ON A CONJECTURE OF ERDÖS ABOUT ADDITIVE FUNCTIONS

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ABSTRACT. For a real-valued additive function $f: \mathbb{N} \rightarrow \mathbb{R}$ and for each $n \in \mathbb{N}$ we define a distribution function

$$F_n(x) := \frac{1}{n} \#\{m \leq n: f(m) \leq x\}.$$

In this paper we prove a conjecture of Erdős, which asserts that in order for the sequence F_n to be (weakly) convergent, it is sufficient that there exist two numbers $a < b$ such that $\lim_{n \rightarrow \infty} (F_n(b) - F_n(a))$ exists and is positive.

The proof is based upon the use of the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ to mimic the behavior of an additive function as a sum of independent random variables.

1. INTRODUCTION

A function $f: \mathbb{N} \rightarrow \mathbb{R}$ is called *additive* if $f(mn) = f(m) + f(n)$ for any coprime integers m and n . Then f is defined by its values $f(p^k)$ on prime powers p^k (p prime, $k \in \mathbb{N}$) and $f(1) = 0$.

Given a real-valued additive function f , one can define, for each $n \in \mathbb{N}$, a distribution function

$$(1.1) \quad F_n(x) := \frac{1}{n} \#\{m \leq n: f(m) \leq x\}.$$

An old conjecture of Erdős in 1947 (see Erdős [4]) asserts that in order for the sequence F_n to be (weakly) convergent (in this case we say that the additive function f possesses a *limit distribution*), it is sufficient that there exist two numbers $a < b$ such that

$$(1.2) \quad \lim_{n \rightarrow \infty} (F_n(b) - F_n(a)) \quad \text{exists and is positive.}$$

In 1992 A. Hildebrand [6] showed that the conclusion of Erdős' conjecture is valid, provided (1.2) is strengthened to

$$(1.3) \quad L_a := \lim_{n \rightarrow \infty} F_n(a) \quad \text{and} \quad L_b := \lim_{n \rightarrow \infty} F_n(b),$$

both exists and $L_a \neq L_b$. Some further discussions are contained in [10] and [11].

In this paper we show that the above conjecture of Erdős holds.

Theorem. *Let $f: \mathbb{N} \rightarrow \mathbb{R}$ be an additive function. In order for the distributions (1.1) to converge, it is sufficient that (1.2) holds for some $a < b$.*

The proof is based upon a method, introduced in [7, 8] using the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ to mimic the behavior of an additive function as a sum of independent random variables.

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2. FINITELY DISTRIBUTED ADDITIVE FUNCTIONS

An additive function f is said to be *finitely distributed* if there are positive constants c_1 and c_2 , and an unbounded sequence $n_1 < n_2 < \dots$ so that for every i there exists a sequence

$$a_1^{(i)} < a_2^{(i)} < \dots < a_{t_i}^{(i)} < n_i$$

satisfying

$$\left| f\left(a_r^{(i)}\right) - f\left(a_s^{(i)}\right) \right| < c_1, \quad t_i > c_2 n_i, \quad 1 \leq r, s \leq t_i.$$

The necessary and sufficient condition that f should be finitely distributed is that there should exist a constant c and an additive function h so that

$$(2.1) \quad f(n) = c \log n + h(n)$$

where both series

$$(2.2) \quad \sum_{|h(p)| > 1} \frac{1}{p}, \quad \sum_{\substack{p \\ |h(p)| \leq 1}} \frac{h^2(p)}{p}$$

converge (Erdős [3], 1946). Further characterizations of finitely distributed additive functions can be found in Chapter 7 of Elliott's book [2]. For our purpose we shall apply the following result ([2, p. 259]).

Proposition. *If an additive function admits representation (2.1) with convergent series (2.2), then, if we define*

$$(2.3) \quad \alpha(n) = c \log n + \sum_{\substack{p \leq n \\ |h(p)| \leq 1}} \frac{h(p)}{p},$$

the distribution functions

$$(2.4) \quad G_n(x) := \frac{1}{n} \#\{m \leq n : f(m) - \alpha(n) \leq x\}$$

weakly converge to some distribution function $G(x)$.

If (1.2) holds, then f is finitely distributed. Now, assume that (2.1) holds and $\alpha(n)$ is unbounded. Then, if $\alpha(n'_k) \rightarrow \infty$, $k \rightarrow \infty$, for some subsequence (n'_k) , then

$$(2.5) \quad \lim_{k \rightarrow \infty} \left\{ G_{n'_k}(b - \alpha(n'_k)) - G_{n'_k}(a - \alpha(n'_k)) \right\} = \lim_{k \rightarrow \infty} (F_{n'_k}(b) - F_{n'_k}(a)) > 0$$

by (1.2), whereas the left side in (2.5) tends to zero since G_n converge weakly to some distribution function. Then, since $\alpha(n) = c \log n + O(\log \log n)$, we conclude $c = 0$, i.e., $f = h$, and

$$(2.6) \quad A(n) := \sum_{\substack{p \leq n \\ |f(p)| \leq 1}} \frac{f(p)}{p} = O(1) \quad \text{for all } n \in \mathbb{N}.$$

In the following we assume that

$$(2.7) \quad \sum_{\substack{p \\ |f(p)| \leq 1}} \frac{f(p)}{p} \quad \text{diverges,}$$

which implies (see [3], Theorem II) that $G(x)$ is continuous and strictly increasing for all $x \in \mathbb{R}$.

For each $n \in \mathbb{N}$ define the additive function f_n by

$$f_n(p^k) = \begin{cases} f(p^k), & \text{if } p \leq n, \\ 0, & \text{otherwise} \end{cases}$$

and put, for $A \subset \mathbb{N}$,

$$\delta_n(A) := \frac{1}{n} \# \{m \leq n : m \in A\}.$$

If the limit

$$(2.8) \quad \delta(A) := \lim_{n \rightarrow \infty} \delta_n(A)$$

exists we say that A possesses the asymptotic density $\delta(A)$.

If some sequence $\{n'_k\}$ is given we write

$$(2.9) \quad \delta'(A) := \lim_{k \rightarrow \infty} \delta_{n'_k}(A)$$

in case the limit (2.9) exists.

With these notations we show

Lemma 1. *Assume that (1.2) holds. Then*

$$(2.10) \quad \lim_{n \rightarrow \infty} \delta(\{m : f_n(m) \in (a, b]\}) = \delta(\{m : f(m) \in (a, b]\}) =: c_0 > 0.$$

Proof. Observe that $\delta(\{m : f_n(m) \in (a, b]\})$ always exists. Assume that (2.10) does not hold. Then there exists a sequence $\{n_k\}$ of natural numbers such that

$$\lim_{k \rightarrow \infty} \delta(\{m : f_{n_k}(m) \in (a, b]\}) = c' \neq c_0.$$

Since $A(n_k) = O(1)$ there exists some subsequence $\{n'_k\}$ of $\{n_k\}$ so that

$$\lim_{k \rightarrow \infty} A(n'_k) =: A'$$

exists. Choose k_1 such that for every $k_0 \geq k_1$,

$$(2.11) \quad \begin{aligned} & \left| \delta \left(\left\{ m : f_{n'_{k_0}}(m) \in (a, b] \right\} \right) - c_0 \right| \\ &= \left| \delta' \left(\left\{ m : f_{n'_{k_0}}(m) \in (a, b] \right\} \right) - \delta' \left(\left\{ m : f(m) \in (a, b] \right\} \right) \right| \\ &\geq \frac{|c_0 - c'|}{2}. \end{aligned}$$

On the other hand, we shall show that

$$(2.12) \quad \lim_{k_0 \rightarrow \infty} \overline{\lim}_{k \rightarrow \infty} \delta_{n'_k} \left(\left\{ m : \left| f(m) - f_{n'_{k_0}}(m) \right| > \varepsilon \right\} \right) = 0$$

for every $\varepsilon > 0$ which contradicts (2.11).

For the proof of (2.12) put

$$\mathcal{P}_0 := \{p : |f(p)| > 1\} \cup \{p^k : k \geq 2\}.$$

Define the functions

$$h'(m) := \sum_{\substack{p^k \parallel m \\ p^k \in \mathcal{P}_0 \\ p > n'_{k_0}}} f(p^k)$$

and

$$j(m) := \sum_{\substack{p \parallel m \\ |f(p)| \leq 1 \\ p > n'_{k_0}}} f(p).$$

We deduce from the definitions of these functions that

$$f(m) - f_{n'_{k_0}}(m) = j(m) - (A(n'_k) - A(n'_{k_0})) + h'(m) + (A(n'_k) - A(n'_{k_0})).$$

We shall prove that for every $\varepsilon > 0$ each of the three expressions

$$L_1(k_0) = \overline{\lim}_{k \rightarrow \infty} \delta_{n'_k}(\{m: |j(m) - (A(n'_k) - A(n'_{k_0}))| > \varepsilon\}),$$

$$L_2(k_0) = \overline{\lim}_{k \rightarrow \infty} \delta_{n'_k}(\{m: |h'(m)| > \varepsilon\})$$

and

$$L_3(k_0) = \overline{\lim}_{k \rightarrow \infty} \delta_{n'_k}(\{m: |A(n'_k) - A(n'_{k_0})| > \varepsilon\})$$

converge to zero as $k_0 \rightarrow \infty$. We may readily estimate the first of these three expressions by appealing to the Turán–Kubilius inequality. In our present circumstances it becomes

$$\frac{1}{n'_k} \sum_{m=1}^{n'_k} |j(m) - (A(n'_k) - A(n'_{k_0}))|^2 \ll \sum_{\substack{n'_{k_0} < p \leq n'_k \\ |f(p)| \leq 1}} \frac{|f(p)|^2}{p}.$$

Appealing to the convergence of the second sum in (2.2) we see that

$$L_1(k_0) \ll \frac{1}{\varepsilon^2} \sum_{\substack{n'_{k_0} < p \\ |f(p)| \leq 1}} \frac{|f(p)|^2}{p} = o(1) \quad \text{as } k_0 \rightarrow \infty.$$

The estimate $L_3(k_0) = o(1)$ as $k_0 \rightarrow \infty$ is obvious.

If an integer m is counted in the expression $L_2(k_0)$ it must satisfy one of two divisibility criteria.

First, it may be divisible by the square of a prime $p > n'_{k_0}$. The frequency of these integers is at most

$$\delta_{n'_k}(\{m: p^2 | m, p > n'_{k_0}\}) \leq \sum_{n'_{k_0} < p} \frac{1}{p^2} = o(1) \quad \text{as } k_0 \rightarrow \infty.$$

Next, it may be exactly divisible by a prime in the range $n'_{k_0} < p$ for which $|f(p)| > 1$. From the hypothesis (2.2) we deduce that the frequencies of such integers is at most

$$\sum_{\substack{n'_{k_0} < p \\ |f(p)| > 1}} \frac{1}{p} = o(1) \quad \text{as } k_0 \rightarrow \infty$$

and thus $L_2(k_0) = o(1)$ as $k_0 \rightarrow \infty$. We have now shown that (2.12) holds and the proof of Lemma 1 is completed. $\square$

In the next step we identify the additive function f with a sum $\sum_{p \text{ prime}} X_p$ of independent random variables.

3. ADDITIVE FUNCTIONS AS A SUM OF INDEPENDENT RANDOM VARIABLES

For the sake of simplicity we restrict ourselves to strongly additive functions. Then f can be written in the form

$$f = \sum_p f(p) \varepsilon_p$$

where

$$\varepsilon_p(n) = \begin{cases} 1, & \text{if } p | n, \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathcal{A}$ denotes the algebra generated by the sets

$$A_p := \{n \in \mathbb{N} : p|n\}, \quad p \text{ prime},$$

then obviously each $A \in \mathcal{A}$ possesses an asymptotic density $\delta(A)$ and $\delta(A_p) = \frac{1}{p}$ (p prime). Thus δ defines a content on $\mathcal{A}$. Now the construction runs as follows. (For details see [7, 8].) We embed $\mathbb{N}$, endowed with the discrete topology, in the Stone-Čech compactification $\beta\mathbb{N}$,

$$\mathbb{N} \hookrightarrow \beta\mathbb{N}$$

and, if for any $A \subset \mathbb{N}$, the closure of A in $\beta\mathbb{N}$ is denoted by $\bar{A}$, then

$$\bar{\mathcal{A}} := \{\bar{A} \subset \beta\mathbb{N} : A \in \mathcal{A}\}$$

is an algebra, too. The extension $\bar{\delta}$ of δ ,

$$\bar{\delta}(\bar{A}) := \delta(A), \quad \bar{A} \in \bar{\mathcal{A}},$$

defines a premeasure on $\bar{\mathcal{A}}$ and leads to a measure P , induced by

$$\delta^*(A) := \overline{\lim}_{n \rightarrow \infty} \delta_n(A) \quad \text{for all } A \subset \mathbb{N},$$

and to a probability space $(\Omega, \sigma(\bar{\mathcal{A}}), P)$ with $\Omega = \beta\mathbb{N}$ and with $P(\bar{A}_p) = 1/p$, p prime.

There is a unique extension of ε_p to a function $\bar{\varepsilon}_p$ on Ω , and putting $X_p = f(p)\bar{\varepsilon}_p$,

$$f = \sum_p f(p)\varepsilon_p \rightarrow X = \sum_p f(p)\bar{\varepsilon}_p = \sum_p X_p,$$

we get

$$f_n \rightarrow S_n := \sum_{p \leq n} X_p$$

with

$$P(X_p = f(p)) = \frac{1}{p}$$

and

$$P(X_p = 0) = 1 - \frac{1}{p}.$$

The $\bar{\varepsilon}_p$ are independent, i.e., $X = \sum_p X_p$ is a sum of independent random variables. If (1.2) holds then, by Lemma 1,

$$\lim_{n \rightarrow \infty} P(S_n \in (a, b]) = c_0 > 0$$

and, by Proposition above, the series $\sum_p X_p$ is essentially convergent (for the definition see [13, p. 262]).

Now writing

$$a_p = E(X_p^c), \quad Y_p = X_p - a_p, \quad T_n := \sum_{p \leq n} Y_p,$$

then $\lim_{n \rightarrow \infty} T_n$ exists almost surely. (Here X_p^c denotes the truncation of X_p at (a positive) c , i.e., we replace X_p by $X_p = X$ or 0 according as $|X_p| < c$ or $|X_p| \geq c$.) Denote $Y := \lim T_n$ a.s.

It is well known that the a.s. convergence of $Y = \sum_p Y_p$ is equivalent to the weak convergence of the distributions of the partial sums of that series. Moreover, by Kolmogorov's three series theorem, $Y = \sum_p Y_p$ converges a.s. if and only if the series

$$(3.1) \quad \sum_p E(Y_p^c), \quad \sum_p P(|Y_p| > c), \quad \sum_p \text{Var}(Y_p^c)$$

converge.

We choose $c = 1$, i.e., $a_p = \mathbb{E}(X_p^1)$ and put (see (2.6))

$$A(n) = \sum_{p \leq n} a_p.$$

Then $A(n) = O(1)$ and, the divergence of the sequence $A(n)$ implies (see [3, Theorem 2]).

Lemma 2. *Let $Y = \sum_p Y_p$ with $Y_p = X_p - a_p$ as above, where the partial sums $\sum_{p \leq N} a_p$ are bounded and divergent. Then the distribution function $G(x) = \mathbb{P}(Y \leq x)$ is continuous and strictly monotone for all $x \in \mathbb{R}$.*

Remark. The divergence of the sequence $A(n)$ implies

$$(3.2) \quad \begin{aligned} \sum_p a_p^- &= -\infty, \\ \sum_p a_p^+ &= +\infty, \end{aligned}$$

where $a_p^+ = \max(a_p, 0)$ and $a_p^- = \max(-a_p, 0)$. Then the strict monotonicity of the distribution function $G(x)$ in Lemma 2 can be directly proved by a result of A. Hildebrand [6].

For this we define, following the notation of Hildebrand in [6, p. 1206], the range of a random variable X as the set

$$R(X) = \{x \in \mathbb{R} : \mathbb{P}(|X - x| \leq \varepsilon) > 0 \text{ for every } \varepsilon > 0\},$$

that is, it is equal to the set of points of increase of the distribution function

$$F(x) = \mathbb{P}(X \leq x).$$

The form of this set was described by A. Hildebrand in Lemma 2 of [6] when X is given as an a.s. convergent series of independent random variables. A special version of this result is contained in the following lemma.

Lemma 3. *Let $\sum_{n=0}^{\infty} X_n$ be an a.s. convergent series of independent random variables and let X denote its sum. Suppose that for every $\varepsilon > 0$ and $n \geq n_0 = n_0(\varepsilon)$ there exist numbers $c_n^- = c_n^-(\varepsilon)$, $c_n^+ = c_n^+(\varepsilon) \in R(X_n)$ with $|c_n^-| \leq \varepsilon$ and $|c_n^+| \leq \varepsilon$ such that*

$$\lim_{N \rightarrow \infty} \sum_{n=n_0}^N c_n^- = -\infty$$

and

$$\lim_{N \rightarrow \infty} \sum_{n=n_0}^N c_n^+ = +\infty.$$

Then $R(X) = \mathbb{R}$.

Now it is easy to prove the assertions of Lemma 2. Put

$$c_p^- = \begin{cases} f(p) - a_p, & \text{if } -\frac{\varepsilon}{2} \leq f(p) < 0, \\ 0, & \text{otherwise} \end{cases}$$

and

$$c_p^+ = \begin{cases} f(p) - a_p, & \text{if } 0 < f(p) \leq \frac{\varepsilon}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Then obviously, $c_p^-, c_p^+ \in R(Y_p)$, $|c_p^-| \leq \frac{\varepsilon}{2} + |a_p| \leq \varepsilon$ and $|c_p^+| \leq \frac{\varepsilon}{2} + |a_p| \leq \varepsilon$ for $p > n_0 = n_0(\varepsilon)$ since $|a_p| \leq 1/p$. Further,

$$\begin{aligned} \sum_{n_0 \leq p \leq N} c_p^- &= \sum_{\substack{n_0 \leq p \leq N \\ -\frac{\varepsilon}{2} < f(p) < 0}} f(p) - \sum_{n_0 \leq p \leq N} a_p \\ &< \sum_{\substack{n_0 \leq p \leq N \\ -\frac{\varepsilon}{2} < f(p) < 0}} \frac{f(p)}{p} + O(1) \\ &< \sum_{\substack{n_0 \leq p \leq N \\ -1 < f(p) < 0}} \frac{f(p)}{p} + O(1) \\ &= \sum_{n_0 \leq p \leq N} a_p^- + O(1) \rightarrow -\infty \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Here the last inequality holds because of the convergence of the second series in (3.1). Similarly,

$$\lim_{N \rightarrow \infty} \sum_{n_0 \leq p \leq N} c_p^+ = +\infty.$$

We use Lemma 3 and recall that the divergence of series (2.7) implies, by Lévy's theorem, the continuity of $G(x)$ to end the proof of Lemma 2.

This ends the remark.

For every subsequence $n' = (n'_k)$ of natural numbers we define

$$\delta'(A) = \lim_{k \rightarrow \infty} \delta_{n'_k}(A)$$

if the limit exists. This leads to a content δ' on $\mathcal{A}$ and a measure P' on $\beta\mathbb{N}$ induced by

$$\delta'^*(A) = \overline{\lim}_{k \rightarrow \infty} \delta_{n'_k}(A) \quad \text{for all } A \subset \mathbb{N}.$$

Obviously, if $\Omega_0 \subset \beta\mathbb{N}$ is $\mathbf{P}$ -measurable it is P' -measurable and $\mathbf{P}(\Omega_0) = P'(\Omega_0)$.

Since every bounded real-valued function g on $\mathbb{N}$ extends uniquely to a (continuous) function $\bar{g}$ on $\beta\mathbb{N}$ (for details see R. Walker [14, p. 8 et seq.]), we conclude

$$\Omega_0 := \overline{\{m: f(m) \in (a, b]\}} = \{\omega: \bar{f}(\omega) \in [a, b]\},$$

where $\bar{f}$ is the unique extension of the (bounded) function $f_{(a,b]}$, defined by

$$f_{(a,b]}(m) = \begin{cases} f(m), & \text{if } f(m) \in (a, b], \\ |a| + |b| + 1, & \text{if } f(m) \notin (a, b]. \end{cases}$$

If (1.2) holds, then

$$\mathbf{P}(\Omega_0) = c_0 > 0.$$

4. PROOF OF THE CONJECTURE OF ERDÖS

We suppose that $A(n)$ is not convergent so that

$$(4.1) \quad \underline{A} := \liminf_{n \rightarrow \infty} A(n) < \limsup_{n \rightarrow \infty} A(n) =: \bar{A},$$

and we shall show that this leads to a contradiction.

We fix two increasing sequences $n' = \{n'_k\}$ and $n'' = \{n''_k\}$ of positive integers so that

$$\underline{A} = \lim_{k \rightarrow \infty} A(n'_k) \quad \text{and} \quad \bar{A} = \lim_{k \rightarrow \infty} A(n''_k).$$

We put

$$g_n = \sum_{p \leq n} (f(p)\varepsilon_p - a_p)$$

and define

$$(4.2) \quad g' = g_{n'_1} + \sum_{k=1}^{\infty} (g_{n'_{k+1}} - g_{n'_k}).$$

Then

$$\{m: g'(m) \in (a - \overline{A}, b - \overline{A}]\} = \{m: f(m) \in (a, b]\}$$

since $g'(m) = f(m) - \overline{A}$ for every $m \in \mathbb{N}$. Further,

$$\delta' \left(\left\{ m: g'(m) \in (a - \overline{A}, b - \overline{A}] \right\} \right) = \lim_{k \rightarrow \infty} \delta' \left(\left\{ m: g_{n'_k}(m) \in (a - \overline{A}, b - \overline{A}] \right\} \right) = c_0.$$

In the same way we define

$$g'' = g_{n''_1} + \sum_{k=1}^{\infty} (g_{n''_{k+1}} - g_{n''_k})$$

with $g''(m) = f(m) - \underline{A}$, $m \in \mathbb{N}$, and obtain

$$\delta'' \left(\left\{ m: g''(m) \in (a - \underline{A}, b - \underline{A}] \right\} \right) = \lim_{k \rightarrow \infty} \delta'' \left(\left\{ m: g_{n''_k}(m) \in (a - \underline{A}, b - \underline{A}] \right\} \right) = c_0.$$

Defining the corresponding extensions $\overline{g'}$ and $\overline{g''}$ and P' and P'' , respectively, we arrive at

$$\Omega_0 = \{\omega: \overline{g'}(\omega) \in [a - \overline{A}, b - \overline{A}]\} = \{\omega: \overline{g''}(\omega) \in [a - \underline{A}, b - \underline{A}]\}$$

together with

$$P'(\{\omega: \overline{g'}(\omega) \in [a - \overline{A}, b - \overline{A}]\}) = P''(\{\omega: \overline{g''}(\omega) \in [a - \underline{A}, b - \underline{A}]\}) = c_0.$$

Since

$$g' \quad \text{corresponds to} \quad Y' = \lim_{k \rightarrow \infty} T_{n'_k},$$

$$g'' \quad \text{corresponds to} \quad Y'' = \lim_{k \rightarrow \infty} T_{n''_k}$$

and since

$$Y = \sum_p Y_p = \lim_{n \rightarrow \infty} T_n$$

converges a.s. with respect to $\mathbf{P}$ and possesses an everywhere continuous distribution function we conclude

- (i) $\{\omega: Y'(\omega) \in [a - \overline{A}, b - \overline{A}]\} = \Omega'_0$ with $P'(\Omega_0 \triangle \Omega'_0) = 0$,
- (ii) $P'(\{\omega: Y'(\omega) \in [a - \overline{A}, a - \underline{A}]\}) \leq P'(\{\omega: Y'(\omega) \neq Y''(\omega)\}) = 0$ and
- (iii) $P'(\{\omega: Y'(\omega) \in [a - \underline{A}, b - \underline{A}]\}) = c_0$.

Observe, that (iii) implies that

$$a - \underline{A} < b - \overline{A}.$$

Since $\mathbf{P}(\{\omega: Y(\omega) \in [a - \overline{A}, a - \underline{A}]\})$ exists it must be zero by (ii), i.e.,

$$(4.3) \quad \mathbf{P}(\{\omega: Y(\omega) \in [a - \overline{A}, a - \underline{A}]\}) = 0.$$

In the same way we show

$$(4.4) \quad \mathbf{P}(\{\omega: Y(\omega) \in [b - \overline{A}, b - \underline{A}]\}) = 0.$$

Equalities (4.3) and (4.4) contradict the monotonicity of $G(x)$, and thus the assertion of Theorem 1 holds.

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BIBLIOGRAPHY

1. P. D. T. A. Elliott, *A conjecture of Kátaı*, Acta Arith. **XXVI** (1974), 11–20. MR0354599 (50:7077)
2. P. D. T. A. Elliott, *Probabilistic Number Theory I/II*, Springer, New York, 1978/1980. MR551361 (82h:10002a)
3. P. Erdős, *On the distribution function of additive functions*, Ann. Math. **47** (1946), 1–20. MR0015424 (7:416c)
4. P. Erdős, *Some remarks and corrections to one of my papers*, Bull. Amer. Math. Soc. **53** (1947), 761–763. MR0021030 (9:12e)
5. A. Hildebrand, *Additive and multiplicative functions on shifted primes*, Proc. London Math. Soc. **59**(2) (1989), 209–232. MR1004429 (90i:11107)
6. A. Hildebrand, *A note on the convergence of sums of independent random variables*, Ann. Probab. **20** (1992), 1204–1212. MR1175258 (93g:60047)
7. K.-H. Indlekofer, *A new method in probabilistic number theory*, Probab. Theory Appl., Math. Appl. **80** (1992), 299–308. MR1211915 (94e:11091)
8. K.-H. Indlekofer, *New approach to probabilistic number theory — compactifications and integration*, Adv. Studies Pure Math. **49** (2005), 133–170. MR2405602 (2009d:11119)
9. I. Kátaı, *On the distribution of arithmetical functions on the set prime plus one*, Composito Math. **19** (1968), 278–289. MR0244181 (39:5498)
10. F. Kong and Q. Tang, *A theorem on the convergence of sums of independent random variables*, Math. Acta Sci. **21 B** (2001), 331–338. MR1849532 (2002f:60052)
11. F. Kong and Q. Tang, *Notes on Erdős’ conjecture*, Math. Acta Sci. **20**(4) (2000), 533–541. MR1805022 (2001k:60032)
12. P. Lévy, *Sur les séries dont les termes sont des variables éventuellement indépendents*, Studia Math. **3** (1931), 119–155.
13. M. Loève, *Probability theory I*, Springer, Heidelberg–New York, 1977. MR0651017 (58:31324a)
14. R. Walker, *The Stone–Čech Compactification*, Springer, Heidelberg–New York, 1974. MR0380698 (52:1595)

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