STORAGE IMPULSIVE PROCESSES ON INCREASING TIME INTERVALS

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ABSTRACT. A storage impulsive process $S(t)$ is a sum of (jointly independent) random variables defined on the embedded Markov chain of a homogeneous Markov process.

A storage impulsive process is considered in the scheme of series on increasing time intervals t/ε , with a small parameter $\varepsilon \rightarrow 0$, $\varepsilon > 0$. A storage impulsive process is investigated in the average and diffusion approximation scheme. The large deviation problem is considered under a corresponding scaling with an asymptotically small diffusion.

1. INTRODUCTION

A storage impulsive process $S(t)$ is a sum of (jointly independent) random variables defined on the embedded Markov chain of a homogeneous Markov process

$$(1) \quad S(t) = u + \sum_{n=1}^{\nu(t)} \alpha_n(x_n), \quad t \geq 0, \quad u \in \mathbb{R}^d.$$

The time homogeneous Markov process $x(t)$, $t \geq 0$, is defined on a standard phase space $(E, \mathcal{E})$ by the generator

$$Q\varphi(x) = q(x) \int_E P(x, dy) [\varphi(y) - \varphi(x)], \quad x \in E,$$

for a real valued test function $\varphi(x)$, $x \in E$, with a bounded sup-norm:

$$\|\varphi(x)\| := \sup_{x \in E} |\varphi(x)|.$$

The embedded Markov chain x_n , $n \geq 0$, is defined by

$$x_n := x(\tau_n), \quad n \geq 0,$$

where the renewal moments of jumps are given by

$$\tau_{n+1} = \tau_n + \theta_{n+1}, \quad n \geq 0, \quad \tau_0 = 0,$$

and the sojourn times θ_{n+1} , $n \geq 0$, are such that

$$P(\theta_{n+1} \geq t \mid x_n = x) = e^{-q(x)t} =: P(\theta_x \geq t).$$

The stochastic kernel $P(x, B)$, $x \in E$, $B \in \mathcal{E}$, defines the transition probabilities of the embedded Markov chain

$$P(x, B) = \mathcal{P}\{x_{n+1} \in B \mid x_n = x\}.$$

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The counting process is defined by

$$\nu(t) := \max\{n > 0: \tau_n \leq t\}, \quad t \geq 0.$$

The random variables in (1) have the distribution functions

$$\Phi_x(dv) = P\{\alpha_n(x) \in dv\} := P\{\alpha_n(x_n) \in dv \mid x_n = x\}, \quad x \in E.$$

A storage impulsive process may be considered as a random evolution process [1, Chapter 2]. The switching Markov process $x(t)$, $t \geq 0$, describes a random environment.

A1: The main assumption is the uniform ergodicity of the Markov process $x(t)$, $t \geq 0$, with the stationary distribution $\pi(B)$, $B \in \mathcal{E}$, satisfying the equation

$$\pi(dx)q(x) = q\rho(dx), \quad q = \int_E \pi(dx)q(x).$$

The stationary distribution $\rho(B)$, $B \in \mathcal{E}$, of the embedded Markov chain x_n , $n \geq 0$, satisfies the equation

$$\rho(B) = \int_E \rho(dx)P(x, B), \quad B \in \mathcal{E}, \quad \rho(E) = 1.$$

If the main assumption **A1** holds, then the potential operator R_0 may be given by a solution of the equation [1, Chapter 2]

$$QR_0 = R_0Q = \Pi - I, \quad \Pi\varphi(x) := \int_E \pi(dx)\varphi(x).$$

2. A STORAGE IMPULSIVE PROCESS ON INCREASING TIME INTERVALS IN AN AVERAGE SCHEME

A storage impulsive process on increasing time intervals in an average scheme is considered in the scheme of series with a small parameter $\varepsilon \rightarrow 0$, $\varepsilon > 0$, in the following scaling:

$$(2) \quad S^\varepsilon(t) = u + \varepsilon \sum_{n=1}^{\nu(t/\varepsilon)} \alpha_n(x_n), \quad t \geq 0, \quad \varepsilon > 0, \quad u \in \mathbb{R}^d.$$

The random evolution approach [1, Chapter 3, 5] is an effective method of asymptotic analysis (2) as $\varepsilon \rightarrow 0$.

Proposition 2.1. *Storage impulsive process (2) in the average scheme convergences weakly,*

$$(3) \quad S^\varepsilon(t) \Rightarrow S^0(t) = u + \hat{a}_0 t, \quad \varepsilon \rightarrow 0,$$

where the average velocity is such that

$$(4) \quad \hat{a}_0 = q\hat{a}, \quad \hat{a} = \int_E \rho(dx)a(x), \quad a(x) = \int_{\mathbb{R}^d} v \Phi_x(dv).$$

Proof of Proposition 2.1 is based on the random evolution approach [1, Chapter 3] by using a solution of the singular perturbation problem [1, Chapter 5].

Remark 2.1. For simplicity and without loss of generality, the proof is given for a storage impulsive process defined on the real line $\mathbb{R}$, $d = 1$.

According to the definition of a random evolution [1, Chapter 2] we consider a two component Markov process,

$$(5) \quad S^\varepsilon(t), \quad x^\varepsilon(t) := x(t/\varepsilon), \quad t \geq 0.$$

Lemma 2.1. *Markov process (5) is characterized by the generator*

$$(6) \quad L^\varepsilon \varphi(u, x) = \varepsilon^{-1} q(x) \int_E P(x, dy) \int_{\mathbb{R}^d} \Phi_y(dv) [\varphi(u + \varepsilon v, y) - \varphi(u, x)].$$

Lemma 2.1 is a direct consequence of the definition of the generator [1, Chapter 3].

Remark 2.2. Generator (6) may be rewritten as

$$(7) \quad L^\varepsilon \varphi(u, x) = \varepsilon^{-1} [Q + Q_0 \Phi_x^\varepsilon] \varphi(u, x),$$

where, by definition,

$$Q_0 \varphi(x) := q(x) \int_E P(x, dy) \varphi(y),$$

$$\Phi_x^\varepsilon \varphi(u) := \int_{\mathbb{R}^d} \Phi_x(dv) [\varphi(u + \varepsilon v) - \varphi(u)].$$

On a test function $\varphi(u)$ being smooth enough,

$$\Phi_x^\varepsilon \varphi(u) = \varepsilon [a(x) \varphi'(u) + \delta^\varepsilon(x) \varphi(u)]$$

with the negligible term

$$\|\delta^\varepsilon(x) \varphi(u)\| \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad \varphi(u) \in C^2(\mathbb{R}).$$

Lemma 2.2. *Generator (7) admits the following asymptotic expansion:*

$$L^\varepsilon \varphi(u, x) = [\varepsilon^{-1} Q + Q_0 \mathbb{A}(x) + \delta^\varepsilon(x)] \varphi(u, x),$$

where

$$\mathbb{A}(x) \varphi(u) := a(x) \varphi'(u),$$

and the negligible term is such that

$$\sup_{x \in E} \|\delta^\varepsilon(x) \varphi(u, x)\| \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad \varphi(u, \cdot) \in C^2(\mathbb{R}).$$

Then a solution of the singular perturbation problem [1, Chapter 5] may be used for the truncated operator

$$(8) \quad L_0^\varepsilon \varphi(u, x) := [\varepsilon^{-1} Q + Q_0 \mathbb{A}(x)] \varphi(u, x).$$

Lemma 2.3. *Truncated operator (8) on a perturbed test function*

$$\varphi^\varepsilon(u, x) = \varphi(u) + \varepsilon \varphi_1(u, x),$$

admits the following asymptotic representation [1, Proposition 5.1]:

$$L_0^\varepsilon \varphi^\varepsilon(u, x) = \widehat{a}_0 \varphi'(u) + \delta^\varepsilon(x) \varphi(u).$$

The negligible term may be written in an explicit form:

$$\delta^\varepsilon(x) \varphi(u) = \varepsilon Q_0 \mathbb{A}(x) R_0 \widehat{\mathbb{A}}(x) \varphi(u),$$

$$\widehat{\mathbb{A}}(x) := \widehat{\mathbb{A}}_0 - Q_0 \mathbb{A}(x), \quad \widehat{\mathbb{A}}_0 := \Pi Q_0 \mathbb{A}(x) \Pi.$$

Conclusion 2.1. *Generator (6) of random evolution (5) admits the asymptotic representation*

$$(9) \quad L^\varepsilon \varphi^\varepsilon(u, x) = \widehat{a}_0 \varphi'(u) + \delta^\varepsilon(x) \varphi(u)$$

with the negligible term $\delta^\varepsilon(x) \varphi(u)$.

Representation (9) implies weak convergence (3)–(4) [1, Chapter 6] since the limit operator,

$$(10) \quad L^0 \varphi(u) := \widehat{a}_0 \varphi'(u), \quad \varphi(u) \in C^1(\mathbb{R}),$$

defines the evolution

$$S^0(t) = u + \hat{a}_0 t, \quad t \geq 0, \quad S^0(0) = u.$$

Remark 2.3. Limit operator (10) in the Euclidean space $\mathbb{R}^d$ has the following representation:

$$\begin{aligned} \hat{a}_0 \varphi'(u) &:= \sum_{k=1}^d \hat{a}_k^0 \varphi'_k(u), \quad \varphi'_k(u) := \partial \varphi(u) / \partial u_k, \\ \hat{a}_k^0 &= q \hat{a}_k, \quad \hat{a}_k = \int_E \rho(dx) a_k(x), \quad a_k(x) = \int_{\mathbb{R}} v_k \Phi_x(dv). \end{aligned}$$

3. A STORAGE IMPULSIVE PROCESS IN A SCHEME OF DIFFUSION APPROXIMATION

It is well known that the diffusion approximation of stochastic systems may be realized under some additional *Balance Condition*.

We consider two different balance conditions for a storage impulsive process, namely the *total* and *local* ones.

3.1. A storage impulsive process under the total balance condition. A storage impulsive process in the scheme of series with the parameter $\varepsilon \rightarrow 0$, $\varepsilon > 0$, in the diffusion approximation scheme under the *Total Balance Condition*,

$$(11) \quad a(x) = \int_{\mathbb{R}^d} v \Phi_x(dv) \equiv 0,$$

is considered in the following scaling:

$$S^\varepsilon(t) = u + \varepsilon \sum_{n=1}^{\nu(t/\varepsilon^2)} \alpha_n(x_n), \quad t \geq 0, \quad \varepsilon > 0.$$

Proposition 3.1. *If total balance condition (11) holds, then*

$$S^\varepsilon(t) \Rightarrow W_\sigma(t), \quad \varepsilon \rightarrow 0.$$

The limit Brownian motion $W_\sigma(t)$, $t \geq 0$, is defined by the variance matrix

$$\begin{aligned} \hat{C} &= \sigma^* \sigma = q \hat{B}, \\ \hat{B} &= \int_E \rho(dx) B(x), \quad B(x) = \int_{\mathbb{R}^d} v^* v \Phi_x(dv). \end{aligned}$$

Proof of Proposition 3.1. As in Section 2, we start by characterizing the coupled Markov process. □

Lemma 3.1. *The Markov process*

$$S^\varepsilon(t), \quad x^\varepsilon(t) := x(t/\varepsilon^2), \quad t \geq 0,$$

is characterized by the generator

$$(12) \quad L^\varepsilon \varphi(u, x) = \varepsilon^{-2} q(x) \int_E P(x, dy) \int_{\mathbb{R}^d} \Phi_x(dv) [\varphi(u + \varepsilon v, y) - \varphi(u, x)].$$

Generator (12) can be rewritten as

$$(13) \quad L^\varepsilon \varphi(u, x) = \varepsilon^{-2} [Q + Q_0 \Phi_x^\varepsilon] \varphi(u, x),$$

where

$$(14) \quad \Phi_x^\varepsilon \varphi(u) := \int_{\mathbb{R}^d} \Phi_x(dv) [\varphi(u + \varepsilon v) - \varphi(u)] = \varepsilon^2 \left[\frac{1}{2} B(x) \varphi''(u) + \delta^\varepsilon(x) \varphi(u) \right],$$

with the negligible term $\delta^\varepsilon(x) \varphi(u)$.

Lemma 3.2. *Generator (13)–(14) admits the following asymptotic expansion*

$$L^\varepsilon \varphi(u, x) = [\varepsilon^{-2}Q + Q_0\mathbb{B}(x)] \varphi(u, x) + \delta^\varepsilon(x)\varphi(u)$$

with the negligible term $\delta^\varepsilon(x)\varphi(u)$. Here, by definition,

$$(15) \quad \mathbb{B}(x)\varphi(u) = \frac{1}{2}B(x)\varphi''(u).$$

Then the solution of singular perturbation problem [1, Chapter 5] can be used for the truncated operator

$$(16) \quad \mathbb{L}_0^\varepsilon \varphi(u, x) = [\varepsilon^{-2}Q + Q_0\mathbb{B}(x)] \varphi(u, x).$$

Lemma 3.3. *Truncated operator (16) on a perturbed test function*

$$(17) \quad \varphi^\varepsilon(u, x) = \varphi(u) + \varepsilon^2\varphi_2(u, x),$$

admits the asymptotic representation

$$L_0^\varepsilon \varphi^\varepsilon(u, x) = \frac{1}{2}\widehat{C}\varphi''(u) + \delta^\varepsilon(x)\varphi(u).$$

Proof. Considering (16) and (17),

$$\begin{aligned} L_0^\varepsilon \varphi^\varepsilon &= [\varepsilon^{-2}Q + Q_0\mathbb{B}(x)] [\varphi(u) + \varepsilon^2\varphi_2(u, x)] \\ &= \varepsilon^{-2}Q\varphi(u) + [Q\varphi_2(u, x) + Q_0\mathbb{B}(x)\varphi(u)] + \delta^\varepsilon(x)\varphi(u). \end{aligned}$$

It is obvious that

$$Q\varphi(u) = 0.$$

The equation

$$Q\varphi_2(u, x) + Q_0\mathbb{B}(x)\varphi(u) = \widehat{L}_0\varphi(u)$$

can be solved under the solvability condition [1, Chapter 5]:

$$\widehat{L}_0\Pi = \Pi Q_0\mathbb{B}(x)\Pi.$$

Transforming (15) gives us

$$\widehat{L}_0\varphi(u) = \frac{1}{2}\widehat{C}\varphi''(u).$$

Indeed,

$$\begin{aligned} \widehat{L}_0\varphi(u) &= \int_E \pi(dx)q(x) \int_E P(x, dy) \frac{1}{2}B(y)\varphi''(u) \\ &= \frac{1}{2}q \int_E \rho(dx)B(x)\varphi''(u) = \frac{1}{2}q\widehat{B}\varphi''(u). \end{aligned} \quad \square$$

Remark 3.1. The limit generator $\widehat{L}_0$ in the Euclidean space $\mathbb{R}^d$ is represented as follows:

$$\begin{aligned} \widehat{L}_0\varphi(u) &= \frac{q}{2} \sum_{k,r=1}^d B_{kr}\varphi''_{kr}(u), \\ \widehat{B} &= [B_{kr}; 1 \leq k, r \leq d], \quad \varphi''_{kr}(u) := \partial^2\varphi(u)/\partial u_k\partial u_r, \\ B_{kr} &= \int_E \rho(dx)B_{kr}(x), \quad B_{kr}(x) = \int_{\mathbb{R}} v_kv_r \Phi_x(dv). \end{aligned}$$

We complete the proof of Proposition 3.1 by using the asymptotic representation

$$(18) \quad L^\varepsilon \varphi^\varepsilon(u, x) = \widehat{L}_0\varphi(u) + \delta^\varepsilon(x)\varphi(u)$$

and convergence Theorem 6.3 in [1, Chapter 6]. The negligible term in (18) can be written in an explicit form. $\square$

3.2. A storage impulsive process under the local balance condition. The local balance condition means that the average value of jumps is such that

$$(19) \quad \widehat{a} := \int_E \rho(dx) a(x) \neq 0.$$

A storage impulsive process in the scheme of series under local balance condition (19) with the parameter $\varepsilon \rightarrow 0$, $\varepsilon > 0$, is considered in the following scaling:

$$(20) \quad S^\varepsilon(t) = u + \varepsilon \sum_{n=1}^{\nu(t/\varepsilon^2)} \alpha_n(x_n) - q\widehat{a}t/\varepsilon, \quad t \geq 0.$$

Proposition 3.2. *If local balance condition (19) holds, then*

$$S^\varepsilon(t) \Rightarrow W_\sigma(t), \quad \varepsilon \rightarrow 0.$$

The limit Brownian motion $W_\sigma(t)$, $t \geq 0$, is defined by the variance matrix

$$(21) \quad \begin{aligned} \widehat{C} &= \sigma^* \sigma = q\widehat{B}, & \widehat{B} &= \widehat{B}_0 + \widehat{B}_1, \\ \widehat{B}_0 &= \int_E \rho(dx) B_0(x), & B_0(x) &= \int_{\mathbb{R}^d} v^* v \Phi_x(dv), \\ \widehat{B}_1 &= \int_E \rho(dx) B_1(x), & B_1(x) &= 2\widehat{a}^*(x) R_0 \widehat{a}(x), \\ \widehat{a}(x) &:= a_0(x) - q\widehat{a}, \\ a_0(x) &:= q(x) \int_E P(x, dy) a(y). \end{aligned}$$

Here the potential operator R_0 is defined as the solution of the equation

$$QR_0 = R_0Q = \Pi - I$$

(see [1, Chapter 3]).

Proof of Proposition 3.2. As in the previous section, we use the generator of the two component Markov process. $\square$

Lemma 3.4. *The two component Markov process $S^\varepsilon(t)$, $x^\varepsilon(t) := x(t/\varepsilon^2)$, $t \geq 0$, is characterized by the generator*

$$(22) \quad L^\varepsilon \varphi(u, x) = \varepsilon^{-2} q(x) \int_E P(x, dy) \int_{\mathbb{R}^d} \Phi(dv) [\varphi(u + \varepsilon v, y) - \varphi(u, x)] - \varepsilon^{-1} \widehat{a}_0 \varphi'(u, x).$$

This generator can be written as

$$(23) \quad L^\varepsilon \varphi(u, x) = \left[\varepsilon^{-2} [Q + Q_0 \Phi_x^\varepsilon] - \varepsilon^{-1} \widehat{\mathbb{A}}_0 \right] \varphi(u, x)$$

with $\widehat{\mathbb{A}}_0 \varphi(u) := \widehat{a}_0 \varphi'(u)$,

$$\begin{aligned} \Phi_x^\varepsilon \varphi(u) &= \int_{\mathbb{R}^d} \Phi_x(dv) [\varphi(u + \varepsilon v) - \varphi(u)] \\ &= \varepsilon a(x) \varphi'(u) + \varepsilon^2 \frac{1}{2} B(x) \varphi''(u) + \varepsilon^2 \delta^\varepsilon(x) \varphi(u). \end{aligned}$$

Lemma 3.5. *Generator (22) admits the following asymptotic expansion:*

$$L^\varepsilon \varphi(u, x) = \left[\varepsilon^{-2} Q + \varepsilon^{-1} \widehat{\mathbb{A}}(x) + Q_0 \mathbb{B}(x) \right] \varphi(u, x) + \delta^\varepsilon(x) \varphi(u, x).$$

Here

$$(24) \quad \begin{aligned} \widehat{\mathbb{A}}(x)\varphi(u) &= \widehat{a}(x)\varphi'(u), \\ \widehat{a}(x) &:= a_0(x) - \widehat{a}_0, \\ a_0(x) &:= q(x) \int_E P(x, dy) a(y). \end{aligned}$$

Note that the following balance condition holds:

$$(25) \quad \Pi \widehat{a}(x) = 0.$$

Now a solution of the singular perturbation problem [1, Chapter 5] can be used for the truncated operator

$$(26) \quad L_0^\varepsilon \varphi(u, x) = \left[\varepsilon^{-2} Q + \varepsilon^{-1} \widehat{\mathbb{A}}(x) + Q_0 \mathbb{B}(x) \right] \varphi(u, x).$$

Lemma 3.6. *Truncated operator (26) on a perturbed test function*

$$\varphi^\varepsilon(u, x) = \varphi(u) + \varepsilon \varphi_1(u, x) + \varepsilon^2 \varphi_2(u, x)$$

admits the following asymptotic representation:

$$L_0^\varepsilon \varphi^\varepsilon(u, x) = \frac{1}{2} \widehat{C} \varphi''(u) + \delta^\varepsilon(x) \varphi(u).$$

Proof. Consider

$$\begin{aligned} L_0^\varepsilon \varphi^\varepsilon(u, x) &= \left[\varepsilon^{-2} Q + \varepsilon^{-1} \widehat{\mathbb{A}}(x) + Q_0 \mathbb{B}(x) \right] [\varphi(u) + \varepsilon \varphi_1(u, x) + \varepsilon^2 \varphi_2(u, x)] \\ &= \varepsilon^{-2} Q \varphi(u) + \varepsilon^{-1} \left[Q \varphi_1 + \widehat{\mathbb{A}}(x) \varphi \right] + \left[Q \varphi_2 + \widehat{\mathbb{A}}(x) \varphi_1 + Q_0 \mathbb{B}(x) \varphi \right] \\ &\quad + \delta^\varepsilon(x) \varphi(u). \end{aligned}$$

Then

$$\begin{aligned} Q \varphi(u) &= 0, \\ Q \varphi_1(u, x) + \widehat{\mathbb{A}}(x) \varphi(u) &= 0, \\ Q \varphi_2(u, x) + \widehat{\mathbb{A}}(x) \varphi_1(u, x) + Q_0 \mathbb{B}(x) \varphi(u) &= \widehat{L}_0 \varphi(u). \end{aligned}$$

The first equation above is obvious. The second equation satisfies the solvability condition (25). Hence

$$\varphi_1(u, x) = R_0 \widehat{\mathbb{A}}(x) \varphi(u).$$

Now the third equation is

$$(27) \quad Q \varphi_2 + \left[\widehat{\mathbb{A}}_0(x) + Q_0 \mathbb{B}(x) \right] \varphi(u) = \widehat{L}_0 \varphi(u),$$

where

$$(28) \quad \widehat{\mathbb{A}}_0(x) \varphi(u) := \widehat{\mathbb{A}}(x) R_0 \widehat{\mathbb{A}}(x) \varphi(u).$$

The solvability condition for (27) implies

$$\widehat{L}_0 \Pi = \Pi \left[\widehat{\mathbb{A}}_0(x) + Q_0 \mathbb{B}(x) \right] \Pi.$$

Using (28), (24), and (15) we evaluate the limit generator

$$\widehat{L}_0 \varphi(u) = \frac{1}{2} \widehat{C} \varphi''(u),$$

where the variance matrix $\widehat{C}$ is given by (21).

Note that (see (24))

$$\widehat{\mathbb{A}}_0(x) \varphi(u) = \widehat{\mathbb{A}}(x) R_0 \widehat{\mathbb{A}}(x) \varphi(u) = \widehat{\mathbb{A}}(x) R_0 \widehat{a}(x) \varphi'(u) = \widehat{a}(x) R_0 \widehat{a}(x) \varphi''(u) = \frac{1}{2} B_1(x) \varphi''(u).$$

Here

$$\widehat{a}(x) = a_0(x) - \widehat{a}_0.$$

The proof of Lemma 3.6 and Proposition 3.2 is complete. $\square$

4. LARGE DEVIATION IN THE SCHEME OF AN ASYMPTOTICALLY SMALL DIFFUSION

A storage impulsive process in the scheme of an asymptotically small diffusion is considered under two different balance conditions, namely total and local ones.

4.1. A storage impulsive process under the total balance condition. The total balance condition means that the mean values of jumps of a storage impulsive process equal, totally, zero:

$$(29) \quad a(x) = \int_{\mathbb{R}^d} v \Phi_x(dv) \equiv 0.$$

A storage impulsive process in the scheme of an asymptotically small diffusion is considered in the following scaling [3]:

$$(30) \quad S^\varepsilon(t) = u + \varepsilon^2 \sum_{n=1}^{\nu(t/\varepsilon^3)} \alpha_n(x_n), \quad t \geq 0, \varepsilon > 0, u \in \mathbb{R}^d.$$

The coupled Markov process

$$S^\varepsilon(t), x^\varepsilon(t) := x(t/\varepsilon^3), \quad t \geq 0,$$

is defined by the generator

$$L^\varepsilon \varphi(u, x) = \varepsilon^{-3} q(x) \int_E P(x, dy) \int_{\mathbb{R}^d} \Phi_y(dv) [\varphi(u + \varepsilon^2 v, y) - \varphi(u, x)],$$

which can be rewritten as

$$(31) \quad L^\varepsilon \varphi(u, x) = \varepsilon^{-3} [Q + Q_0 \Phi_x^\varepsilon] \varphi(u, x),$$

where, by definition,

$$\Phi_x^\varepsilon \varphi(u) := \int_{\mathbb{R}^d} \Phi_x(dv) [\varphi(u + \varepsilon^2 v) - \varphi(u)] = \varepsilon^4 [\mathbb{B}(x) \varphi(u) + \delta^\varepsilon(x) \varphi(u)].$$

Here

$$\mathbb{B}(x) \varphi(u) := \frac{1}{2} B(x) \varphi''(u).$$

Hence generator (31) admits the asymptotic expansion

$$(32) \quad \begin{aligned} L^\varepsilon \varphi(u, x) &= L_0^\varepsilon \varphi(u, x) + \delta^\varepsilon(x) \varphi(u, x), \\ L_0^\varepsilon \varphi(u, x) &= [\varepsilon^{-3} Q + \varepsilon Q_0 \mathbb{B}(x)] \varphi(u, x). \end{aligned}$$

Truncated operator (32) on a perturbed test function,

$$\varphi^\varepsilon(u, x) = \varphi(u) + \varepsilon^4 \varphi_1(u, x),$$

admits the asymptotic representation

$$(33) \quad L_0^\varepsilon \varphi^\varepsilon(u, x) = \varepsilon [Q \varphi_1 + Q_0 \mathbb{B}(x) \varphi(u)] + \delta^\varepsilon(x) \varphi(u).$$

Representations (32) and (33) give

$$L^\varepsilon \varphi^\varepsilon(u, x) = \varepsilon \left[\widehat{\mathbb{C}} \varphi(u) + \delta^\varepsilon(x) \varphi(u, x) \right],$$

where the main part

$$\varepsilon \widehat{\mathbb{C}} \varphi(u) = \varepsilon \frac{1}{2} \widehat{C} \varphi''(u)$$

is the generator of a small diffusion.

4.2. Large deviation for a storage impulsive process under the total balance condition. We investigate the large deviation problem for a storage impulsive process by using the asymptotic analysis of the exponential generator of the large deviation

$$(34) \quad H^\varepsilon \varphi(u, x) = e^{-\varphi/\varepsilon} \varepsilon L^\varepsilon e^{\varphi/\varepsilon}$$

(see [2, Part I]).

Proposition 4.1. *The large deviation of a storage impulsive process (30) under the total balance condition (29) is realized by the exponential generator of a small diffusion,*

$$(35) \quad H\varphi(u) = \frac{1}{2} \widehat{C} [\varphi'(u)]^2,$$

$$\widehat{C} = q \int_E \rho(dx) B(x), \quad B(x) = \int_{\mathbb{R}^d} v^* v \Phi_x(dv).$$

Proof of Proposition 4.1. The proof is based on the following auxiliary result. □

Lemma 4.1. *Exponential generator (34) on a perturbed test function,*

$$\varphi^\varepsilon(u, x) = \varphi(u) + \varepsilon \ln [1 + \varepsilon^2 \varphi_1(u, x)],$$

admits the asymptotic representation

$$H^\varepsilon \varphi^\varepsilon(u, x) = Q\varphi_1 + \frac{1}{2} Q_0 B(x) [\varphi'(u)]^2 + h^\varepsilon(x) \varphi(u)$$

with the negligible term

$$\|h^\varepsilon(x) \varphi(u)\| \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad \varphi(u) \in C^3(\mathbb{R}).$$

Proof of Lemma 4.1. It is clear that

$$\begin{aligned} H^\varepsilon \varphi^\varepsilon &= e^{-\varphi/\varepsilon} [1 + \varepsilon^2 \varphi_1]^{-1} \varepsilon L^\varepsilon [1 + \varepsilon^2 \varphi_1] e^{\varphi/\varepsilon} \\ &= e^{-\varphi/\varepsilon} [1 - \varepsilon^2 \varphi_1] \varepsilon L_0^\varepsilon [1 + \varepsilon^2 \varphi_1] e^{\varphi/\varepsilon} + h^\varepsilon(x) \varphi(u) \\ &= e^{-\varphi/\varepsilon} [1 - \varepsilon^2 \varphi_1] \varepsilon^{-2} Q [1 + \varepsilon^2 \varphi_1] e^{\varphi/\varepsilon} + e^{-\varphi/\varepsilon} \varepsilon^{-2} Q_0 \Phi_x^\varepsilon e^{\varphi/\varepsilon} + h^\varepsilon(x) \varphi(u) \\ &= Q\varphi_1 + \frac{1}{2} Q_0 B(x) [\varphi'(u)]^2 + h^\varepsilon(x) \varphi(u). \end{aligned} \quad \square$$

Turning back to the proof of Proposition 4.1, the solution of the singular perturbation problem [1, Chapter 5] gives

$$(36) \quad H^\varepsilon \varphi^\varepsilon(u, x) = H\varphi(u) + h^\varepsilon(x) \varphi(u).$$

The asymptotic representation (36) completes the proof of Proposition 4.1. □

Remark 4.1. The exponential generator of small diffusion (35) in the Euclidean space $\mathbb{R}^d$, $d \geq 2$, is represented as

$$H\varphi(u) = \frac{1}{2} \varphi'^*(u) \widehat{C} \varphi'(u),$$

where $\varphi'^*(u) = (\varphi'_k(u), 1 \leq k \leq d)$ is a vector-row, $\varphi'(u) = (\varphi'_k(u), 1 \leq k \leq d)$ is a vector-column, and $\widehat{C} = [\widehat{C}_{kr}; 1 \leq k, r \leq d]$ is the variance matrix.

4.3. Large deviation for a storage impulsive process under the local balance condition. The *Local Balance Condition* means that the average value of jumps is not equal to zero:

$$(37) \quad \widehat{a} := \int_E \rho(dx) a(x) \neq 0.$$

A storage impulsive process under local balance condition (37) is considered in the following scaling:

$$(38) \quad S^\varepsilon(t) = u + \varepsilon^2 \sum_{n=1}^{\nu(t/\varepsilon^3)} \alpha_n(x_n) - \widehat{a}_0 t / \varepsilon.$$

Lemma 4.2. *The coupled Markov process $S^\varepsilon(t)$, $x^\varepsilon(t) := x(t/\varepsilon^3)$, $t \geq 0$, is determined by the generator (cf. (22))*

$$L^\varepsilon \varphi(u, x) = \varepsilon^{-3} q(x) \int_E P(x, dy) \int_{\mathbb{R}^d} \Phi_y(dv) [\varphi(u + \varepsilon^2 v, y) - \varphi(u, x)] - \varepsilon^{-1} \widehat{a}_0 \varphi'_u(u, x).$$

Or, in a different form,

$$\begin{aligned} L^\varepsilon \varphi(u, x) &= \left[\varepsilon^{-3} [Q + Q_0 \Phi_x^\varepsilon] - \varepsilon^{-1} \widehat{\mathbb{A}}_0 \right] \varphi(u, x), \\ \Phi_x^\varepsilon \varphi(u) &= \int_{\mathbb{R}^d} \Phi_x(dv) [\varphi(u + \varepsilon^2 v) - \varphi(u)]. \end{aligned}$$

Proposition 4.2. *The large deviation for a storage impulsive process (38) under local balance condition (37) is realized by the exponential generator of a small diffusion*

$$(39) \quad \begin{aligned} H\varphi(u) &= \frac{1}{2} \widehat{C} [\varphi'(u)]^2, \\ \widehat{C} &= q [\widehat{B}_1 + \widehat{B}_2], \end{aligned}$$

$$(40) \quad \begin{aligned} \widehat{B}_k &= \int_E \rho(dx) B_k(x), \quad k = 1, 2, \\ B_1(x) &= \int_{\mathbb{R}^d} v^* v \Phi_x(dv), \quad B_2(x) = 2\widehat{a}(x) R_0 \widehat{a}(x), \\ \widehat{a}(x) &= a_0(x) - \widehat{a}_0, \quad a_0(x) := q(x) \int_E P(x, dy) a(y). \end{aligned}$$

The exponential generator of large deviation (39)–(40) contains two components. One of them is the variance matrix of the second moment of jumps. The second component $\widehat{B}_2$ is defined by the fluctuation of the first moment of jumps.

Proof of Proposition 4.2. The proof is based on the following auxiliary result. $\square$

Lemma 4.3. *Exponential generator (34) under local balance condition (37) on the perturbed test function*

$$\varphi^\varepsilon(u, x) = \varphi(u) + \varepsilon \ln [1 + \varepsilon \varphi_1(u, x) + \varepsilon^2 \varphi_2(u, x)]$$

admits the asymptotic representation

$$(41) \quad \begin{aligned} H^\varepsilon \varphi^\varepsilon(u, x) &= \varepsilon^{-1} \left[Q \varphi_1 + \widetilde{\mathbb{A}}(x) \varphi(u) \right] \\ &+ \left[Q \varphi_2 - \varphi_1 Q \varphi_1 + \frac{1}{2} Q_0 B(x) [\varphi'(u)]^2 \right] + h^\varepsilon(x) \varphi(u) \end{aligned}$$

with the negligible term

$$\|h^\varepsilon(x) \varphi(u)\| \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad \varphi(u) \in C^3(\mathbb{R}).$$

Proof. Note the following asymptotic representations:

$$\begin{aligned} H_Q^\varepsilon \varphi^\varepsilon(u, x) &:= e^{-\varphi^\varepsilon/\varepsilon} \varepsilon^{-2} Q e^{\varphi^\varepsilon/\varepsilon} = \varepsilon^{-1} Q \varphi_1 + [Q \varphi_2 - \varphi_1 Q \varphi_1] + h_q^\varepsilon(x) \varphi(u), \\ H_\varphi^\varepsilon \varphi^\varepsilon(u, x) &:= e^{-\varphi^\varepsilon/\varepsilon} \varepsilon^{-2} Q_0 \Phi_x^\varepsilon e^{\varphi^\varepsilon/\varepsilon} = \varepsilon^{-1} Q_0 \mathbb{A}(x) \varphi(u) + Q_0 \mathbb{A}(x) \varphi_1(u, x) + h_\varphi^\varepsilon(x) \varphi(u), \\ H_a^\varepsilon \varphi^\varepsilon(u, x) &:= e^{-\varphi^\varepsilon/\varepsilon} \widehat{\mathbb{A}}_0 e^{\varphi^\varepsilon/\varepsilon} = \varepsilon^{-1} \widehat{a}_0 \varphi'(u) + h_a^\varepsilon(x) \varphi(u). \end{aligned}$$

Thus, the relation

$$H^\varepsilon \varphi^\varepsilon(u, x) = [H_Q^\varepsilon + H_\varphi^\varepsilon - H_a^\varepsilon] \varphi^\varepsilon(u, x)$$

implies (41) with

$$\begin{aligned} \widetilde{\mathbb{A}}(x) \varphi(u) &:= \widetilde{a}(x) \varphi'(u), \\ \widetilde{a}(x) &:= Q_0 a(x) - \widehat{a}_0 \end{aligned}$$

(see (24)–(25)). □

Now the solution of the singular perturbation problem [1, Chapter 5] may be used for the equations

$$(42) \quad \begin{aligned} Q \varphi_1 + \widetilde{A}(x) \varphi(u) &= 0, \quad \Pi \widetilde{A}(x) = 0; \\ Q \varphi_2 - \varphi_1 Q \varphi_1 + \frac{1}{2} B_1(x) [\varphi'(u)]^2 &= \widehat{H} \varphi(u). \end{aligned}$$

The first equation in (42) has the solution

$$\varphi_1(u, x) = R_0 \widetilde{a}(x) \varphi'(u), \quad Q \varphi_1 = \widetilde{a}(x) \varphi'(u).$$

Hence, the second equation in (42) can be rewritten as

$$Q \varphi_2 + \frac{1}{2} [B_1(x) + B_2(x)] [\varphi'(u)]^2 = \widehat{H} \varphi(u)$$

with $B_2(x)$ given in (40).

The solvability condition [1, Chapter 5] for the latter equation implies Proposition 4.2. □

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