally we show that the sequence of processes

$$Y'_n = \left\{ \frac{1}{\sqrt{n}} Y'(nt), t \geq 0 \right\}$$

weakly converges to the skew Brownian motion and check that the limit of the sequence $\{X_n\}$ is the same.

Note that there exists only a single nonsymmetric point for the walk Y' . Nevertheless one cannot apply the Harrison–Shepp result [3] to Y' , since Y' is not a Markov chain. Indeed, the transition probabilities from the origin for Y' depend on a semiaxis from where the walk comes to the origin.

3.2. Weak convergence of the auxiliary sequence. We consider the distributions of Y' . Let $0 < a < b$. Denote by $r(n)$ the number of visits of the walk Y to the origin over a time n . Then

$$\begin{aligned} \mathbb{P}(Y'(n) \in [a, b]) &= \mathbb{P}(Y(n) \in [a, b], \text{sign}(Z(\zeta_{r(n)})) = +1) \\ &= \sum_{k=0}^n \mathbb{P}(Y(n) \in [a, b], \text{sign}(Z(\zeta_k)) = +1, r(n) = k). \end{aligned}$$

Here and in what follows the sums over the number of returns contain terms whose indices run from 0 to n , however only at most a half of the summands are nonzero.

The first term of the latter sum corresponding to $k = 0$ represents the probability that $\{Y(n)\}$ reaches $[a, b]$ omitting the origin. This probability can be found with the help of the reflection principle (see, for example, [1], Chapter III, §2):

$$\mathbb{P}(Y(n) \in [a, b], r(n) = 0) = \mathbb{P}(S(n) \in [a, b]) - \mathbb{P}(S(n) \in [-b, -a]).$$

Then

$$\mathbb{P}\left(\frac{1}{\sqrt{n}}Y([nt]) \in [a, b], r(nt) = 0\right) \rightarrow \int_a^b (\varphi_t(x - x_0) + \varphi_t(x + x_0)) dx, \quad n \rightarrow \infty.$$

To evaluate the limit of the expression

$$\begin{aligned} &\sum_{k=1}^n \mathbb{P}\left(\frac{1}{\sqrt{n}}Y([nt]) \in [a, b], \text{sign}(Z(\zeta_k)) = +1, r(nt) = k\right) \\ &= \sum_{k=1}^n \mathbb{P}\left(\frac{1}{\sqrt{n}}Y([nt]) \in [a, b], r(nt) = k\right) \cdot \mathbb{P}(\text{sign}(Z(\zeta_k)) = +1), \end{aligned}$$

we use a Toeplitz theorem. Below we formulate this result in a form being convenient for further applications in this paper.

Theorem 2. *Let s_n and $a_{n,k}$ be nonnegative numbers such that*

- 1) $\lim_{n \rightarrow \infty} s_n = s$;
- 2) $\lim_{n \rightarrow \infty} \sum_{k=1}^n a_{n,k} = A$;
- 3) $\lim_{n \rightarrow \infty} a_{n,k} = 0, \quad k \geq 1$.

Then

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n a_{n,k} s_k = As.$$

Put $s_k = P(\text{sign}(Z(\zeta_k)) = +1)$ and

$$a_{n,k} = P\left(\frac{1}{\sqrt{n}}Y([nt]) \in [a, b], r(nt) = k\right)$$

in the Toeplitz theorem. It is not complicated to check that the sequence

$$\{Z'(k) = \text{sign } Z(\zeta_k), k \in \mathbb{Z}_+\}$$

is a Markov chain. Let the probabilities α and β (see Theorem 1) be positive and strictly less than unity (other cases are trivial). Then the chain $\{Z'(k)\}$ is homogeneous and aperiodic, whence we conclude that there exists a stationary distribution (q, p) . Moreover $\lim_{n \rightarrow \infty} s_n = p$. We will determine this distribution at the end of the proof.

Now we consider the sum $\sum_{k=1}^n a_{n,k}$. Note that

$$\sum_{k=1}^n a_{n,k} = P\left(\frac{1}{\sqrt{n}}Y([nt]) \in [a, b], r(nt) > 0\right).$$

Applying the reflection principle, we get

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n a_{n,k} = 2 \int_a^b \varphi_t(x + x_0) dx.$$

Finally we show that $\lim_{n \rightarrow \infty} a_{n,k} = 0$. Indeed, the probability

$$a_{n,k} = P\left(\frac{1}{\sqrt{n}}Y([nt]) \in [a, b], r(nt) = k\right)$$

does not exceed the probability $P(r(nt) = k)$. Let $\tau_1 = \tau_1(n)$ be the first moment when the walk $\{Y_n(k), k \geq 0\}$ reaches the origin from the point $[x_0\sqrt{n}]$. Denote by $r_0 = r_0(n)$ the number of visits of the walk $\{S'(k) = S(\tau_1 + k), k \geq 0\}$ to the origin over the time n (recall that this walk starts from the origin). Then

$$\begin{aligned} a_{n,k} &\leq P(r(nt) = k) \leq P(1 \leq r(nt) \leq k) \\ &= P(1 \leq r(nt) \leq k, \tau_1(n) < n(t - \delta)) + P(1 \leq r(nt) \leq k, \tau_1(n) \geq n(t - \delta)) \\ &\leq P(r_0(n\delta) \leq k) + P(n(t - \delta) \leq \tau_1(n) \leq nt). \end{aligned}$$

From the results of [1, Chapter III §4, §6] we conclude that the right-hand side of the latter inequality can be made as small as one wishes if n is sufficiently large, that is, $\lim_{n \rightarrow \infty} a_{n,k} = 0$.

Now the above Toeplitz theorem implies

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n a_{n,k} s_k = 2p \int_a^b \varphi_t(x + x_0) dx.$$

Therefore

$$(1) \quad \lim_{n \rightarrow \infty} P\left(\frac{1}{\sqrt{n}}Y'([nt]) \in [a, b]\right) = \int_a^b (\varphi_t(x - x_0) + (2p - 1)\varphi_t(x + x_0)) dx.$$

The corresponding limit distribution for negative a and b is found similarly. Since the probability that the walk reaches the interval $[a, b]$ omitting the origin is equal to zero, we obtain

$$(2) \quad \lim_{n \rightarrow \infty} P\left(\frac{1}{\sqrt{n}}Y'([nt]) \in [a, b]\right) = 2q \int_a^b \varphi_t(x - x_0) dx, \quad a < b < 0,$$

where $q = \lim_{n \rightarrow \infty} P(\text{sign}(Z(\zeta_n)) = -1)$.

Note that $2p - 1 = p - q$. Then (1) and (2) imply that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbf{P} \left(\frac{1}{\sqrt{n}} Y'([nt]) \in [a, b] \right) \\ &= \int_a^b (\varphi_t(x - x_0) + (p - q) \operatorname{sign}(x) \varphi_t(|x| + |x_0|)) dx, \quad -\infty \leq a \leq b \leq \infty. \end{aligned}$$

The two dimensional distributions are studied similarly. The main idea again is to represent the probability $\mathbf{P}(Y'(nt_1) \in [a_1, b_1], Y'(nt_2) \in [a_2, b_2])$ as a double sum over possible numbers k_1 and k_2 of returns to the origin over the time nt_1 and then over the time $n(t_2 - t_1)$, respectively. As above we consider the terms with $k_i = 0$ separately:

$$\sum_{k_1, k_2=0}^n = \sum_{k_1=0}^0 \sum_{k_2=0}^0 + \sum_{k_1=0}^0 \sum_{k_2=1}^n + \sum_{k_2=0}^0 \sum_{k_1=1}^n + \sum_{k_1, k_2=1}^n.$$

The limits for the first three sums are found in the same way as in the case of one dimensional distributions (namely, with the help of the reflection principle and Toeplitz theorem). The following analog of the Toeplitz theorem for double sums is needed to treat the last sum.

Theorem 3. *Let s'_n , s''_n , and $a_n(k_1, k_2)$ be nonnegative numbers such that*

- 1) $\lim_{n \rightarrow \infty} s'_n = s'$, $\lim_{n \rightarrow \infty} s''_n = s''$;
- 2) $\lim_{n \rightarrow \infty} \sum_{k_1=1}^n \sum_{k_2=1}^n a_n(k_1, k_2) = A$;
- 3) $\lim_{n \rightarrow \infty} \sum_{k_1=1}^n a_n(k_1, k_2) = 0$ for all $k_2 \geq 1$ and $\lim_{n \rightarrow \infty} \sum_{k_2=1}^n a_n(k_1, k_2) = 0$ for all $k_1 \geq 1$.

Then

$$\lim_{n \rightarrow \infty} \sum_{k_1=1}^n \sum_{k_2=1}^n a_n(k_1, k_2) s'_{k_1} s''_{k_2} = A s' s''.$$

Proof. Let

$$a'_{n, k_1} = \sum_{k_2=1}^n a_n(k_1, k_2)$$

and

$$a''_{n, k_2} = \sum_{k_1=1}^n a_n(k_1, k_2) s'_{k_1}.$$

The assumptions of the Toeplitz theorem are satisfied for the numbers s'_{k_1} and a'_{n, k_1} . Hence

$$\lim_{n \rightarrow \infty} \sum_{k_1=1}^n a'_{n, k_1} s'_{k_1} = A s'.$$

Note that $\sum_{k_1=1}^n a'_{n, k_1} s'_{k_1} = \sum_{k_2=1}^n a''_{n, k_2}$. Moreover, by the Toeplitz theorem

$$\lim_{n \rightarrow \infty} a''_{n, k_2} = \lim_{n \rightarrow \infty} \sum_{k_1=1}^n a_n(k_1, k_2) s'_{k_1} = 0 \cdot s' = 0, \quad k_2 = 1, \dots, n.$$

Applying the Toeplitz theorem once more, we get

$$\lim_{n \rightarrow \infty} \sum_{k_2=1}^n \sum_{k_1=1}^n a_n(k_1, k_2) s'_{k_1} s''_{k_2} = \lim_{n \rightarrow \infty} \sum_{k_2=1}^n a''_{n, k_2} s''_{k_2} = A s' s''.$$

The theorem is proved. $\square$

The latter result can be viewed as a Toeplitz theorem for double sums. It can be used to prove an analogous result for triple sums in exactly the same way as the Toeplitz theorem is used to prove Theorem 3. Proceeding further by induction, we prove a result in the general case for multiple sums.

Now the finite dimensional distributions can be studied similarly to the one dimensional case, namely the probabilities of interest are represented by multiple sums over the numbers of visits to the origin and the corresponding version of the generalized Toeplitz theorem is used to find the limits of the multiple sums.

It remains to check that the sequence

$$\left\{ \frac{1}{\sqrt{n}} Y'(nt), t \in [0, 1] \right\}$$

is relatively compact, and this will prove that this sequence weakly converges in the space $C[0, 1]$.

The following two conditions are necessary and sufficient in order that a sequence $\{V_n(t), t \in [0, 1]\}$ is relatively compact in $C[0, 1]$:

(i) for an arbitrary $\varepsilon > 0$, there exists a number a such that, for all $n \geq 1$,

$$P(|V_n(0)| > a) < \varepsilon;$$

(ii) for arbitrary $\alpha > 0$ and $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that

$$P(w_{V_n}(\delta) > \alpha) < \varepsilon, \quad n \geq n_0,$$

where $w_f(\delta) = \sup_{|t-s| < \delta} |f(t) - f(s)|$ is the modulus of continuity of a function f (see [2, Theorem 8.2]). Condition (i) obviously holds in our case. It remains to check condition (ii). By construction of Y' (see Section 3.1), it follows that $|Y'| = Y = |S|$, where S is a usual symmetric random walk. Moreover, Y' may change its sign only if Y is equal to zero. Thus

$$w_{Y'_n}(\delta) \leq 2w_{Y_n}(\delta) = 2w_{|S_n|}(\delta) \leq 2w_{S_n}(\delta).$$

This means that condition (ii) for the sequence Y'_n follows from the corresponding condition for the sequence S_n . Recall that the latter sequence is relatively compact by the Donsker theorem.

Therefore the convergence of the sequence of processes

$$\left\{ \frac{1}{\sqrt{n}} Y'(nt), t \in [0, 1] \right\}$$

to the skew Brownian motion $\{W_\gamma(t), t \in [0, 1]\}$ is proved.

3.3. The limit of the sequence X_n . To show that the sequence of processes

$$X_n = \left\{ \frac{1}{\sqrt{n}} X(nt), t \in [0, 1] \right\}$$

converges to the same limit process as that in the case of the sequence

$$\left\{ \frac{1}{\sqrt{n}} Y'(nt), t \in [0, 1] \right\}, \quad n \geq 1,$$

we need the following auxiliary results.

Lemma 1. *Let $V_n(\cdot) \Rightarrow V(\cdot)$ in the space $C[0, 1]$. Assume that $\{\eta_n(t), t \in [0, 1]\}$ is a sequence of continuous processes such that $\sup_t |\eta_n(t)| \xrightarrow{P} 0$ as $n \rightarrow \infty$. Then the sequence of stochastic processes $\{V'_n(t) = V_n(t) + \eta_n(t), t \in [0, 1]\}$ weakly converges to $\{V(t), t \in [0, 1]\}$ in $C[0, 1]$.*

Lemma 2. *Let $V_n(\cdot) \Rightarrow V(\cdot)$ in the space $C[0, 1]$. Assume that $\{\theta_n(t), t \in [0, 1]\}$ is a sequence of continuous processes such that $0 \leq \theta_n(t) \leq t$ for all $t \in [0, 1]$ and that $\sup_t \theta_n(t) \xrightarrow{P} 0$ as $n \rightarrow \infty$. Then the sequence of stochastic processes*

$$\{V'_n(t) = V_n(t - \theta_n(t)), t \in [0, 1]\}$$

weakly converges to $\{V(t), t \in [0, 1]\}$ in $C[0, 1]$.

To prove Lemma 2, it is sufficient to check that $\sup_t |V_n(t - \theta_n(t)) - V_n(t)| \xrightarrow{P} 0$ as $n \rightarrow \infty$ and then to use Lemma 1.

Since

$$|V_n(t) - V_n(s)| \leq \sup_{|t-s| < \delta} |V_n(t) - V_n(s)| = w_{V_n}(\delta)$$

for all $|t - s| < \delta$, the bound

$$\begin{aligned} & \mathbb{P} \left(\sup_t |V_n(t - \theta_n(t)) - V_n(t)| > \alpha \right) \\ &= \mathbb{P} \left(\sup_t |V_n(t - \theta_n(t)) - V_n(t)| > \alpha, \sup_t \theta_n(t) \leq \delta \right) \\ &+ \mathbb{P} \left(\sup_t |V_n(t - \theta_n(t)) - V_n(t)| > \alpha, \sup_t \theta_n(t) > \delta \right) \\ &\leq \mathbb{P}(w_{V_n}(\delta) > \alpha) + \mathbb{P} \left(\sup_t \theta_n(t) > \delta \right) \end{aligned}$$

holds for all α and δ . Now, given $\alpha > 0$ and $\varepsilon > 0$, we choose $\delta > 0$ and n_1 such that

$$\mathbb{P}(w_{V_n}(\delta) > \alpha) < \frac{\varepsilon}{2}$$

for all $n \geq n_1$ (see (ii) in Section 3.2). Further, we choose n_2 for these numbers ε and δ such that

$$\mathbb{P} \left(\sup_t \theta_n(t) > \delta \right) < \frac{\varepsilon}{2}$$

for all $n \geq n_2$ (such a number n_2 exists in view of the assumptions of the lemma). Then

$$\mathbb{P} \left(\sup_t |V_n(t - \theta_n(t)) - V_n(t)| > \alpha \right) < \varepsilon$$

for all $n \geq n_1 \vee n_2$, and thus $V'_n(\cdot)$ converges to $V(\cdot)$ by Lemma 1, since $V'_n(\cdot)$ is the sum of $V_n(\cdot)$ and $V_n(\cdot - \theta_n(\cdot)) - V_n(\cdot)$.

The lemma is proved.

Next we apply Lemma 2 to $V_n(t) = n^{-1/2}Y'(nt)$, where $\theta_n(t)$ is the portion of time spent by the process X in the membrane. To do that we have to show first that $\sup_t \theta_n(t)$ approaches zero in probability as $n \rightarrow \infty$.

Let $r(n)$ be the number of visits to the origin of the walk $S(k) = S([\sqrt{n}x_0], k)$ during n steps if the walk starts from the point $[\sqrt{n}x_0]$. Also let $\zeta'_k = \zeta_{k+1} - \zeta_k$ be the lengths of excursions of the process Z in the membrane. Also let $r_0(n)$ be the number of returns to the origin of the walk $S'(k) := S(\tau_1 + k)$, $k \geq 0$, where $\tau_1 = \tau_1(n)$ is the first moment when the walk $S([\sqrt{n}x_0], k)$ reaches the origin. Since $r(n) = r_0(n - \tau_1) \leq r_0(n)$, we obtain the bound

$$\theta_n(t) \leq \frac{1}{n} \sum_{k=1}^{r(n)} \zeta'_k \leq \frac{1}{n} \sum_{k=1}^{r_0(n)} \zeta'_k.$$

Now we show that the expression

$$\frac{1}{n} \sum_{k=1}^{r_0(n)} \zeta'_k = \frac{r_0(n)}{n} \frac{1}{r_0(n)} \sum_{k=1}^{r_0(n)} \zeta'_k$$

on the right-hand side of the latter bound tends to zero in probability.

Since $r_0(n)/\sqrt{n}$ converges in distribution to the absolute value of a normal random variable (see [1]), the ratio $r_0(n)/n$ approaches zero in probability as $n \rightarrow \infty$.

Now we study the asymptotic behavior of

$$\frac{1}{r_0(n)} \sum_{k=1}^{r_0(n)} \zeta'_k.$$

We split the latter sum into a sum of groups of its terms such that each group

$$\tilde{\zeta}'_k = \zeta'_{i_k} + \dots + \zeta'_{i_{k+1}-1}$$

represents the time when the walk exits the membrane in the upward direction. The total number of the groups does not exceed $r_0(n)$, whence

$$\frac{1}{r_0(n)} \sum_{k=1}^{r_0(n)} \zeta'_k \leq \frac{1}{r_0(n)} \sum_{k=1}^{r_0(n)} \tilde{\zeta}'_k.$$

The right-hand side of the latter inequality is the sum of independent identically distributed random variables $\tilde{\zeta}'_k$. Since $\mathbf{E} \tilde{\zeta}'_k < \infty$, the random variables $\tilde{\zeta}'_k$ do not depend on $r_0(n)$. Since $r_0(n) \rightarrow \infty$ almost surely as $n \rightarrow \infty$,

$$\frac{1}{r_0(n)} \sum_{k=1}^{r_0(n)} \tilde{\zeta}'_k \xrightarrow{\mathbf{P}} \mathbf{E} \tilde{\zeta}'_1, \quad n \rightarrow \infty.$$

Therefore $n^{-1} \sum_{k=1}^{r_0(n)} \zeta'_k$ approaches zero in probability.

Now Lemmas 1 and 2 imply that

$$\frac{1}{\sqrt{n}} X(nt) = \frac{1}{\sqrt{n}} Y'(n(t - \theta_n(t))) + \frac{1}{\sqrt{n}} (X(nt) - Y'(n(t - \theta_n(t))))$$

converges to $W_\gamma(t)$, since $|X(nt) - Y'(n(t - \theta_n(t)))| \leq m$ by construction.

It remains to evaluate the parameters of the skew Brownian motion.

If all the states of the chain $\{X(k), k \in \mathbb{Z}_+\}$ communicate with each other, then the stationary distribution (q, p) of the Markov chain

$$Z'(k) = \text{sign}(Z(\zeta_k)) = \frac{Z(\zeta_k)}{m+1}$$

can be found as a solution of the linear system

$$(q, p) \begin{pmatrix} 1 - \rho_{-m} & \rho_{-m} \\ 1 - \rho_m & \rho_m \end{pmatrix} = (q, p)$$

such that $q + p = 1$. Here ρ_i , $i = -m, \dots, m$, is the probability that the walk reaches $m+1$ from i omitting the point $m-1$. These probabilities are found from the system

$$\rho_i = p_{i, m+1} + \sum_{j=-m}^m p_{i, j} \rho_j, \quad i = -m, \dots, m.$$

Therefore

$$p = \frac{\rho_{-m}}{1 - \rho_m + \rho_{-m}}, \quad q = \frac{1 - \rho_m}{1 - \rho_m + \rho_{-m}}.$$

Using the notation $\rho_{-m} = \alpha$ and $1 - \rho_m = \beta$ as in Theorem 1, we obtain the result required, that is

$$p = \frac{\alpha}{\alpha + \beta}, \quad q = \frac{\beta}{\alpha + \beta}.$$

If both assumptions 1) and 2) of Theorem 1 hold only for the state $m + 1$, then clearly $p = 1$.

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