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PROPERTIES OF THE STOCHASTIC ORDERING FOR DISCRETE DISTRIBUTIONS AND THEIR APPLICATIONS TO THE RENEWAL SEQUENCE GENERATED BY A NONHOMOGENEOUS MARKOV CHAIN

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Abstract. The generalized stochastic ordering is studied for which the dominating sequence is not necessarily a probability distribution so that its total mass may exceed unity. We study the stochastic ordering for sums as well as random sums of independent as well as dependent random variables. A stochastic ordering is constructed for the renewal sequence generated by a nonhomogeneous Markov chain. The consideration is restricted to the case of discrete random variables.

1. Introduction

In this paper, some properties of the stochastic ordering are considered in a somewhat generalized sense for discrete distributions. The classical definition of the stochastic ordering reads as follows.

We say that a random variable ξ stochastically dominates a random variable η if $\mathsf{P}\{\xi > x\} \geq \mathsf{P}\{\eta > x\}$ for all x. In other words, a random variable ξ stochastically dominates a random variable η if the distribution function of ξ does not exceed the distribution function of η pointwise.

This definition for nonnegative discrete random variables, that are the main object of this paper, can be reformulated as follows: let the distribution of ξ be $\{p_i\}$ and that of η be $\{g_i\}$. Then ξ stochastically dominates η if

$$\sum_{k \geq n} p_k \geq \sum_{k \geq n} g_k$$

for all n.

The notion of the stochastic ordering is useful in various applications, in particular in finance mathematics and mathematical economics where it is a way to compare different types of lotteries.

Some related questions concerning the stochastic ordering are studied in the paper [11] for some discrete distributions such as the Bernoulli distribution. Some typical methods used to prove the defining property of the stochastic ordering are also discussed in [11].

The stochastic ordering plays an important role in the analysis of nonhomogeneous Markov chains as well as of renewal sequences generated by these chains. If θ_k denotes the time between two visits of a certain set by a Markov chain, then this sequence is

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not homogeneous for nonhomogeneous chains. Moreover the distribution of θ_k depends on the sum $\sum_{j=0}^{k-1}\theta_j$ (more detail is given in [4]). The stochastic ordering is used to treat such sequences. In particular, the stochastic ordering is used in [4] as a condition for the existence of the expectation for the coupling time. Other applications of the stochastic ordering are described in [5] (for the generalization of Daley's inequality in the nonhomogeneous case) and in [6] (for estimating the expectation of the coupling time). Further, the stochastic ordering plays the key role for constructing bounds for the stability of perturbed nonhomogeneous Markov chain if the conditions on the uniform ergodicity does not hold (see [7]). Note that a bound of such a stability can be obtained in a simpler way under the condition of the uniform ergodicity (see [8–10]).

When analyzing Markov chains, it is often convenient to weaken the defining property of the stochastic ordering by abandoning the condition that a dominating sequence is a probability distribution and by allowing the total mass to exceed 1 (see, for example, [6]). By the way, such a weaker property is used in the papers [4–7] mentioned above.

Clearly, if the total mass of a dominating sequence exceeds 1, then one cannot speak about a random variable that stochastically dominates another one. Instead, one can speak about a sequence that dominates the distribution of a random variable.

It turns out that the well-known properties of the stochastic ordering require a separate proof if the total mass of a dominating sequence exceeds 1 (recall that a dominating sequence is a probability distribution in the classical case).

In addition, a stochastic dominating sequence for sums of a random number of dependent random variables appeared naturally in [7] in the studies of the stability of perturbed Markov chains (the random variables constitute the renewal sequence for some nonhomogeneous Markov chain).

The aim of the current paper is to study the questions described above. The paper is organized as follows. Section 2 contains the main definitions and introduces the notation of the stochastic ordering in the discrete case used throughout the paper. Some properties of the stochastic ordering are proved in Section 3, namely we find a dominating sequence for a sum of two independent random variables and for the sum of the random number of independent random variables.

Some questions concerning the stochastic order for the renewal sequence constructed from a nonhomogeneous Markov chain are considered in Section 4.

2. Main notation

We say that a sequence $\{s_n, n\geq 0\}$ stochastically dominates a probability distribution $\{g_n, n\geq 0\}$ if $\sum_{k>n} s_k \geq \sum_{k>n} g_k$.

Throughout the paper, the tails of sequences are denoted by capital letters, for example,

$$S_n = \sum_{k>n} s_k, \qquad G_n = \sum_{k>n} g_k$$

for $n\geq -1$, where S_{-1} or G_{-1} denotes the total sum of elements of the corresponding sequence.

The dominating sequence $\{s_n, n\geq 0\}$ is assumed to be such that

$$1 \leq \sum_{n=0}^{\infty} s_n < \infty.$$

Given two summable sequences $\{g_n, n \geq 0\}$ and $\{p_n, n \geq 0\}$ we define their convolution by the following equality:

$$(g \star p)_n = \sum_{k=0}^{n} g_k p_{n-k}.$$

Then the convolution of a sequence with itself is given by

$$g_n^{\star 2} = \sum_{k=0}^{n} g_k g_{n-k}.$$

The convolution of order m of a sequence with itself is defined recursively,

$$g_n^{\star m} = (g^{\star m-1} \star g)_n.$$

The symbol $G_n^{\star 2}$ stands for the tail of the convolution,

$$G_n^{\star 2} = \sum_{k>n} g_k^{\star 2}.$$

Note that the nth element of the convolution of a sequence $\{g_k\}$ with identity is equal to the sum of elements of this sequence whose indices run from 0 to n,

$$(g \star 1)_n = \sum_{k=0}^{n} g_k.$$

3. Stochastic dominating sequence for sums of random variables

Below we prove that the convolution of two dominating sequences for two random variables is a dominating sequence of the sum of the random variables. The result below is stated for sequences whose total sum may exceed 1.

The cases of nonnegative and sign alternating random variables are considered separately, since the proofs are different. However, we are mainly interested in the case of nonnegative random variables in the rest of the paper. Several formulas obtained in the course of the proof will be useful for further reference.

Theorem 3.1. *Consider the following four nonnnegative sequences:*

$$\{s_n, n \geq 0\}, \qquad \{r_n, n \geq 0\}, \qquad \{g_n, n \geq 0\}, \qquad \{p_n, n \geq 0\},$$

such that

$$1 \leq \sum_{k=0}^{\infty} s_n = S, \qquad 1 \leq \sum_{k=0}^{\infty} r_n = R < \infty,$$

$$1 \leq \sum_{k=0}^{\infty} g_n = G < \infty, \qquad 1 \leq \sum_{k=0}^{\infty} p_n = P < \infty.$$

We assume that, for every $n \geq -1$,

$$S_n = \sum_{k>n} s_n \geq G_n = \sum_{k>n} g_n,$$

$$R_n = \sum_{k>n} r_n \geq P_n = \sum_{k>n} p_n.$$

Then the convolution $\{(s \star r)_n, n \geq 0\}$ is a dominating sequence for the convolution $\{(g \star p)_n, n \geq 0\}$. In other words,

$$\sum_{k>n} (s \star r)_k \geq \sum_{k>n} (g \star p)_k$$

for all $n \geq -1$.

Proof. Applying Lemma 3.1 for the difference

$$\sum_{k>n}\sum_{j=0}^{k}(s\star r)_j - \sum_{k>n}\sum_{j=0}^{k}(g\star p)_j,$$

we obtain

$$\sum_{k>n}(s\star r)_k - \sum_{k>n}(g\star p)_k = \sum_{k=0}^{n} s_k R_{n-k} - \sum_{k=0}^{n} g_k P_{n-k} + S_n R - G_n P. \tag{1}$$

By condition, $R_{n-k} \ge P_{n-k}$, whence

$$\begin{aligned}
\sum_{k=0}^{n} s_k R_{n-k} - \sum_{k=0}^{n} g_k P_{n-k} + S_n R - G_n P &\ge \sum_{k=0}^{n}(s_k - g_k)P_{n-k} + S_n R - G_n P \\
&= \sum_{k=0}^{n}\big(S_{k-1} - S_k - (G_{k-1} - G_k)\big)P_{n-k} + S_n R - G_n P \\
&= \sum_{k=0}^{n}(S_{k-1} - G_{k-1})P_{n-k} - \sum_{k=0}^{n}(S_k - G_k)P_{n-k} + S_n R - G_n P \\
&= (S_{-1} - G_{-1})P_n + \sum_{k=1}^{n-1}(S_k - G_k)(P_{n-k} - P_{n-k+1}) - (S_n - G_n)P_0 + S_n R - G_n P.
\end{aligned}$$

Note that $S_{-1} = S \ge G = G_{-1}$ and $R > P$. Thus

$$\begin{aligned}
(S_{-1} - G_{-1})P_n &+ \sum_{k=1}^{n-1}(S_k - G_k)(P_{n-k} - P_{n-k+1}) - (S_n - G_n)P_0 + S_n R - G_n P \\
&\ge (S - G)P_n + \sum_{k=1}^{n-1}(S_k - G_k)p_{n-k} - (S_n - G_n)P_0 + S_n P - G_n P \\
&= (S - G)P_n + \sum_{k=1}^{n-1}(S_k - G_k)p_{n-k} + (S_n - G_n)(P - P_0) \\
&= (S - G)P_n + (S_n - G_n)p_0 + \sum_{k=1}^{n-1}(S_k - G_k)p_{n-k} \ge 0,
\end{aligned}$$

since $S_n \ge G_n$ for all $n \ge 0$. Hence both the second and third terms are nonnegative. The same result for the first term follows from the inequality $S \ge G$. We complete the proof by substituting the results obtained into equality (1). □

A similar result is valid for two-sided sequences, as well.

Theorem 3.2. *Consider the following four nonnegative two sided sequences:*

$$\{s_n, -\infty < n < \infty\}, \qquad \{r_n, -\infty < n < \infty\},$$
$$\{g_n, -\infty < n < \infty\}, \qquad \{p_n, -\infty < n < \infty\},$$

such that

$$1 \le \sum_{k=-\infty}^{\infty} s_n = S < \infty, \qquad 1 \le \sum_{k=-\infty}^{\infty} r_n = R < \infty,$$
$$1 \le \sum_{k=-\infty}^{\infty} g_n = G < \infty, \qquad 1 \le \sum_{k=-\infty}^{\infty} p_n = P < \infty.$$

Assume that, for all $n \geq -\infty$,

$$S_n = \sum_{k>n} s_n \geq G_n = \sum_{k>n} g_n,$$
$$R_n = \sum_{k>n} r_n \geq P_n = \sum_{k>n} p_n.$$

Then the convolution $\{(s \star r)_n, n \geq 0\}$ is a dominating sequence for the convolution $\{(g \star p)_n, n \geq 0\}$. In other words,

$$\sum_{k>n} (s \star r)_k \geq \sum_{k>n} (g \star p)_k$$

for all $n \geq -\infty$.

Proof. First we rewrite the formula for the convolution,

$$\sum_{k>n} (s \star r)_k = \sum_{k>n} \sum_{j=-\infty}^{\infty} s_j r_{n-j} = \sum_{j=-\infty}^{\infty} s_j R_{n-j}. \tag{2}$$

Then

$$\begin{aligned}\sum_{k>n} (s \star r)_k - \sum_{k>n} (g \star p)_k &= \sum_{j=-\infty}^{\infty} s_j R_{n-j} - \sum_{j=-\infty}^{\infty} g_j P_{n-j} \\ &= \lim_{N\to\infty} \left(\sum_{j=-N}^{N} (s_j R_{n-j} - g_j P_{n-j}) \right).\end{aligned}$$

Since $R_{n-j} \geq P_{n-j}$, the latter expression does not exceed

$$\begin{aligned}&\lim_{N\to\infty} \left(\sum_{j=-N}^{N} (s_j - g_j) P_{n-j} \right) \\ &\quad = \lim_{N\to\infty} \left(\sum_{j=-N}^{N} \big(S_{j-1} - S_j - (G_{j-1} - G_j)\big) P_{n-j} \right) \\ &\quad = \lim_{N\to\infty} \left(\sum_{j=-N}^{N} (S_{j-1} - G_{j-1}) P_{n-j} - \sum_{j=-N}^{N} (S_j - G_j) P_{n-j} \right) \\ &\quad = \lim_{N\to\infty} \left(\sum_{j=-N+1}^{N-1} (S_j - G_j)(P_{n-j} - P_{n-j+1}) \right) \\ &\qquad + \lim_{N\to\infty} \big((S_{-N-1} - G_{-N-1}) P_{-N-j} - (S_N - G_N) P_{N-j} \big) \\ &\quad = \sum_{j=-\infty}^{\infty} (S_j - G_j) p_{n-j} + (S_{-\infty} - G_{-\infty}) P_{-\infty} - (S_\infty - G_\infty) P_\infty \\ &\quad = \sum_{j=-\infty}^{\infty} (S_j - G_j) p_{n-j} + (S - G)P - 0 \geq 0.\end{aligned}$$

The result just obtained is valid, since all the terms are nonnegative. □

Corollary 3.1. *If two independent discrete random variables ξ and η are dominated by the sequences $\{s_n\}$ and $\{r_n\}$, respectively, then their sum $\xi + \eta$ is dominated by the convolution $(s \star r)_n$.*

Corollary 3.2. *Consider a sequence of independent random variables* ξ_n*,* $n \geq 0$*. Assume that each random variable* ξ_i *is dominated by a sequence* $\{s_n^{(i)}\}$*. Then the sum* $\sum_{j=0}^n \xi_j$ *is dominated by the convolution* $\left(s^{(0)} \star s^{(1)} \star \cdots \star s^{(n)}\right)$*,* $n \geq 1$*.*

Lemma 3.1. *Let* $\{a_n, n \geq 0\}$ *and* $\{b_n, n \geq 0\}$ *be two summable nonnegative sequences. Put* $A_n = \sum_{k>n} a_k$*,* $B_n = \sum_{k>n} b_k$*, and*

$$A = A_{-1} = \sum_{k=0}^{\infty} a_n, \qquad B = B_{-1} = \sum_{k=0}^{\infty} b_n.$$

Then

$$\sum_{k>n} (a \star b)_k = \sum_{k=0}^{n} a_k B_{n-k} + A_n B = \sum_{k=0}^{n+1} a_k B_{n-k} + A_{n+1} B. \tag{3}$$

Proof. We have

$$\begin{aligned} \sum_{k>n} (a \star b)_k &= \sum_{k=n+1}^{\infty} \sum_{j=0}^{k} a_j b_{k-j} = \sum_{k=0}^{n+1} a_k \sum_{j=n+1-k}^{\infty} b_j + \sum_{k=n+2}^{\infty} a_k \sum_{j=0}^{\infty} b_j \\ &= \sum_{k=0}^{n+1} a_k B_{n-k} + \sum_{k>n+1} a_k B = \sum_{k=0}^{n+1} a_k B_{n-k} + A_{n+1} B. \end{aligned}$$

This proves the second equality in (3).

To prove the first equality in (3), note that the last term in the sum $\sum\limits_{k=0}^{n+1} a_k B_{n-k}$ is equal to

$$a_{n+1} B_{n-(n+1)} = a_{n+1} B_{-1} = a_{n+1} B.$$

Substituting this expression into the formula just obtained, we conclude that

$$\begin{aligned} \sum_{k>n} (a \star b)_k &= \sum_{k=0}^{n+1} a_k B_{n-k} + A_{n+1} B = \sum_{k=0}^{n} a_k B_{n-k} + a_{n+1} B + A_{n+1} B \\ &= \sum_{k>n} a_k B_{n-k} + (a_{n+1} + A_{n+1}) B = \sum_{k>n} a_k B_{n-k} A_n B. \end{aligned}$$

This proves the first equality in (3). □

4. Stochastically dominating sequence for random sums

In this section, we construct a dominating sequence for random sums of random variables. The random variables that constitute the sum are not necessarily assumed to be independent. Note that random sums appear naturally when dealing with renewal sequences generated by nonhomogeneous Markov chains (more detail is given in Section 5).

Let $(\mathcal{U}, \mathbb{U})$ be a measurable space and let $\xi(u)$, $u \in \mathcal{U}$, be a family of independent random variables whose indices belong to $\mathcal{U}$. Further let

$$\{\nu_n, n \geq 0\}$$

be a sequence of random variables assuming values in the space $(\mathcal{U}, \mathbb{U})$ and such that $\xi(u)$ does not depend on ν_n for all n and u.

Put

$$\xi_n = \xi(\nu_n).$$

Note that the random variables in the family $\{\xi_i, i \geq 0\}$ as well as in the sequence $\{\nu_n, n \geq 0\}$ are not supposed to be independent. Moreover, the random variables in the sequence $\{\xi_i, i \ge 0\}$ are not supposed to be independent of the random variables in the sequence $\{\nu_n, n \ge 0\}$.

Let ζ be a random variable that does not depend on every one of the random variables ξ_i and ν_n. Denote

$$p_m = \mathsf{P}\{\zeta = m\}.$$

Our aim is to construct a dominating sequence for the sum

$$\sum_{k=0}^{\zeta} \xi_k. \tag{4}$$

We assume that all random variables are defined on a common probability space $(\Omega, \mathbb{F}, \mathsf{P})$. The symbol E stands in the sequel for the mean value with respect to the probability measure P.

Theorem 4.1. *Let $\{s_n(u), n \ge 0\}$ be a sequence that dominates a random variable $\xi(u)$, $u \in \mathfrak{U}$. We further assume that*

$$\sup_{u\in\mathfrak{U}} \sum_{j\ge 0} s_j(u) < \infty. \tag{5}$$

Then the sequence

$$\hat{s}_n = \mathsf{E}\left[\big(s(\nu_0) \star s(\nu_1) \star \cdots \star s(\nu_\zeta)\big)_n\right] \tag{6}$$

stochastically dominates random sum (4).

Proof. Consider the sum

$$S = \sum_{k=0}^{\zeta} \xi_k = \sum_{k=0}^{\zeta} \xi(\nu_k).$$

Let $\hat{\imath}(m) = (\hat{\imath}_0, \hat{\imath}_1, \ldots, \hat{\imath}_m)$ be a vector of dimension $m+1$ whose coordinates are integer. Consider the set

$$A\big(\hat{\imath}(m)\big) = \{\nu_0 = \hat{\imath}_0, \nu_1 = \hat{\imath}_1, \ldots, \nu_m = \hat{\imath}_m\}.$$

Then

$$S = \sum_{m=0}^{\infty} \Bigg(\sum_{\hat{\imath}(m)} S 1\!\!\!\!\not\,\,_{A(\hat{\imath}(m))}\Bigg) p_m = \sum_{m=0}^{\infty} \Bigg(\sum_{\hat{\imath}(m)} \Bigg(\sum_{k=0}^{m} \xi(\hat{\imath}_k)\Bigg) 1\!\!\!\!\not\,\,_{A(\hat{\imath}(m))}\Bigg) p_m.$$

Hence

$$\mathsf{P}\{S > n\} = \sum_m \sum_{\hat{\imath}(m)} \mathsf{P}\big\{S > n \;\big|\; \zeta = m, A\big(\hat{\imath}(m)\big)\big\}\, \mathsf{P}\big\{A\big(\hat{\imath}(m)\big), \zeta = m\big\}. \tag{7}$$

Note that ζ and random variables ν_i are independent, whence we conclude that

$$\mathsf{P}\left\{\zeta = m, A\big(\hat{\imath}(m)\big)\right\} = \mathsf{P}\{\zeta = m\}\, \mathsf{P}\left\{A\big(\hat{\imath}(m)\big)\right\} = \mathsf{P}\left\{A\big(\hat{\imath}(m)\big)\right\} p_m. \tag{8}$$

Moreover, the random variable $\xi(u)$ (u is fixed) is independent of ζ and of all ν_n. Thus $\xi(u)$ for a fixed u is independent of $A(\hat{\imath}(m))$. Therefore

$$\begin{aligned} \mathsf{P}\left\{S > n \mid \zeta = m, A\big(\hat{\imath}(m)\big)\right\} &= \mathsf{P}\Bigg\{\sum_{k=0}^{m} \xi\big(\hat{\imath}_k(m)\big) > n \;\Bigg|\; \zeta = m, A\big(\hat{\imath}(m)\big)\Bigg\} \\ &= \mathsf{P}\Bigg\{\sum_{k=0}^{m} \xi\big(\hat{\imath}_k(m)\big) > n\Bigg\}. \end{aligned} \tag{9}$$

Substituting (8) and (9) into (7), we obtain

$$\mathsf{P}\{S > n\} = \sum_{m=0}^{\infty} \sum_{\hat{\imath}(m)} \mathsf{P}\Big\{\sum_{k=0}^{m} \xi\big(\hat{\imath}_k(m)\big) > n\Big\} \mathsf{P}\left\{A\big(\hat{\imath}(m)\big)\right\} p_m. \tag{10}$$

On the other hand, $\xi(\hat{\imath}_k)$ are independent random variables and the theorem on the dominating sequence of a sum of independent random variables implies that

$$\mathsf{P}\Big\{\sum_{k=0}^{m} \xi\big(\hat{\imath}_k(m)\big) > n\Big\} \le \sum_{j>n} \big(s(\hat{\imath}_0) \star \cdots \star s(\hat{\imath}_m)\big)_j. \tag{11}$$

Substituting (11) into (10), we get

$$\begin{aligned}
\mathsf{P}\{S > n\} &\le \sum_{m\ge 0} \Bigg[\sum_{\hat{\imath}(m)} \Bigg(\sum_{j>n} \big(s(\hat{\imath}_0) \star \cdots \star s(\hat{\imath}_m)\big)_j\Bigg) \mathsf{P}\left\{A\big(\hat{\imath}(m)\big)\right\}\Bigg] p_m \\
&= \sum_{m\ge 0} \Bigg[\sum_{j>n} \Bigg(\sum_{\hat{\imath}(m)} \big(s(\hat{\imath}_0) \star \cdots \star s(\hat{\imath}_m)\big)_j \mathsf{P}\left\{A\big(\hat{\imath}(m)\big)\right\}\Bigg)\Bigg] p_m \\
&= \sum_{j>n} \Bigg[\sum_{m\ge 0} \Bigg(\sum_{\hat{\imath}(m)} \big(s(\hat{\imath}_0) \star \cdots \star s(\hat{\imath}_m)\big)_j \mathsf{P}\left\{A\big(\hat{\imath}(m)\big)\right\}\Bigg) p_m\Bigg] \\
&= \sum_{j>n} \Bigg(\sum_{m\ge 0} \mathsf{E}\left[\big(s(\nu_0) \star \cdots \star s(\nu_m)\big)_j \mid \zeta = m\right] p_m\Bigg) \\
&= \sum_{j>n} \mathsf{E}\left[\big(s(\nu_0) \star \cdots \star s(\nu_m)\big)_j\right].
\end{aligned}$$

This proves Theorem 4.1.

Note that the change of order of summation is justified by condition (5). □

The following result is an obvious corollary of Theorem 4.1.

Theorem 4.2. *Let ξ_n, $n \ge 0$, be a sequence of independent discrete random variables. Let ζ be another nonnegative discrete random variable being independent of all $(\xi_n, n \ge 0)$. The distribution of ζ is denoted by $p_n = \mathsf{P}\{\zeta = n\}$, $n \ge 0$.*

Further let ξ_i be stochastically dominated by the sequence $\{s_n^{(i)}\}$. Then $\sum_{j=0}^{\zeta} \xi_j$ is stochastically dominated by the sequence

$$s_n = \sum_{k=0}^{\infty} p_k \left(s^{(0)} \star \cdots \star s^{(k)}\right)_n.$$

Proof. The result follows from Theorem 4.1 with

$$\mathcal{U} = \{0, 1, 2, \dots\}$$

and $\nu_n = n$. □

5. Stochastically dominating sequence of a renewal sequence generated by a nonhomogeneous Markov chain

The method described in the preceding section is useful for constructing a coupling of two different nonhomogeneous Markov chains. In doing so one needs to analyze the renewal sequence generated by a nonhomogeneous Markov chain and to find a stochastic dominating sequence for a random number of renewals.

We construct the renewal sequence mentioned above and show how Theorem 4.1 is used to find a stochastically dominating sequence for a random number of renewals.

Let a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ be given. All random variables considered below are assumed to be defined on this probability space.

Consider some time nonhomogeneous Markov chain X_n assuming values in a measurable space $(E, \mathcal{E})$. Let $P_t(x, A)$ denote its transition probability at step t. The initial distribution of the chain is denoted by μ_0.

Also let $\mu_{t,x}(\cdot)$, $x \in E$, $t > 0$, be a family of probability measures.

Let $C \in \mathcal{E}$ be some set. The renewal sequence generated by a nonhomogeneous chain X_n is constructed as follows. First we put

$$\theta_0 = \inf_t \{t > 0\colon X_t \in C\}.$$

Now we define the random variable θ_1. Consider a Markov chain $X_t^{(1)} = X_{\theta_0 + t}$, $t \geq 0$, with the initial distribution $\mu_{\theta_0, X_{\theta_0}}(\cdot)$.

Then

$$\theta_1 = \inf_t \left\{t > 0\colon X_t^{(1)} \in C\right\}.$$

Put

$$\tau_k = \sum_{j=0}^{k} \theta_k, \qquad k \geq 0.$$

The chains $X_n^{(k)} = X_{\tau_{k-1}+t}^{(k-1)}$, $k \geq 1$, with the initial distribution $\mu_{\tau_{k-1}, X_{\theta_{k-1}}^{(k-1)}}$ are defined similarly.

Let $s_n^{(t)}(x)$ be a stochastically dominating sequence for the chain that starts at moment t from the state x. Put

$$s_n^{(t)}(\mu, x) = \int_E s^{(t)}(y)\, \mu_{t,x}(dy).$$

Also let

$$S_n^{(t)}(x) = \sum_{k>n} s_k^{(t)}(x), \qquad S_n^{(t)}(\mu, x) = \sum_{k>n} s_k^{(t)}(\mu, x).$$

The sequence $\hat{s}_n^{(k)}(x)$, $n \geq 0$, is defined recursively for $k \geq 0$.

Let $\hat{s}_n^{(1)}(x) = s_n^{(0)}(x)$ and assume that the sequence $\hat{s}_n^{(k)}(x)$ is defined for some k. Then the sequence $\hat{s}_n^{(k+1)}(x)$ is defined by

$$\hat{s}_n^{(k+1)}(x) = \mathsf{E}\left[\hat{s}^{(k)}(x) \star s_{\mu, X_0^{(k+1)}}^{(\tau_k)}\right]. \tag{12}$$

With the notation introduced above we obtain the following result.

Theorem 5.1. *The sequence* $\left\{\hat{s}_n^{(k)}(x), n \geq 0\right\}$ *is a stochastically dominating sequence for the sum* $\sum_{j=0}^{k} \theta_j$ *provided the chain starts from the point* $x \in E$.

Proof. We apply Theorem 4.1 in the proof.

The set $\mathcal{U}$ is taken as the set of all possible pairs (t, x), $t \geq 0$, $x \in E$. Then $\nu_n = (t, x)$ if the nth visit to the set C happens at the moment t and the value of the chain at that time is $x \in C$.

Then the statement of Theorem 5.1 follows from Theorem 4.1. $\square$

6. An application

Consider an example of applications of the theorems proved in the preceding sections. The following coupling construction is introduced in the paper [7].

Let X_n, X'_n, $n \geq 0$, be two time nonhomogeneous Markov chains defined on a common probability space $(\Omega, \mathcal{E}, \mathsf{P})$. The chains assume values in a measurable space $(E, \mathcal{E})$. Let $P_t(x, A)$ and $P'_t(x, A)$ be the transient probability at step t for the chains X and X', respectively. Some conditions on the closeness of the transient probabilities are known from [7], however these conditions are not useful for our example.

Let $C \in \mathcal{E}$ be some set such that

$$\min\{P_t(x, A), P'_t(x, A)\} \geq \alpha\nu(A) \qquad \text{for all } x \in C,$$

where $\alpha \in (0, 1]$ is a constant and $\nu(\cdot)$ is a probability distribution.

The initail value of the stochastic process (Y_n, Y'_n) considered in [7] is such that

$$Y_0 = X_0, \qquad Y'_0 = X'_0.$$

If $(Y_n, Y'_n) \notin C \times C$, then $(Y_{n+1}, Y'_{n+1}) \sim (P_n(Y_n, \cdot), P'_n(Y'_n, \cdot))$. In other words, the distribution of the pair of processes (Y_n, Y'_n) is the same as that of (X_n, X'_N) up to the moment when the chain enters the set $C \times C$.

If $(Y_n, Y'_n) \in C \times C$, then the chains couple with probability α and $Y_{n+1} = Y'_{n+1}$ in what follows with a certain distribution; the processes do not couple with probability $1-\alpha$ and the distribution of the pair (Y_{n+1}, Y'_{n+1}) is $(P_\alpha(Y_n, \cdot), P'_\alpha(Y'_n, \cdot))$ in what follows, where P_α and P'_α are certain transient probabilities.

The stochastically dominating sequence for the coupling moment is an important tool when constructing a bound for the deviation between transient probabilities over n steps for the chains X and X'.

It is clear that if a stochastic dominating sequence is known for the moment when the pair (X_n, X'_n) enters the set $(C \times C)$, then a stochastically dominating sequence for the coupling moment is easy to construct by using Theorem 5.1. This theorem allows one not only to show that a stochastically dominating sequence exists for the coupling moment but also it provides some properties of that sequence under certain conditions. For example, one can estimate the mean value of the coupling time in terms of the mean value of the stochastically dominating sequence for the moment when the chain enters the set $C \times C$.

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