QUANTITATIVE AND QUALITATIVE LIMITS FOR EXPONENTIAL ASYMPTOTICS OF HITTING TIMES FOR BIRTH-AND-DEATH CHAINS IN A SCHEME OF SERIES

UDC 519.21

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ABSTRACT. We consider a time-homogeneous discrete birth-and-death Markov chain (X_t) and investigate the asymptotics of the hitting time $\tau_n = \inf(t \geq 1: X_t \geq n)$ as well as the chain position before this time in the scheme of series as $n \rightarrow \infty$. In our case, one-step probabilities of the chain vary simultaneously with n . The proofs are based on the explicit two-sided inequalities with numerical bounds for the survival probability $P(\tau_n > t)$. These inequalities can be used also for the pre-limit finite-time schemes. We have applied the results obtained for constructing the uniform asymptotic representations of the corresponding risk function.

1. INTRODUCTION

A detailed study of the stability of distributions for general Markov chains is conducted in the monograph [2] under rather wide assumptions about the nature of jumps. Some applications of the theory are discussed there as well. The proofs developed in [2] are based on the analytical operator method. The book [2] contains some new inequalities for asymptotics of a renewal process and for solutions of the renewal equation.

Foundations of the stability theory for stochastic models are set in the monograph by Zolotarev [5]. Important achievements in the stability theory are presented in the book by Mayn and Tweedie [4].

This paper is based on the author results of [2, Chapter 7]. The results of this paper were obtained earlier, but they have not been published yet. A comparison with the paper [3] can be useful. Similar but not identical results were obtained independently in [6].

2. MAIN RESULTS

Consider a time-homogeneous birth-and-death Markov chain

$$X = (X_t, t = 0, 1, \dots)$$

with values in a discrete space $E = \mathbb{Z}_+$. The matrix of one-step transition probabilities $P = (p_{ij}, i, j \in E)$ has entries $p_{i,i-1} = q_i$, $p_{i,i+1} = p_i$, $p_{ii} = r_i = 1 - p_i - q_i$ for $i \geq 1$, and $p_{01} = p_0$, $p_{00} = 1 - p_0 = q_0$. We assume that the chain is not reducible, $p_i, q_i > 0$. The symbols $P_i(\cdot)$ and $E_i(\cdot)$ denote the conditional probability and expectation given $\{X_0 = i\}$, respectively.

Now we define the hitting moment of a “distant” level by

$$(1) \quad \tau_n = \inf(t \geq 1: X_t \geq n).$$

2010 *Mathematics Subject Classification.* Primary 60J45; Secondary 60A05, 60K05.

Key words and phrases. Birth-and-death Markov chains, hitting times, exponential asymptotics.

We investigate the asymptotic behavior of the moment τ_n in a scheme of series as $n \rightarrow \infty$. In our case, the one-step transition probabilities (p_i, q_i) may change, in general. For instance, they may depend on n .

The following notation is useful throughout the paper: if $t \geq 0$,

$$(2) \quad \theta_t = \prod_{1 \leq i \leq t} (q_i/p_i), \quad \theta_0 = 1, \quad \sigma_t = \sum_{0 \leq i < t} \theta_i, \quad \varkappa_t = 1/(p_t \theta_t), \quad t \geq 0.$$

Consider the aggregate parameters

$$(3) \quad \begin{aligned} \lambda_n &= \left(1 + \sum_{i \leq j \in E_n} \varkappa_i \theta_j \right)^{-1} = \left(1 + \sum_{i \in E_n} \varkappa_i (\sigma_n - \sigma_i) \right)^{-1}, \\ \omega_n &= \lambda_n - \lambda_n^2 + \lambda_n^2 \sum_{i \leq j < k \leq l \in E_n} \varkappa_i \theta_j \varkappa_k \theta_l \\ &= \lambda_n - \lambda_n^2 + \lambda_n^2 \sum_{i < k \in E_n} \varkappa_i (\sigma_k - \sigma_i) \varkappa_k (\sigma_n - \sigma_k). \end{aligned}$$

Hereafter the sign of summation without upper and lower indices means the summation for the whole index set $E_n \equiv \{0, 1, \dots, n-1\}$. It is worth mentioning that the continuity of the process allows us to completely determine the distribution of the moment τ_n in terms of $(p_i, q_i, i \in E_n)$.

The following bound applies to any scheme of series as well as for a fixed n .

Theorem 2.1. *The following inequality holds:*

$$(4) \quad \sup_{t \geq 0} \left| \mathbb{P}_0(\tau_n > t) - (1 - m_n^{-1})^t \right| \leq 2\omega_n(1 + \lambda_n)p_0/\lambda_n\sigma_n(1 - \omega_n),$$

where

$$m_n^{-1} = \lambda_n/(1 + \omega_n).$$

Remark 2.1. It follows from the definitions (3) that $0 < \omega_n \leq 1/2$ in (4).

Corollary 2.1. *Let $n \rightarrow \infty$ in a scheme of series in such a way that $\lambda_n \rightarrow 0$ and*

$$(5) \quad \omega_n p_0 = o(\lambda_n \sigma_n), \quad n \rightarrow \infty.$$

Then

$$\sup_{x \geq 0} |\mathbb{P}_0(\tau_n/m_n > x) - \exp(-x)| \rightarrow 0, \quad n \rightarrow \infty.$$

This convergence is uniform in the scheme of series if relation (5) holds uniformly, too.

Corollary 2.2. *Let a chain X be unchangeable for the scheme of series, irreducible and ergodic, and let $n \rightarrow \infty$. Then $\lambda_n \rightarrow 0$, $\omega_n \rightarrow 0$, and*

$$(6) \quad \sup_{x \geq 0} |\mathbb{P}_0(\lambda_n \tau_n > x) - \exp(-x)| = O(\omega_n), \quad n \rightarrow \infty.$$

Corollary 2.3. *Let n and the distribution of the chain X be fixed except for $p_0 \rightarrow 0$. Then*

$$(7) \quad \sup_{x \geq 0} |\mathbb{P}_0(p_0 \tau_n > x) - \exp(-x)| = O(p_0), \quad p_0 \rightarrow 0.$$

One can obtain the limit results for the specially structured schemes of series at one time from (4). Here are some examples.

Corollary 2.4. *Let the transition probabilities in a scheme of series for the birth-and-death chain be such that*

$$(8) \quad p_i = \varepsilon_n v_i + o(\varepsilon_n), \quad q_i = \varepsilon_n u_i + o(\varepsilon_n), \quad i \geq 1, \quad n \rightarrow \infty,$$

for some $\varepsilon_n \rightarrow 0$, and $v_i, u_i > 0$. Let

$$(9) \quad \theta_t = \prod_{i=1}^t (u_i/v_i), \quad \sigma_t = \sum_{s=1}^{t-1} \theta_s, \quad \chi_t = 1/(v_t \theta_t), \quad t \geq 1.$$

We assume that, in a scheme of series,

$$(10) \quad \sigma_n \rightarrow \infty, \quad \sum_{t \geq 1} \chi_t \equiv \chi = O(1), \quad n \rightarrow \infty.$$

Then the uniform convergence is true,

$$\sup_{t \geq 0} \left| \mathbb{P}_0(\tau_n > t) - (1 - m_n^{-1})^t \right| = O(\bar{\omega}_n + \sigma_n^{-1}) = o(1), \quad n \rightarrow \infty,$$

provided that

$$\bar{\omega}_n \equiv \sigma_n^{-1} \sum_{1 \leq i < k < n} \chi_i (\sigma_k - \sigma_i) \chi_k = o(1), \quad n \rightarrow \infty.$$

Remark 2.2. If the coefficients v_i, u_i are bounded and separated from zero, then conditions (10) are equivalent to the ergodicity of the birth-and-death chain with jump probabilities $(u_i/(u_i + v_i), v_i/(u_i + v_i))$, $i \geq 1$.

To analyze the asymptotic behavior of the joint distribution of the moment τ_n and the chain value X till this moment, we additionally assume that there is a systematic shift to zero,

$$(11) \quad q_i > p_i, \quad i \geq 1,$$

and that the state 0 in a scheme of series is asymptotically positive and attainable:

$$(12) \quad \varliminf_{n \rightarrow \infty} \lambda_n \sigma_n > 0, \quad 0 < \varliminf_{n \rightarrow \infty} p_0 \leq \overline{\varliminf}_{n \rightarrow \infty} p_0 < 1$$

(cf. [6]).

In the proof of Corollary 2.2 (see limit relation (38)) we obtain that conditions (12) hold for any fixed irreducible ergodic birth-and-death chain.

We define the speed-of-mixing indicator as

$$(13) \quad \delta_n = \min_{1 \leq i < n} (q_i - p_i) > 0.$$

Theorem 2.2. *Let conditions (11), (12) hold and let $\lambda_n \rightarrow 0$, $\omega_n \rightarrow 0$ in a scheme of series as $n \rightarrow \infty$ in such a way that*

$$(14) \quad \lambda_n \ln \lambda_n^{-1} = o(\delta_n^4), \quad n \rightarrow \infty.$$

Then, for every $s_0 > 0$,

$$(15) \quad \sup_{s \geq s_0, B \subset E} \left| \mathbb{P}_0(\lambda_n \tau_n > s, X_{s/\lambda_n} \in B) - \pi^n(B) \exp(-s) \right| \\ = O(\omega_n + \lambda_n \delta_n^{-4} \ln(1/\lambda_n \delta_n)) = o(1), \quad n \rightarrow \infty,$$

where the discrete distribution $\pi^n = (\pi_i^n, i \in E^n)$ is determined from (2), (3) as follows:

$$(16) \quad \pi_n^n = \lambda_n, \quad \pi_i^n = \lambda_n \varkappa_i(\sigma_n - \sigma_i), \quad i < n, \quad \pi^n(B) = \sum_{i \in B} \pi_i^n.$$

Corollary 2.5. *Let the chain X not change in a scheme of series and be irreducible and ergodic. Then, the sufficient condition for the convergence to zero of the left-hand part of (15) is*

$$(17) \quad \lim_{n \rightarrow \infty} n\delta_n / \ln n > 3/2.$$

Remark 2.3. For comparison with (17), note that

- (a) in a class of birth-and-death chains satisfying conditions (13) and

$$\delta_n \rightarrow 0, \quad n \rightarrow \infty,$$

the sufficient condition for the ergodicity is

$$\lim_{n \rightarrow \infty} n\delta_n / \ln n > 1;$$

- (b) under additional assumptions $r_i \equiv 0$ and $q_i - p_i \downarrow 0$, $i \rightarrow \infty$, it follows from the condition

$$\lim_{n \rightarrow \infty} n\delta_n < 1/2$$

that the chain is not ergodic.

3. PROOFS

The proofs in this section are based on Corollary 7.5 in [2, Chapter VII].

In order to apply Corollary 7.5 [2, Chapter VII], we consider an auxiliary finite chain $X^n = (X_t^n, t \geq 0)$ with the set of states $E^n \equiv \{0, 1, \dots, n\} = E_n \cup \{n\}$ and the transition probabilities

$$P_n = (p_{ij}(n), i, j \in E^n),$$

where $p_{ij}(n) = p_{ij}$ for $i \in E_n$ and

$$p_{n0}(n) = 1.$$

It is evident that the distributions of the moments τ_n in (1) for chains (X_t) and (X_t^n) are equal.

Lemma 3.1. *The chain X^n has the unique invariant probability $\pi^n = (\pi_i^n, i \in E^n)$ where*

$$(18) \quad \pi_n^n = \lambda_n, \quad \pi_j^n = \lambda_n \varkappa_j (\sigma_n - \sigma_j) = \lambda_n \sum_{i < j \in E_n} \theta_i \varkappa_j, \quad j < n.$$

Proof. The system of equations for $x_i \equiv \pi_i^n$ is given by

$$(19) \quad \begin{aligned} x_0 q_0 + x_1 q_1 + x_n &= x_0, \\ x_{i-1} p_{i-1} + x_i r_i + x_{i+1} q_{i+1} &= x_i, \quad 1 \leq i < n-1, \\ x_{n-2} p_{n-2} + x_{n-1} r_{n-1} &= x_{n-1}, \\ x_{n-1} p_{n-1} &= x_n. \end{aligned}$$

We obtain the following equations from the first and second lines of the system above:

$$(20) \quad x_{i-1} p_{i-1} - x_i q_i = x_i p_i - x_{i+1} q_{i+1} = x_n, \quad 1 \leq i < n-1.$$

Finally, using the third and forth lines of the system above we recurrently calculate by using (20) that, for $0 \leq k < n$,

$$\begin{aligned}
 (21) \quad x_k &= x_n q_k^{-1} \theta_{k-1}^{-1} \left[\sum_{i=k}^{n-3} \theta_i + \theta_{n-3} q_{n-2} (p_{n-1} + q_{n-1}) / p_{n-2} p_{n-1} \right] \\
 &= x_n q_k^{-1} \theta_{k-1}^{-1} \left[\sum_{i=k}^{n-3} \theta_i + \theta_{n-2} (1 + q_{n-1} / p_{n-1}) \right] = x_n \sum_{i=k}^{n-1} \varkappa_k \theta_i.
 \end{aligned}$$

Now the normalization condition $\sum_{k=0}^n x_k = 1$ implies (18). $\square$

In order to prove Theorem 2.1, we use Corollary 7.5 of [2, Chapter VII] for the chain X^n with the set E^n and invariant probability π^n . Note that every transition kernel $Q(x, A)$ and the corresponding linear operator in the discrete space can be defined by the matrix $Q(x, A) = \sum_{y \in A} Q_{xy}$, $Q_{xy} = Q(x, \{y\})$. The multiplication of kernels by measures or that of functions by kernels corresponds to the multiplication of matrices by rows or columns and matrices.

In particular, the system for the kernel $\bar{R} = (\bar{R}_{xy}, x, y \in E^n)$ in the statement of Corollary 7.5 in [2, Chapter VII] is as follows:

$$\begin{aligned}
 (22) \quad \bar{R}_{xy} &= \sum_{k \in E^n} P_{xk}(n) \bar{R}_{ky} + P_{xy}(n) - \pi_y^n, \\
 \sum_{k \in E^n} \pi_k^n \bar{R}_{ky} &= 0, \quad x, y \in E^n,
 \end{aligned}$$

where the last equation arises from Lemma 3.1 since π^n is the eigenvector of the matrix P_n .

Moreover, it follows from the definition of $\bar{R}$ as a sum of series of powers of P_n (see Corollary 7.5 in [2, Chapter VII]) that the operators $\bar{R}$ and P_n commute, so equations (22) are equivalent to the system

$$\begin{aligned}
 (23) \quad \bar{R}_{xy} &= \sum_{k \in E^n} \bar{R}_{xk} P_{ky}(n) + P_{xy}(n) - \pi_y^n, \\
 \sum_{k \in E^n} \bar{R}_{xk} &= 0, \quad x, y \in E^n.
 \end{aligned}$$

Lemma 3.2. *The solutions of systems (22), (23) for $x = n$ or $y = n$ are given by*

$$(24) \quad \bar{R}_{nn} = -\omega_n,$$

$$(25) \quad \bar{R}_{kn} = \lambda_n \sum_{i \leq j < k} \varkappa_i \theta_j + \lambda_n - \omega_n, \quad k < n,$$

$$(26) \quad \bar{R}_{nk} = \varkappa_k (\sigma_n - \sigma_k) (\lambda_n - \omega_n) + \lambda_n \sum_{k < i < n} \varkappa_k (\sigma_n - \sigma_i) \varkappa_i (\sigma_i - \sigma_k), \quad k < n.$$

Proof. Denote $x_k = \bar{R}_{kn}$. Taking into account (18) we use (22) with $y = n$ and obtain the system

$$\begin{aligned}
 (27) \quad x_0 &= q_0 x_0 + p_0 x_1 - \lambda_n, \\
 x_i &= q_i x_{i-1} + r_i x_i + p_i x_{i+1} - \lambda_n, \quad 0 < i < n-1, \\
 x_{n-1} &= q_{n-1} x_{n-2} + r_{n-1} x_{n-1} + p_{n-1} x_n + p_{n-1} - \lambda_n, \\
 x_n &= x_0 - \lambda_n.
 \end{aligned}$$

The following equalities are deduced from the first and second lines above:

$$\begin{aligned} p_0(x_1 - x_0) &= \lambda_n, \\ (x_{i+1} - x_i)/\theta_i &= \lambda_n \varkappa_i + (x_i - x_{i-1})/\theta_{i-1}, \quad 1 \leq i < n-1. \end{aligned}$$

By recurrence,

$$\begin{aligned} x_{k+1} - x_k &= \lambda_n \sum_{i=0}^k \varkappa_i \theta_k, \quad 0 \leq k < n-1, \\ (28) \quad x_k &= x_0 + \lambda_n \sum_{j < k} \sum_{i=0}^j \varkappa_i \theta_j, \quad 0 \leq k < n. \end{aligned}$$

Substituting equalities (28) and (27) into the second equation in (22) we obtain

$$\begin{aligned} 0 &= \sum_{k \in E^n} \pi_k^n x_k = x_0 - \lambda_n^2 + \lambda_n \sum_{k \in E^n} \sum_{j < k} \sum_{i=0}^j \varkappa_i \theta_j \\ &= x_0 - \lambda_n^2 + \lambda_n^2 \sum_{i \leq j < k \in E^n} \varkappa_i \theta_j \varkappa_k (\sigma_n - \sigma_k) = x_0 - \lambda_n^2 + \omega_n - \lambda_n + \lambda_n^2 \\ &= x_0 + \omega_n - \lambda_n \end{aligned}$$

and thus deduce the identities in (24), (25) from (27).

To prove (26) we denote $x_k \equiv \bar{R}_{nk}$ and use (23) with $x = n$:

$$\begin{aligned} x_0 &= q_0 x_0 + q_1 x_1 + x_n + 1 - \pi_0^n, \\ x_k &= p_{k-1} x_{k-1} + r_k x_k + q_{k+1} x_{k+1} - \pi_k^n, \quad 1 \leq k < n-1, \\ x_{n-1} &= p_{n-2} x_{n-2} + r_{n-1} x_{n-1} - \pi_{n-1}^n, \\ (29) \quad x_n &= p_{n-1} x_{n-1} - \pi_n^n, \end{aligned}$$

where the probabilities π_k^n are defined in Lemma 3.1.

Using the first two equations of (29) we deduce by recurrence that

$$(30) \quad p_k x_k - q_{k+1} x_{k+1} = - \sum_{i=0}^k \pi_i^n + 1 + x_n = \sum_{i=k+1}^n \pi_i^n + x_n, \quad 0 \leq k < n-2.$$

Multiplying (30) by θ_k and summing over $k = 0, \dots, n-3$ we obtain

$$x_k \theta_k p_k = x_{n-2} \theta_{n-2} p_{n-2} + \sum_{j=k}^{n-3} \theta_j \left(x_n + \sum_{i=j+1}^n \pi_i^n \right).$$

Taking into account the last two equations in (29) and the identity

$$\theta_{n-2} (p_{n-1} + q_{n-1}) / p_{n-1} = \theta_{n-2} + \theta_{n-1}$$

we deduce that

$$(31) \quad x_k = x_n (\sigma_n - \sigma_k) + \sum_{j=k}^{n-1} \varkappa_k \theta_j \sum_{i=j+1}^n \pi_i^n, \quad 0 \leq k < n.$$

Finally, substituting the values π_i^n from (16) and $x_n = \bar{R}_{nn}$ from (24) we complete the proof of Lemma 3.2. $\square$

Proof of Theorem 2.1. We are going to apply inequality (7.40) of Corollary 7.5 in [2, Chapter VII] to the chain $X = X^n$ on $E = E^n$ with the moment $\tau_H = \tau_n$ and the set $H = \{n\}$. An invariant measure and the chain potential are calculated in Lemmas 3.1 and 3.2.

Using notation (7.39) [2, Chapter VII],

$$(32) \quad r_{HH} = \sup_{x \in H} |\overline{R}|(x, H) = |\overline{R}_{nn}| = \omega_n$$

under (24) since $\omega_n > 0$. It follows from (25), (18), and (3) that

$$(33) \quad \begin{aligned} r_{\pi H} &= \int \pi(dx) |\overline{R}|(x, H) = \sum_{k=0}^n \pi_k^n |\overline{R}_{kn}| \\ &\leq \lambda_n |\overline{R}_{nn}| + \sum_{k=0}^{n-1} \lambda_n \varkappa_k (\sigma_n - \sigma_k) \max \left(\omega_n, \lambda_n + \lambda_n \sum_{i \leq j < k} \varkappa_i \theta_j \right) \\ &= \lambda_n \omega_n + \max \left[\lambda_n \omega_n \sum_{k=0}^{n-1} \varkappa_k (\sigma_n - \sigma_k), \right. \\ &\quad \left. \lambda_n^2 \sum_{k=0}^{n-1} \varkappa_k (\sigma_n - \sigma_k) \left(1 + \sum_{i \leq j < k} \varkappa_i \theta_j \right) \right] \\ &\leq \lambda_n \omega_n + \max [\lambda_n \omega_n (\lambda_n^{-1} - 1), \lambda_n^2 (\lambda_n^{-1} - 1)] + \omega_n - \lambda_n + \lambda_n^2 \\ &= \lambda_n \omega_n + \max [\omega_n (1 - \lambda_n), \omega_n] = \omega_n (1 + \lambda_n). \end{aligned}$$

Furthermore, according to equality (7.41) of [2, Chapter VII],

$$(34) \quad m_H^{-1} = (E_\pi \tau_H)^{-1} = \pi_n^n \sum_{t \geq 0} (-1)^t (\overline{R}_{nn})^t = \lambda_n (1 + \omega_n)^{-1} = m_n^{-1}.$$

Finally, the constant a in Corollary 7.5 of [2, Chapter VII] is the upper limit for the density of the initial distribution of α (it is concentrated in 0) with respect to the measure π^n :

$$(35) \quad a = 1/\pi_0^n = 1/\lambda_n \varkappa_0 \sigma_n = p_0/\lambda_n \sigma_n.$$

Substituting relations (32), (33), (34) and (35) into inequality (7.40) of Corollary 7.5 [2, Chapter VII] we complete the proof of estimate (4) of Theorem 1. $\square$

Proof of Remark 1. The inequality $\omega_n > 0$ follows from the condition $\lambda_n < 1$ in definition (3). Denote by

$$s_n = \sum_{i \leq j \in E_n} \varkappa_i \theta_j > 0$$

the sum involved in (3). Then

$$(36) \quad \begin{aligned} \lambda_n &= (1 + s_n)^{-1}, \\ \omega_n &= \left(s_n + \sum_{i \leq j < k \leq l \in E_n} \varkappa_i \theta_j \varkappa_k \theta_l \right) / (1 + s_n)^2 \\ &\leq \left(s_n + 1/2 \sum_{i \leq j \in E_n} \varkappa_i \theta_j \sum_{k \leq l \in E_n} \varkappa_k \theta_l \right) / (1 + s_n)^2 \\ &= (s_n + s_n^2/2) / (1 + s_n)^2 \leq 1/2 < 1. \end{aligned}$$

$\square$

Proof of Corollary 2.1. The proof follows from inequality (4), since the right-hand side of (4) is equal to $O(p_0\omega_n/\lambda_n\sigma_n)$ in view of (36). On the other hand, after the substitution $t = [xm_n]$, the relation on the left-hand side is equivalent to

$$(37) \quad (1 - m_n^{-1})^{[xm_n]} \rightarrow \exp(-x), \quad n \rightarrow \infty,$$

uniformly for $x \geq 0$, since $m_n^{-1} \leq \lambda_n \rightarrow 0$. $\square$

Proof of Corollary 2.2. The well-known recurrence and positivity criteria for a birth-and-death chain [1] correspond to the divergence $\sigma_n \rightarrow \infty$ and convergence $\varkappa = \sum_{i \geq 0} \varkappa_i < \infty$. Hence

$$(38) \quad \begin{aligned} (\lambda_n \sigma_n)^{-1} &= \sigma_n^{-1} \left(1 + \sum_{i \leq j \in E_n} \varkappa_i \theta_j \right) = \sigma_n^{-1} + \sum_{i \geq 0} \varkappa_i (1 - \sigma_i \sigma_n^{-1}) \mathbb{1}_{i < n} \\ &\rightarrow \sum_{i \geq 0} \varkappa_i = \varkappa \in (0, \infty), \quad n \rightarrow \infty, \end{aligned}$$

by the Lebesgue dominated convergence theorem.

Thus $\lambda_n \sim 1/\varkappa \sigma_n \rightarrow 0$, $n \rightarrow \infty$.

Similarly, the representation

$$\begin{aligned} \omega_n &= \lambda_n - \lambda_n^2 + \lambda_n^2 \sum_{0 \leq i < k < n} \varkappa_i (\sigma_k - \sigma_i) \varkappa_k (\sigma_n - \sigma_k) \\ &\leq \lambda_n + (\lambda_n \sigma_n)^2 \sum_{i, k \in E_n} \varkappa_i \varkappa_k (\sigma_k - \sigma_i)^+ \sigma_n^{-1} \mathbb{1}_{i < k < n} \end{aligned}$$

and the monotonicity of σ_n imply that $\overline{\lim}_{n \rightarrow \infty} \omega_n = 0$ in view of the Lebesgue dominated convergence theorem.

Taking into account (38) and Remark 1 we can conclude that the right-hand side of (4), $2\omega_n(1 + \lambda_n)p_0/\lambda_n\sigma_n(1 - \omega_n)$, is equal to $O(\omega_n)$.

Using approximation (37), the convergence $\omega_n \rightarrow 0$, and the estimate

$$|\exp(-x - x\varepsilon) - \exp(-x)| \leq \varepsilon, \quad x, \varepsilon \geq 0,$$

we prove (6). $\square$

Proof of Corollary 2.3. We use the representation (4) of Theorem 2.1, where n is fixed. Since p_0 is involved in (3) only implicitly via $\varkappa_0$, we have $\lambda_n = 1/(1 + L/p_0) \sim p_0/L$, $p_0 \rightarrow 0$, $\omega_n = p_0/L + o(p_0)$, $p_0 \rightarrow 0$, $\sigma_n^2 = C$ for some constants $L, C > 0$. Thus, (7) follows from (4). $\square$

Proof of Corollary 2.4. It follows from definitions (3), (8), and (9) that

$$\lambda_n^{-1} = 1 + \sigma_n/p_0 + \varepsilon_n^{-1} \sum_{1 \leq i < n} \chi_i (\sigma_n - \sigma_i) \sim \sigma_n \chi / \varepsilon_n \rightarrow \infty, \quad n \rightarrow \infty.$$

At the same time,

$$\begin{aligned} \omega_n &= \lambda_n - \lambda_n^2 + \lambda_n^2 \sum_{1 \leq i < k < n} \varepsilon_n^{-2} \chi_i (\sigma_k - \sigma_i) \chi_k (\sigma_n - \sigma_k) \sim \lambda_n + \lambda_n^2 \sigma_n^2 \overline{\omega}_n / \varepsilon_n^2 \\ &\sim \lambda_n + \chi^{-2} \overline{\omega}_n = o(1), \quad n \rightarrow \infty. \end{aligned} \quad \square$$

Proof of Theorem 2.2. We apply inequality (7.43) of Corollary 7.5 of [2, Chapter VII] to the chain $X = X^n$ on $E = E^n$ with the moment $\tau_H = \tau_n$ and set $H = \{n\}$. The invariant probability and potential of the chain X^n are found in Lemmas 3.1 and 3.2.

A bound for the new variation of the potential in (7.42) can be derived from equalities (24) and (26), since

$$\begin{aligned}
 r_H &= 1 + \sup_{x \in H} |\bar{R}|(x, E) = 1 + \sum_{k=0}^n |\bar{R}_{nk}| \\
 (39) \quad &\leq 1 + |\bar{R}_{nn}| + (\lambda_n + \omega_n) \sum_{k < n} \mathcal{X}_k(\sigma_n - \sigma_k) + \lambda_n \sum_{k < i < n} \mathcal{X}_k(\sigma_i - \sigma_k) \mathcal{X}_i(\sigma_n - \sigma_k) \\
 &= 1 + \omega_n + (\lambda_n + \omega_n) (\lambda_n^{-1} - 1) + (\omega_n - \lambda_n + \lambda_n^2) \lambda_n^{-1} = 1 + 2\omega_n \lambda_n^{-1}.
 \end{aligned}$$

Here we have used relations (3).

Since $\pi(H) = \pi_n^n = \lambda_n$, we apply (32) to the first term on the right-hand side of (7.43) in [2, Chapter VII] and obtain

$$(40) \quad \pi(H) 2r_H / (1 - r_{HH}) \leq \lambda_n 2 (1 + 2\omega_n \lambda_n^{-1}) / (1 - \omega_n) = O(\omega_n), \quad n \rightarrow \infty,$$

by Remark 2.1. The representation $\lambda_n = O(\omega_n)$ is an obvious corollary of (3).

In order to apply (7.43) we need to find two constants $r_\alpha \in (0, 1)$ and $b < \infty$ such that

$$(41) \quad |\mathbf{P}_0(X_t^n \in B) - \pi^n(B)| \leq b(1 - r_\alpha)^t$$

for all $t > 0$ and $B \subset E$.

Now we use Theorem 3.6 of [2, Chapter III].

We define the following norm on the space of measures $\mu = (\mu_i, i \in E^n)$:

$$(42) \quad \|\mu\| = \sum_{i \in E^n} v^i |\mu_i|,$$

where the constant $v > 1$ will be chosen later. The corresponding operator norm on $L(E^n)$ is given in [2, p. 1.1]. Note that

$$\begin{aligned}
 (43) \quad |\mathbf{P}_0(X_t^n \in B) - \pi^n(B)| &\leq |\alpha P_n^t - \pi^n|(E^n) \leq \|\alpha P_n^t - \pi^n\| = \|\alpha(P_n^t - \Pi_n)\| \\
 &\leq \|\alpha\| \|P_n^t - \Pi_n\| = \|P_n^t - \Pi_n\|,
 \end{aligned}$$

since $v > 1$, where $P_n^t \equiv (P_n)^t$ and $\alpha_i = \delta_{i0}$ is the initial distribution of the matrix Π_n that contains rows of the type π^n .

We represent the matrix P_n as $P_n = T_n + h \circ \beta$, where

$$h = (\delta_{i0}, i \in E^n)$$

is a function, $\beta = (p_0, 1 - p_0, 0, \dots, 0) = (P_{0j}(n), j \in E^n)$ is a measure, and

$$T_n = (P_{ij}(n) 1_{i>0}, i, j \in E^n)$$

is a matrix. Therefore condition (C) of [2, p. 3.3] holds with $n = 1$ (if one uses the notation of [2]).

Now we calculate the operator norm $\rho_n \equiv \|T_n\|$:

$$\begin{aligned}
 (44) \quad \rho_n &= \max_{i \in E^n} v^{-i} \sum_{j \in E^n} v^j P_{ij}(n) 1_{i>0} \\
 &= \max \left\{ \max_{1 \leq i < n} v^{-i} (q_i v^{i-1} + r_i v^i + p_i v^{i+1}), v^{-n} \right\} \\
 &= \max_{1 \leq i < n} (1 - (v - 1) (q_i v^{-1} - p_i)) = 1 - (v - 1) v^{-1} \min_{1 \leq i < n} (q_i - p_i - (v - 1) p_i) \\
 &\leq 1 - (v - 1) v^{-1} (\delta_n - (v - 1)/2)
 \end{aligned}$$

by (13) and (11) which yield $\delta_n > 0$ and $p_i < 1/2$.

Then (44) implies

$$\rho_n \leq 1 - \delta_n^2 / 2(1 + \delta_n) < 1 - \delta_n^2 / 4.$$

Condition (T) of [2, p. 3.3] holds with $m = 1$ (if one uses the notation of [2]) and

$$(45) \quad (1 - \rho_n)^{-1} = O(\delta_n^{-2}), \quad n \rightarrow \infty,$$

uniformly in the scheme of series.

Thus, all the assumptions of Theorem 3.6 of [2, Chapter III] hold and $n = m = 1$, $\alpha = \beta$, $h = h$, $P = P_n$, $\pi = \pi_n$, $\Pi = \Pi_n$, $T = T_n$, $\rho = \rho_n$, and the norm $\|\cdot\|$ is defined in (42). In particular,

$$(46) \quad \sigma \leq k \|\alpha\| / (1 - \rho) = O(\delta_n^{-2}), \quad n \rightarrow \infty,$$

where the parameter σ is defined in (3.31) in [2].

In order to apply (3.30) we choose (3.29) according to (3.32):

$$(47) \quad \omega \leq \omega_1 = 2 \exp(-(1 - \pi h) \ln(\alpha h) / \pi h (1 - (\alpha h))) - 1 = O(1), \quad n \rightarrow \infty.$$

Here we used the equalities $\pi h = \pi_0^n = \lambda_n \sigma_n / p_0$ and $\alpha h = p_0$ and condition (12).

The asymptotics for the parameter θ_0 in (3.29) in [2] is calculated from (45), (46), and (47) as follows:

$$(48) \quad (1 - \theta_0)^{-1} = O((1 - \rho_n)^{-1}) O(\sigma \omega) = O(\delta_n^{-4}), \quad n \rightarrow \infty.$$

Now we choose $\theta = (1 + \theta_0)/2$ in (3.30) [2, p. 3.3].

Since $\theta - \theta_0 = (1 - \theta_0)/2$ and $1 - \theta = (1 - \theta_0)/2$, we derive inequality (41),

$$(49) \quad \|P_n^t - \Pi_n\| \leq b_n (1 - r_n)^t,$$

from (46), (48) and from (3.30) [2, p. 3.3], where

$$(50) \quad r_n^{-1} = \max((1 - \rho_n)^{-1}, (1 - \theta)^{-1}) = O(\delta_n^{-4}), \quad n \rightarrow \infty,$$

$$(51) \quad b_n = (1 + \sigma) / (\theta - \theta_0) = O(\delta_n^{-6}), \quad n \rightarrow \infty.$$

For the second term on the right-hand side of (7.43) in [2, Chapter VII], we get

$$(52) \quad \begin{aligned} \pi(H) a r_\alpha^{-1} \ln(1 + b e / a \pi(H)) &= \lambda_n a r_n^{-1} \ln(1 + b_n e / a \lambda_n) \\ &\leq \lambda_n O(1) O(\delta_n^{-4}) \ln(O(\delta_n^{-6}) \lambda_n^{-1}) \\ &= O(\lambda_n \delta_n^{-4} \ln(1 / \delta_n \lambda_n)), \quad n \rightarrow \infty, \end{aligned}$$

by using identity (35), boundedness conditions (12), and estimates (50), (51).

Relation (14) is equivalent to the convergence to zero of the last term in (15):

$$\lambda_n \delta_n^{-4} \ln(\lambda_n^{-1} \delta_n^{-1}) \rightarrow 0, \quad n \rightarrow \infty.$$

Indeed, (14) follows from (15) since $\lambda_n \rightarrow 0$ and $\delta_n \rightarrow 0$. On the other hand, relation (14) implies that $\delta_n^{-4} = o(1 / \lambda_n \ln \lambda_n^{-1})$, whence

$$\lambda_n \delta_n^{-4} \ln \delta_n^{-1} = \lambda_n o\left((\lambda_n \ln \lambda_n^{-1})^{-1} \ln(\lambda_n \ln \lambda_n^{-1})^{-1}\right) = o(1), \quad n \rightarrow \infty,$$

which proves (15).

Substituting $t = \lambda_n^{-1} s$ in (7.43) [2, Chapter VII] we obtain from the inequality $s > s_0 > 0$ that

$$\lambda_n^{-1} s_0 \geq t_0 \equiv r_\alpha^{-1} \ln^+(b / a \pi(H)) = O(\delta_n^{-4} \ln(1 / \delta_n \lambda_n)), \quad n \rightarrow \infty,$$

by the convergence to zero of the right-hand side of (15), whence $t \geq t_0$ in Corollary 7.5.

Taking into account (40) and (52) we prove (15). $\square$

Proof of Corollary 2.5. The convergence $\lambda_n \rightarrow 0$, $\omega_n \rightarrow 0$ is proved in Corollary 2.2. Relation (12) follows from (38).

If $\lim \delta_n > 0$, then uniform ergodicity holds and the result needed is obvious, since $\lambda_n + \omega_n \rightarrow 0$, $n \rightarrow \infty$.

Now let $\delta_n \rightarrow 0$.

We find from relation (38) that

$$(53) \quad \lambda_n \ln \lambda_n^{-1} \sim \sigma_n^{-1} \ln \sigma_n, \quad n \rightarrow \infty.$$

Further, definition (13) implies that

$$\begin{aligned} \theta_t &= \prod_{i=1}^t (1 + (q_i - p_i)/p_i) \geq (1 + 2\delta_n)^t, \\ \sigma_n &= \sum_{t < n} \theta_t \geq (2\delta_n)^{-1} ((1 + 2\delta_n)^n - 1). \end{aligned}$$

The latter relations together with (17) yield

$$\begin{aligned} (1 + 2\delta_n)^n &\geq \exp((2 - \varepsilon)n\delta_n) \geq \exp((3 + \varepsilon) \ln n) = n^{3+\varepsilon}, \\ \sigma_n^{-1} &= O(\delta_n n^{-3-\varepsilon}) \end{aligned}$$

for some $\varepsilon > 0$.

So, condition (14) follows from (53):

$$\lambda_n \ln \lambda_n^{-1} = O(\delta_n n^{-3-\varepsilon} \ln(\delta_n^{-1} n^{3+\varepsilon})) = o(\delta_n^4), \quad n \rightarrow \infty,$$

since $\delta_n^{1-\alpha} n^{1+\varepsilon/3} \rightarrow \infty$ for all sufficiently small $\alpha, \varepsilon > 0$ provided (17) holds. Therefore

$$(\delta_n n^{-3-\varepsilon} \ln \delta_n^{-1}) / \delta_n^4 = (\delta_n n^{1+\varepsilon/3} (\ln \delta_n^{-1})^{-1/3})^{-3} \rightarrow 0, \quad n \rightarrow \infty. \quad \square$$

Proof of Remark 2.3. According to [1], the ergodicity of the chain is equivalent to the convergence of the series $\sum \varkappa_t$, which corresponds to the convergence of the series $\sum_{n \geq 1} \prod_{i=1}^n (p_i/q_i)$. By definition (13), the convergence of the latter series follows from the convergence of

$$\sum_{n \geq 1} \prod_{i=1}^n (1 - \delta_n/q_i) \leq \sum_{n \geq 1} (1 - \delta_n)^n < \infty.$$

The equality $\delta_{i+1} = q_i - p_i$ in conditions (b) holds, whence $q_i > 1/2$ and

$$\sum_{n \geq 1} \prod_{i=1}^n (p_i/q_i) \geq \sum_{n \geq 1} \prod_{i=1}^n (1 - 2\delta_{i+1}) = \infty. \quad \square$$

ACKNOWLEDGEMENT

The author is grateful to an anonymous referee for the insightful comments that significantly improved the paper.

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Received 04/SEP/2013
Originally published in English