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RATE OF CONVERGENCE OF OPTION PRICES FOR APPROXIMATIONS OF THE GEOMETRIC ORNSTEIN–UHLENBECK PROCESS BY BERNOULLI JUMPS OF PRICES ON ASSETS

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Abstract. We consider the discrete approximation scheme for the price of an asset that is modeled by the geometric Ornstein–Uhlenbeck process. The approximation scheme corresponds to Euler type discrete-time approximations where the increments of the Wiener process are changed by independent identically distributed Bernoulli random variables. The rate of convergence of both objective and fair option prices is estimated by using the classical results on the rate of convergence to the normal law of the distribution functions of sums of identically distributed random variables. We analyze option prices and specific changes in a model where the martingale measure is used instead of the objective measure.

1. Introduction

A number of papers in finance mathematics is devoted to the convergence of models with discrete time to those with continuous time. This is explained by the fact that analytical evaluations are easier for models with continuous time as compared to models with discrete time, despite that the real financial operations are performed in discrete time. In doing so, a natural question is about the rate of convergence of option prices.

A variety of choices for pre-limit and limit models is known in the literature, however, the rate of convergence is studied mainly for binomial or trinomial pre-limit models and Black–Scholes limit model [1, 2, 4, 10] that follows from rather deep results on the rate of convergence of the binomial distribution to the standard Gaussian distribution. These results allow one to improve the rate of convergence up to $O(n^{-1})$, where n is the number of trade periods on a fixed time interval in the pre-limit model.

More general pre-limit as well as limit models are considered in the papers [5]–[9]. General conditions for the weak convergence of a sequence of stochastic processes with discrete time to a diffusion process are applied in [5] to study the weak convergence of discrete models of financial markets to diffusion models with continuous time. The Ornstein–Uhlenbeck process is viewed as the limit model (no rate of convergence of option prices is obtained in [5]). The case where a general martingale scheme with discrete time approximates the standard Black–Scholes model is considered in [6]. For this case, the rate of convergence of option prices is not slower than $O(n^{-1/8})$. Applying the asymptotic

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expansion of the distribution function, it is proved in [6] that the rate of convergence is at least $O\left(n^{-1/2}\right)$.

A discrete approximation scheme for the Ornstein–Uhlenbeck process is considered in [7]. This scheme is based on Euler approximations where increments of the Wiener process are changed by independent identically distributed bounded symmetric random variables with uniform distribution. It is proved in [7] that the rate of convergence for objective and fair option prices is at least $O(n^{-1/3})$. The paper [9] studies the rate of convergence of prices of put and call options if the risk assets in a model with discrete time weakly converge to the Black–Scholes model. The rate of convergence is of order $O(n^{-1})$, where n is the number of trading periods in a fixed time interval for the pre-limit model. This rate of convergence is achieved in [9] by using a bound for the rate of convergence in the central limit theorem for identically distributed random variables obtained in [8] with the help of pseudomoments. The main result of [9] follows by using the method of pseudomoments; however, a specific method of proof of that bound does not matter when obtaining the main result in [9]: the main conclusion remains the same even if the rate of convergence in that bound could be improved, since the proof in [9] is based on some extra bounds that obviously are of order $O(n^{-1})$ and therefore an improvment of the rate of convergence of option prices seems to be impossible for this model.

In the current paper, we use the following approach to estimate the rate of convergence of option prices. The limit model for the asset prices generated by a geometric Ornstein–Uhlenbeck process is introduced in Section 2.

Section 3 contains a description of properties of the pre-limit discrete price process. We consider a discrete approximation for the Ornstein–Uhlenbeck process based on Euler's approximations where increments of the Wiener process are changed by independent identically distributed Bernoulli random variables.

Sections 4 and 5 contain the main results of the paper. We provide some sufficient conditions under which the rate of convergence of objective and fair option prices are of order $O(n^{-1/2})$. We analyze the changes in the distribution of prices in the market when substituting the martingale measure for the objective measure.

2. Description and properties of the limit continuous price process

Let $T > 0$, $\mathbb{T} = [0,T]$, and let $\Omega_{\mathcal{F}} = (\Omega, \mathcal{F}, (\mathcal{F}_t, t \in \mathbb{T}), \mathsf{P})$ be a complete standard stochastic basis. Furthermore, let $W = \{W_t, \mathcal{F}_t, t \in \mathbb{T}\}$ be an adapted Wiener process. In this stochastic basis, consider an adapted Ornstein–Uhlenbeck process $X = \{X_t, \mathcal{F}_t, t \in \mathbb{T}\}$ with constant parameters. This Ornstein–Uhlenbeck process is a unique solution of the following stochastic differential equation:

$$dX_t = (\mu - X_t)\,dt + \sigma\,dW_t, \qquad X_0 = x_0 \in \mathbb{R}, \quad t \in \mathbb{T}, \tag{1}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. The process X can be written explicitly as follows:

$$X_t = x_0 e^{-t} + \mu\left(1 - e^{-t}\right) + \sigma e^{-t} \int_0^t e^s\,dW_s.$$

Finally, we assume that the price S_t of an asset is such that

$$S_t = \exp\left\{X_t - \frac{\sigma^2}{2} t\right\}, \qquad t \in \mathbb{T}, \tag{2}$$

where the nonrandom constant $-\frac{1}{2}\sigma^2 t$ is used to make further transformations easier.

It is proved in [5] that the market with the bond $B_t = e^{rt}$ and share S_t is arbitrage free and complete. Moreover, a unique probability measure $\mathsf{P}^* \sim \mathsf{P}$ is such that $Z_t := \frac{S_t}{B_t}$

is an $\mathcal{F}_t$-martingale with respect to P^*, possesses the Radon–Nikodym derivative $\frac{d\,\mathsf{P}^*}{d\,\mathsf{P}}\Big|_T$, where

$$\frac{d\,\mathsf{P}^*}{d\,\mathsf{P}}\bigg|_t = \exp\left\{-\int_0^t \frac{\mu - r - X_s}{\sigma}\,dW_s - \frac{1}{2}\int_0^t \frac{(\mu - r - X_s)^2}{\sigma^2}\,ds\right\},$$

and Z_t is represented with respect to P^* as

$$Z_t = \exp\left\{x_0 + \sigma\widetilde{W}_t - \frac{\sigma^2}{2}t\right\}$$

with some Wiener process $\widetilde{W}$.

3. Description and properties of the pre-limit discrete price process

We construct a discrete scheme that weakly converges to the geometric Ornstein–Uhlenbeck process (2). First we consider the following discrete approximation scheme for the Ornstein–Uhlenbeck process itself that is based on Euler approximations of a solution of stochastic differential equation (1) where increments of the Wiener process are changed by independent identically distributed Bernoulli random variables. More precisely, we assume that a sequence of probability spaces $(\Omega_n, \mathcal{F}_n, \mathsf{P}_n)$, $n \geq 1$, is given and let $\{q_k^{(n)}, n \geq 1, 0 \leq k \leq n\}$ be a sequence of independent identically distributed random variables such that $q_k^{(n)} = \pm\sqrt{T/n}$ with probability $\frac{1}{2}$.

Let $n > T$. Consider the recurrence scheme

$$x_0^{(n)} \in \mathbb{R}, \qquad R_k^{(n)} := x_k^{(n)} - x_{k-1}^{(n)} = \frac{\left(\mu - x_{k-1}^{(n)}\right)T}{n} + \sigma q_k^{(n)}, \quad 1 \leq k \leq n. \tag{3}$$

Let $\mathcal{F}_0^n = \{\varnothing, \Omega\}$ and $\mathcal{F}_k^n = \sigma\{R_i^{(n)}, 1 \leq i \leq k\}$. Put

$$X_t^n = x_0^{(n)}\mathbb{1}_{t<\frac{T}{n}} + \left(x_0^{(n)} + \sum_{1\leq k\leq\left[\frac{tn}{T}\right]} R_k^{(n)}\right)\mathbb{1}_{t\geq\frac{T}{n}} = x^{(n)}_{\left[\frac{tn}{T}\right]}.$$

Throughout the rest of the paper we use the convention that

$$\sum_{1\leq k\leq\left[\frac{tn}{T}\right]} = 0, \qquad \prod_{1\leq k\leq\left[\frac{tn}{T}\right]} = 1$$

for $t < \frac{T}{n}$. Then we construct the corresponding multiplicative scheme for the pre-limit price process as follows:

$$S_t^n = \exp\left\{x_0^{(n)}\right\}\prod_{1\leq k\leq\left[\frac{tn}{T}\right]}\left(1 + R_k^{(n)}\right), \qquad t \in \mathbb{T}. \tag{4}$$

The following results for the scheme (3)–(4) are proved in the paper [7].

Lemma 3.1. *Let the sequence $x_0^{(n)}$ be bounded. Then*

(i) *there exist a positive integer number $n_0 \in \mathbb{N}$ and a constant $C > 0$ that does not depend on n such that*

$$\left|R_k^{(n)}\right| \leq \frac{C}{\sqrt{n}} < 1$$

for all $n > n_0$ and all $1 \leq k \leq n$;

(ii) *there exists a constant $C > 0$ that does not depend on n and such that*

$$\mathsf{E}\left(x_k^{(n)}\right)^4 \leq C$$

for $n > T$ and $1 \leq k \leq n$.

As shown in the paper [5], if $q_k^{(n)}$ is a Bernoulli random variable, then the financial market with the bond

$$B_t^n = \prod_{1\le k\le [\frac{nt}{T}]} \left(1 + r_k^{(n)}\right)$$

and share S_t^n defined by (4) is arbitrage free and complete under the following additional assumptions that

$$r_k^{(n)} = o\left(n^{-1/2}\right)$$

and $\left|x_0^{(n)}\right| \le C$.

It is also proved in [5] that there exists a unique equivalent martingale measure $\mathsf{P}^{n,*}$, $\mathsf{P}^{n,*} \sim \mathsf{P}^n$, whose Radon–Nikodym derivative is given by

$$\frac{d\,\mathsf{P}^{n,*}}{d\,\mathsf{P}^n} = \prod_{k=1}^{n} \left(1 + \rho_{k-1}^{(n)} q_k^{(n)}\right), \tag{5}$$

where the random variables $\rho_{k-1}^{(n)}$ are such that

$$\rho_{k-1}^{(n)} = \frac{n r_k^{(n)} - \left(\mu - x_{k-1}^{(n)}\right) T}{\sigma T} \tag{6}$$

and

$$x_k^{(n)} - \mu = \left(x_0^{(n)} - \mu\right)\left(1 - \frac{T}{n}\right)^k + \sigma \sum_{i=1}^{k} q_i^{(n)} \left(1 - \frac{T}{n}\right)^{k-i}. \tag{7}$$

We collect the above observations in the following theorem.

Theorem 3.1. *Let $\left\{q_k^{(n)}, n \ge 1, 0 \le k \le n\right\}$ be a sequence of independent identically distributed random variables such that $q_k^{(n)} = \pm\sqrt{T/n}$ with probability $\frac{1}{2}$ and*

$$r_k^{(n)} = o\big(n^{-1/2}\big), \qquad \big|x_0^{(n)}\big| \le C.$$

Then the market (B_t^n, S_t^n) is asymptotically arbitrage free which means that there exists a positive integer number $n_0 \in \mathbb{N}$ such that the market (B_t^n, S_t^n) is arbitrage free for every integer number $n \ge n_0$.

If $n \ge n_0$, then the market (B_t^n, S_t^n) is complete and the Radon–Nikodym derivative of a unique equivalent martingale measure $\mathsf{P}^{n,*}$ is given by relations (5)–(6).

Now we are going to prove that

$$\sum_{1\le k\le n} \left(R_k^{(n)}\right)^2 \to \sigma^2 T$$

in $L_2(\mathsf{P})$ and find the corresponding rate of convergence.

Lemma 3.2. *For $n > T$,*

$$\mathsf{E}\bigg(\sum_{1\le k\le n} \left(R_k^{(n)}\right)^2 - \sigma^2 T\bigg)^2 \le \frac{C}{n^2}.$$

Proof. We start with an obvious observation. Using the equality

$$\left(q_k^{(n)}\right)^2 = T/n,$$

we conclude that

$$\begin{aligned}
\Sigma_n &:= \mathsf{E}\Bigg(\sum_{1\le k\le n}\Big(R_k^{(n)}\Big)^2-\sigma^2T\Bigg)^2\\
&= \mathsf{E}\left(\sum_{1\le k\le n}\left(\frac{\Big(\mu-x_{k-1}^{(n)}\Big)T}{n}+\sigma q_k^{(n)}\right)^2-\sigma^2T\right)^2\\
&= \mathsf{E}\Bigg(\sum_{1\le k\le n}\left(\frac{\Big(\mu-x_{k-1}^{(n)}\Big)T}{n}\right)^2\\
&\qquad +2\sigma\sum_{1\le k\le n}\frac{\Big(\mu-x_{k-1}^{(n)}\Big)T}{n}q_k^{(n)}+\sigma^2\sum_{1\le k\le n}\Big(q_k^{(n)}\Big)^2-\sigma^2T\Bigg)^2\\
&= \mathsf{E}\left(\sum_{1\le k\le n}\left(\frac{\Big(\mu-x_{k-1}^{(n)}\Big)T}{n}\right)^2+2\sigma\sum_{1\le k\le n}\frac{\Big(\mu-x_{k-1}^{(n)}\Big)T}{n}q_k^{(n)}\right)^2\\
&\le 3\,\mathsf{E}\left(\sum_{1\le k\le n}\left(\frac{\Big(\mu-x_{k-1}^{(n)}\Big)T}{n}\right)^2\right)^2+12\sigma^2\,\mathsf{E}\left(\sum_{1\le k\le n}\frac{\Big(\mu-x_{k-1}^{(n)}\Big)T}{n}q_k^{(n)}\right)^2.
\end{aligned}$$

Taking into account Lemma 3.1 and the inequality

$$\Bigg(\sum_{1\le k\le n}a_k\Bigg)^2\le n\sum_{1\le k\le n}a_k^2,$$

we get

$$\begin{aligned}
\mathsf{E}\left(\sum_{1\le k\le n}\left(\frac{\Big(\mu-x_{k-1}^{(n)}\Big)T}{n}\right)^2\right)^2 &\le n\sum_{1\le k\le n}\mathsf{E}\left(\frac{\Big(\mu-x_{k-1}^{(n)}\Big)T}{n}\right)^4\\
&\le Cn^2n^{-4}\le\frac{C}{n^2}.
\end{aligned}$$

Recalling that the random variables $q_k^{(n)}$ are independent, we deduce from Lemma 3.1 that

$$\mathsf{E}\Big(\mu-x_{k-1}^{(n)}\Big)^2\le C,$$

whence

$$\begin{aligned}
\mathsf{E}\left(\sum_{1\le k\le n}\frac{\Big(\mu-x_{k-1}^{(n)}\Big)T}{n}q_k^{(n)}\right)^2 &= \sum_{1\le k\le n}\mathsf{E}\left(\frac{\Big(\mu-x_{k-1}^{(n)}\Big)T}{n}q_k^{(n)}\right)^2\\
&= \sum_{1\le k\le n}\mathsf{E}\left(\frac{\Big(\mu-x_{k-1}^{(n)}\Big)T}{n}\right)^2\mathsf{E}\Big(q_k^{(n)}\Big)^2\\
&\le Cnn^{-3}\le Cn^{-2}.
\end{aligned}$$

The lemma is proved. □

4. Main theorem on the rate of convergence of objective option prices in the Bernoulli scheme

Denote by $\mathbf{C}_n$ and $\mathbf{C}$ a standard call option and by $\mathbf{P}_n$ and $\mathbf{P}$ the standard put option for pre-limit and limit assets, respectively. Denote by $K \geq 0$ the strike price and by T the expiration date. Denote the corresponding discounted objective prices by $\pi(\mathbf{C}_n)$, $\pi(\mathbf{C})$, $\pi(\mathbf{P}_n)$, and $\pi(\mathbf{P})$, and the fair prices by $\pi^*(\mathbf{C}_n)$, $\pi^*(\mathbf{C})$, $\pi^*(\mathbf{P}_n)$, and $\pi^*(\mathbf{P})$, respectively.

For the sake of simplicity, assume that the price of a bond for the pre-limit model is equal to

$$B_t^{(n)} = \left(1 + \frac{rT}{n}\right)^{[tn/T]},$$

while the limit price of a bond equals $B_t = e^{rt}$. Then

$$\pi(\mathbf{C}_n) = \mathsf{E}\left(\prod_{1\le k\le n}\left(1+R_k^{(n)}\right) - K\right)^+ \left(1+\frac{rT}{n}\right)^{-n}, \qquad n \geq 1,$$

$$\pi(\mathbf{C}) = \mathsf{E}\left(\exp\left\{X_T - \frac{1}{2}\sigma^2 T\right\} - K\right)^+ e^{-rT},$$

$$\pi(\mathbf{P}_n) = \mathsf{E}\left(K - \prod_{1\le k\le n}\left(1+R_k^{(n)}\right)\right)^+ \left(1+\frac{rT}{n}\right)^{-n}, \qquad n \geq 1,$$

$$\pi(\mathbf{P}) = \mathsf{E}\left(K - \exp\left\{X_T - \frac{1}{2}\sigma^2 T\right\}\right)^+ e^{-rT}.$$

Theorem 4.1. *Assume that*

(i)
$$\left|x_0^{(n)} - x_0\right| \leq \frac{C_0}{n^{1/2}}$$
with some constant $C_0 > 0$;

(ii) *independent identically distributed random variables* $q_k^{(n)}$ *assume values* $\pm\sqrt{T/n}$ *with probability* $\frac{1}{2}$.

Then, starting with some $n_0 \in \mathbb{N}$,

$$|\pi(\mathbf{D}) - \pi(\mathbf{D}_n)| \leq \frac{C_1}{n^{1/2}} \tag{8}$$

for some $C_1 > 0$ *and* $\mathbf{D} = \mathbf{C}, \mathbf{P}$.

Proof. We consider the put options, since their payoff function is bounded. Then the proof for call options follows from the result for put options in view of the put-call parity. To make further reasoning simpler we assume that $x_0^{(n)} = x_0 = 1$. We are going to prove an upper bound for the difference of prices

$$|\pi(\mathbf{P}) - \pi(\mathbf{P}_n)| = \Bigg|\mathsf{E}\left(K - \prod_{1\le k\le n}\left(1+R_k^{(n)}\right)\right)^+ \left(1+\frac{rT}{n}\right)^{-n} - \mathsf{E}\left(K - \exp\left\{X_T - \frac{1}{2}\sigma^2 T\right\}\right)^+ e^{-rT}\Bigg|.$$

Applying Lemma A.2 of [7], we get after simple algebra that

$$
\begin{aligned}
&|\pi(\mathbf{P}) - \pi(\mathbf{P}_n)| \\
&\quad \le \left(1+\frac{rT}{n}\right)^{-n} \Bigg| \mathsf{E}\left(K - \prod_{1\le k\le n}\left(1+R_k^{(n)}\right)\right)^+ \\
&\qquad\qquad - \mathsf{E}\left(K - \exp\left\{X_T - \frac{1}{2}\sigma^2 T\right\}\right)^+ \Bigg| \\
&\quad + \mathsf{E}\left(K - \exp\left\{X_T - \frac{1}{2}\sigma^2 T\right\}\right)^+ \left| \left(1+\frac{rT}{n}\right)^{-n} - e^{-rT} \right| \qquad (9) \\
&\quad \le \left| \mathsf{E}\left(K - \prod_{1\le k\le n}\left(1+R_k^{(n)}\right)\right)^+ - \mathsf{E}\left(K - \exp\left\{X_T - \frac{1}{2}\sigma^2 T\right\}\right)^+ \right| \\
&\quad + \frac{K(rT)^2}{2n}.
\end{aligned}
$$

Integrating by parts we conclude that

$$
\mathsf{E}(K-\xi)^+ = \int_{-\infty}^{K} \mathsf{P}(\xi \le x)\, dx
$$

for all integrable random variables ξ. Thus,

$$
\begin{aligned}
&\left| \mathsf{E}\left(K - \prod_{1\le k\le n}\left(1+R_k^{(n)}\right)\right)^+ - \mathsf{E}\left(K - \exp\left\{X_T - \frac{1}{2}\sigma^2 T\right\}\right)^+ \right| \\
&\quad = \left| \int_0^K \left(\mathsf{P}\left(\exp\left\{X_T - \frac{1}{2}\sigma^2 T\right\} \le z\right) - \mathsf{P}\left(\prod_{1\le k\le n}\left(1+R_k^{(n)}\right) \le z\right)\right) dz \right| \\
&\quad = \Bigg| \int_0^K \Bigg(\mathsf{P}\left(X_T - \frac{1}{2}\sigma^2 T \le \log z\right) \\
&\qquad\qquad - \mathsf{P}\left(\log\left(\prod_{1\le k\le n}\left(1+R_k^{(n)}\right)\right) \le \log z\right)\Bigg) dz \Bigg| \qquad (10) \\
&\quad \le \Bigg| \int_0^K \Bigg(\mathsf{P}\left(X_T - \frac{1}{2}\sigma^2 T \le \log z\right) \\
&\qquad\qquad - \mathsf{P}\left(\sum_{1\le k\le n} R_k^{(n)} - \frac{1}{2}\sum_{1\le k\le n}\left(R_k^{(n)}\right)^2 \le \log z\right)\Bigg) dz \Bigg| \\
&\quad + \Bigg| \int_0^K \Bigg(\mathsf{P}\left(\sum_{1\le k\le n} \log\left(1+R_k^{(n)}\right) \le \log z\right) \\
&\qquad\qquad - \mathsf{P}\left(\sum_{1\le k\le n} R_k^{(n)} - \frac{1}{2}\sum_{1\le k\le n}\left(R_k^{(n)}\right)^2 \le \log z\right)\Bigg) dz \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&= \left| \int_{-\infty}^{\log K} e^y \left(\mathsf{P}\left(X_T - \frac{1}{2}\sigma^2 T \le y \right) \right.\right. \\
&\qquad\qquad \left.\left. - \mathsf{P}\left(\sum_{1\le k\le n} R_k^{(n)} - \frac{1}{2} \sum_{1\le k\le n} \left(R_k^{(n)}\right)^2 \le y \right)\right) dy \right| \\
&\quad + \left| \int_{-\infty}^{\log K} e^y \left(\mathsf{P}\left(\sum_{1\le k\le n} \log\left(1 + R_k^{(n)}\right) \le y \right) \right.\right. \\
&\qquad\qquad \left.\left. - \mathsf{P}\left(\sum_{1\le k\le n} R_k^{(n)} - \frac{1}{2} \sum_{1\le k\le n} \left(R_k^{(n)}\right)^2 \le y \right)\right) dy \right| \\
&=: I_1^n + I_2^n.
\end{aligned}
$$

We rewrite both probabilities in the latter inequality in order to estimate I_1^n from above. Put

$$
D(y) = \frac{\sqrt{2}\left(y - \mu\left(1 - e^{-T}\right) - x_0 e^{-T} + \frac{\sigma^2 T}{2}\right)}{\sigma\sqrt{1 - e^{-2T}}}.
$$

It is obvious that

$$
\mathsf{P}\left(X_T - \frac{1}{2}\sigma^2 T \le y \right) = \Phi\left(D(y)\right).
$$

Considering the inequality

$$
|\Phi(x) - \Phi(y)| \le |x - y|/\sqrt{2\pi}
$$

for the standard normal distribution function, we get

$$
\begin{aligned}
&\left| \mathsf{P}\left(\sum_{1\le k\le n} R_k^{(n)} - \frac{1}{2}\sum_{1\le k\le n}\left(R_k^{(n)}\right)^2 \le y \right) - \mathsf{P}\left(\sum_{1\le k\le n} R_k^{(n)} - \frac{\sigma^2 T}{2} \le y \right) \right| \\
&\quad \le \mathsf{P}\left(\left| \sum_{1\le k\le n} \left(R_k^{(n)}\right)^2 - \sigma^2 T \right| > \frac{2}{n^{1/2}} \right) \\
&\qquad + \left| \mathsf{P}\left(\sum_{1\le k\le n} R_k^{(n)} - \frac{\sigma^2 T}{2} \le y \right) - \mathsf{P}\left(\sum_{1\le k\le n} R_k^{(n)} - \frac{\sigma^2 T}{2} \le y \pm \frac{1}{n^{1/2}} \right) \right| \\
&\quad \le \mathsf{P}\left(\left| \sum_{1\le k\le n} \left(R_k^{(n)}\right)^2 - \sigma^2 T \right| > \frac{2}{n^{1/2}} \right) \\
&\qquad + \left| \mathsf{P}\left(\sum_{1\le k\le n} R_k^{(n)} - \frac{\sigma^2 T}{2} \le y \right) - \Phi(D(y)) \right| \\
&\qquad + \left| \mathsf{P}\left(\sum_{1\le k\le n} R_k^{(n)} - \frac{\sigma^2 T}{2} \le y \pm \frac{1}{n^{1/2}} \right) - \Phi\left(D\left(y \pm \frac{1}{n^{1/2}}\right)\right) \right| + \frac{C}{n^{1/2}} \\
&\quad =: J_1^n + J_2^n + J_3^n + \frac{C}{n^{1/2}}.
\end{aligned} \tag{11}
$$

It follows from (11) that

$$\begin{aligned}
&\left|\mathsf{P}\left(X_T-\frac{1}{2}\sigma^2T\le y\right)-\mathsf{P}\left(\sum_{1\le k\le n}R_k^{(n)}-\frac{1}{2}\sum_{1\le k\le n}\left(R_k^{(n)}\right)^2\le y\right)\right|\\
&\qquad\le\left|\Phi(D(y))-\mathsf{P}\left(\sum_{1\le k\le n}R_k^{(n)}-\frac{1}{2}\sigma^2T\le y\right)\right|+J_1^n+J_2^n+J_3^n+\frac{C}{n^{1/2}}\\
&\qquad\le J_1^n+2J_2^n+J_3^n+\frac{C}{n^{1/2}}.
\end{aligned}$$

Now Lemma 3.2 implies

$$J_1^n\le Cn\,\mathsf{E}\left(\sum_{1\le k\le n}\left(R_k^{(n)}\right)^2-\sigma^2T\right)^2\le\frac{C}{n}.$$

The reasoning needed to establish bounds for J_i^n, $i=2,3$, is the same, so that we treat the case of J_2^n only. Put $X_k=\sqrt{n}q_k^{(n)}(1-T/n)^{n-k}$ and

$$B_n=\sum_{1\le k\le n}\mathsf{E}\,X_k^2=\frac{n\left(1-\left(1-\frac{T}{n}\right)^{2n}\right)}{2-\frac{T}{n}}.$$

Hence, definition (3) yields

$$x_k^{(n)}=x_0^{(n)}\left(1-\frac{T}{n}\right)^k+\mu\left(1-\left(1-\frac{T}{n}\right)^k\right)+\sigma\sum_{i=1}^k q_i^{(n)}\left(1-\frac{T}{n}\right)^{k-i}.$$

For $k=n$, we obtain

$$J_2^n=\left|\mathsf{P}\left(B_n^{-1/2}\sum_{1\le k\le n}X_k\le D_n(y)\right)-\Phi(D(y))\right|,$$

where

$$D_n(y)=\frac{\sqrt{2-\frac{T}{n}}\left(y-(\mu-x_0)\left(1-\left(1-\frac{T}{n}\right)^n\right)+\frac{\sigma^2T}{2}\right)}{\sigma\sqrt{1-\left(1-\frac{T}{n}\right)^{2n}}}.$$

Similarly to the proof of Lemma A.2 of [7], one can see that $|D(y)-D_n(y)|\le(C+|y|)/\sqrt{n}$. Moreover, applying the Berry–Esseen inequality we conclude that

$$J_2^n\le\left|\mathsf{P}\left(B_n^{-1/2}\sum_{1\le k\le n}X_k\le D_n(y)\right)-\Phi(D_n(y))\right|+\frac{C+|y|}{\sqrt{n}}\le\frac{C+|y|}{\sqrt{n}}.$$

Then I_1^n admits the bound

$$I_1^n\le\int_{-\infty}^{\log K}e^y\left(\frac{C}{n}+\frac{C+|y|}{\sqrt{n}}\right)dy\le\frac{C}{\sqrt{n}}. \tag{12}$$

To estimate I_2^n from above, we note that

$$\sum_{1\le k\le n}\log\left(1+R_k^{(n)}\right)=\sum_{1\le k\le n}R_k^{(n)}-\frac{1}{2}\sum_{1\le k\le n}\left(R_k^{(n)}\right)^2+\frac{1}{3}\alpha_n\sum_{1\le k\le n}\left(R_k^{(n)}\right)^3.$$

Then we obtain from Lemma 3.1 that $\sum_{1\le k\le n}\left|R_k^{(n)}\right|^3 \le C^3/\sqrt{n}$ and thus Taylor's expansion provides a bound for α_n:

$$|\alpha_n| \le \frac{1}{\left(1-\max_{1\le k\le n}\left|R_k^{(n)}\right|\right)^3} \le \frac{1}{\left(1-\frac{C}{\sqrt{n}}\right)^3} \le 8,$$

where n is such that $C/\sqrt{n} \le 1/2$. Then using inequality (11) we get

$$\begin{aligned}
&\left|\mathsf{P}\left(\sum_{1\le k\le n}\log\left(1+R_k^{(n)}\right)\le y\right)-\mathsf{P}\left(\sum_{1\le k\le n}R_k^{(n)}-\frac12\sum_{1\le k\le n}\left(R_k^{(n)}\right)^2\le y\right)\right|\\
&\quad\le\left|\mathsf{P}\left(\sum_{1\le k\le n}R_k^{(n)}-\frac12\sum_{1\le k\le n}\left(R_k^{(n)}\right)^2\le y\right)\right.\\
&\qquad\qquad\left.-\mathsf{P}\left(\sum_{1\le k\le n}R_k^{(n)}-\frac12\sum_{1\le k\le n}\left(R_k^{(n)}\right)^2\le y\pm\frac{C}{\sqrt{n}}\right)\right|\\
&\quad\le\left|\mathsf{P}\left(\sum_{1\le k\le n}R_k^{(n)}-\frac12\sum_{1\le k\le n}\left(R_k^{(n)}\right)^2\le y\right)-\Phi(D(y))\right|\\
&\qquad+\left|\mathsf{P}\left(\sum_{1\le k\le n}R_k^{(n)}-\frac12\sum_{1\le k\le n}\left(R_k^{(n)}\right)^2\le y\pm\frac{C}{\sqrt{n}}\right)-\Phi\left(D\left(y\pm\frac{C}{\sqrt{n}}\right)\right)\right|\\
&\qquad+\left|\Phi(D(y))-\Phi\left(D\left(y\pm\frac{C}{\sqrt{n}}\right)\right)\right|\le\frac{C+|y|}{\sqrt{n}}.
\end{aligned}$$

Similarly to the proof of (12), one can prove that $I_2^n \le C/\sqrt{n}$. Finally, (9), (10), (12), and bounds obtained above yield inequality (8) and this completes the proof. □

Remark 4.1. A sequence of independent random variables $\left\{q_k^{(n)}, n\ge 1, 0\le k\le n\right\}$ with the uniform distribution in the interval $(-\sqrt{3T/n}, \sqrt{3T/n})$ is considered in the paper [7]. The bound

$$\mathsf{E}\left(\sum_{1\le k\le n}\left(R_k^{(n)}\right)^2-\sigma^2T\right)^2\le\frac{C}{n^2}+Cn\left(\mathsf{E}\left(q_1^{(n)}\right)^4-\frac{T^2}{n^2}\right)$$

is used in [7] to estimate the integral $J_1^{(n)}$ similarly to the proof of Theorem 4.1 above. An extra property $n\big(\mathsf{E}\big(q_1^{(n)}\big)^4-T^2/n^2\big)=4T^2/(5n)$ used in [7] is of a slower order of decrease to zero as compared to the term $O\left(n^{-2}\right)$ and thus the rate of convergence of option prices is of order $O\left(n^{-1/3}\right)$ in [7].

In the current paper, $q_k^{(n)}$ are independent identically distributed random variables assuming values $\pm\sqrt{T/n}$ with probability $\frac12$. This yields $\mathsf{E}\big(q_1^{(n)}\big)^4-T^2/n^2=0$ and, as a result, a better rate of convergence of option prices of order $n^{-1/2}$.

5. Changing the objective measure by martingale measure and a result on the rate of convergence of fair prices of options

In the preceding section, we obtained the rate of convergence of objective option prices under the assumption that $q_k^{(n)}$ assume values $\pm\sqrt{T/n}$ with probability $\frac12$ with respect to the objective measure. If one wishes to get a rate of convergence of the same order with respect to the martingale measure, then one needs to ensure that the probabilities for the joint distribution $\mathsf{P}_n\big(\bigcap_{k=1}^n\{q_k^{(n)}=\pm\sqrt{T/n}\}\big)$ are such that the random variables $q_k^{(n)}$ are jointly independent and assume values $\pm\sqrt{T/n}$ with probability $\frac{1}{2}$ with respect to the martingale measure P_n^*. In contrast to the results of the paper [9] where the independence of random factors is preserved when passing from the objective measure to martingale measure in the discrete Black–Scholes model, our case here does not possess such a property. Therefore, we first consider the model of price process (3)–(4) without the assumption on the joint independence of the random variables $\{q_k^{(n)}, 1\le k\le n\}$.

Let $\mathsf{P}_{k,n}^{\pm}=\mathsf{P}_n\big(q_k^{(n)}=\pm\sqrt{T/n}\mid \mathcal{F}_{k-1}^n\big)$. If the independence is not assumed, properties of the pre-limit model depend essentially on the behavior of $\mathsf{P}_{k,n}^{\pm}$. This is explained in the next result. Note that $\mathsf{P}_{k,n}^{+}+\mathsf{P}_{k,n}^{-}=1$.

Put

$$h_{k,n}^{\pm}=\frac{\left(\mu-x_{k-1}^{(n)}\right)T}{n}\pm\sigma\sqrt{\frac{T}{n}}$$

and

$$\rho_{k,n}=\frac{r_k^{(n)}-h_{k,n}^{+}\,\mathsf{P}_{k,n}^{+}-h_{k,n}^{-}\,\mathsf{P}_{k,n}^{-}}{4\sigma\frac{T}{n}\,\mathsf{P}_{k,n}^{+}\,\mathsf{P}_{k,n}^{-}}.$$

Theorem 5.1. (i) *Let every series be such that, for $n>T$:*

(*a*) $\mathsf{P}_{k,n}^{\pm}>0$ *with probability one and* $\mathsf{E}\big|\rho_{k,n}\big(q_k^{(n)}-\mathsf{E}\big(q_k^{(n)}\mid\mathcal{F}_{k-1}^n\big)\big)\big|<\infty$ *for* $1\le k\le n$;

(*b*) *there exists a constant $C>0$ that does not depend on k and n for which*

$$\left|2\,\mathsf{P}_{k,n}^{+}-1\right|<\frac{C}{n^{1/2}},\qquad r_k^{(n)}\le\frac{C}{n},\qquad \left|x_0^{(n)}-x_0\right|\le C,$$
$$1\le k\le n.$$

Then there exists a number of a series $n_0>T$ for which market (3)–(4) *is arbitrage free and complete starting with this number.*

(ii) *Assume that, for some number $n>T$ of a series, $\mathsf{P}_{k,n}^{\pm}>0$ with probability one for $1\le k\le n$, and there exists k such that*

$$\mathsf{E}\left|\rho_{k,n}\left(q_k^{(n)}-\mathsf{E}\left(q_k^{(n)}\mid\mathcal{F}_{k-1}^n\right)\right)\right|=\infty.$$

Then there is no equivalent martingale measure and thus the market is not arbitrage free (the question on completeness is not discussed for this case).

(iii) *Let $\mathsf{P}_{k,n}^{+}=0$ with positive probability or $\mathsf{P}_{k,n}^{-}=0$ with positive probability.*

(*c*) *If*

$$h_{k,n}^{+}=r_k^{(n)}\tag{13}$$

in the set $A_{k,n}^{+}:=\{\omega\in\Omega\colon\ \mathsf{P}_{k,n}^{+}=0\}$ provided that $\mathsf{P}_{k,n}^{+}=0$ with positive probability, or

$$h_{k,n}^{-}=r_k^{(n)}\tag{14}$$

in the set $A_{k,n}^{-}:=\{\omega\in\Omega\colon\ \mathsf{P}_{k,n}^{-}=0\}$ provided that $\mathsf{P}_{k,n}^{-}=0$ with positive probability, then

$$|2\,\mathsf{P}_{k,n}^{+}-1|<C/n^{1/2}$$

in the set $\Omega\setminus A_{k,n}^{+}$, while

$$|2\,\mathsf{P}_{k,n}^{-}-1|<C/n^{1/2}$$

in the set $\Omega \setminus A^-_{k,n}$*. If, in addition,* $\mathsf{E}\big|\rho_{k,n}\big(q^{(n)}_k - \mathsf{E}\big(q^{(n)}_k|\mathcal{F}^n_{k-1}\big)\big)\big| < \infty$ *for* $1 \le k \le n$*, then the market is arbitrage free and incomplete.*

(*d*) *The market is not arbitrage free if* $\mathsf{P}^+_{k,n} = 0$ *with positive probability, and equality* (13) *does not hold in the set* $A^+_{k,n}$*, or if* $\mathsf{P}^-_{k,n} = 0$ *with positive probability, and equality* (14) *does not hold in the set* $A^-_{k,n}$*.*

Proof. According to the general theory of financial markets in the discrete time (see, for example, [3]), a martingale measure P^*_n for the pre-limit market is a probability measure whose Radon–Nikodym derivative is given by

$$\frac{d\,\mathsf{P}^{n,*}}{d\,\mathsf{P}^n} = \prod_{k=1}^{n}\left(1 + \Delta M^{(n)}_k\right), \tag{15}$$

where $M^{(n)} = \left\{M^{(n)}_k, 0 \le k \le n\right\}$ is a martingale with respect to the objective measure. In this case, the random variables $\Delta M^{(n)}_k = M^{(n)}_k - M^{(n)}_{k-1}$ are measurable with respect to the σ-algebra $\mathcal{F}^n_k$ and thus there exists a Borel function $f(x_1, x_2, \dots, x_k)$ such that

$$\begin{aligned}\Delta M^{(n)}_k &= f\left(q^{(n)}_1, q^{(n)}_2, \dots, q^{(n)}_k\right)\\ &:= f\left(\overline{q}^{(n)}_{k-1}, q^{(n)}_k\right) = f\left(\overline{q}^{(n)}_{k-1}, \sqrt{\frac{T}{n}}\right)\mathbb{1}_{k,n,+} + f\left(\overline{q}^{(n)}_{k-1}, -\sqrt{\frac{T}{n}}\right)\mathbb{1}_{k,n,-},\end{aligned}$$

where $\mathbb{1}_{k,n,\pm} = \mathbb{1}_{\{q^{(n)}_k = \pm\sqrt{\frac{T}{n}}\}}$.

Let $g^{\pm}_{k,n} = f\big(\overline{q}^{(n)}_{k-1}, \pm\sqrt{T/n}\big)\,\mathsf{P}^{\pm}_{k,n}$. Then the condition that the process $M^{(n)}$ is a martingale is rewritten as follows:

$$g^+_{k,n} + g^-_{k,n} = 0. \tag{16}$$

Now we provide the condition that the discounted price process is a martingale with respect to the measure $\mathsf{P}^{n,*}$: For all $1 \le k \le n$,

$$\mathsf{E}_{\mathsf{P}^{n,*}}\left(\prod_{i=1}^{k}\frac{1+R^{(n)}_i}{1+r^{(n)}_i}\;\Big|\;\mathcal{F}^n_{k-1}\right) = \prod_{i=1}^{k-1}\frac{1+R^{(n)}_i}{1+r^{(n)}_i}.$$

Using the standard equality

$$\mathsf{E}_{\mathsf{Q}}(\xi \mid G) = \frac{\mathsf{E}_{\mathsf{P}}\big(\frac{d\mathsf{Q}}{d\,\mathsf{P}}\xi \mid G\big)}{\mathsf{E}_{\mathsf{P}}\big(\frac{d\mathsf{Q}}{d\,\mathsf{P}} \mid G\big)}$$

we rewrite the above condition in the following form:

$$\frac{\mathsf{E}\left(\prod_{j=1}^n\left(1+\Delta M^{(n)}_j\right)\prod_{i=1}^k\frac{1+R^{(n)}_i}{1+r^{(n)}_i}\;\Big|\;\mathcal{F}^n_{k-1}\right)}{\mathsf{E}\left(\prod_{j=1}^n\left(1+\Delta M^{(n)}_j\right)\;\Big|\;\mathcal{F}^n_{k-1}\right)} = \prod_{i=1}^{k-1}\frac{1+R^{(n)}_i}{1+r^{(n)}_i}$$

or

$$\mathsf{E}\left(\left(1+\Delta M^{(n)}_k\right)\left(1+R^{(n)}_k\right)\;\Big|\;\mathcal{F}^n_{k-1}\right) = 1 + r^{(n)}_k.$$

The latter relation is equivalent to the equality

$$\mathsf{E}\left(R^{(n)}_k\left(1+\Delta M^{(n)}_k\right)\;\Big|\;\mathcal{F}^n_{k-1}\right) = r^{(n)}_k. \tag{17}$$

Recalling the definition of all terms on the left-hand side of this equality and taking into account recurrent scheme (3), we obtain

$$h^+_{k,n}g^+_{k,n} + h^-_{k,n}g^-_{k,n} + h^+_{k,n}\,\mathsf{P}^+_{k,n} + h^-_{k,n}\,\mathsf{P}^-_{k,n} = r^{(n)}_k.$$

Combining this result with equality (16) we get a system of two linear equations with two unknowns, $g^{+}_{k,n}$ and $g^{-}_{k,n}$. A solution of this system exists, is unique, and can be written as follows:

$$g^{+}_{k,n} = \frac{r^{(n)}_{k} - h^{+}_{k,n}\,\mathsf{P}^{+}_{k,n} - h^{-}_{k,n}\,\mathsf{P}^{-}_{k,n}}{2\sigma\sqrt{\frac{T}{n}}}, \qquad g^{-}_{k,n} = -g^{+}_{k,n}. \tag{18}$$

Now we distinguish between the following three cases.

(i) If $\mathsf{P}^{+}_{k,n} > 0$ and $\mathsf{P}^{-}_{k,n} > 0$ with probability one, then we get a unique formula for $\Delta M^{(n)}_{k}$ of the form

$$\begin{aligned} \Delta M^{(n)}_{k} &= f\left(\overline{q}^{(n)}_{k-1}, \sqrt{\frac{T}{n}}\right) \mathbb{1}_{k,n,+} + f\left(\overline{q}^{(n)}_{k-1}, -\sqrt{\frac{T}{n}}\right) \mathbb{1}_{k,n,-} \\ &= \frac{r^{(n)}_{k} - h^{+}_{k,n}\,\mathsf{P}^{+}_{k,n} - h^{-}_{k,n}\,\mathsf{P}^{-}_{k,n}}{2\sigma\sqrt{\frac{T}{n}}} \left(\frac{\mathbb{1}_{k,n,+}}{\mathsf{P}^{+}_{k,n}} - \frac{\mathbb{1}_{k,n,-}}{\mathsf{P}^{-}_{k,n}}\right). \end{aligned} \tag{19}$$

Recalling the notation,

$$\rho_{k,n} = \frac{r^{(n)}_{k} - h^{+}_{k,n}\,\mathsf{P}^{+}_{k,n} - h^{-}_{k,n}\,\mathsf{P}^{-}_{k,n}}{4\sigma\frac{T}{n}\,\mathsf{P}^{+}_{k,n}\,\mathsf{P}^{-}_{k,n}},$$

we rewrite equality (19) as follows:

$$\Delta M^{(n)}_{k} = \rho_{k,n}\left(q^{(n)}_{k} - \mathsf{E}\left(q^{(n)}_{k} \,\middle|\, \mathcal{F}^{n}_{k-1}\right)\right). \tag{20}$$

Note that the random variables $\rho_{k,n}$ is $\mathcal{F}^{n}_{k-1}$-measurable. If condition (a) of Theorem 5.1 holds, then equality (20) defines a martingale, indeed.

Next we check the condition $\Delta M^{(n)}_{k} > -1$. The following relation is proved in [5]:

$$\begin{aligned} \left(\frac{1}{\sigma}\sqrt{\frac{T}{n}}\left(x^{(n)}_{0} - \mu\right) + 1\right)\left(1 - \frac{T}{n}\right)^{k} - 1 &\le \frac{\left(x^{(n)}_{k-1} - \mu\right)}{\sigma}\sqrt{\frac{T}{n}} \\ &\le \left(\frac{1}{\sigma}\sqrt{\frac{T}{n}}\left(x^{(n)}_{0} - \mu\right) - 1\right)\left(1 - \frac{T}{n}\right)^{k} + 1. \end{aligned} \tag{21}$$

Note that inequality (21) holds without any assumption on the independence. Using the latter inequality in condition (b) we simplify the inequalities on the left- and right-hand sides of (21) as follows:

$$\frac{\left(x^{(n)}_{k-1} - \mu\right)}{\sigma}\sqrt{\frac{T}{n}} \ge -1 + e^{-T} + O\left(n^{-1/2}\right) \tag{22}$$

and

$$\frac{\left(x^{(n)}_{k-1} - \mu\right)}{\sigma}\sqrt{\frac{T}{n}} \le 1 - e^{-T} + O\left(n^{-1/2}\right). \tag{23}$$

Then we use relations (22)–(23) to estimate the right-hand side of (19). For those elementary events ω, where $\mathbb{1}_{k,n,+} = 1$ and hence $\mathbb{1}_{k,n,-} = 0$, we obtain

(24)
$$\begin{aligned}\Delta M_k^{(n)} &= \frac{r_k^{(n)} - h_{k,n}^+ \,\mathsf{P}_{k,n}^+ - h_{k,n}^- \,\mathsf{P}_{k,n}^-}{2\sigma\sqrt{\frac{T}{n}}\,\mathsf{P}_{k,n}^+} \\ &= \frac{\left(x_{k-1}^{(n)} - \mu\right)}{2\sigma\,\mathsf{P}_{k,n}^+}\sqrt{\frac{T}{n}} + \frac{r_k^{(n)}}{2\sigma\,\mathsf{P}_{k,n}^+\sqrt{\frac{T}{n}}} + \frac{1 - 2\,\mathsf{P}_{k,n}^+}{2\,\mathsf{P}_{k,n}^+}.\end{aligned}$$

By assumption (b), we have $2\,\mathsf{P}_{k,n}^+ = 1 + O(n^{-1/2})$ and $O(n^{-1/2})$ is estimated by $\frac{C}{n^{1/2}}$ with a constant C that does not depend on both k and n. Thus,

$$\frac{1}{2\,\mathsf{P}_{k,n}^+} = 1 + O\left(n^{-1/2}\right),$$

where the latter term $O(n^{-1/2})$ is also bounded by $C/n^{1/2}$ with the constant that does not depend on both k and n. Then inequality (22) implies

(25)
$$\frac{(x_{k-1}^{(n)} - \mu)}{2\sigma\,\mathsf{P}_{k,n}^+}\sqrt{\frac{T}{n}} \geq -1 + e^{-T} + O\left(n^{-1/2}\right).$$

The second and third terms on the right-hand side of (24) are bounded by $O(n^{-1/2})$. For those elementary events ω where $\mathbb{1}_{k,n,-} = 1$ and correspondingly $\mathbb{1}_{k,n,+} = 0$, the transformations and reasoning are the same. Therefore, there exists a number n_0 for which the market is arbitrage free and complete for $n > n_0$.

(ii) If condition (ii) holds, then the process $M_k^{(n)}$, $1 \leq k \leq n$, is not integrable and thus is not a martingale. On the other hand, the preceding reasoning makes it clear that there is no other martingales that generate martingale measures if $2\,\mathsf{P}_{k,n}^{\pm} > 0$ with probability one. Therefore, the market in not arbitrage free and the question on its completeness is not discussed at all.

(iii) (c) If equality (13) holds in the set $A_{k,n}^+$ provided that $\mathsf{P}_{k,n}^+ = 0$ with positive probability, then let $f\left(\overline{q}_{k-1}^{(n)}, \sqrt{T/n}\right)$ be equal to an arbitrary constant and let $f\left(\overline{q}_{k-1}^{(n)}, -\sqrt{T/n}\right)$ be equal to zero on this set. Thus, equalities (18) hold on this set and one can choose $\Delta M_k^{(n)}$ to be equal to an arbitrary constant on this set. For the complement of $A_{k,n}^+$, we repeat the same reasoning as that used for the case of (i) and obtain an arbitrage free and incomplete market. We follow a similar approach in the case where $\mathsf{P}_{k,n}^- = 0$ with positive probability.

(d) If equality (13) does not hold in the set $A_{k,n}^+$, provided that $\mathsf{P}_{k,n}^+ = 0$ with positive probability, or if equality (14) does not hold in the set $A_{k,n}^-$, provided that $\mathsf{P}_{k,n}^- = 0$ with positive probability, then equality (18) does not hold on these sets, that is, $\Delta M_k^{(n)}$ cannot be defined on these sets and thus the market is not arbitrage free.

The theorem is proved. □

Now we assume that condition (i) of Theorem 5.1 holds, that is, the market is arbitrage free and complete. We are going to find sufficient conditions that random variables $\{q_k^{(n)}, 1 \leq k \leq n\}$ are independent symmetric identically distributed with respect to the unique martingale measure P_n^*.

We introduce the notation for the set of all possible values of the families of random variables $\{q_k^{(n)}, 1 \le k \le n\}$: $\Xi = \{\xi = \sqrt{T/n}(\pm 1, \dots, \pm 1)\}$. Let $\omega(\xi)$ be the elementary events for which a family $\{q_k^{(n)}, 1 \le k \le n\}$ assumes value ξ and denote the probability of every family with this property with respect to the objective measure by $\mathsf{P}_n(\xi)$. Finally, denote by $\overline{q}^{(n)}$ the family of random variables $\{q_k^{(n)}, 1 \le k \le n\}$.

Lemma 5.1. *If, for every $\omega(\xi)$,*

$$\prod_{k=1}^{n} \left(1 + \Delta M_k^{(n)}(\omega(\xi))\right) \mathsf{P}_n(\xi) = 2^{-n}, \tag{26}$$

then the random variables $\{q_k^{(n)}, 1 \le k \le n\}$ are independent symmetric and identically distributed with respect to the martingale measure P_n^*.

Proof. We have $\mathsf{P}_n^*(\xi) = 2^{-n}$ with respect to the martingale measure and for every family ξ, or $\mathsf{P}_n\left(\prod_{k=1}^n \left(1 + \Delta M_k^{(n)}\right) \mathbb{1}_{\omega(\xi)}\right) = 2^{-n}$, whence the lemma follows. $\square$

Remark 5.1. Equalities (26) may simultaneously hold, that is, they do not contradict each to other. This becomes obvious if one interchanges the measures. More precisely, define a measure Q_n for which a family of independent symmetric identically distributed random variables $\{q_k^{(n)}, 1 \le k \le n\}$, exists. Then determine increments $\Delta M_k^{(n)}$ of the martingale with respect to the measure Q_n in the form of $\Delta M_k^{(n)} = \rho_k^{(n)} q_k^{(n)}$ and such that

$$\mathsf{E}_{\mathsf{Q}_n}\left(\left(1 + R_k^{(n)}\right)\left(1 + \rho_k^{(n)} q_k^{(n)}\right) \,\Big|\, \mathcal{F}_{k-1}^n\right) = 1 + r_k^{(n)}.$$

Finally, put $\mathsf{P}_n^* = \mathsf{Q}_n$ and choose P_n to be the measure whose Radon–Nikodym derivative is given by

$$\frac{d\mathsf{Q}_n}{d\,\mathsf{P}_n} = \prod_{k=1}^{n}\left(1 + \Delta M_k^{(n)}\right) = \prod_{k=1}^{n}\left(1 + \rho_k^{(n)} q_k^{(n)}\right).$$

The following result follows directly from Theorems 4.1 and 5.1 and Lemma 5.1.

Theorem 5.2. *Assume that*

(i) *there exists a constant $C > 0$ such that*

$$\left|x_0^{(n)} - x_0\right| \le \frac{C}{n^{1/2}};$$

(ii) *conditions (a) and (b) of the case (i) in Theorem* 5.1 *hold as well as all assumptions of Lemma* 5.1.

Then

$$\left|\pi^*(\mathbf{D}) - \pi^*(\mathbf{D}_n)\right| \le \frac{C_1}{n^{1/2}}$$

starting with some $n_0 > T$ for some $C_1 > 0$ and $\mathbf{D} = \mathbf{C}, \mathbf{P}$.

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