ACCURACY AND RELIABILITY OF A MODEL FOR A GAUSSIAN HOMOGENEOUS AND ISOTROPIC RANDOM FIELD IN THE SPACE $L_p(\mathbb{T})$, $p \geq 1$

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ABSTRACT. A model is constructed for a Gaussian homogeneous isotropic random field that approximates it with a given accuracy and reliability in the space $L_p(\mathbb{T})$, $p \geq 1$. The theory of the spaces $\text{Sub}(\Omega)$ is used for studying such a model.

1. INTRODUCTION

There are many methods for modeling random fields (an interested reader may consult [1, 2] and [3, 4]). The methods for modeling random fields with a given accuracy and reliability in some function spaces, including Orlicz spaces $L_U(\mathbb{T})$ and $C(\mathbb{T})$, are considered in [5] and [6]–[11]. Models of some stochastic processes are constructed in [12] and [13] for the spaces $D_{V,W}$ and in [14] for the space $F_\psi(\Omega)$.

This paper aims at constructing models for a homogeneous and isotropic Gaussian random field X that approximate X with a given reliability and accuracy in the space $L_p(\mathbb{T})$, $p \geq 1$. We construct models of a mean square continuous homogeneous isotropic Gaussian random field.

The paper consists of four main sections. Section 2 contains main definitions and known results that are used in the paper. Section 3 is devoted to constructing models of a homogeneous isotropic random field. In Section 4, we find new bounds for Bessel functions. The accuracy and reliability of models for random fields are discussed in Section 5.

2. MAIN DEFINITIONS AND PROPERTIES

2.1. Sub-Gaussian random variables and their properties. Let $\{\Omega, \mathbf{B}, \mathbf{P}\}$ be a standard probability space.

Definition 2.1 ([15]). A random variable ξ is called sub-Gaussian if there exists a constant $a \geq 0$ such that

$$\mathbf{E} \exp\{\lambda \xi\} \leq \exp \left\{ \frac{a^2 \lambda^2}{2} \right\}$$

for all $\lambda \in \mathbf{R}$.

The space of all sub-Gaussian random variables defined on a standard probability space $\{\Omega, \mathbf{B}, \mathbf{P}\}$ is denoted by $\text{Sub}(\Omega)$. Note that $\text{Sub}(\Omega)$ is a Banach space with respect

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to the norm

$$\tau(\xi) = \sup_{\lambda \neq 0} [(2 \ln \mathbb{E} \exp\{\lambda \xi\})/\lambda^2]^{1/2}.$$

Lemma 2.1 ([15]). *Let $\xi_1, \xi_2, \dots, \xi_n$ be independent sub-Gaussian random variables. Then*

$$\tau^2 \left(\sum_{k=1}^n \xi_k \right) \leq \sum_{k=1}^n \tau^2(\xi_k).$$

Lemma 2.2 ([15]). *Let ξ be a centered random variable such that $\mathbb{E} \xi^{2k+1} = 0$ and*

$$\theta(\xi) = \sup_{k \geq 1} \left[\frac{2^k k!}{(2k)!} \mathbb{E} \xi^{2k} \right]^{\frac{1}{2k}} < \infty.$$

Then $\xi \in \text{Sub}(\Omega)$ and $\tau(\xi) \leq \theta(\xi)$.

2.2. Homogeneous and isotropic random fields.

Definition 2.2 ([16]). A random field $X = \{X(t), t \in \mathbf{R}^2\}$ is called wide sense homogeneous in the space $\mathbf{R}^2$ if $\mathbb{E} X(t) = \text{const}$, $t \in \mathbf{R}^2$, and

$$\mathbb{E} X(t)X(s) = B(t-s) = \int_{\mathbf{R}^2} e^{i(\lambda, t-s)} dF(\lambda), \quad t, s \in \mathbf{R}^2.$$

Definition 2.3 ([16]). Let $SO(2)$ be the group of rotations in $\mathbf{R}^2$. A homogeneous random field $X(t)$, $t \in \mathbf{R}^2$, is called isotropic if

$$\mathbb{E} X(t)X(s) = \mathbb{E} X(gt)X(gs)$$

for any element g of the group $SO(2)$ and for all $t, s \in \mathbf{R}^2$.

3. CONSTRUCTING A MODEL OF A HOMOGENEOUS AND ISOTROPIC RANDOM FIELD

Let $X = \{X(t, x), t \in \mathbf{R}, x \in [0, 2\pi]\}$ be a square mean continuous real valued homogeneous isotropic random field defined on $\mathbf{R}^2$. Then, similarly to the case of a complex valued field considered in [16], the following representation:

$$(1) \quad X(t, x) = \sum_{k=1}^{\infty} \cos(kx) \int_0^{\infty} J_k(t\lambda) d\eta_{1,k}(\lambda) + \sum_{k=1}^{\infty} \sin(kx) \int_0^{\infty} J_k(t\lambda) d\eta_{2,k}(\lambda)$$

holds where $\eta_{i,k}(\lambda)$, $i = 1, 2$, $k = 1, 2, \dots$, are independent Gaussian processes with independent increments, $\mathbb{E} \eta_{i,k}(\lambda) = 0$, $\mathbb{E}(\eta_{i,k}(b) - \eta_{i,k}(c))^2 = F(b) - F(c)$, $b > c$, $F(\lambda)$ is the spectral function of the field X , and $J_k(u)$ is the Bessel function. It is known that

$$J_k(u) = \frac{1}{\pi} \int_0^{\pi} \cos(k\varphi - u \sin \varphi) d\varphi$$

(see [17]).

Consider a certain partition $L = \{\lambda_0, \dots, \lambda_N\}$ of the set $[0, \infty)$ such that $\lambda_0 = 0$, $\lambda_l < \lambda_{l+1}$, $\lambda_{N-1} = \Lambda$, $\lambda_N = \infty$ and $C = \max_{0 < l \leq N-2} \frac{\lambda_{l+1}}{\lambda_l} < \infty$.

Then

$$\hat{X}(t, x) = \sum_{k=1}^M \cos(kx) \sum_{l=0}^{N-1} \eta_{1,k,l} J_k(t\zeta_l) + \sum_{k=1}^M \sin(kx) \sum_{l=0}^{N-1} \eta_{2,k,l} J_k(t\zeta_l)$$

is viewed as a model for the field $X(t, x)$, where

$$\eta_{i,k,l} = \int_{\lambda_l}^{\lambda_{l+1}} d\eta_{i,k}(\lambda), \quad i = 1, 2,$$

are independent Gaussian random variables such that $\mathbb{E} \eta_{i,k,l} = 0$,

$$\mathbb{E} \eta_{i,k,l}^2 = F(\lambda_{l+1}) - F(\lambda_l) = b_l^2,$$

ζ_l are independent random variables being independent of $\eta_{i,k,l}$ and that assume values in the intervals $[\lambda_l, \lambda_{l+1}]$, and where $b_l^2 > 0$ are such that

$$F_l(\lambda) = \mathbb{P}\{\zeta_l < \lambda\} = \frac{F(\lambda) - F(\lambda_l)}{F(\lambda_{l+1}) - F(\lambda_l)}.$$

If $b_l^2 = 0$, then $\zeta_l = 0$ with probability one. For the sake of simplicity, we assume that $b_l^2 > 0$, $l = 0, 1, \dots, N-1$.

Thus $\hat{X}(t, x)$ is given by

$$(2) \quad \begin{aligned} \hat{X}(t, x) = & \sum_{k=1}^M \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} J_k(t\zeta_l) d\eta_{1,k}(\lambda) \\ & + \sum_{k=1}^M \sin(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} J_k(t\zeta_l) d\eta_{2,k}(\lambda). \end{aligned}$$

It is clear that $X(t, x)$ can be rewritten as follows:

$$\begin{aligned} X(t, x) = & \sum_{k=1}^{\infty} \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} J_k(t\lambda) d\eta_{1,k}(\lambda) \\ & + \sum_{k=1}^{\infty} \sin(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} J_k(t\lambda) d\eta_{2,k}(\lambda). \end{aligned}$$

4. PRELIMINARY RESULTS

Put

$$(3) \quad \begin{aligned} \chi_M(t, x) &= X(t, x) - \hat{X}(t, x) \\ &= \sum_{k=1}^M \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) \\ &\quad + \sum_{k=M+1}^{\infty} \cos(kx) \int_0^{\infty} J_k(t\lambda) d\eta_{1,k}(\lambda) \\ &\quad + \sum_{k=1}^M \sin(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{2,k}(\lambda) \\ &\quad + \sum_{k=M+1}^{\infty} \sin(kx) \int_0^{\infty} J_k(t\lambda) d\eta_{2,k}(\lambda) \\ &=: \chi_{M,1}(t, x) + \chi_{M,2}(t, x). \end{aligned}$$

Then

$$(4) \quad \tau(\chi_M(t, x)) \leq \tau(\chi_{M,1}(t, x)) + \tau(\chi_{M,2}(t, x))$$

and

$$\begin{aligned}\tau^2(\chi_{M,1}(t, x)) &\leq \tau^2 \left(\sum_{k=1}^M \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) \right) \\ &\quad + \tau^2 \left(\sum_{k=M+1}^{\infty} \cos(kx) \int_0^{\infty} J_k(t\lambda) d\eta_{1,k}(\lambda) \right), \\ \tau^2(\chi_{M,2}(t, x)) &\leq \tau^2 \left(\sum_{k=1}^M \sin(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{2,k}(\lambda) \right) \\ &\quad + \tau^2 \left(\sum_{k=M+1}^{\infty} \sin(kx) \int_0^{\infty} J_k(t\lambda) d\eta_{2,k}(\lambda) \right).\end{aligned}$$

Lemma 4.1. For all $\frac{1}{2} < \alpha \leq 1$,

$$|J_k(u)| \leq 2^{1-\alpha} |u|^\alpha \pi^\alpha \frac{1}{k^\alpha}.$$

Proof. We have

$$\begin{aligned}|J_k(u)| &= \frac{1}{\pi} \left| \int_0^\pi \cos(k\varphi - u \sin \varphi) d\varphi \right| \\ &= \frac{1}{\pi} \left| \int_0^\pi \cos(k\varphi) \cos(u \sin \varphi) d\varphi + \int_0^\pi \sin(k\varphi) \sin(u \sin \varphi) d\varphi \right| \\ &= \frac{1}{\pi} |I_1 + I_2| \\ &\leq \frac{1}{\pi} (|I_1| + |I_2|).\end{aligned}$$

Since the integrand in the integral I_1 is an even and periodic function with period 2π , we can transform I_1 as follows:

$$\begin{aligned}I_1 &= \int_0^\pi \cos(k\varphi) \cos(u \sin \varphi) d\varphi \\ &= \frac{1}{2} \int_{-\pi}^\pi \cos(k\varphi) \cos(u \sin \varphi) d\varphi \\ &= \frac{1}{2} \int_{-\pi}^\pi \cos\left(k\left(\varphi + \frac{\pi}{k}\right)\right) \cos\left(u \sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi \\ &= \frac{1}{2} \int_{-\pi}^\pi \cos(k\varphi + \pi) \cos\left(u \sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi \\ &= -\frac{1}{2} \int_{-\pi}^\pi \cos(k\varphi) \cos\left(u \sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi.\end{aligned}$$

Then the integral I_1 is written as follows:

$$\begin{aligned}I_1 &= -\frac{1}{4} \int_{-\pi}^\pi \cos(k\varphi) \cos\left(u \sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi \\ &\quad + \frac{1}{4} \int_{-\pi}^\pi \cos(k\varphi) \cos(u \sin \varphi) d\varphi.\end{aligned}$$

We are going to obtain a bound for $|I_1|$. Indeed,

$$\begin{aligned}
|I_1| &= \left| -\frac{1}{4} \int_{-\pi}^{\pi} \cos(k\varphi) \cos\left(u \sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi + \frac{1}{4} \int_{-\pi}^{\pi} \cos(k\varphi) \cos(u \sin \varphi) d\varphi \right| \\
&\leq \frac{1}{4} \int_{-\pi}^{\pi} \left| \cos(k\varphi) \left(\cos(u \sin \varphi) - \cos\left(u \sin\left(\varphi + \frac{\pi}{k}\right)\right) \right) \right| d\varphi \\
&= \frac{1}{4} \int_{-\pi}^{\pi} \left| \cos(k\varphi) \cdot 2 \sin\left(\frac{u(\sin(\varphi + \frac{\pi}{k}) - \sin \varphi)}{2}\right) \right. \\
&\quad \left. \times \sin\left(\frac{u(\sin(\varphi + \frac{\pi}{k}) + \sin \varphi)}{2}\right) \right| d\varphi \\
&\leq \frac{1}{2} \int_{-\pi}^{\pi} \left| \sin\left(\frac{u(\sin(\varphi + \frac{\pi}{k}) - \sin \varphi)}{2}\right) \right| d\varphi \\
&\leq \frac{1}{2} \int_{-\pi}^{\pi} \left| \frac{u}{2} \right|^\alpha \left| \sin\left(\varphi + \frac{\pi}{k}\right) - \sin \varphi \right|^\alpha d\varphi \\
&= \frac{1}{2} \int_{-\pi}^{\pi} \left| \frac{u}{2} \right|^\alpha \left| 2 \cos\left(\frac{2\varphi + \frac{\pi}{k}}{2}\right) \cdot \sin \frac{\pi}{2k} \right|^\alpha d\varphi \\
&\leq \frac{1}{2} \int_{-\pi}^{\pi} |u|^\alpha \left| \sin \frac{\pi}{2k} \right|^\alpha d\varphi \\
&= |u|^\alpha \cdot \pi \left| \sin \frac{\pi}{2k} \right|^\alpha \\
&\leq \pi \cdot |u|^\alpha \left(\frac{\pi}{2k} \right)^\alpha.
\end{aligned}$$

For the integral I_2 ,

$$\begin{aligned}
I_2 &= \int_0^\pi \sin(k\varphi) \sin(u \sin \varphi) d\varphi \\
&= \frac{1}{2} \int_{-\pi}^{\pi} \sin(k\varphi) \sin(u \sin \varphi) d\varphi \\
&= \frac{1}{2} \int_{-\pi}^{\pi} \sin\left(k\left(\varphi + \frac{\pi}{k}\right)\right) \sin\left(u \sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi \\
&= \frac{1}{2} \int_{-\pi}^{\pi} \sin(k\varphi - \pi) \sin\left(u \sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi \\
&= -\frac{1}{2} \int_{-\pi}^{\pi} \sin(k\varphi) \sin\left(u \sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi.
\end{aligned}$$

As in the case of I_1 , the integral I_2 is transformed to the following form:

$$\begin{aligned}
I_2 &= -\frac{1}{4} \int_{-\pi}^{\pi} \sin(k\varphi) \sin\left(u \sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi \\
&\quad + \frac{1}{4} \int_{-\pi}^{\pi} \sin(k\varphi) \sin(u \sin \varphi) d\varphi.
\end{aligned}$$

Then $|I_2|$ admits the following bound:

$$|I_2| \leq \pi \cdot |u|^\alpha \left(\frac{\pi}{2k} \right)^\alpha,$$

whence

$$\begin{aligned}
 |J_k(u)| &= \frac{1}{\pi} |I_1 + I_2| \\
 &\leq \frac{1}{\pi} \left(\pi \cdot |u|^\alpha \left(\frac{\pi}{2k} \right)^\alpha + \pi \cdot |u|^\alpha \left(\frac{\pi}{2k} \right)^\alpha \right) \\
 &= 2^{1-\alpha} \cdot |u|^\alpha \pi^\alpha \cdot \frac{1}{k^\alpha},
 \end{aligned}$$

where $\frac{1}{2} < \alpha \leq 1$. □

Lemma 4.2. *For all $\frac{1}{2} < \alpha \leq 1$,*

$$|J_k(t\lambda) - J_k(tu)| \leq 4^{1-\alpha} t^\alpha |\lambda - u|^\alpha \pi^\alpha \cdot \frac{1}{k^\alpha} \left(1 + \frac{t^\alpha |\lambda + u|^\alpha}{2^\alpha} \right).$$

Proof. Using the integrals I_1 and I_2 (evaluated in the proof of Lemma 4.1), we get

$$\begin{aligned}
 |J_k(t\lambda) - J_k(tu)| &= \frac{1}{\pi} \left| \left(-\frac{1}{4} \int_{-\pi}^{\pi} \cos(k\varphi) \cos \left(t\lambda \sin \left(\varphi + \frac{\pi}{k} \right) \right) d\varphi \right. \right. \\
 &\quad + \frac{1}{4} \int_{-\pi}^{\pi} \cos(k\varphi) \cos(t\lambda \sin \varphi) d\varphi \\
 &\quad + \frac{1}{4} \int_{-\pi}^{\pi} \cos(k\varphi) \cos \left(tu \sin \left(\varphi + \frac{\pi}{k} \right) \right) d\varphi \\
 &\quad \left. - \frac{1}{4} \int_{-\pi}^{\pi} \cos(k\varphi) \cos(tu \sin \varphi) d\varphi \right) \\
 &\quad + \left(-\frac{1}{4} \int_{-\pi}^{\pi} \sin(k\varphi) \sin \left(t\lambda \sin \left(\varphi + \frac{\pi}{k} \right) \right) d\varphi \right. \\
 &\quad + \frac{1}{4} \int_{-\pi}^{\pi} \sin(k\varphi) \sin(t\lambda \sin \varphi) d\varphi \\
 &\quad + \frac{1}{4} \int_{-\pi}^{\pi} \sin(k\varphi) \sin \left(tu \sin \left(\varphi + \frac{\pi}{k} \right) \right) d\varphi \\
 &\quad \left. - \frac{1}{4} \int_{-\pi}^{\pi} \sin(k\varphi) \sin(tu \sin \varphi) d\varphi \right) \Big| \\
 &= \frac{1}{\pi} |S_1 + S_2| \\
 &\leq \frac{1}{\pi} (|S_1| + |S_2|).
 \end{aligned}$$

Then we find a bound for $|S_1|$:

$$\begin{aligned}
 |S_1| &= \left| -\frac{1}{4} \int_{-\pi}^{\pi} \cos(k\varphi) \cos \left(t\lambda \sin \left(\varphi + \frac{\pi}{k} \right) \right) d\varphi \right. \\
 &\quad + \frac{1}{4} \int_{-\pi}^{\pi} \cos(k\varphi) \cos(t\lambda \sin \varphi) d\varphi + \frac{1}{4} \int_{-\pi}^{\pi} \cos(k\varphi) \cos \left(tu \sin \left(\varphi + \frac{\pi}{k} \right) \right) d\varphi \\
 &\quad \left. - \frac{1}{4} \int_{-\pi}^{\pi} \cos(k\varphi) \cos(tu \sin \varphi) d\varphi \right| \\
 &\leq \frac{1}{4} \int_{-\pi}^{\pi} |\cos(k\varphi)| \left| \cos(t\lambda \sin \varphi) - \cos(tu \sin \varphi) \right. \\
 &\quad \left. - \left(\cos \left(t\lambda \sin \left(\varphi + \frac{\pi}{k} \right) \right) - \cos \left(tu \sin \left(\varphi + \frac{\pi}{k} \right) \right) \right) \right| d\varphi
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int_{-\pi}^{\pi} |\cos(k\varphi)| \left| \sin \frac{t(u+\lambda) \sin \varphi}{2} \sin \frac{t(u-\lambda) \sin \varphi}{2} \right. \\
&\quad \left. - \sin \frac{t(u+\lambda) \sin(\varphi + \frac{\pi}{k})}{2} \sin \frac{t(u-\lambda) \sin(\varphi + \frac{\pi}{k})}{2} \right| d\varphi \\
&= \frac{1}{2} \int_{-\pi}^{\pi} |\cos(k\varphi)| \left| \sin \frac{t(u+\lambda) \sin \varphi}{2} \left(\sin \frac{t(u-\lambda) \sin \varphi}{2} - \sin \frac{t(u-\lambda) \sin(\varphi + \frac{\pi}{k})}{2} \right) \right. \\
&\quad \left. + \sin \frac{t(u-\lambda) \sin(\varphi + \frac{\pi}{k})}{2} \right. \\
&\quad \left. \times \left(\sin \frac{t(u+\lambda) \sin \varphi}{2} - \sin \frac{t(u+\lambda) \sin(\varphi + \frac{\pi}{k})}{2} \right) \right| d\varphi \\
&= \int_{-\pi}^{\pi} |\cos(k\varphi)| \left| \sin \frac{t(u+\lambda) \sin \varphi}{2} \cdot \cos \frac{t(u-\lambda)(\sin \varphi + \sin(\varphi + \frac{\pi}{k}))}{4} \right. \\
&\quad \times \sin \frac{t(u-\lambda)(\sin \varphi - \sin(\varphi + \frac{\pi}{k}))}{4} \\
&\quad + \sin \frac{t(u-\lambda) \sin(\varphi + \frac{\pi}{k})}{2} \cdot \cos \frac{t(u+\lambda)(\sin \varphi + \sin(\varphi + \frac{\pi}{k}))}{4} \\
&\quad \left. \times \sin \frac{t(u+\lambda)(\sin \varphi - \sin(\varphi + \frac{\pi}{k}))}{4} \right| d\varphi \\
&\leq \int_{-\pi}^{\pi} \left(\left| \sin \frac{t(u-\lambda)(\sin \varphi - \sin(\varphi + \frac{\pi}{k}))}{4} \right| \right. \\
&\quad \left. + \left| \sin \frac{t(u-\lambda) \sin(\varphi + \frac{\pi}{k})}{2} \sin \frac{t(u+\lambda)(\sin \varphi - \sin(\varphi + \frac{\pi}{k}))}{4} \right| \right) d\varphi \\
&\leq 2\pi \left(\frac{t^\alpha |\lambda - u|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha} + \frac{t^\alpha |\lambda - u|^\alpha t^\alpha |\lambda + u|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha \cdot 2^\alpha} \right) \\
&= 2\pi \left(\frac{t}{2} \right)^\alpha |\lambda - u|^\alpha \left(\frac{\pi}{2k} \right)^\alpha \left(1 + \frac{t^\alpha |\lambda + u|^\alpha}{2^\alpha} \right).
\end{aligned}$$

Similarly, we obtain a bound for $|S_2|$:

$$\begin{aligned}
|S_2| &= \left| -\frac{1}{4} \int_{-\pi}^{\pi} \sin(k\varphi) \sin \left(t\lambda \sin \left(\varphi + \frac{\pi}{k} \right) \right) d\varphi \right. \\
&\quad + \frac{1}{4} \int_{-\pi}^{\pi} \sin(k\varphi) \sin(t\lambda \sin \varphi) d\varphi \\
&\quad + \frac{1}{4} \int_{-\pi}^{\pi} \sin(k\varphi) \sin \left(tu \sin \left(\varphi + \frac{\pi}{k} \right) \right) d\varphi - \frac{1}{4} \int_{-\pi}^{\pi} \sin(k\varphi) \sin(tu \sin \varphi) d\varphi \left. \right| \\
&\leq \frac{1}{4} \int_{-\pi}^{\pi} |\sin(k\varphi)| \left| (\sin(t\lambda \sin \varphi) - \sin(tu \sin \varphi)) \right. \\
&\quad \left. - \left(\sin \left(t\lambda \sin \left(\varphi + \frac{\pi}{k} \right) \right) - \sin \left(tu \sin \left(\varphi + \frac{\pi}{k} \right) \right) \right) \right| d\varphi \\
&= \frac{1}{2} \int_{-\pi}^{\pi} |\sin(k\varphi)| \left| \cos \frac{t(\lambda + u) \sin \varphi}{2} \sin \frac{t(\lambda - u) \sin \varphi}{2} \right. \\
&\quad \left. - \cos \frac{t(\lambda + u) \sin(\varphi + \frac{\pi}{k})}{2} \sin \frac{t(\lambda - u) \sin(\varphi + \frac{\pi}{k})}{2} \right| d\varphi
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int_{-\pi}^{\pi} |\sin(k\varphi)| \left| \cos \frac{t(\lambda+u)\sin\varphi}{2} \left(\sin \frac{t(\lambda-u)\sin\varphi}{2} - \sin \frac{t(\lambda-u)\sin(\varphi+\frac{\pi}{k})}{2} \right) \right. \\
&\quad \left. + \sin \frac{t(\lambda-u)\sin(\varphi+\frac{\pi}{k})}{2} \right. \\
&\quad \left. \times \left(\cos \frac{t(\lambda+u)\sin\varphi}{2} - \cos \frac{t(\lambda+u)\sin(\varphi+\frac{\pi}{k})}{2} \right) \right| d\varphi \\
&\leq \int_{-\pi}^{\pi} |\sin(k\varphi)| \left(\left| \cos \frac{t(\lambda+u)\sin\varphi}{2} \cos \frac{t(\lambda-u)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4} \right. \right. \\
&\quad \left. \times \sin \frac{t(\lambda-u)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4} \right| \\
&\quad \left. + \left| \sin \frac{t(\lambda-u)\sin(\varphi+\frac{\pi}{k})}{2} \sin \frac{t(\lambda+u)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4} \right. \right. \\
&\quad \left. \times \sin \frac{t(\lambda+u)(\sin(\varphi+\frac{\pi}{k})-\sin\varphi)}{4} \right| \Bigg) d\varphi \\
&\leq \int_{-\pi}^{\pi} \left(\left| \sin \frac{t(\lambda-u)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4} \right| \right. \\
&\quad \left. + \left| \sin \frac{t(\lambda-u)\sin(\varphi+\frac{\pi}{k})}{2} \sin \frac{t(\lambda+u)(\sin(\varphi+\frac{\pi}{k})-\sin\varphi)}{4} \right| \right) d\varphi \\
&\leq 2\pi \left(\frac{t^\alpha |\lambda-u|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha} + \frac{t^\alpha |\lambda-u|^\alpha t^\alpha |\lambda+u|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha \cdot 2^\alpha} \right) \\
&= 2\pi \left(\frac{t}{2} \right)^\alpha |\lambda-u|^\alpha \left(\frac{\pi}{2k} \right)^\alpha \left(1 + \frac{t^\alpha |\lambda+u|^\alpha}{2^\alpha} \right).
\end{aligned}$$

Then

$$\begin{aligned}
|J_k(t\lambda) - J_k(tu)| &\leq 2 \left(\frac{t}{2} \right)^\alpha |\lambda-u|^\alpha \left(\frac{\pi}{2k} \right)^\alpha \left(1 + \frac{t^\alpha |\lambda+u|^\alpha}{2^\alpha} \right) \\
&\quad + 2 \left(\frac{t}{2} \right)^\alpha |\lambda-u|^\alpha \left(\frac{\pi}{2k} \right)^\alpha \left(1 + \frac{t^\alpha |\lambda+u|^\alpha}{2^\alpha} \right) \\
&= 4^{1-\alpha} t^\alpha |\lambda-u|^\alpha \pi^\alpha \cdot \frac{1}{k^\alpha} \left(1 + \frac{t^\alpha |\lambda+u|^\alpha}{2^\alpha} \right). \quad \square
\end{aligned}$$

Lemma 4.3. For all $\frac{1}{2} < \alpha \leq 1$,

$$\begin{aligned}
&\tau^2 \left(\sum_{k=1}^M \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) \right) \\
&\leq \frac{1}{2\alpha-1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) 2 \cdot 4^{2(1-\alpha)} \pi^{2\alpha} t^{2\alpha} \\
&\quad \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{t(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) \\
&\quad + 4M(F(+\infty) - F(\Lambda)),
\end{aligned}$$

$$\begin{aligned}
& \tau^2 \left(\sum_{k=1}^M \sin(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{2,k}(\lambda) \right) \\
& \leq \frac{1}{2\alpha - 1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) 2 \cdot 4^{2(1-\alpha)} \pi^{2\alpha} t^{2\alpha} \\
& \quad \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{t(1+C')}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) \\
& \quad + 4M(F(+\infty) - F(\Lambda)),
\end{aligned}$$

where $C' = \max_{0 < l \leq N-2} \frac{\lambda_{l+1}}{\lambda_l}$.

Proof. We have

$$\begin{aligned}
& \tau^2 \left(\sum_{k=1}^M \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) \right) \\
& \leq \sum_{k=1}^M \sum_{l=0}^{N-1} \tau^2 \left(\int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) \right) \\
& \leq \sum_{k=1}^M \sum_{l=0}^{N-1} \theta^2 \left(\int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) \right) \\
& = \sum_{k=1}^M \sum_{l=0}^{N-1} \sup_{m \geq 1} \left[\frac{2^m \cdot m!}{(2m)!} E \left(\int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) \right)^{2m} \right]^{\frac{1}{m}}.
\end{aligned}$$

Since $E\xi = 0$, $E\xi^{2k+1} = 0$, and

$$E\xi^{2k} = \frac{(2k)!}{2^k \cdot k!} \sigma^{2k}$$

for any centered Gaussian random variable ξ and since the random variables ζ_l do not depend on $\eta_{i,k}(\lambda)$, $i = 1, 2$, we obtain from Fubini's theorem and Lemma 4.2 that

$$\begin{aligned}
& E \left(\int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) \right)^{2m} \\
& = E_{\zeta_l} \left(\int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) \right)^{2m} \\
& \leq \frac{(2m)!}{2^m \cdot m!} E \left(\int_{\lambda_l}^{\lambda_{l+1}} |J_k(t\lambda) - J_k(t\zeta_l)|^2 dF(\lambda) \right)^m \\
& \leq \frac{(2m)!}{2^m \cdot m!} E \left(\int_{\lambda_l}^{\lambda_{l+1}} \left(4^{1-\alpha} t^\alpha |\lambda - \zeta_l|^\alpha \pi^\alpha \cdot \frac{1}{k^\alpha} \left(1 + \frac{t^\alpha |\lambda + \zeta_l|^\alpha}{2^\alpha} \right) \right)^2 dF(\lambda) \right)^m \\
& = \frac{(2m)!}{2^m \cdot m!} 4^{2m(1-\alpha)} t^{2m\alpha} \pi^{2m\alpha} \left(\frac{1}{k} \right)^{2m\alpha} \\
& \quad \times E \left(\int_{\lambda_l}^{\lambda_{l+1}} |\lambda - \zeta_l|^{2\alpha} \left(1 + \frac{t^\alpha |\lambda + \zeta_l|^\alpha}{2^\alpha} \right)^2 dF(\lambda) \right)^m
\end{aligned}$$

$$\begin{aligned}
&= \frac{(2m)!}{2^m \cdot m!} \cdot 4^{2m(1-\alpha)} t^{2m\alpha} \pi^{2m\alpha} \left(\frac{1}{k}\right)^{2m\alpha} \\
&\quad \times \int_{\lambda_l}^{\lambda_{l+1}} \left(\int_{\lambda_l}^{\lambda_{l+1}} |\lambda - u|^{2\alpha} \left(1 + \frac{t^\alpha |\lambda + u|^\alpha}{2^\alpha}\right)^2 dF(\lambda) \right)^m dF_l(u) \\
&\leq \frac{(2m)!}{2^m \cdot m!} \cdot 4^{2m(1-\alpha)} t^{2m\alpha} \pi^{2m\alpha} \left(\frac{1}{k}\right)^{2m\alpha} |\lambda_{l+1} - \lambda_l|^{2m\alpha} \\
&\quad \times \int_{\lambda_l}^{\lambda_{l+1}} \left(\int_{\lambda_l}^{\lambda_{l+1}} \left(1 + \frac{t^\alpha \lambda^\alpha |1 + \frac{u}{\lambda}|^\alpha}{2^\alpha}\right)^2 dF(\lambda) \right)^m dF_l(u) \\
&\leq \frac{(2m)!}{2^m \cdot m!} \cdot 4^{2m(1-\alpha)} t^{2m\alpha} \pi^{2m\alpha} \left(\frac{1}{k}\right)^{2m\alpha} |\lambda_{l+1} - \lambda_l|^{2m\alpha} \\
&\quad \times \left(\int_{\lambda_l}^{\lambda_{l+1}} \left(1 + \frac{t^\alpha \lambda^\alpha |1 + \frac{\lambda_{l+1}}{\lambda_l}|^\alpha}{2^\alpha}\right)^2 dF(\lambda) \right)^m \\
&\leq \frac{(2m)!}{2^m \cdot m!} \cdot 4^{2m(1-\alpha)} t^{2m\alpha} \pi^{2m\alpha} \left(\frac{1}{k}\right)^{2m\alpha} |\lambda_{l+1} - \lambda_l|^{2m\alpha} \\
&\quad \times \left(\int_{\lambda_l}^{\lambda_{l+1}} \left(2 + 2 \cdot \frac{t^{2\alpha} \lambda^{2\alpha} (1+C)^{2\alpha}}{2^{2\alpha}}\right) dF(\lambda) \right)^m \\
&= \frac{(2m)!}{2^m \cdot m!} \cdot 4^{2m(1-\alpha)} t^{2m\alpha} \pi^{2m\alpha} \left(\frac{1}{k}\right)^{2m\alpha} |\lambda_{l+1} - \lambda_l|^{2m\alpha} \\
&\quad \times \left(2b_l^2 + 2 \cdot \left(\frac{t(1+C)}{2}\right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right)^m.
\end{aligned}$$

Here the symbol E_{ζ_l} denotes the conditional expectation with respect to ζ_l . Note that

$$\begin{aligned}
&\left(\int_{\Lambda}^{\infty} \left(\int_{\Lambda}^{\infty} |J_k(t\lambda) - J_k(tu)|^2 dF(\lambda) \right)^m dF_l(u) \right)^{\frac{1}{m}} \\
&= \left(\int_{\Lambda}^{\infty} \left(\int_{\Lambda}^{\infty} \left| \frac{1}{\pi} \left(\int_0^{\pi} \cos(k\varphi - t\lambda \sin \varphi) d\varphi \right. \right. \right. \\
&\quad \left. \left. \left. - \int_0^{\pi} \cos(k\varphi - tu \sin \varphi) d\varphi \right) \right|^2 dF(\lambda) \right)^m dF_l(u) \right)^{\frac{1}{m}} \\
&\leq \left(\int_{\Lambda}^{\infty} \left(\int_{\Lambda}^{\infty} \left(\frac{1}{\pi} \int_0^{\pi} |\cos(k\varphi - t\lambda \sin \varphi) \right. \right. \right. \\
&\quad \left. \left. \left. - \cos(k\varphi - tu \sin \varphi) | d\varphi \right)^2 dF(\lambda) \right)^m dF_l(u) \right)^{\frac{1}{m}} \\
&= \left(\int_{\Lambda}^{\infty} \left(\int_{\Lambda}^{\infty} \left(\frac{1}{\pi} \int_0^{\pi} \left| 2 \sin \left(k\varphi - \frac{t(\lambda + u) \sin \varphi}{2} \right) \right. \right. \right. \right. \\
&\quad \left. \left. \left. \times \sin \left(\frac{t(\lambda - u) \sin \varphi}{2} \right) \right| d\varphi \right)^2 dF(\lambda) \right)^m dF_l(u) \right)^{\frac{1}{m}}.
\end{aligned}$$

Therefore

$$\begin{aligned} & \left(\int_{\Lambda}^{\infty} \left(\int_{\Lambda}^{\infty} |J_k(t\lambda) - J_k(tu)|^2 dF(\lambda) \right)^m dF_l(u) \right)^{\frac{1}{m}} \\ & \leq 4 \left(\int_{\Lambda}^{\infty} \left(\int_{\Lambda}^{\infty} dF(\lambda) \right)^m dF_l(u) \right)^{\frac{1}{m}} = 4(F(+\infty) - F(\Lambda)), \end{aligned}$$

whence

$$\begin{aligned} & \tau^2 \left(\sum_{k=1}^M \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) \right) \\ & \leq 4^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \\ & \quad \times \sum_{k=1}^M \sum_{l=0}^{N-1} \frac{1}{k^{2\alpha}} \\ & \quad \times \sup_{m \geq 1} \left(\int_{\lambda_l}^{\lambda_{l+1}} \left(\int_{\lambda_l}^{\lambda_{l+1}} |\lambda - u|^{2\alpha} \left(1 + \frac{t^\alpha |\lambda + u|^\alpha}{2^\alpha} \right)^2 dF(\lambda) \right)^m dF_l(u) \right)^{\frac{1}{m}} \\ & \leq \sum_{k=1}^M \frac{1}{k^{2\alpha}} \cdot 2 \cdot 4^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \\ & \quad \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{t(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) \\ & \quad + 4 \sum_{k=1}^M (F(+\infty) - F(\Lambda)). \end{aligned}$$

The sum $\sum_{k=1}^M \frac{1}{k^{2\alpha}}$ with $\frac{1}{2} < \alpha \leq 1$ can be estimated as follows:

$$\begin{aligned} \sum_{k=1}^M \frac{1}{k^{2\alpha}} & \leq 1 + \sum_{k=2}^M \int_{k-1}^k \frac{1}{x^{2\alpha}} dx = 1 + \int_1^M \frac{1}{x^{2\alpha}} dx = 1 + \frac{x^{1-2\alpha}}{1-2\alpha} \Big|_1^M \\ & = \frac{2\alpha}{2\alpha-1} - \frac{1}{(2\alpha-1)M^{2\alpha-1}}. \end{aligned}$$

Then

$$\begin{aligned} & \tau^2 \left(\sum_{k=1}^M \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) \right) \\ & \leq \frac{1}{2\alpha-1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) 2 \cdot 4^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \\ & \quad \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{t(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) \\ & \quad + 4M(F(+\infty) - F(\Lambda)). \end{aligned}$$

The second inequality is proved similarly. □

Lemma 4.4. *Let the integral $\int_0^\infty \lambda^{2\alpha} dF(\lambda) < \infty$ converge for $\frac{1}{2} < \alpha \leq 1$. Then*

$$\begin{aligned} & \tau^2 \left(\sum_{k=M+1}^{\infty} \cos(kx) \int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right) \\ & \leq 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \frac{1}{(2\alpha-1)M^{2\alpha-1}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right), \\ & \tau^2 \left(\sum_{k=M+1}^{\infty} \sin(kx) \int_0^\infty J_k(t\lambda) d\eta_{2,k}(\lambda) \right) \\ & \leq 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \frac{1}{(2\alpha-1)M^{2\alpha-1}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right). \end{aligned}$$

Proof. Indeed,

$$\begin{aligned} & \tau^2 \left(\sum_{k=M+1}^{\infty} \cos(kx) \int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right) \\ & \leq \sum_{k=M+1}^{\infty} \tau^2 \left(\int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right) \leq \sum_{k=M+1}^{\infty} \theta^2 \left(\int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right) \\ & = \sum_{k=M+1}^{\infty} \sup_{m \geq 1} \left[\frac{2^m \cdot m!}{(2m)!} \mathbb{E} \left(\int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right)^{2m} \right]^{\frac{1}{m}}. \end{aligned}$$

Applying Lemma 4.1,

$$\begin{aligned} \mathbb{E} \left(\int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right)^{2m} & \leq \frac{(2m)!}{2^m \cdot m!} \left(\int_0^\infty |J_k(t\lambda)|^2 dF(\lambda) \right)^m \\ & \leq \frac{(2m)!}{2^m \cdot m!} \left(\int_0^\infty \left(2^{1-\alpha} |t\lambda|^\alpha \pi^\alpha \cdot \frac{1}{k^\alpha} \right)^2 dF(\lambda) \right)^m \\ & = \frac{(2m)!}{2^m \cdot m!} \cdot \frac{2^{2m(1-\alpha)} t^{2m\alpha} \pi^{2m\alpha}}{k^{2m\alpha}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right)^m, \end{aligned}$$

whence

$$\tau^2 \left(\sum_{k=M+1}^{\infty} \cos(kx) \int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right) \leq 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \sum_{k=M+1}^{\infty} \frac{1}{k^{2\alpha}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right).$$

The tail $\sum_{k=M+1}^{\infty} \frac{1}{k^{2\alpha}}$ with $\frac{1}{2} < \alpha \leq 1$ can be estimated as follows:

$$\sum_{k=M+1}^{\infty} \frac{1}{k^{2\alpha}} \leq \sum_{k=M+1}^{\infty} \int_{k-1}^k \frac{1}{x^{2\alpha}} dx = \int_M^\infty \frac{1}{x^{2\alpha}} dx = \frac{x^{1-2\alpha}}{1-2\alpha} \Big|_M^\infty = \frac{1}{(2\alpha-1)M^{2\alpha-1}}.$$

Thus

$$\begin{aligned} & \tau^2 \left(\sum_{k=M+1}^{\infty} \cos(kx) \int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right) \\ & \leq 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \frac{1}{(2\alpha-1)M^{2\alpha-1}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right). \end{aligned}$$

The second inequality is proved similarly. □

Theorem 4.1. Let $X(t, x)$ and $\hat{X}(t, x)$ be defined by (1) and (2), respectively. Assume that the integral $\int_0^\infty \lambda^{2\alpha} dF(\lambda)$ with $\frac{1}{2} < \alpha \leq 1$ converges. Then

$$\begin{aligned} \tau^2(X(t, x) - \hat{X}(t, x)) &\leq \frac{4}{2\alpha - 1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) 2 \cdot 4^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \\ &\quad \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{t(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) \\ &\quad + 16M(F(+\infty) - F(\Lambda)) \\ &\quad + 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \frac{4}{(2\alpha - 1)M^{2\alpha-1}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right), \end{aligned}$$

where $C = \max_{0 \leq l \leq N-2} \frac{\lambda_{l+1}}{\lambda_l}$.

Proof. The proof of Theorem 4.1 follows from relations (3) and (4) in view of Lemmas 4.3 and 4.4. $\square$

5. ACCURACY AND RELIABILITY OF MODELS FOR RANDOM FIELDS

Definition 5.1 ([5]). Let $\{\mathbb{T}, \mathfrak{B}, \mu\}$ be a measurable space and let X and $\hat{X}$ be two random fields defined on $\mathbb{T}$. We say that $\hat{X}(t)$ approximates $X(t)$ with reliability $1 - \delta$, $0 < \delta < 1$, and accuracy $\varepsilon > 0$ in the space $L_p(\mathbb{T})$ if

$$\mathbb{P} \left\{ \left(\int_{\mathbb{T}} |X(t) - \hat{X}(t)|^p d\mu(t) \right)^{\frac{1}{p}} > \varepsilon \right\} \leq \delta.$$

Theorem 5.1. Let $X = \{X(t), t \in \mathbb{T}\}$ be a sub-Gaussian random field such that

$$\mathbb{E} X(t) = 0, \quad \tau^2(t) = \tau^2(X(t)) = \mathbb{E}(X(t))^2.$$

Assume that

$$\int_{\mathbb{T}} (\mathbb{E}(X(t))^2)^{\frac{p}{2}} d\mu(t) < \infty, \quad p \geq 1.$$

Then

$$\int_{\mathbb{T}} |X(t)|^p d\mu(t) < \infty$$

with probability one. Moreover

$$\mathbb{P} \left\{ \|X(t)\|_{L_p} > \varepsilon \right\} \leq 2 \exp \left\{ -\frac{\varepsilon^2}{2c_p^p} \right\}$$

for all ε such that $\varepsilon > c_p^{\frac{1}{p}} p^{\frac{1}{2}}$, where $c_p = \int_{\mathbb{T}} (\tau(t))^p d\mu(t)$.

Theorem 5.1 is a partial case of Theorem 2.1 in [18].

Theorem 5.2. Let $\frac{1}{2} < \alpha \leq 1$ and let

$$\int_0^\infty \lambda^{2\alpha} dF(\lambda) < \infty.$$

Assume that a partition L used to construct a model $\hat{X}(t, x)$, $t \in [0, T]$, $x \in [0, 2\pi]$, according to (2) is such that

$$I \leq \frac{\varepsilon^p}{\max \left(\left(2 \ln \frac{2}{\delta} \right)^{\frac{p}{2}}, p^{\frac{p}{2}} \right)},$$

where

$$\begin{aligned}
 I = & \frac{T^{p\alpha+1}}{p\alpha+1} \left(\frac{2^p D_p^3}{(2\alpha-1)^{\frac{p}{2}}} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right)^{\frac{p}{2}} 2^{\frac{p}{2}+1} \cdot 4^{p(1-\alpha)} \pi^{p\alpha+1} \left(\sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} b_l^2 \right)^{\frac{p}{2}} \right. \\
 & \left. + D_p 2^{p(1-\alpha)+1} \pi^{p\alpha+1} \left(\frac{4}{(2\alpha-1)M^{2\alpha-1}} \right)^{\frac{p}{2}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right)^{\frac{p}{2}} \right) \\
 & + \frac{T^{2p\alpha+1}}{2p\alpha+1} \frac{2^p D_p^3}{(2\alpha-1)^{\frac{p}{2}}} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right)^{\frac{p}{2}} 2^{\frac{p}{2}+1} \cdot 4^{p(1-\alpha)} \pi^{p\alpha+1} \left(\frac{1+C}{2} \right)^{p\alpha} \\
 & \times \left(\sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \cdot \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right)^{\frac{p}{2}} \\
 & + T \cdot 2^{2p+1} \pi D_p^2 M^{\frac{p}{2}} (F(+\infty) - F(\Lambda))^{\frac{p}{2}}
 \end{aligned}$$

and

$$C = \max_{0 < l \leq N-2} \frac{\lambda_{l+1}}{\lambda_l}, \quad D_p = \begin{cases} 1, & \text{if } 0 < \frac{p}{2} \leq 1, \\ 2^{\frac{p}{2}-1}, & \text{if } \frac{p}{2} > 1. \end{cases}$$

Then the model $\hat{X}(t, x)$ approximates the Gaussian field $X(t, x)$ with reliability $1 - \delta$, $0 < \delta < 1$, and accuracy $\varepsilon > 0$ in the space $L_p(\mathbb{T})$, $p \geq 1$.

Proof. If

$$\varepsilon > \left(\int_0^T \int_0^{2\pi} \left(\tau(X(t, x) - \hat{X}(t, x)) \right)^p dx dt \right)^{\frac{1}{p}} \cdot p^{\frac{1}{2}},$$

then Theorem 5.1 and Definition 5.1 imply that

$$\mathbf{P} \left\{ \|X(t, x) - \hat{X}(t, x)\|_{L_p} > \varepsilon \right\} \leq 2 \exp \left\{ -\frac{\varepsilon^2}{2c_p^{\frac{p}{2}}} \right\} \leq \delta,$$

where

$$c_p = \int_0^T \int_0^{2\pi} (\tau(X(t, x) - \hat{X}(t, x)))^p dx dt.$$

The latter inequality holds if

$$\int_0^T \int_0^{2\pi} (\tau(X(t, x) - \hat{X}(t, x)))^p dx dt \leq \frac{\varepsilon^p}{(2 \ln \frac{2}{\delta})^{\frac{p}{2}}}.$$

Since

$$(a + b)^{\frac{p}{2}} \leq D_p \left(a^{\frac{p}{2}} + b^{\frac{p}{2}} \right)$$

with the constant

$$D_p = \begin{cases} 1, & \text{if } \frac{p}{2} \leq 1, \\ 2^{\frac{p}{2}-1}, & \text{if } \frac{p}{2} > 1, \end{cases}$$

Theorem 4.1 implies that

$$\begin{aligned}
& \left(\tau(X(t, x) - \hat{X}(t, x)) \right)^p \\
& \leq \left[\frac{4}{2\alpha - 1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) 2 \cdot 4^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \right. \\
& \quad \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{t(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) \\
& \quad + 16M(F(+\infty) - F(\Lambda)) \\
& \quad \left. + 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \frac{4}{(2\alpha - 1)M^{2\alpha-1}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right) \right]^{\frac{p}{2}} \\
& \leq D_p \left(\frac{4}{2\alpha - 1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) 2 \cdot 4^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \right. \\
& \quad \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{t(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) \\
& \quad \left. + 16M(F(+\infty) - F(\Lambda)) \right)^{\frac{p}{2}} \\
& \quad + D_p 2^{p(1-\alpha)} t^{p\alpha} \pi^{p\alpha} \left(\frac{4}{(2\alpha - 1)M^{2\alpha-1}} \right)^{\frac{p}{2}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right)^{\frac{p}{2}} \\
& \leq D_p^2 \left(\left(\frac{4}{2\alpha - 1} \right)^{\frac{p}{2}} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right)^{\frac{p}{2}} 2^{\frac{p}{2}} \cdot 4^{p(1-\alpha)} t^{p\alpha} \pi^{p\alpha} \right. \\
& \quad \times \left(\sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{t(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) \right)^{\frac{p}{2}} \\
& \quad \left. + 4^p M^{\frac{p}{2}} (F(+\infty) - F(\Lambda))^{\frac{p}{2}} \right)^{\frac{p}{2}} \\
& \quad + D_p 2^{p(1-\alpha)} t^{p\alpha} \pi^{p\alpha} \left(\frac{4}{(2\alpha - 1)M^{2\alpha-1}} \right)^{\frac{p}{2}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right)^{\frac{p}{2}} \\
& \leq D_p^2 \left(\frac{2^p D_p}{(2\alpha - 1)^{\frac{p}{2}}} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right)^{\frac{p}{2}} 2^{\frac{p}{2}} \cdot 4^{p(1-\alpha)} t^{p\alpha} \pi^{p\alpha} \right. \\
& \quad \times \left(\left(\sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} b_l^2 \right)^{\frac{p}{2}} \right. \\
& \quad \left. + t^{p\alpha} \left(\frac{1+C}{2} \right)^{p\alpha} \left(\sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right)^{\frac{p}{2}} \right. \\
& \quad \left. \left. + 4^p M^{\frac{p}{2}} (F(+\infty) - F(\Lambda))^{\frac{p}{2}} \right) \right)^{\frac{p}{2}}
\end{aligned}$$

$$+ D_p 2^{p(1-\alpha)} t^{p\alpha} \pi^{p\alpha} \left(\frac{4}{(2\alpha-1)M^{2\alpha-1}} \right)^{\frac{p}{2}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right)^{\frac{p}{2}}.$$

Therefore

$$\begin{aligned} & \left(\tau(X(t, x) - \hat{X}(t, x)) \right)^p \\ & \leq \left(\frac{2^p D_p^3}{(2\alpha-1)^{\frac{p}{2}}} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right)^{\frac{p}{2}} 2^{\frac{p}{2}} \cdot 4^{p(1-\alpha)} \pi^{p\alpha} \left(\sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} b_l^2 \right)^{\frac{p}{2}} \right. \\ & \quad \left. + D_p 2^{p(1-\alpha)} \pi^{p\alpha} \left(\frac{4}{(2\alpha-1)M^{2\alpha-1}} \right)^{\frac{p}{2}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right)^{\frac{p}{2}} \right) t^{p\alpha} \\ & + \frac{2^p D_p^3}{(2\alpha-1)^{\frac{p}{2}}} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right)^{\frac{p}{2}} 2^{\frac{p}{2}} \cdot 4^{p(1-\alpha)} \pi^{p\alpha} \left(\frac{1+C}{2} \right)^{p\alpha} \\ & \quad \times \left(\sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right)^{\frac{p}{2}} t^{2p\alpha} \\ & + 4^p D_p^2 M^{\frac{p}{2}} (F(+\infty) - F(\Lambda))^{\frac{p}{2}}. \end{aligned}$$

Then

$$\begin{aligned} & \int_0^T \int_0^{2\pi} \left(\tau(X(t, x) - \hat{X}(t, x)) \right)^p dx dt \\ & \leq \frac{T^{p\alpha+1}}{p\alpha+1} \left(\frac{2^p D_p^3}{(2\alpha-1)^{\frac{p}{2}}} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right)^{\frac{p}{2}} \right. \\ & \quad \times 2^{\frac{p}{2}+1} \cdot 4^{p(1-\alpha)} \pi^{p\alpha+1} \left(\sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} b_l^2 \right)^{\frac{p}{2}} \\ & \quad \left. + D_p 2^{p(1-\alpha)+1} \pi^{p\alpha+1} \left(\frac{4}{(2\alpha-1)M^{2\alpha-1}} \right)^{\frac{p}{2}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right)^{\frac{p}{2}} \right) \\ & + \frac{T^{2p\alpha+1}}{2p\alpha+1} \frac{2^p D_p^3}{(2\alpha-1)^{\frac{p}{2}}} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right)^{\frac{p}{2}} 2^{\frac{p}{2}+1} \cdot 4^{p(1-\alpha)} \pi^{p\alpha+1} \left(\frac{1+C}{2} \right)^{p\alpha} \\ & \quad \times \left(\sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \cdot \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right)^{\frac{p}{2}} \\ & + T \cdot 2^{2p+1} \pi D_p^2 M^{\frac{p}{2}} (F(+\infty) - F(\Lambda))^{\frac{p}{2}} \\ & = I. \end{aligned}$$

□

Corollary 5.1. *Let a partition $L = \{\lambda_0, \dots, \lambda_N\}$ of the set $[0, \infty)$ be such that $\lambda_l < \lambda_{l+1}$ and $\lambda_{l+1} - \lambda_l = \frac{\Lambda}{N-1}$. Then Theorem 5.2 holds with*

$$I = \left(\frac{\Lambda}{N-1} \right)^{p\alpha} \cdot A + \left(\frac{1}{M^{2\alpha-1}} \right)^{\frac{p}{2}} \cdot B + (F(+\infty) - F(\Lambda))^{\frac{p}{2}} \cdot H,$$

where

$$\begin{aligned}
 A &= 2^p D_p^3 \left(\frac{2\alpha}{2\alpha-1} \right)^{\frac{p}{2}} 2^{\frac{p}{2}+1} \cdot 4^{p(1-\alpha)} \pi^{p\alpha+1} \\
 &\quad \times \left(\frac{T^{p\alpha+1}}{p\alpha+1} + \left(\frac{3}{2} \right)^{p\alpha} \left(\int_0^\Lambda \lambda^{2\alpha} dF(\lambda) \right)^{\frac{p}{2}} \frac{T^{2p\alpha+1}}{2p\alpha+1} \right), \\
 B &= \frac{2^p D_p}{(2\alpha-1)^{\frac{p}{2}}} 2^{p(1-\alpha)+1} \pi^{p\alpha+1} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right)^{\frac{p}{2}} \cdot \frac{T^{p\alpha+1}}{p\alpha+1}, \\
 H &= 2^{2p+1} \cdot D_p^2 \cdot \pi \cdot M^{\frac{p}{2}} \cdot T.
 \end{aligned}$$

6. CONCLUDING REMARKS

A model for a homogeneous isotropic Gaussian random field is constructed. Bounds are found for the probability that a model approximates a random field with a given accuracy and reliability in the space $L_p(\mathbb{T})$, $p \geq 1$.

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