A BAXTER TYPE ESTIMATOR OF AN UNKNOWN PARAMETER OF THE COVARIANCE FUNCTION IN THE NON-GAUSSIAN CASE

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O. O. SYNYAVS'KA

ABSTRACT. The problem of estimation of a parameter of the covariance function is studied for a non-Gaussian stochastic process. Non-asymptotic confidence intervals for the estimator are constructed by using the Baxter statistics.

1. INTRODUCTION

Limit theorems for the mean square convergence or for the almost sure convergence of Baxter sums to a positive constant are called the Lévy–Baxter theorems or Baxter type theorems. These limit theorems for Baxter sums constructed from stochastic processes and random fields were considered by P. Lévy [16], G. Baxter [10], E. G. Gladyshev [3], S. M. Berman [11], T. Kawada [14], S. M. Krasnitskii [5], and others.

E. P. Besklins'ka and Yu. V. Kozachenko [1] proved several Baxter type theorems for pre-Gaussian stochastic processes. Lévy–Baxter theorems for jointly strictly sub-Gaussian and jointly pseudo-Gaussian stochastic processes were studied in the monograph by V. V. Buldygin and Yu. V. Kozachenko [2]. O. O. Kurchenko [7] proved a Baxter type theorem for jointly sub-Gaussian random fields. Yu. V. Kozachenko and O. O. Kurchenko [15] obtained some Lévy–Baxter theorems for a certain class of non-Gaussian stochastic processes.

Various Baxter type theorems for stochastic processes and random fields were used by R. E. Mañboroda [8], Yu. V. Kozachenko and O. O. Kurchenko [4], O. O. Kurchenko [6], J.-C. Breton, I. Nourdin, and G. Peccati [12] and others for the estimation of parameters of covariance functions.

2. SETTING OF THE PROBLEM

Let (Ω, F, P) be a probability space.

Definition 2.1 ([15]). We say that a random vector $(\xi, \eta) \in L_4(\Omega) \times L_4(\Omega)$ possesses property K if

- (1) $E\xi = E\eta = 0$,
- (2) $E(\xi \pm \eta)^4 \leq 3(E(\xi \pm \eta)^2)^2$.

The subclass K_1 is defined as the set of all vectors of the class K for which

$$E(\xi \pm \eta)^4 = 3(E(\xi \pm \eta)^2)^2.$$

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Definition 2.2 ([15]). A centered stochastic process $\{X(t), t \in [0, 1]\}$ is called a process with the first and second order increments of the class K if the random vectors (ξ_1, η_1) and (ξ_2, η_2) possess property K for all $0 \leq s \leq t \leq u \leq v \leq 1$, where

$$\begin{aligned}\xi_1 &= X(t) - X(s), & \eta_1 &= X(v) - X(u), & \xi_2 &= X(t) - 2X\left(\frac{t+s}{2}\right) + X(s), \\ \eta_2 &= X(v) - 2X\left(\frac{u+v}{2}\right) + X(u).\end{aligned}$$

A stochastic process $\{X(t), t \in [0, 1]\}$ is called a process with the first and second order increments of the class K_1 if the random vectors (ξ_1, η_1) and (ξ_2, η_2) defined as above belong to the class K_1 .

For example, centered Gaussian stochastic processes have increments of the first and second order of the class K_1 .

Note that if increments of the first and second order of a stochastic process

$$\{\xi(t), t \in [0, 1]\}$$

belong to the class K_1 , then the increments ξ_i , $i = 1, 2$, are such that

$$\mathbf{E} \xi_i^4 = 3 (\mathbf{E} \xi_i^2)^2, \quad i = 1, 2.$$

Let $\{\xi(t), t \in [0, 1]\}$ be a centered stochastic process with the covariance function depending on a parameter $H \in (0, 1)$,

$$(1) \quad r(t, s) = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \quad t, s \in [0, 1],$$

and let its increments of the first and second order belong to the class K_1 . We want to construct an estimator of the unknown parameter H and a non-asymptotic confidence interval by using the observations of a stochastic process $\{\xi(t), t \in [0, 1]\}$ at points

$$\left\{ \frac{k}{a_n}, \frac{k+0.5}{a_n} \mid 0 \leq k \leq a_n, n \geq 1 \right\},$$

where the sequence $(a_n) \subset \mathbf{N}$ is such that $a_n \rightarrow +\infty$ as $n \rightarrow \infty$.

In what follows we assume that the sequence (a_n) has the property that the series $\sum_{n=1}^{\infty} a_n^{-\alpha}$ converges for all numbers $\alpha > 0$.

The fractional Brownian motion is an example of a stochastic process with covariance function (1) and with increments of the first and second order of the class K_1 . Below we provide an example of a non-Gaussian stochastic process with covariance function (1) and with increments of the first and second order of the class K_1 .

Example 2.1. It is shown in [13] that the fractional Brownian motion $\{B_H(t), t \in [0, 1]\}$ with the Hurst parameter $H \in (\frac{1}{2}, 1)$ admits the following representation

$$B_H(t) = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1 - \cos(y_n t)}{y_n} Y_n, \quad t \in [0, 1],$$

where (X_n) and (Y_n) are two independent sequences, each consisting of independent Gaussian random variables such that $\mathbf{E} X_n = \mathbf{E} Y_n = 0$, $n \geq 1$, and

$$(2) \quad \text{Var } X_n = 2c_H^2 x_n^{-2H} J_{1-H}^{-2}(x_n),$$

$$(3) \quad \text{Var } Y_n = 2c_H^2 y_n^{-2H} J_{-H}^{-2}(y_n);$$

$c_H^2 = \pi^{-1} \Gamma(1 + 2H) \sin \pi H$; J_ν , $\nu \neq -1, -2, \dots$, is the Bessel function of the first kind and of order ν ; (x_n) is the sequence of positive zeros of the Bessel function J_{-H} written in ascending order; (y_n) is the sequence of positive zeros of the Bessel function J_{1-H} written in ascending order.

Further let (ξ_n) and (η_n) be two independent sequences each consisting of independent random variables such that

$$\begin{aligned} \mathbb{E} \xi_n &= \mathbb{E} \eta_n = 0, & \mathbb{E} \xi_n^4 &= 3 (\mathbb{E} \xi_n^2)^2, & \mathbb{E} \eta_n^4 &= 3 (\mathbb{E} \eta_n^2)^2, \\ \text{Var } \xi_n &= \text{Var } X_n, & \text{Var } \eta_n &= \text{Var } Y_n. \end{aligned}$$

Then the stochastic process

$$(4) \quad \xi(t) = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n} \xi_n + \sum_{n=1}^{\infty} \frac{1 - \cos(y_n t)}{y_n} \eta_n, \quad t \in [0, 1]$$

has covariance function (1) and its increments of the first and second order belong to the class K_1 . Now we prove that the first series on the right-hand side of (4) converges in $L_4(\Omega)$ for an arbitrary $t \in [0, 1]$. The proof of the convergence of the second series is the same. Let $1 \leq N < M$. Applying the Hölder inequality with $p = \frac{4}{3}$ and $q = 4$, we get

$$\left| \sum_{n=N}^M \frac{\sin x_n t}{x_n} \xi_n \right| \leq \sum_{n=N}^M \frac{1}{x_n^{1-\lambda}} \frac{|\xi_n|}{x_n^\lambda} \leq \left(\sum_{n=N}^M \frac{1}{x_n^{4(1-\lambda)/3}} \right)^{3/4} \left(\sum_{n=N}^M \frac{\xi_n^4}{x_n^{4\lambda}} \right)^{1/4}$$

for all $\lambda \in (0, \frac{1}{4})$. Since the series $\sum_{n=1}^{\infty} 1/x_n^{1+p}$ converges for all $p > 0$ [13] and since $\frac{4}{3}(1-\lambda) > 1$, the series $\sum_{n=1}^{\infty} 1/x_n^{4(1-\lambda)/3}$ converges for all $\lambda < \frac{1}{4}$. Further,

$$\mathbb{E} \left| \sum_{n=N}^M \frac{\sin x_n t}{x_n} \xi_n \right|^4 \leq \left(\sum_{n=N}^M \frac{1}{x_n^{4(1-\lambda)/3}} \right)^3 \mathbb{E} \left(\sum_{n=N}^M \frac{\xi_n^4}{x_n^{4\lambda}} \right).$$

Recall that $\mathbb{E} \xi_n^4 = 3(\mathbb{E} \xi_n^2)^2$, whence we get

$$\mathbb{E} \left(\sum_{n=N}^M \frac{\xi_n^4}{x_n^{4\lambda}} \right) = 3 \sum_{n=N}^M \frac{(\mathbb{E} \xi_n^2)^2}{x_n^{4\lambda}} = 3 \sum_{n=N}^M \frac{4c_H^4}{x_n^{4\lambda+4H} J_{1-H}^4(x_n)}.$$

Taking into account that $J_{1-H}^2(x_n) \sim \frac{2}{\pi x_n}$ as $n \rightarrow \infty$ [13], we obtain

$$\frac{1}{x_n^{4\lambda+4H} J_{1-H}^4(x_n)} \sim \frac{\pi^2}{4} \frac{x_n^2}{x_n^{4\lambda+4H}} = \frac{\pi^2}{4} \frac{1}{x_n^{4\lambda+4H-2}}.$$

If $H > \frac{1}{2}$, then there exists a number $\lambda < \frac{1}{4}$ such that $4\lambda+4H-2 > 1$. Now we conclude from $\sum_{n=1}^{\infty} 1/x_n^{1+p} < +\infty$ for all $p > 0$ that the series

$$\sum_{n=1}^{\infty} (x_n^{4\lambda+4H} J_{1-H}^4(x_n))^{-1}$$

converges. Hence

$$\mathbb{E} \left(\sum_{n=N}^M \frac{\sin x_n t}{x_n} \xi_n \right)^4 \rightarrow 0, \quad N, M \rightarrow \infty,$$

and the series $\sum_{n=1}^{\infty} (\sin x_n t / x_n) \xi_n$ converges in $L_4(\Omega)$ by the Cauchy criterion.

Next we prove that the increments of the first and second order of the series

$$\sum_{n=1}^{\infty} (\sin x_n t / x_n) \xi_n$$

belong to the class K_1 . Let

$$X(t) = \sum_{n=1}^{\infty} \varphi_n(t) \xi_n, \quad t \in [0, 1],$$

where $\varphi_n(t) = \sin x_n t / x_n$, $n \geq 1$. For arbitrary segments $[s, t], [u, v] \subset [0, 1]$ without joint inner points put

$$X_{1,N} = \sum_{n=1}^N \rho_n \xi_n, \quad \text{where } \rho_n = \varphi_n(t) - \varphi_n(s),$$

$$X_{2,N} = \sum_{n=1}^N \tau_n \xi_n, \quad \text{where } \tau_n = \varphi_n(v) - \varphi_n(u), \quad n \geq 1.$$

Our aim is to prove that

$$(5) \quad \mathbf{E} (X_{1,N} \pm X_{2,N})^4 = 3 \left(\mathbf{E} (X_{1,N} \pm X_{2,N})^2 \right)^2.$$

Since $X_{1,N} \pm X_{2,N} = \sum_{n=1}^N \mu_n \xi_n$, where $\mu_n = \rho_n \pm \tau_n$, $1 \leq n \leq N$, we get

$$\begin{aligned} \mathbf{E} \left(\sum_{n=1}^N \mu_n \xi_n \right)^4 &= \sum_{n=1}^N \mu_n^4 \mathbf{E} \xi_n^4 + 6 \sum_{\substack{n,m=1 \\ n < m}}^N \mu_n^2 \mu_m^2 \mathbf{E} \xi_n^2 \mathbf{E} \xi_m^2 \\ &= 3 \left(\sum_{n=1}^N \mu_n^4 (\mathbf{E} \xi_n^2)^2 + 2 \sum_{\substack{n,m=1 \\ n < m}}^N \mu_n^2 \mu_m^2 \mathbf{E} \xi_n^2 \mathbf{E} \xi_m^2 \right) \\ &= 3 \left(\sum_{n=1}^N \mu_n^2 \mathbf{E} \xi_n^2 \right)^2 = 3 \left(\mathbf{E} \left(\sum_{n=1}^N \mu_n \xi_n \right)^2 \right)^2. \end{aligned}$$

Let $X_i = \lim_{N \rightarrow \infty} X_{i,N}$ in $L_4(\Omega)$ for $i = 1, 2$. Passing to the limit in (5) as $N \rightarrow \infty$ we conclude that $\mathbf{E} (X_1 \pm X_2)^4 = 3(\mathbf{E} (X_1 \pm X_2)^2)^2$. Thus $X(t)$ is a stochastic process with increments of the first and second order belonging to the class K_1 . Similarly,

$$Y(t) = \sum_{n=1}^{\infty} ((1 - \cos y_n t) / y_n) \eta_n, \quad t \in [0, 1],$$

is a stochastic process with increments of the first order of the class K_1 . Then,

$$\xi(t) = X(t) + Y(t), \quad t \in [0, 1],$$

is a stochastic process with increments of the first order belonging to the class K_1 . Indeed, for $X_1 = X(t) - X(s)$, $X_2 = X(v) - X(u)$ and $Y_1 = Y(t) - Y(s)$, $Y_2 = Y(v) - Y(u)$ we have

$$\begin{aligned} \mathbf{E} ((X_1 + Y_1) \pm (X_2 + Y_2))^4 &= \mathbf{E} ((X_1 \pm X_2) + (Y_1 \pm Y_2))^4 \\ &= 3 \left(\left(\mathbf{E} (X_1 \pm X_2)^2 \right)^2 + 2 \mathbf{E} (X_1 \pm X_2)^2 \mathbf{E} (Y_1 \pm Y_2)^2 + \left(\mathbf{E} (Y_1 \pm Y_2)^2 \right)^2 \right) \\ &= 3 \left(\mathbf{E} (X_1 \pm X_2)^2 + \mathbf{E} (Y_1 \pm Y_2)^2 \right)^2 = 3 \left(\mathbf{E} ((X_1 \pm X_2) + (Y_1 \pm Y_2))^2 \right)^2 \\ &= 3 \left(\mathbf{E} ((X_1 + Y_1) \pm (X_2 + Y_2))^2 \right)^2. \end{aligned}$$

We prove similarly that $\xi(t) = X(t) + Y(t)$, $t \in [0, 1]$, is a stochastic process with increments of the second order of the class K_1 .

3. STRONGLY CONSISTENT ESTIMATOR OF THE PARAMETER H

For a stochastic process $\{\xi(t), t \in [0, 1]\}$, we introduce the following notation

$$\begin{aligned}\xi_{k,n}^{(1)} &= \xi\left(\frac{k+1}{a_n}\right) - \xi\left(\frac{k}{a_n}\right), \\ \xi_{k,n}^{(2)} &= \xi\left(\frac{k+1}{a_n}\right) - 2\xi\left(\frac{k+0.5}{a_n}\right) + \xi\left(\frac{k}{a_n}\right), \quad 0 \leq k \leq a_n - 1.\end{aligned}$$

Consider the sequence of related Baxter sums

$$S_n^{(i)} = \sum_{k=0}^{a_n-1} \left(\xi_{k,n}^{(i)}\right)^2, \quad \hat{S}_n^{(i)} = a_n^{2H-1} S_n^{(i)}, \quad i = 1, 2.$$

The following auxiliary result is needed to estimate the variances of the Baxter sums $\hat{S}_n^{(i)}$, $i = 1, 2$.

Lemma 3.1 ([15]). *Let a random vector (ξ, η) belong to the class K . Then*

$$\mathbb{E}(\xi^2 \eta^2) \leq 2(\mathbb{E} \xi \eta)^2 + \mathbb{E} \xi^2 \mathbb{E} \eta^2 + \frac{1}{2} \left((\mathbb{E} \xi^2)^2 - \frac{1}{3} \mathbb{E} \xi^4 \right) + \frac{1}{2} \left((\mathbb{E} \eta^2)^2 - \frac{1}{3} \mathbb{E} \eta^4 \right).$$

Lemma 3.1 implies that

$$(6) \quad \mathbb{E}(\xi_i^2 \eta_i^2) \leq 2(\mathbb{E}(\xi_i \eta_i))^2 + \mathbb{E} \xi_i^2 \mathbb{E} \eta_i^2, \quad i = 1, 2,$$

where the random vectors (ξ_1, η_1) and (ξ_2, η_2) are defined in Definition 2.2 and where a stochastic process $\{\xi(t), t \in [0, 1]\}$ has increments of the first and second order belonging to the class K_1 .

Lemma 3.2. *Let $H^* \in (0, 1)$. Then*

$$\sup_{H \in (0, H^*]} \text{Var } \hat{S}_n^{(1)} \leq \begin{cases} \frac{2}{a_n} (3 + 2\zeta(4 - 4H^*)), & \text{if } H^* \in (0, \frac{3}{4}), \\ \frac{2}{a_n} (3 + 2(1 + \ln a_n)), & \text{if } H^* = \frac{3}{4}, \\ \frac{2}{a_n} (3 + 2\frac{a_n^{4H^*-3}}{4H^*-3}), & \text{if } H^* \in (\frac{3}{4}, 1), \end{cases}$$

where

$$(7) \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1.$$

Proof. We have

$$\begin{aligned}\text{Var } \hat{S}_n^{(1)} &= a_n^{4H-2} \text{Var } S_n^{(1)}, \\ \text{Var } S_n^{(1)} &= \mathbb{E} \left(S_n^{(1)} - \mathbb{E} S_n^{(1)} \right)^2 = \mathbb{E} \left(\sum_{k=0}^{a_n-1} \left(\xi_{k,n}^{(1)} \right)^2 - \sum_{k=0}^{a_n-1} \mathbb{E} \left(\xi_{k,n}^{(1)} \right)^2 \right)^2 \\ &= \mathbb{E} \left(\sum_{k,j=0}^{a_n-1} \left(\left(\xi_{k,n}^{(1)} \right)^2 - \mathbb{E} \left(\xi_{k,n}^{(1)} \right)^2 \right) \left(\left(\xi_{j,n}^{(1)} \right)^2 - \mathbb{E} \left(\xi_{j,n}^{(1)} \right)^2 \right) \right) \\ &= \sum_{k,j=0}^{a_n-1} \left(\mathbb{E} \left(\xi_{k,n}^{(1)} \right)^2 \left(\xi_{j,n}^{(1)} \right)^2 - \mathbb{E} \left(\xi_{k,n}^{(1)} \right)^2 \mathbb{E} \left(\xi_{j,n}^{(1)} \right)^2 \right).\end{aligned}$$

By inequality (6),

$$\text{Var } S_n^{(1)} \leq 2 \sum_{k,j=0}^{a_n-1} \left(\mathbb{E} \xi_{k,n}^{(1)} \xi_{j,n}^{(1)} \right)^2.$$

Further,

$$\mathbb{E} \xi_{k,n}^{(1)} \xi_{j,n}^{(1)} = \frac{1}{2} \left| \frac{(k-j)+1}{a_n} \right|^{2H} - \left| \frac{k-j}{a_n} \right|^{2H} + \frac{1}{2} \left| \frac{(k-j)-1}{a_n} \right|^{2H}, \quad 0 \leq k, j \leq a_n - 1.$$

Put $v_l := (l+1)^{2H} - 2l^{2H} + (l-1)^{2H}$, $l \geq 1$. Then

$$\text{Var } S_n^{(1)} \leq 2 \left(a_n^{1-4H} + \frac{a_n^{-4H}}{2} \sum_{\substack{k,j=0, \\ k < j}}^{a_n-1} v_{j-k}^2 \right) \leq 2a_n^{1-4H} \left(1 + \frac{1}{2} \sum_{l=1}^{a_n-1} v_l^2 \right).$$

Since $v_1^2 = (2^{2H} - 2)^2 \leq 4$ for $H \in (0, 1)$, we obtain

$$\text{Var } S_n^{(1)} \leq 2a_n^{1-4H} \left(3 + \frac{1}{2} \sum_{l=2}^{a_n-1} v_l^2 \right).$$

Now v_l is an increment of the second order of the function $f(x) = x^{2H}$, $x \geq 1$, corresponding to the interval $[l-1, l+1]$. Thus the representation obtained in [9, p. 244] for increments of an arbitrary order implies with $n = 2$ that $v_l = 2H(2H-1)\theta_l^{2H-2}$, where $\theta_l \in (l-1, l+1)$. Recalling that $\sup_{H \in (0,1)} |2H(2H-1)| = 2$ we derive

$$v_l^2 \leq \frac{4}{(l-1)^{4-4H}}, \quad l \geq 2,$$

whence

$$\text{Var } S_n^{(1)} \leq 2a_n^{1-4H} \left(3 + 2 \sum_{l=2}^{a_n-1} \frac{1}{(l-1)^{4-4H}} \right).$$

Therefore

$$\text{Var } \hat{S}_n^{(1)} \leq \frac{2}{a_n} \left(3 + 2 \sum_{l=2}^{a_n-1} \frac{1}{(l-1)^{4-4H}} \right)$$

and

$$(8) \quad \sup_{H \in (0, H^*]} \text{Var } \hat{S}_n^{(1)} \leq \frac{2}{a_n} \left(3 + 2 \sum_{l=2}^{a_n-1} \frac{1}{(l-1)^{4-4H^*}} \right).$$

Since

$$\sum_{l=1}^{a_n} \frac{1}{l^{4-4H^*}} \leq \begin{cases} \zeta(4-4H^*), & H^* \in (0, \frac{3}{4}); \\ 1 + \ln a_n, & H^* = \frac{3}{4}; \\ \frac{a_n^{4H^*-3}}{4H^*-3}, & \text{if } H^* \in (\frac{3}{4}, 1), \end{cases}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, $s > 1$, inequality (8) completes the proof of Lemma 3.2. $\square$

Theorem 3.1. $\hat{S}_n^{(1)} \rightarrow 1$ with probability one as $n \rightarrow \infty$.

Proof. Note that

$$(9) \quad \mathbb{E} \hat{S}_n^{(1)} = a_n^{2H-1} a_n \mathbb{E} \left(\xi_{0,n}^{(1)} \right)^2 = 1, \quad n \geq 1.$$

In view of the convergence for all $\alpha > 0$ of the series $\sum_{n=1}^{\infty} a_n^{-\alpha}$, Lemma 3.2 implies that the series $\sum_{n=1}^{\infty} \text{Var } \hat{S}_n^{(1)}$ converges for an arbitrary $H \in (0, 1)$. Considering (9), we complete the proof of the theorem. $\square$

Lemma 3.3. For an arbitrary $H \in (0, 1)$,

$$\text{Var } \hat{S}_n^{(2)} \leq \frac{2}{a_n} \left(9 + \frac{c}{2} + \frac{c_*^2}{2} \zeta(4) \right),$$

where $\zeta(4) = \pi^4/90$,

$$c = \sup_{H \in (0,1)} \left(\frac{6 \cdot 2^{2H} - 4 - 4 \cdot 3^{2H} + 2^{4H}}{2^{2H}} \right)^2,$$

$$c_* = \sup_{H \in (0,1)} \left| \frac{2H(2H-1)(2H-2)(2H-3)}{2^4} \right|.$$

Proof. We have

$$\text{Var } \hat{S}_n^{(2)} = a_n^{4H-2} \text{Var } S_n^{(2)}.$$

As in the proof of Lemma 3.2, we obtain

$$\text{Var } S_n^{(2)} \leq 2 \sum_{k,j=0}^{a_n-1} \left(\mathbb{E} \xi_{k,n}^{(2)} \xi_{j,n}^{(2)} \right)^2.$$

Further,

$$\begin{aligned} \mathbb{E} \left(\xi_{k,n}^{(2)} \xi_{j,n}^{(2)} \right) &= \frac{1}{2} \left(- \left| \frac{(k-j)-1}{a_n} \right|^{2H} + 4 \left| \frac{(k-j)-1/2}{a_n} \right|^{2H} - 6 \left| \frac{k-j}{a_n} \right|^{2H} \right. \\ &\quad \left. + 4 \left| \frac{(k-j)+1/2}{a_n} \right|^{2H} - \left| \frac{(k-j)+1}{a_n} \right|^{2H} \right), \\ &0 \leq k, j \leq a_n - 1. \end{aligned}$$

If $k = j$, then

$$\mathbb{E} \left(\xi_{k,n}^{(2)} \right)^2 = \mathbb{E} \left(\xi_{0,n}^{(2)} \right)^2 = a_n^{-2H} (2^{2-2H} - 1).$$

Put

$$w_l := \left(|l-1|^{2H} - 4 \left| l - \frac{1}{2} \right|^{2H} + 6l^{2H} - 4 \left| l + \frac{1}{2} \right|^{2H} + |l+1|^{2H} \right), \quad l \geq 1.$$

If $l = 1$, then $w_1 = (6 \cdot 2^{2H} - 4 - 4 \cdot 3^{2H} + 2^{4H})/(2^{2H})$. Thus

$$\begin{aligned} \text{Var } S_n^{(2)} &\leq 2 \left(a_n^{1-4H} (2^{2-2H} - 1)^2 + \frac{a_n^{-4H}}{2} \sum_{\substack{k,j=0, \\ k < j}}^{a_n-1} w_{j-k}^2 \right) \\ &\leq 2 \left(a_n^{1-4H} (2^{2-2H} - 1)^2 \right. \\ &\quad \left. + \frac{a_n^{1-4H}}{2} \left(\frac{6 \cdot 2^{2H} - 4 - 4 \cdot 3^{2H} + 2^{4H}}{2^{2H}} \right)^2 + \frac{a_n^{1-4H}}{2} \sum_{l=2}^{a_n-1} w_l^2 \right). \end{aligned}$$

Since $\sup_{H \in (0,1)} (2^{2-2H} - 1)^2 = 9$, we conclude that

$$\text{Var } S_n^{(2)} \leq 2a_n^{1-4H} \left(9 + \frac{c}{2} + \frac{1}{2} \sum_{l=2}^{a_n-1} w_l^2 \right),$$

where

$$(10) \quad c = \sup_{H \in (0,1)} \left(\frac{6 \cdot 2^{2H} - 4 - 4 \cdot 3^{2H} + 2^{4H}}{2^{2H}} \right)^2.$$

Further, w_l is an increment of the fourth order of the function $f(x) = x^{2H}$, $x \geq 1$, corresponding to the interval $[l-1, l+1]$. Thus the representation for increments of order n with $n = 4$ [9, p. 244] implies that

$$w_l = \frac{2H(2H-1)(2H-2)(2H-3)}{2^4} \theta_l^{2H-4},$$

where $\theta_l \in (l-1, l+1)$. Taking into account that

$$(11) \quad c_* = \sup_{H \in (0,1)} \left| \frac{2H(2H-1)(2H-2)(2H-3)}{2^4} \right|,$$

we establish

$$w_l^2 \leq c_*^2 \frac{1}{(l-1)^{8-4H}}, \quad l \geq 2.$$

Hence

$$\text{Var } S_n^{(2)} \leq 2a_n^{1-4H} \left(9 + \frac{c}{2} + \frac{c_*^2}{2} \sum_{l=2}^{a_n-1} \frac{1}{(l-1)^{8-4H}} \right) \leq 2a_n^{1-4H} \left(9 + \frac{c}{2} + \frac{c_*^2}{2} \zeta(4) \right),$$

where

$$\zeta(4) = \sum_{n=1}^{\infty} n^{-4} = \frac{\pi^4}{90} = 1.08232323 \dots$$

Therefore

$$\sup_{H \in (0,1)} \text{Var } \hat{S}_n^{(2)} \leq \frac{2}{a_n} \left(9 + \frac{c}{2} + \frac{c_*^2}{2} \zeta(4) \right).$$

Lemma 3.3 is proved. □

Theorem 3.2. $\hat{S}_n^{(2)} \rightarrow 2^{2-2H} - 1$ with probability one as $n \rightarrow \infty$.

Proof. Note that

$$\mathbb{E} \hat{S}_n^{(2)} = a_n^{2H-1} a_n \mathbb{E} \left(\xi_{0,n}^{(2)} \right)^2 = 2^{2-2H} - 1.$$

Since the series $\sum_{n=1}^{\infty} a_n^{-1}$ converges, Lemma 3.3 implies that the series $\sum_{n=1}^{\infty} \text{Var } \hat{S}_n^{(2)}$ converges, too, for an arbitrary $H \in (0,1)$. Theorem 3.2 is proved. □

Theorems 3.1 and 3.2 imply the following result.

Corollary 3.1.

$$(12) \quad \frac{S_n^{(2)}}{S_n^{(1)}} = \frac{\hat{S}_n^{(2)}}{\hat{S}_n^{(1)}} \rightarrow 2^{2-2H} - 1, \quad H \in (0,1),$$

with probability one as $n \rightarrow \infty$.

Put

$$(13) \quad \theta(H) := 2^{2-2H} - 1, \quad H \in (0,1).$$

Then

$$(14) \quad \varphi(\theta) = 1 - \frac{1}{2} \log_2(\theta + 1), \quad \theta \in (0,3),$$

is the inverse function to $\theta(H)$, $H \in (0,1)$.

Theorem 3.3. *The statistic*

$$\hat{H}_n = 1 - \frac{1}{2} \log_2(\theta_n + 1), \quad n \geq 1,$$

where $\theta_n = S_n^{(2)}/S_n^{(1)}$, $n \geq 1$, is a strongly consistent estimator of the parameter H .

Proof. Since (14) is the inverse function to (13), convergence (12) implies that $\hat{H}_n \rightarrow H$ with probability one as $n \rightarrow \infty$. $\square$

4. CONFIDENCE INTERVALS

Lemma 4.1. *Let $\{X_k \mid 0 \leq k \leq a_n, a_n \in N\}$ and $\{Y_k \mid 0 \leq k \leq a_n, a_n \in N\}$ be two sequences of centered random variables with finite moments of fourth order. Assume that*

$$\begin{aligned} \mathbb{E} X_k = \mathbb{E} Y_k = 0, \quad \mathbb{E} X_k^2 = \mathbb{E} X_0^2, \quad \mathbb{E} Y_k^2 = \mathbb{E} Y_0^2, \quad 0 \leq k \leq a_n; \\ S_1 = \sum_{k=0}^{a_n} X_k^2, \quad S_2 = \sum_{k=0}^{a_n} Y_k^2, \quad \theta = \frac{\mathbb{E} X_0^2}{\mathbb{E} Y_0^2}. \end{aligned}$$

Then

$$\mathbb{P} \left\{ \left| \frac{S_1}{S_2} - \theta \right| \geq \varepsilon \right\} \leq \frac{\text{Var } Q_1}{(\mathbb{E} Q_1)^2} + \frac{\text{Var } Q_2}{(\mathbb{E} Q_2)^2}$$

for an arbitrary $\varepsilon > 0$, where $Q_1 = (\theta - \varepsilon) S_2 - S_1$ and $Q_2 = S_1 - (\theta + \varepsilon) S_2$.

Proof. We have

$$\begin{aligned} \mathbb{P} \left\{ \left| \frac{S_1}{S_2} - \theta \right| \geq \varepsilon \right\} &= \mathbb{P} \{ |S_1 - \theta S_2| \geq \varepsilon S_2 \} \\ &= \mathbb{P} \{ S_1 - \theta S_2 \leq -\varepsilon S_2 \} + \mathbb{P} \{ S_1 - \theta S_2 \geq \varepsilon S_2 \} \\ &= \mathbb{P} \{ Q_1 \geq 0 \} + \mathbb{P} \{ Q_2 \geq 0 \}. \end{aligned}$$

Note that

$$\begin{aligned} \mathbb{E} Q_1 &= (\theta - \varepsilon) \mathbb{E} S_2 - \mathbb{E} S_1 = (\theta - \varepsilon) (a_n + 1) \mathbb{E} Y_0^2 - (a_n + 1) \mathbb{E} X_0^2 \\ &= (a_n + 1)(\theta - \varepsilon - \theta) = -\varepsilon(a_n + 1) \mathbb{E} Y_0^2 < 0, \\ \mathbb{E} Q_2 &= (a_n + 1) \mathbb{E} X_0^2 - (\theta + \varepsilon) (a_n + 1) \mathbb{E} Y_0^2 = -\varepsilon(a_n + 1) \mathbb{E} Y_0^2 < 0. \end{aligned}$$

The probability $\mathbb{P} \{ Q_1 \geq 0 \}$ is estimated from above by the Chebyshev inequality:

$$\mathbb{P} \{ Q_1 \geq 0 \} = \mathbb{P} \{ Q_1 - \mathbb{E} Q_1 \geq -\mathbb{E} Q_1 \} \leq \mathbb{P} \{ |Q_1 - \mathbb{E} Q_1| \geq -\mathbb{E} Q_1 \} \leq \frac{\text{Var } Q_1}{(\mathbb{E} Q_1)^2}.$$

The probability $\mathbb{P} \{ Q_2 \geq 0 \}$ is estimated similarly. $\square$

Using Lemma 4.1 we construct the confidence interval for the parameter $\theta = 2^{2-2H} - 1$, $H \in (0, 1)$. Let $1 - p \in (0, 1)$ be a given confidence level. A positive number $\gamma_n(p)$ is defined as follows

$$\mathbb{P} \{ |\theta_n - \theta| \geq \gamma_n(p) \} < p.$$

Put

$$Q_1 = (\theta - \gamma_n(p)) \hat{S}_n^{(1)} - \hat{S}_n^{(2)}, \quad Q_2 = \hat{S}_n^{(2)} - (\theta + \gamma_n(p)) \hat{S}_n^{(1)},$$

where $\gamma_n(p)$ is not specified yet. The expectations of the random variables Q_1 and Q_2 are such that

$$\mathbb{E} Q_1 = \mathbb{E} Q_2 = -\gamma_n(p).$$

We use the inequality $(a + b)^2 \leq 2(a^2 + b^2)$, $a, b \in \mathbf{R}$, to estimate the variances of the random variables Q_1 and Q_2 :

$$\text{Var } Q_1 \leq 2(\theta - \gamma_n(p))^2 \text{Var } \hat{S}_n^{(1)} + 2 \text{Var } \hat{S}_n^{(2)},$$

$$\text{Var } Q_2 \leq 2(\theta + \gamma_n(p))^2 \text{Var } \hat{S}_n^{(1)} + 2 \text{Var } \hat{S}_n^{(2)}.$$

The estimates for $\text{Var } \hat{S}_n^{(1)}$ and $\text{Var } \hat{S}_n^{(2)}$ are proved in Lemmas 3.2 and 3.3, respectively. Recall that $\theta \in (0, 3)$.

We choose $\gamma_n(p)$ such that

$$(15) \quad \frac{\text{Var } Q_1}{(\mathbb{E} Q_1)^2} \leq \frac{2(\theta - \gamma_n(p))^2 \text{Var } \hat{S}_n^{(1)} + 2 \text{Var } \hat{S}_n^{(2)}}{\gamma_n^2(p)} \leq \frac{p}{2},$$

$$(16) \quad \frac{\text{Var } Q_2}{(\mathbb{E} Q_2)^2} \leq \frac{2(\theta + \gamma_n(p))^2 \text{Var } \hat{S}_n^{(1)} + 2 \text{Var } \hat{S}_n^{(2)}}{\gamma_n^2(p)} \leq \frac{p}{2}.$$

For $H^* \in (0, 1)$, put

$$\alpha_n = \begin{cases} \frac{4}{a_n} (3 + 2\zeta(4 - 4H^*)), & H^* \in (0, \frac{3}{4}); \\ \frac{4}{a_n} (3 + 2(1 + \ln a_n)), & H^* = \frac{3}{4}; \\ \frac{4}{a_n} \left(3 + 2\frac{a_n^{4H^*-3}}{4H^*-3}\right), & H^* \in (\frac{3}{4}, 1), \end{cases}$$

$$\beta_n = \frac{4}{a_n} \left(9 + \frac{c}{2} + \frac{c_*}{2} \zeta(4)\right),$$

where $c, c_*, \zeta(s)$, $s > 1$, are defined by equalities (10), (11), and (7), respectively.

Now Lemmas 3.2 and 3.3 imply that

$$\alpha_n \leq \sup_{H \in (0, H^*]} \text{Var } \hat{S}_n^{(1)}, \quad \beta_n \leq \sup_{H \in (0, H^*]} \text{Var } \hat{S}_n^{(2)}.$$

If $H \in (0, H^*]$, then inequality (16) follows from the inequality

$$\frac{(3 + \gamma_n(p))^2 \alpha_n + \beta_n}{\gamma_n^2(p)} \leq \frac{p}{4}.$$

Solving this inequality with respect to $\gamma_n(p)$ under the condition that $\frac{p}{4} - \alpha_n > 0$ (which is true for all n sufficiently large) we get

$$\gamma_n(p) \geq \frac{6\alpha_n + \sqrt{D_n}}{2\left(\frac{p}{4} - \alpha_n\right)},$$

where

$$D_n = 36\alpha_n^2 + 4\left(\frac{p}{4} - \alpha_n\right)(9\alpha_n + \beta_n).$$

Put

$$(17) \quad \gamma_n(p) = \frac{6\alpha_n + \sqrt{D_n}}{2\left(\frac{p}{4} - \alpha_n\right)}.$$

The above reasoning implies the following result.

Theorem 4.1. *Let $H \in (0, H^*]$, where $H^* \in (0, 1)$ is a fixed number, and let*

$$\frac{p}{4} - \alpha_n > 0.$$

Then $(H_{n,l}, H_{n,r})$ is a confidence interval for the parameter H with the confidence probability $1 - p \in (0, 1)$, where

$$H_{n,l} = \varphi(\min(\theta_n + \gamma_n(p), 3)), \quad H_{n,r} = \varphi(\max(\theta(H^*), \theta_n - \gamma_n(p))),$$

$\theta_n = S_n^{(2)}/S_n^{(1)}$, the function $\varphi(H)$ is defined by equality (14), and $\gamma_n(p)$ is evaluated by (17).

Note that $\gamma_n(p) = O(\sqrt{\alpha_n})$ as $n \rightarrow \infty$.

5. CONCLUDING REMARKS

A Baxter type estimator of the unknown parameter H of the covariance function (1) of a stochastic process whose first and second order increments belong to the class K_1 is studied in this paper. Non-asymptotic confidence intervals for H are also found.

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DEPARTMENT OF PROBABILITY THEORY, STATISTICS, AND ACTUARIAL MATHEMATICS, FACULTY FOR MECHANICS AND MATHEMATICS, NATIONAL TARAS SHEVCHENKO UNIVERSITY, VOLODYMYRS'KA STREET, 64, KYIV 01601, UKRAINE

E-mail address: olja_sunjavska@ua.fm

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